On the decomposition of excursions measures of processes whose generators have diffusion coefficients discontinuous at one point

Antoine Lejay[1] — Mathematical Institute (Oxford University)

Abstract: The coefficients in the decomposition of the excursions measure as convex combination of excursions measures of reflected processes are computed in order to characterize the discontinuity at one point of the diffusion coefficient. In some sense, this result extends to general diffusions a similar one for the skew Brownian motion, and we advocate it may be used in Monte Carlo methods for discontinuous media.

Keywords: scale function and speed measure, excursions theory, skew Brownian motion, Monte Carlo methods

AMS Classification: 60J60, (60J55) .

Financial support: This work has been partially supported by European Union's TMR Stochastic Analysis Network (grant reference 960075)

Published in *Markov Processes and Related Fields.* **8**:1, pp. 117-126, 2002

Archives, links & reviews:

- MR Number: MR1897608

[1]Current address: Projet OMEGA (INRIA Lorraine)
IECN, Campus scientifique
BP 239
54506 Vandœuvre-lès-Nancy CEDEX (France)
E-mail: `Antoine.Lejay@iecn.u-nancy.fr`

 11 pages

Introduction and motivations

Being related to diffusion's properties, second-order differential operators in divergence form or non-divergence form

$$\frac{1}{2}\rho_{i,j}(x)\frac{\partial}{\partial x_i}\left(a_{i,j}(x)\frac{\partial}{\partial x_j}\right)$$

are present in many modelisation problems. One may think for example to the Darcy law, which is satisfied by the pressure of a fluid in a porous media. However, discontinuities of the coefficients of such differential operators reflects the ones of the media, due for example to fissures or rock inclusions.

This causes great difficulties to implement Monte Carlo methods in which one should simulate processes generated by such differential operators. The standard algorithms to simulate solutions of Stochastic Differential Equations (SDE) requires that the coefficients of its infinitesimal generator are at least continuous. Besides, divergence form operators with a coefficient discontinuous at a hypersurface generates process which is solution of a SDE with a local time (See *e.g.*, [Lej00, chapter5] for the one-dimensional case). Being not absolutely continuous, the local time is an object hard to approximate numerically. Of course, one may think to simulate the trajectory up to it hit the discontinuity's hypersurface, and to start it afresh. But the trajectories of the particles are very irregular, and they cross the hypersurface in rapid succession. In one dimension, this problem is studied by the theory of excursions, initiated first by K. Itô. Although the zeros of the trajectories of a one-dimensional Brownian motion may not be ordered, the pieces of trajectories between two zeros with positive length may be described by an infinite measure, called the Itô's excursions measure.

Heuristically, the discontinuities may be seen as "permeable barriers" which change the probability that the particle goes to one side or to the other, though this description is mathematically a non-sense. In this article, Itô's excursions measures will be helpful to characterize the discontinuities of the coefficients of the differential operators generating a one-dimensional diffusion process. Alongside with a theoretical interest, it may lead to Monte Carlo methods for discontinuous media by modifying the one given by E. Hausenblas in [Hau99a, Hau99b]. In these articles, a multidimensional reflected Brownian motion is simulated by the following method: (a) The trajectory is simulated by whatever method up to it hit the boundary. (b) A parameter η is fixed. We wait for some small time during which the particle only does excursions of length smaller than η. (c) After this time, the particle starts an excursion whose length is at least η. (d) The position of the particle after a time η is simulated using the entrance law of the reflected Brownian motion, and we come back to the first step. In this method, the particle jumps both in space and time, to start away from the discontinuity's hypersurface. In some sense, this method gives a criteria to decided whether a particle is "really" on one side or the other of the discontinuity and has been used by in [CL01] to develop a Monte Carlo method for fissured media in which no grid is generated.

We give now a more precise description of our result. Let us consider a one-dimensional process whose infinitesimal generator have the form

$$A = \frac{\rho}{2}\frac{\mathrm{d}}{\mathrm{d}x}\left(a\frac{\mathrm{d}}{\mathrm{d}x}\right) + b\frac{\mathrm{d}}{\mathrm{d}x},$$

where ρ is continuous on $\mathbb{R} \setminus \{0\}$, and a is continuous with a continuous derivative on $\mathbb{R}\setminus\{0\}$. Away from 0, this process behaves as the one generated by the differential operator

$$\frac{a\rho}{2}\frac{\mathrm{d}^2}{\mathrm{d}x^2} + \left(b + \frac{\rho a'}{2}\right)\frac{\mathrm{d}}{\mathrm{d}x}.$$

However, these two processes have different behavior at 0. If a and ρ are constant on $\mathbb{R}_+$ and $\mathbb{R}_-$ and $a = 1/\rho$, $b = 0$, then the process generated by A is the skew Brownian motion (see [IM74, HS81]), and the discontinuity of a and ρ changes the probability that a particle whose trajectories are the ones of our process goes to one side or to the other side of 0. To be rigorous, the probability for an excursion to be positive or negative is just chosen at random using an independent Bernoulli random variable.

Unfortunately, the construction of the skew Brownian motion relies on the two important facts that the diffusion coefficients are piecewise constant and that the law of the Brownian motion is symmetric with respect to 0.

Another way to describe the skew Brownian motion X^α of parameter α is to express its Itô's excursions measure $\widehat{\mathbb{P}}^{\mathsf{X}^\alpha}$ as the sum

$$\widehat{\mathbb{P}}^{\mathsf{X}^\alpha} = \alpha\widehat{\mathbb{P}}^{\mathsf{B}^+} + (1-\alpha)\widehat{\mathbb{P}}^{\mathsf{B}^-} \tag{1}$$

of the Itô's excursions measures of the positive and negative reflected Brownian motions. In this case, the value of α may be simply computed using the fact that the distribution of the Brownian motion is symmetric with respect to 0. See [IM74, Problem 1, p. 115] for example. When a and ρ remains constant on $\mathbb{R}_+$ and $\mathbb{R}_-$ but $a \neq 1/\rho$, the Itô-Tanaka formula used as in [Ouk90] allows to identify such a process as a skew Brownian motion on which a piecewise linear function is applied. Once, the coefficient α may be easily identified.

The decomposition (1) may be extended to the process generated by A when the coefficients a and ρ are not constant on $\mathbb{R}_+$ and $\mathbb{R}_-$. This question may also be reversed: given a process killed when hitting zero, one may sometimes reconstruct a Feller process by this way. The article [V-A96] deals with the possible extensions depending on a parameter α of symmetric processes killed at 0.

In this article, we compute the value of α for diffusion processes whose generator has not necessarily piecewise constant nor symmetric coefficients.

As we have seen it, an identification of α could help to develop Monte Carlo methods. Furthermore, our result shows that the discontinuity acts in an opposite manner for divergence form operators and non-divergence form operators, and this may be interesting for interpreting physical phenomenons.

Finally, this result allows to construct a process whose Itô's excursions measure may be decomposed as in (1) for an arbitrary $\alpha \in (0, 1)$. For that, it is sufficient to multiply its speed measure and scale function by a constant on $\mathbb{R}_+$ and leaving them unchanged on $\mathbb{R}_-$.

The value of α is given by the first derivative at 0 of a function deduced from the 1-potential of the local time. Unfortunately, except in trivial cases, the value of α may not be computed directly. However, using some perturbation theory's results, one may give a good approximation of α when the coefficients have slow variations around the discontinuity point at 0. This approximation may be sufficient for numerical applications.

1 Decomposition of the excursions measures

In fact, we work on a slightly more general setting than the one given in the introduction. In dimension 1, the speed measure and the scale function allow to characterize a broad class of stochastic processes (see [IM74, Bre81], ... for example). Let us consider a process X with scale function $S(x)$ and its speed measure $m(\mathrm{d}x)$ having a density $V(x)$. We assume that

1. The process X a Feller, conservative process on $\mathbb{R}$.
2. The process is regular on $\mathbb{R}$, *i.e.,* the probability to reach any point starting from any given point is positive.
3. The functions S and V are defined on $\mathbb{R}$ and $V(x)$ is positive for any x in $\mathbb{R}$. We may also assume without loss of generality that $S(0) = 0$.
4. The scale function S is continuous on $\mathbb{R}$ with a continuous derivative on $\mathbb{R} \setminus \{0\}$.

5. The density V of the speed measure is continuous on $\mathbb{R}\setminus\{0\}$.

6. Both V and the derivative S' of S have a right and a left limit at 0.

7. The function V/S' is bounded. The boundedness of V/S' corresponds to some uniform ellipticity of the infinitesimal generator of X (See (8)).

A few examples of stochastic processes satisfying these conditions are given in Section 3.

It is well known that there exists a Brownian motion B and a random time change T such that $\mathsf{X}_t = S^{-1}(\mathsf{B}_{\mathsf{T}(t)})$ for any $t\geq 0$. Under our hypotheses, the function $t\mapsto \mathsf{T}(t)$ is continuous on $\mathbb{R}_+$ almost surely.

If Y is a Feller, conservative process on $\mathbb{R}$ for which $\{0\}$ is regular, the complementary set of the closure of $\{t\geq 0\,|\,\mathsf{Y}_t=0\}$ is a denumerable union $\cup_{n\in\mathbb{N}}\mathcal{B}_n$ of maximal open intervals. If (r,s) is such an interval, the piece of trajectory defined by $t\mapsto \mathsf{Y}_{t+r}$ for $t<r-s$ and $t\mapsto 0$ for $t\geq r-s$ is called an *excursion*. Let $\sigma_0(\mathsf{Y})=\inf\{t>0\,|\,\mathsf{Y}_t=0\}$ be the first hitting time of 0 for Y. The function $\psi(x)=\mathbb{E}_x\left[e^{-\sigma_0}\right]$ is the 1-potential of a unique continuous additive functional $\mathsf{L}(\mathsf{Y})$, *i.e.,* $\psi(x)=\mathbb{E}_x\int_0^{+\infty}e^{-t}\,\mathrm{d}\mathsf{L}_t(\mathsf{Y})$ for any x (see Theorem III.(3.7) and III.3.(c) in [Blu92] for example). This functional $\mathsf{L}(\mathsf{Y})$ is called the *local time* of Y. This allows us to construct (see Theorem III.(3.24) in [Blu92] for example) a σ-finite measure $\widehat{\mathbb{P}}^{\mathsf{Y}}$ on the set of excursions, called the *Itô's characteristic measure* or the *excursions measure*. This infinite measure gives the average number of excursions in a given (functional) set during time 0 and time 1 (see [IM74, Blu92], ...). If f is a measurable function on the space of excursions, that is the space of continuous functions starting at 0 and equal to 0 once they reach this point for the first time. Let Z be a progressively measurable, non-negative process. Let θ_t be the shift for the process Y, and G^{Y} the (random) set of left-end points of the $\mathcal{B}_n$, that is the times at which Y starts some excursion. Then

$$\mathbb{E}_0\left[\sum_{s\in G^{\mathsf{Y}}}\mathsf{Z}_s f(\mathsf{Y}_{\cdot\wedge\sigma_0}\circ\theta_s)\right]=\left(\int f\,\mathrm{d}\widehat{\mathbb{P}}^{\mathsf{Y}}\right)\mathbb{E}_0\left[\int_0^{+\infty}\mathsf{Z}_s\,\mathrm{d}\mathsf{L}_s(\mathsf{Y})\right]. \tag{2}$$

Now, let us consider the processes X^+ and X^- characterized by the scale function $S(x)$ and the density of the speed measure $V(x)$ respectively on $\mathbb{R}_+$ and $\mathbb{R}_-$ instantaneously reflected at 0, *i.e.,* $m(\{0\})=0$. Then, there exists two Brownian motions B^+ and B^- reflected respectively on $\mathbb{R}_+$ and $\mathbb{R}_-$ and two continuous time changes T^+ and T^- such that $\mathsf{X}^\pm_t=S^{-1}(\mathsf{B}^\pm_{\mathsf{T}^\pm(t)})$.

The Itô's excursions measures for the strong Markov processes X, X^+ and X^- given by the previous construction are denoted by $\widehat{\mathbb{P}}^{\mathsf{X}}$, $\widehat{\mathbb{P}}^{\mathsf{X}^+}$ and $\widehat{\mathbb{P}}^{\mathsf{X}^-}$. The

Itô's excursions measure of X may be decomposed as

$$\widehat{\mathbb{P}}^{\mathsf{X}} = \alpha \widehat{\mathbb{P}}^{\mathsf{X}^+} + (1-\alpha)\widehat{\mathbb{P}}^{\mathsf{X}^-}. \tag{3}$$

Let σ_x be the first hitting time of some point x. Now, let us denote by $\psi_\pm(x) = \mathbb{E}_x\left[e^{-\sigma_0(\mathsf{X}^\pm)}\right]$ the 1-potential giving rise to the local time $\mathsf{L}(\mathsf{X}^\pm)$ at 0. Let also $\phi_\pm(x) = \mathbb{E}_x\left[e^{-\sigma_0(\mathsf{B}^\pm_{\mathsf{T}^\pm})}\right]$. We remark that $\psi_\pm(x) = \phi_\pm(S(x))$.

Let us also denote by C^+ and C^- the following values:

$$C^\pm = \lim_{x\to 0\pm} \frac{S(x)}{1-\psi_\pm(x)} = \frac{-1}{\phi'_\pm(0)}. \tag{4}$$

The following theorem gives the value of α.

Theorem 1. *The value of α is*

$$\alpha = \frac{-C^-}{C^+ - C^-}.$$

In fact, we may admit in the proof that both V and S' may be discontinuous at other points than 0, but 0 shall not be an accumulation point of discontinuities, to ensure the existence of $\phi'_\pm(0)$.

The proof of Theorem 1 relies on two Lemmas.

Lemma 1. *The constants C^+ and C^- defined by* (4) *are also equal to*

$$S(x)\widehat{\mathbb{P}}^{\mathsf{X}^+}[\,\sigma_x < \sigma_0\,] = C^+ \text{ for any } x > 0, \tag{5}$$

$$S(x)\widehat{\mathbb{P}}^{\mathsf{X}^-}[\,\sigma_x < \sigma_0\,] = C^- \text{ for any } x < 0, \tag{6}$$

Proof of Lemma 1. We have only to use similar computations as in the proof the point (a) of Theorem IV.1.1 in [Blu92, p. 110], where this Lemma is proved for the reflected Brownian motion. □

In the case of the Brownian motion, $S(x) = x$ and Theorem IV.1.1 in [Blu92, p. 110] gives the explicit values $C^+ = 1/\sqrt{2}$ and $C^- = -1/\sqrt{2}$. In fact, for coefficients constant on $\mathbb{R}_+$ and $\mathbb{R}_-$, the values of C^+ and C^- may also be directly computed (see Section 2).

Lemma 2. *For any $x > 0$,*

$$\widehat{\mathbb{P}}^{\mathsf{X}}[\,\sigma_x < \sigma_0\,] = \widehat{\mathbb{P}}^{\mathsf{X}}\left[\sigma_{S^{-1}(-S(x))} < \sigma_0\right]. \tag{7}$$

Proof of Lemma 2. Let x be a fixed point, $x > 0$. Let $\beta_1 = \inf\{t \geq 0 \,|\, \mathsf{L}_t(\mathsf{X}) \geq 1\}$. We set $\Gamma_x = \{\sigma_x < \sigma_0\}$. According to (2) with $\mathsf{Z}_s = \mathbf{1}_{\{s<\beta_1\}}$ and $f = \mathbf{1}_{\Gamma_x}$,

$$\widehat{\mathbb{P}}^{\mathsf{X}}\left[\,\Gamma_x\,\right] = \mathbb{E}_0\left[\sum_{s\in G^{\mathsf{X}},s<\beta_1} \mathbf{1}_{\Gamma_x}(\mathsf{X}_{\cdot\wedge\sigma_0}\circ\theta_s)\right]$$

As $\mathsf{X} = S^{-1}(\mathsf{B}_{\mathsf{T}})$ and T is continuous, we obtain that

$$\begin{aligned}\widehat{\mathbb{P}}^{\mathsf{X}}\left[\,\Gamma_x\,\right] &= \mathbb{E}_0\left[\sum_{s\in\mathsf{T}^{-1}(G^{\mathsf{X}}),s<\mathsf{T}(\beta_1)} \mathbf{1}_{\Gamma_{S(x)}}(\mathsf{B}_{\cdot\wedge\sigma_0}\circ\theta_s)\right]\\ &= \widehat{\mathbb{P}}^{\mathsf{B}}\left[\,\Gamma_{S(x)}\,\right]\mathbb{E}_0\left[\,\mathsf{L}_{\mathsf{T}(\beta_1)}(\mathsf{B})\,\right]\end{aligned}$$

as the event Γ_x is not sensitive to continuous time change. Equality (7) follows because $\widehat{\mathbb{P}}^{\mathsf{B}}\left[\,\Gamma_y\,\right] = \widehat{\mathbb{P}}^{\mathsf{B}}\left[\,\Gamma_{-y}\,\right]$ for any y, which is true thanks to the symmetry of the law of the Brownian motion. □

Proof of Theorem 1. Combining (7), (5) and (6), we have for any $x > 0$,

$$\begin{aligned}\alpha\frac{C^+}{x} &= \alpha\widehat{\mathbb{P}}^{\mathsf{X}^+}\left[\,\Gamma_{S^{-1}(x)}\,\right] = \widehat{\mathbb{P}}^{\mathsf{X}}\left[\,\Gamma_{S^{-1}(x)}\,\right]\\ &= \widehat{\mathbb{P}}^{\mathsf{X}}\left[\,\Gamma_{S^{-1}(-x)}\,\right] = (1-\alpha)\widehat{\mathbb{P}}^{\mathsf{X}^-}\left[\,\Gamma_{S^{-1}(-x)}\,\right] = -(1-\alpha)\frac{C^-}{x}\end{aligned}$$

Hence, α may be computed. □

2 A remark on the value of α

The value of α in Theorem 1 is given by the value of the left and right derivatives of the 1-potential $\phi_\pm$ at the point 0, but is in general not explicit. However, we give in this section an approximation one may also obtain using perturbation methods.

Let us deal with ϕ_+, the result being similar for ϕ_-. According to Theorem 16.69 in [Bre81, p. 380], ϕ_+ is continuous on $\mathbb{R}\setminus\{0\}$, convex, decreasing on $\mathbb{R}_+$. Furthermore, it solves the equation

$$\frac{1}{2\widetilde{a}(x)}\frac{\mathrm{d}^2\phi_+(x)}{\mathrm{d}x^2} = \phi_+(x) \text{ on } [0,\infty), \text{ where } \widetilde{a}(x) = \frac{V\circ S^{-1}(x)}{S'\circ S^{-1}(x)}. \qquad (8)$$

The differential operator $\frac{1}{2\widetilde{a}(x)}\frac{\mathrm{d}^2}{\mathrm{d}x^2}$ generates the process $\mathsf{B}^+_{\mathsf{T}^+} = S(\mathsf{X}^+)$. Since $\widetilde{a}$ is continuous on $[0,+\infty)$, it is clear that ϕ_+ is of class $\mathcal{C}^2$ on $[0,+\infty)$. Moreover, the function ϕ_+ is bounded.

If $\widetilde{a}$ is constant on $\mathbb{R}_+$, then $\phi_+(x) = \exp(-\sqrt{2\widetilde{a}(0+)}x)$ and in this case, $\phi'_+(0+) = -\sqrt{2\widetilde{a}(0+)}$.

If $\widetilde{a}$ is not constant, we face in some sense an inverse problem, which is to estimate the first derivative at 0 of the unique function ϕ_+ converging to 0 at infinity among all the possible solutions of (8). These solutions may be written as linear combination of two independent solutions. We use here Laplace transforms to give an approximation of this first derivative, but the result we obtain is the same as the one deduced using perturbation technics for differential equations (see [Har64] or [Eas70] for example).

Let us set $\gamma = \sqrt{2\widetilde{a}(0+)}$ and $r(x) = 2\widetilde{a}(x) - \gamma^2$. So $\phi''_+(x) = (\gamma^2 + r(x))\phi_+(x)$. Using the property of the Laplace transform and the equality $\phi_+(0) = 1$,

$$\phi'_+(0) = -\gamma - \int_0^{+\infty} e^{-\gamma x} r(x)\phi_+(x)\,\mathrm{d}x.$$

The fact that the diffusion coefficient $1/\widetilde{a}$ is not constant leads to add a perturbation on the value of $\phi'_+(0)$. But ϕ_+ and r are bounded, so $\phi'_+(0)$ will be close to $-\gamma$. More precisely, if $2\widetilde{a}(x) = \gamma^2 + \mathrm{O}(\varepsilon)$, then it is clear that $\phi'_+(0) = -\gamma + \mathrm{O}(\varepsilon)$. Furthermore, if $2\widetilde{a}(x) = \gamma^2$ on $[0, \ell]$, then $|\phi'_+(0) - \gamma| \leq \gamma^{-1} e^{-\gamma\ell} \|a\|_\infty$, which goes quickly to zero as ℓ increases. For numerical applications, if the variations of $\widetilde{a}$ at the left and the right of 0 are slow, $-\sqrt{2\widetilde{a}(0+)}$ remains a good approximation of $\phi'_+(0)$. So, one may write $C^+ \simeq -\sqrt{2S'(0+)/V(0+)}$ in this case.

In fact, a perturbation formula gives an integral formulation of ϕ_+ (See [Eas70, Property 3.1.2]):

$$\phi_+(x) = c_1 e^{-\gamma x} + c_2 e^{\gamma x} + \frac{1}{2\gamma}\int_0^x (e^{\gamma(x-t)} - e^{-\gamma(x-t)}) r(t)\phi_+(t)\,\mathrm{d}t,$$

with $c_2 = -\frac{1}{2\gamma}\int_0^{+\infty} e^{\gamma x} r(x)\phi_+(x)\,\mathrm{d}x$ and $c_1 = 1 - c_2$.

If $G(x, y)$ is the Green function of the operator $\frac{1}{2\widetilde{a}(x)}\frac{\mathrm{d}^2}{\mathrm{d}x^2}$ on $\mathbb{R}_+^*$ with a Dirichlet boundary condition at 0, then ϕ_+ is equal to $\phi_+(x) = 1 - \int_0^{+\infty} G(x, y)\mathrm{d}y$. In fact, giving an exact formula for ϕ_+, or equivalently the Green function G remains an open problem. However, in the case of piecewise constant coefficients (for example, $\widetilde{a}$ may take three different values in $(-\infty, 0)$, $[0, \ell)$ and $[\ell, +\infty)$), the Green function may be explicitly computed using transfer matrices, as it is explained in [GOO86, GOO87].

3 Examples and applications

Example 1 (Skew Brownian Motion). The *skew Brownian motion* of parameter α is a process obtained by changing independently the sign of each excursions of some reflected Brownian motion using the following rule: the excursion remains positive with probability α and becomes negative with probability $1-\alpha$ (see [IM74, Problem 1, p. 115] or [HS81] for example). The skew Brownian motion of parameter α is characterized by the functions (S, V), with

$$S(x) = \begin{cases} \alpha x & \text{if } x \geq 0, \\ (1-\alpha)x & \text{if } x < 0 \end{cases} \text{ and } V(x) = \begin{cases} \dfrac{1}{\alpha} & \text{if } x \geq 0, \\ \dfrac{1}{1-\alpha} & \text{if } x < 0. \end{cases}$$

Hence, we obtain that $\widehat{\mathbb{P}}^{\mathsf{X}} = \alpha \widehat{\mathbb{P}}^{\mathsf{B}^+} + (1-\alpha)\widehat{\mathbb{P}}^{\mathsf{B}^-}$. This result is given in Example V.b.(b) in [Blu92] and may be proved by a much more elementary way, using the fact that S' and V are piecewise constant.

Example 2 (Non divergence form operators). Let a and b be measurable functions such that there exists positive constants λ and Λ for which $\lambda \leq a(x) \leq \Lambda$ and $b \leq \Lambda$ for any x. We assume that a is continuous except at 0. The stochastic process generated by the operator $A^{\mathsf{X}} = \frac{a}{2}\frac{\mathrm{d}^2}{\mathrm{d}x^2} + b\frac{\mathrm{d}}{\mathrm{d}x}$ is characterized by the scale function S and density V of speed measure equal to

$$S^{\mathsf{X}}(x) = \int_0^x \exp(-h(y))\,\mathrm{d}y \text{ and } V^{\mathsf{X}}(x) = \frac{\exp h(x)}{a(x)}$$

where $h(x) = 2\int_0^x \frac{b(y)}{a(y)}\,\mathrm{d}y$. If we assume that the coefficient have slow variations at the left and right of 0, Theorem 1 implies that, using the result of Section 2,

$$\alpha \simeq \frac{\sqrt{a(0-)}}{\sqrt{a(0+)} + \sqrt{a(0-)}}. \tag{9}$$

We remark that the first-order differential term play almost no role (or no role in the case of piecewise constant coefficients), and that this formula is exact if a and b are constant on $\mathbb{R}_+$ and $\mathbb{R}_-$.

Example 3 (Divergence form operators). Let us assume that a and b are as previously and that a has a first derivative in $\mathbb{R}_+$ which is also continuous on $\mathbb{R}^*$ and has left and right limit at 0. Then, the Itô's excursions measure $\widehat{\mathbb{P}}^{\mathsf{Y}}$ of the process Y generated by $A^{\mathsf{Y}} = \frac{1}{2}\frac{\mathrm{d}}{\mathrm{d}x}\left(a\frac{\mathrm{d}}{\mathrm{d}x}\right) + b\frac{\mathrm{d}}{\mathrm{d}x}$ may be decomposed as $\widehat{\mathbb{P}}^{\mathsf{Y}} = \beta\widehat{\mathbb{P}}^{\mathsf{Y}^+} + (1-\beta)\widehat{\mathbb{P}}^{\mathsf{Y}^-}$, where $\beta \simeq \sqrt{a(0+)}/(\sqrt{a(0+)} + \sqrt{a(0-)})$. This value has to be compared with that of (9). In fact, we remark that outside 0, $A^{\mathsf{Y}} = A^{\mathsf{X}}$, hence X and Y have the same behavior excepted when they reach zero. This means that $\widehat{\mathbb{P}}^{\mathsf{Y}^\pm} = \widehat{\mathbb{P}}^{\mathsf{X}^\pm}$. Only the decomposition of the excursions measure changes. The scale function S^{Y} and the density of the speed measure V^{Y} are given by

$$S^{\mathsf{Y}}(x) = \begin{cases} a(0+)^{-1}S^{\mathsf{X}}(x) & \text{if } x \geq 0, \\ a(0-)^{-1}S^{\mathsf{X}}(x) & \text{if } x < 0, \end{cases} \text{ and } V^{\mathsf{Y}}(x) = \begin{cases} a(0+)V^{\mathsf{X}}(x) & \text{if } x \geq 0, \\ a(0-)V^{\mathsf{X}}(x) & \text{if } x < 0. \end{cases}$$

Example 4 (Construction of an arbitrary decomposition). More generally, consider two conservative processes X^+ and X^- on $\mathbb{R}_+$ and $\mathbb{R}_-$ characterized by the functions (S^+, V^+) and (S^-, V^-) and with excursions measures $\widehat{\mathbb{P}}^{\mathsf{X}^+}$ and $\widehat{\mathbb{P}}^{\mathsf{X}^-}$. Under some good conditions (see Chapter V in [Blu92]), it is possible to construct a process whose excursions measure will be given by $\alpha\widehat{\mathbb{P}}^{\mathsf{X}^+} + (1-\alpha)\widehat{\mathbb{P}}^{\mathsf{X}^-}$. This process will be characterized by the functions (S, V), and it is clear that there exists some positive constant γ such that

$$S(x) = \begin{cases} \gamma S^+(x) & \text{if } x \geq 0, \\ S^-(x) & \text{if } x < 0 \end{cases} \text{ and } V(x) = \begin{cases} \gamma^{-1}V^+(x) & \text{if } x \geq 0, \\ V^-(x) & \text{if } x < 0. \end{cases}$$

For that, we have to remember that (S, V) and $(\gamma S, \gamma^{-1}V)$ characterized the same process for any constant γ. Theorem 1 allows to compute γ.

Acknowledgment. I wish to thank the anonymous referees for suggesting improvements of this article.

References

[Blu92] R.M. Blumenthal. *Excursions of Markov Processes.* Probability and Its Applications. Birkhäuser, 1992. 5, 6, 9, 10

[Bre81] L. Breiman. *Probability.* Addison-Wesley, 1981. 4, 7

[CL01] F. Campillo and A. Lejay. A Monte Carlo method to compute the exchange coefficient in the double porosity model. In: Monte Carlo and probabilistic methods for partial differential equations (Monte Carlo, 2000). *Monte Carlo Methods Appl.*, vol. 7, no. 1–2, pp. 65–72, 2001 3

[Eas70] M.S.P. Eastham. *Theory of ordinary differential equations.* Van Nostrand Reinhold Company, 1970. 8

[GOO86] B. Gaveau, M. Okada and T. Okada. Opérateurs du second ordre à coefficients irréguliers en une dimension et leur calcul fonctionnel. *C. R. Acad. Sci. Paris Sér. I Math.*, vol. 302, no. 1, pp. 21–24, 1986. 8

[GOO87] B. Gaveau, M. Okada and T. Okada. Second order differential operators and Dirichlet integrals with singular coefficients. I. Functional calculus of one-dimensional operators. *Tohoku Math. J. (2)* vol. 39, no. 4, pp. 465–504, 1987. 8

[Hau99a] E. Hausenblas. A numerical scheme using Itô excursions for simulating local time resp. stochastic differential equations with reflection. *Osaka J. Math.*, vol. 26, no. 1, pp. 105–137, 1999. 2

[Hau99b] E. Hausenblas. A Monte-Carlo Method with Inherent Parallelism for Numerical Solving Partial Differential Equations with Boundary Conditions. In: *Parallel Computations*, P. Zinterhof, M. Vajtersic and A. Uhl eds., Lecture Notes in Computers Science, vol. 1557, Springer-Verlag, 1999. [url: `http://link.springer.de/link/service/series/0558/tocs/t1557.htm`] 2

[HS81] J.M. Harrison and L.A. Shepp. On Skew Brownian Motion. *Ann. Probab.*, vol. 9, no. 2, pp. 309–313, 1981. 3, 9

[Har64] P. Hartman. *Ordinary Differential Equations.* John Wiley & Sons, 1964. 8

[IM74] K. Itô and H.P. McKean. *Diffusion and Their Sample Paths.* Springer-Verlag, 2^{nd} edition, 1974. 3, 4, 5, 9

[Lej00] A. Lejay. *Méthodes probabilistes pour l'homogénéisation des opérateurs sous forme divergence : cas linéaires et semi-linéaires.* Ph.D. thesis, Université de Provence (Marseille, France), 2000. 2

[Ouk90] Y. Ouknine. Skew-Brownian motion and associated processes. *Theory Probab. Appl.*, vol. 35, no. 1, pp. 163–169, 1991. 3

[V-A96] J. Vuolle-Apalia. Skew Brownian Motion-type of extensions. *J. Theoret. Probab.*, vol. 9, no. 4, pp. 853–861, 1996. 4