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Rapports Techniques

N° 140

Programme 2

Calcul symbolique, Programmation et Génie logiciel

THE CAML NUMBERS REFERENCE MANUAL

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Juillet 1992

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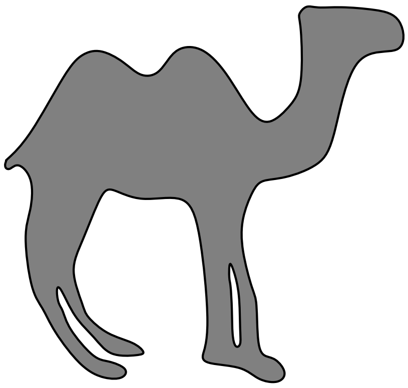

Projet Formel

INRIA-ENS

Version 3.1

The CAML Numbers Reference Manual

Valérie Ménissier-Morain[1]

Abstract

This is the reference manual for the arithmetic of the functional language CAML V3.1. It is available by anonymous ftp from `ftp.inria.fr` as a compressed PostScript file `lang/caml/V3.1/doc_arith.tar.Z`. This arithmetic includes floating point numbers, arbitrary large integers and rationals, and a complete set of primitives. Its implementation relies on the efficient `BigNum` package [2].

Résumé

Ceci est le manuel de référence de l'arithmétique du langage fonctionnel CAML V3.1. Il est disponible sous forme de fichier PostScript compressé `lang/caml/V3.1/doc_arith.tar.Z` par ftp anonyme sur `ftp.inria.fr`. Cette arithmétique contient des nombres en virgule flottante, des entiers et rationnels de longueur arbitraire, et un ensemble complet de primitives. Cette implémentation s'appuie sur le package `BigNum` [2].

[1] This work is partly supported by Allocation de Recherche de troisième Cycle of Ministère de la Recherche et de la Technologie (contrat 89553) and Institut National de Recherche en Informatique et Automatique.

A Jean-Louis Thomas

It makes me nervous to fly on airplanes since I know
they are designed using floating point arithmetic.

(Anston Householder)

Acknowledgements

I thank Gérard Huet for the interest he took first in the implementation and then in the documentation of this arithmetic.

While trying to write an alternative implementation of CAML, Damien Doligez and Pierre Weis asked me to implement a new arbitrary precision rational arithmetic for CAML using the BigNum Package.

This CAML implementation is still under development but Pierre Weis, the great meharist, asked me to insert this new arithmetic in the V3.1 CAML version as the main change. He believes in this new implementation and supported me all along this work.

This work would have been much harder without the constant help of François Morain and Bernard Serpette.

Thierry Coquand made me aware of the problem of normalization vs. non-normalization of rational numbers.

With my grateful thanks to Ian Jacobs for his TEX-nical contribution to this documentation.

I thank the CAML Team and the Formel Project for the interest to my work and quality of the working and programming environment.

I am thankful to all of them for their technical contribution, their judicious advice and their friendly support.

And last but not least, I thank all the rereaders for their invaluable help to find and correct typos and other errors.

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Chapitre 1

Introduction

1.1 Casual user

First of all, here are information for the casual user. It can be used in the three following modes (for more details see chapter 8).

In the standard arithmetic mode (which is the default one), numbers are supposed to be either integers in the range $[monster\_int = -32768 \ldots biggest\_int = 2^{length\_of\_int} - 1 = 32767]$ or floating point numbers. Standard arithmetic operations (`+`, `-`, `*`, `/`, ...) are defined on integers.
Example

```
#1+2;;
3 : int

#1.0+2.0;;

line 1: ill-typed phrase, the constant 1.0 of type float
cannot be used with type instance int in 1.0+2.0

line 1: ill-typed phrase, the constant 2.0 of type float
cannot be used with type instance int in 1.0+2.0
2 errors in typechecking

Typecheck Failed
```

If you want to use arithmetic operators for both integers and floating point numbers, you can overload these operators using the pragma:
Example

```
##arith overloading;;
Symbol prefix + overloaded with
       add_num : num * num -> num,
       add_float : float * float -> float,
       add_int : int * int -> int
Symbol prefix * overloaded with
       mult_num : num * num -> num,
       mult_float : float * float -> float,
       mult_int : int * int -> int
```

```
Symbol prefix ** overloaded with
       power_num : num * num -> num,
       power_float : float * float -> float,
       power_int : int * int -> int
Symbol prefix - overloaded with
       sub_num : num * num -> num,
       sub_float : float * float -> float,
       sub_int : int * int -> int
Symbol prefix / overloaded with
       div_num : num * num -> num,
       div_float : float * float -> float,
       div_int : int * int -> int
Symbol prefix <= overloaded with
       le_num : num * num -> bool,
       le_float : float * float -> bool,
       le_int : int * int -> bool
Symbol prefix < overloaded with
       lt_num : num * num -> bool,
       lt_float : float * float -> bool,
       lt_int : int * int -> bool
Symbol prefix >= overloaded with
       ge_num : num * num -> bool,
       ge_float : float * float -> bool,
       ge_int : int * int -> bool
Symbol prefix > overloaded with
       gt_num : num * num -> bool,
       gt_float : float * float -> bool,
       gt_int : int * int -> bool
Symbol - overloaded with
       minus_num : num -> num,
       minus_float : float -> float,
       minus_int : int -> int
Symbol float overloaded with
       float_of_num : num -> float,
       float_of_int : int -> float
Symbol pred overloaded with
       pred_num : num -> num,
       pred_float : float -> float,
       pred_int : int -> int
Symbol succ overloaded with
       succ_num : num -> num,
       succ_float : float -> float,
       succ_int : int -> int
Symbol incr overloaded with
       incr_num : num ref -> num,
       incr_float : float ref -> float,
       incr_int : int ref -> int
Symbol decr overloaded with
       decr_num : num ref -> num,
       decr_float : float ref -> float,
       decr_int : int ref -> int
```

```
Symbol prefix quo overloaded with
       quo_num : num * num -> num,
       quo_int : int * int -> int
Symbol prefix mod overloaded with
       mod_num : num * num -> num,
       mod_int : int * int -> int
Symbol max overloaded with
       max_num : num -> num -> num,
       max_float : float -> float -> float,
       max_int : int -> int -> int
Symbol min overloaded with
       min_num : num -> num -> num,
       min_float : float -> float -> float,
       min_int : int -> int -> int
Symbol abs overloaded with
       abs_num : num -> num,
       abs_float : float -> float,
       abs_int : int -> int
Symbol prefix power overloaded with
       power_num : num * num -> num,
       power_float : float * float -> float,
       power_int : int * int -> int

#1+2;;
3 : int

#1.0/3.0 + 2.0/3.0;;
0.9999999 : float

#sin 1;;

line 1: ill-typed phrase, the constant 1 of type int
cannot be used with type instance float in sin 1
1 error in typechecking

Typecheck Failed

#sin 1.0;;
0.841471 : float
```

CAML features an exact rational arithmetic package: the last arithmetic mode is thus extended arithmetic: numbers are supposed to be of type `num` and all available arithmetic operations always return exact results (up to 78903 decimal digits!).

Example

```
##standard arith false;;
Pragma false : bool

#1+2;;
#{3} : num
```

```
#1.0/3.0 + 2.0/3.0;;
#{1/1} : num

#sin 1.0;;

line 1: ill-typed phrase, the constant 1 of type num
cannot be used with type instance float in sin 1
1 error in typechecking

Typecheck Failed
```

About numerical directives, we can already mention the `arith_status ();;` instruction that prints the current status of each arithmetic flag and recall how to manage it.

1.2 Regular user

Numbers consist of:

type int: integers of type verb"int" with less than $(length\_of\_int = 15)$ bits, in the range $(monster\_int = -32768, biggest\_int = 2^{length\_of\_int} - 1 = 32767)$,

type float: floating point representations of reals,

type nat: integers of type verb"nat", on which every operation deals with arbitrary large unsigned integers, works by side-effects, allocates no storage and returns generally unit or a carry while storing results in its first argument.

In fact such a number is represented as a string. Since a string cannot contain more than $biggest\_int - 2$ characters and since each character allocates a byte, thus 8 bits, a number of type `nat` can have at most

$$\left\lfloor \frac{biggest\_int - 2}{\left\lceil \frac{length\_of\_digit}{8} \right\rceil} \right\rfloor = 8191$$

digits, and is therefore itself less than or equal to

$$2^{length\_of\_digit \times \left\lfloor \frac{biggest\_int - 2}{\left\lceil \frac{length\_of\_digit}{8} \right\rceil} \right\rfloor} - 1 = 2^{262112} - 1,$$

using classical notation for floor ($\lfloor\ \rfloor$) and ceiling ($\lceil\ \rceil$) functions. This number has 78903 decimal digits, that is to say it would spread on about 25 pages if it were printed in this document.

type big_int: arbitrary large signed integers (with the same limitation as for "natural" numbers), for which every operation allocates storage for the result,

type ratio: arbitrary large rational numbers, represented by a pair of big integers and a boolean indicating if this rational number is known to be normalized or not.

infinity (`1/0`) and *undefined* (`0/0`) are included as regular rational values. On request, computation with these numbers does not lead to runtime errors.

type num: symbolic representations of small integers, arbitrary large integers and rational numbers, with optimization of the representation of a number generally speaking. Most of the time the casual user will manipulate `num` values without worrying about their representations. More sophisticated users will take advantage of the definition of the `num` type:

```
type num = Int of int | Big_int of big_int | Ratio of ratio;;
```

Some functions are only available in a special arithmetic mode (for more information on this mode see chapter 8 on numerical directives). These functions are pointed out by a † after their name in the function index and when they are documented.

The CAML arithmetic implementation uses the BigNum package developed jointly by INRIA and Digital PRL.

This package provides nat operations and is structured in two layers:

- the kernel which contains code for the time-critical low-level operations,
- a set of functions written in high level languages which call the kernel.

A lisp implementation of the kernel exists. However, to obtain a truly efficient implementation on a given machine, it must be written in assembly code. Unfortunately this assembly code implementation is not provided for all architectures.

The distinction between kernel and non-kernel operations is defined so that the time penalty for running the mixture of assembly and high level code, as opposed to pure assembly code, is less than 20% on typical benchmarks.

The kernel functions will be indicated in the corresponding chapter by a $*$ character.

Chapitre 2

Operations on numbers of type int

length_of_int[†]**:** int
biggest_int : int
least_int : int
monster_int : int

Elements of type `int` are small integers represented on `length_of_int` bits, in the range (`least_int, biggest_int`), more exactly the ring of the integers between `least_int - 1` and `biggest_int`, since there is no notion of overflow.

This special value `monster_int = least_int - 1` will be further discussed (see section 2.2).

Integer values of type `int` are manipulated for the sake of efficiency (they are not allocated, hence there is no need to garbage collect them).

The maximum number of bits, the maximum, minimum and extremum values of an `int` are:

```
#length_of_int;;
15 : int

#biggest_int;;
32767 : int

#least_int;;
-32767 : int

#monster_int;;
-32768 : int
```

2.1 Arithmetic operations for ints

2.1.1 Incrementing and decrementing

Incrementing and decrementing `ints`

pred_int : int → int
succ_int : int → int
monster_int : int

`pred_int n` (resp. `succ_int n`) is a primitive equivalent to `n-1` (resp. `n+1`).

`pred_int least_int` and `succ_int biggest_int` have the same value `monster_int`.
`pred_int monster_int` is `biggest_int` and `succ_int monster_int` is `least_int`.

The lexical convention for type `int` will be presented later in this chapter (section 2.3.3) but the essential thing to understand the following example is that elements of type `int` are preceded by a **#**.

Example

```
#pred_int #1;;
0 : int

#pred_int #0;;
-1 : int

#pred_int least_int;;
-32768 : int

#pred_int biggest_int;;
32766 : int

#pred_int monster_int;;
32767 : int

#succ_int #1;;
2 : int

#succ_int #0;;
1 : int

#succ_int least_int;;
-32766 : int

#succ_int biggest_int;;
-32768 : int

#succ_int monster_int;;
-32767 : int
```

Incrementing and decrementing counters

incr_int : int ref → int
decr_int : int ref → int

When one has created a reference containing a number (whose type is `int`), it is convenient to increment and decrement its value, using these two (primitive) functions semantically equivalent to:

```
let incr_int i = i:= succ_int !i and decr_int i = i:= pred_int !i;;
```

2.1.2 Arithmetic operations for ints

add_int : int × int → int
minus_int : int → int
sub_int : int × int → int
mult_int : int × int → int
quo_int : int × int → int
mod_int : int × int → int
gcd_int†: int × int → int

`add_int`, `sub_int`, `mult_int` perform respectively the addition, subtraction and multiplication on numbers of type `int`.

`minus_int x` is the opposite of `x`. We have :

$$minus\_int\ monster\_int = monster\_int.$$

For division on numbers of type `int`, `quo_int (x, y)` and `mod_int (x, y)` are respectively the quotient and the remainder of the Euclidian division of `x` by `y`.
Beware:

→ In fact, in case of a negative argument, `quo_int (x, y)` is one more greater than the result of Euclidian division (i.e. each partial function of `quo_int` is an odd function) and `mod_int` doesn't always produce positive integers.

`gcd_int (n1, n2)` computes the gcd of the two numbers of type `int n1` and `n2`. We have the following relation:

$$gcd\_int(n_1, n_2) = gcd\_int(|n_1|, |n_2|) = gcd\_int(n_2, n_1) \leq 0$$

Example

```
#add_int (#43, #-12);;
31 : int

#minus_int biggest_int;;
-32767 : int

#minus_int least_int;;
32767 : int

#minus_int monster_int;;
```

```
-32768 : int

#mult_int (biggest_int, #2);;
-2 : int

#quo_int (#123, #3);;
41 : int

#mod_int (#123, #3);;
0 : int

#quo_int (#123, #-3);;
-41 : int

#mod_int (#123, #-3);;
0 : int

#quo_int (#123, #4);;
30 : int

#mod_int (#123, #4);;
3 : int

#quo_int (#123, #-4);;
-30 : int

#mod_int (#123, #-4);;
3 : int

#gcd_int (#4, #6);;
2 : int

#gcd_int (#2, #3);;
1 : int

#gcd_int (#-6, #4);;
2 : int
```

Notice that these arithmetic operations do not check for overflow.

Using type **num**, a careful (but less efficient) version of multiplication may be defined as in:

```
#let mult_int_care (x, y) =
# match mult_num (Int x, Int y)
# with Int result -> result
#   |      _      -> failwith "overflow"
#;;
Value mult_int_care is <fun> : int * int -> int

#mult_int_care(biggest_int,#2);;

Evaluation Failed: failure "overflow"
```

2.2 Comparisons between ints

eq_int : int $\times$ int $\rightarrow$ bool
compare_int†**:** int $\times$ int $\rightarrow$ int
lt_int : int $\times$ int $\rightarrow$ bool
le_int : int $\times$ int $\rightarrow$ bool
gt_int : int $\times$ int $\rightarrow$ bool
ge_int : int $\times$ int $\rightarrow$ bool
sign_int†**:** int $\rightarrow$ int
abs_int : int $\rightarrow$ int
minus_int : int $\rightarrow$ int
max_int : int $\rightarrow$ int $\rightarrow$ int
min_int : int $\rightarrow$ int $\rightarrow$ int

`eq_int` is the semantic and syntactic equality on numbers of type `int`.

`lt_int`, `le_int`, `gt_int`, `ge_int` are arithmetic comparisons corresponding respectively to `<`, `<=`, `>`, `>=`.

The primitive function for these functions is `compare_int`, that is semantically (and almost really) equivalent to

```
let compare_int (int1, int2) =
  when (int1 = int2) -> #0
     | (int1 < int2) -> #-1
     |       _       -> #1;;
```

Example

```
#compare_int (#0, #1);;
-1 : int

#compare_int (#0, #0);;
0 : int

#compare_int (#0, #-1);;
1 : int
```

`sign_int` is completely equivalent to

```
let sign_int n = compare_int (n, 0);;
```

`abs_int n` is the absolute value of number `n`.
Beware:

→ *abs_int monster_int = minus_int monster_int = monster_int* and `monster_int` is negative. So `abs_int` and `minus_int` of a negative number of type `int` don't always produce positive numbers.

→ *pred_int monster_int = biggest_int* is the greatest number of type `int`, but `monster_int` is the least one. In fact `least_int` is the smallest "normal" number of type `int`.

`max_int n1 n2` and `min_int n1 n2` are respectively the greater and the smaller of the numbers `n1` and `n2`.

Example

```
#eq_int (succ_int biggest_int, monster_int);;
true : bool

#gt_int (succ_int biggest_int, biggest_int);;
false : bool

#sign_int (succ_int biggest_int);;
-1 : int

#sign_int monster_int;;
-1 : int

#sign_int #0;;
0 : int

#sign_int #2;;
1 : int

#sign_int #-3;;
-1 : int

#abs_int #-3;;
3 : int

#abs_int monster_int;;
-32768 : int

#max_int #1 #2;;
2 : int

#min_int #1 #2;;
1 : int
```

2.3 Coercion functions on ints

2.3.1 Coercion between type float and int

int_of_float : float → int
float_of_int : int → float

`int_of_float f` is the truncation of `f` to a number of type `int`, closest to its floating point representation.

Example

```
#int_of_float #1.2;;
1 : int
```

```
#int_of_float #1.0;;
1 : int

#int_of_float #0.9999999;;
0 : int

#int_of_float #0.99999999;;
1 : int
```

`float_of_int i` is the floating representation of the number of type `int i`.

Example

```
#float_of_int #0;;
0.0 : float

#float_of_int #1;;
1.0 : float

#float_of_int #-1;;
-1.0 : float
```

2.3.2 Coercion between characters and ints

int_of_char : char → int
char_of_int : int → char

`int_of_char c` is the ASCII code for `c`.

`char_of_int i` is the character with ASCII code `i`.

The two following relations hold for each `i` from type `int` and each `c` from type `char` :

$$int\_of\_char\,(char\_of\_int\;i) = i$$
$$char\_of\_int\,(int\_of\_char\;c) = c$$

Example

```
#char_of_int #49;;
‘1‘ : char

#int_of_char ‘1‘;;
49 : int
```

2.3.3 Coercion between strings and ints

Reading `ints`

sys_int_of_string†: int × string × int × int → int
int_of_string : string → int

CAML is able to read `int`s entered directly, when you type in a number preceded by a sharp "`#`", following the lexical convention:

```
INT ::= '#' {'-' | '+'} Digit+
```

Example

```
##1;;
1 : int

##-1;;
-1 : int

##+1;;
1 : int

##32767;;
32767 : int

##32768;;
32768.0 : float

##-32767;;
-32767 : int

##-32768;;
-32768.0 : float
```

Using standard arithmetic, the sharp character is optional (see chapter 8 for more information).

`sys_int_of_string (base, s, off, len)` maps the substring $s_{off,len}$ in base `base` ($2 \leq$ `base` ≤ 16) onto a number of type `int`.

`int_of_string s` is semantically equivalent to:

```
let int_of_string s =
  sys_int_of_string (#10, s, #0, length_string s)
;;
```

Example

```
#sys_int_of_string (#10, "1234567890123456789", #11, #5);;
23456 : int

#int_of_string "12345";;
12345 : int
```

Printing ints

string_of_int : int → string
string_for_read_of_int : int → string
sys_string_of_int†**:** int × string × int × string → string
display_int : int → unit
print_int : int → unit
print_int_for_read : int → unit

`sys_string_of_int (b, before, i, after)` writes into the resulting string first the string `before`, then the representation of `i` in base `b`, and lastly the string `after`.

`string_of_int` is semantically equivalent to:

```
let string_of_int i = sys_string_of_int (#10, "", i, "");;
```

`string_for_read_of_int` maps elements of type `int` onto corresponding elements of type `string` as well as `string_of_int`, but the sharp character `#` is added before the number according to the lexical convention for numbers of type `int`.

This function is semantically equivalent to:

```
let string_for_read_of_int i = sys_string_of_int (#10, "#", i, "");;
```

`display_int` prints in a trivial way (i.e. without formatting its argument) numbers of type `int`.

`print_int` is the formatting function for numbers of type `int`.

`print_int_for_read` prints numbers of type `int` such that they are readable by the CAML system.

On the one hand, `display_int` and `print_int` print numbers of type `int` without printing sharp character `#`. On the other hand, `print_int_for_read` prints numbers of type `int` with a sharp character `#`.

For more details on the differences between `display`- and `print`- and `print_*_for_read`-like functions see chapter "Formatting Functions" in the reference manual.

Example

```
#sys_string_of_int (#2, "begin ", #20, " end");;
"begin 10100 end" : string

#sys_string_of_int (#16, "hello ", #20, " world");;
"hello 14 world" : string

#string_of_int #12345;;
"12345" : string

#string_for_read_of_int #12345;;
"#12345" : string

#print_int #10;;
10() : unit

#print_int_for_read #10;;
#10() : unit
```

2.4 Logical operations

land_int : int × int → int
lor_int : int × int → int
lxor_int : int × int → int
lshift_int : int × int → int
lnot_int : int → int

These are "logical" functions which operate on numbers *bitwise* without modification of the arguments. All these functions are expanded on the fly into one or a few lines of assembly code.

Their meaning is:

land_int: The nth bit of `land_int (n1,n2)` has value 1 if and only if the nth bit of `n1` and the one of `n2` have both value 1, else 0.

Example

```
#land_int (#2, #4);;
0 : int

#land_int (#3, #8);;
0 : int
```

For instance to find if a number is a power of 2, you may define

```
#let is_power2 n = (n = land_int (n, -n)) ;;
Value is_power2 is <fun> : int -> bool

#is_power2 #4;;
true : bool

#is_power2 #3;;
false : bool
```

lor_int: The nth bit of `lor_int (n1,n2)` has value 0 if and only if the nth bit of `n1` and the one of `n2` have both value 0, else 1.

lxor_int: The nth bit of `lxor_int (n1,n2)` has value 0 if and only if the nth bit of `n1` and the one of `n2` have different values.

lshift_int: `lshift_int (n1,n2)` has value `n1` shifted of `n2` bits. If `n2` is positive the shift is done to the left (towards the most significant bits, hence it is a multiplication by $2^{\mathtt{n2}}$). If `n2` is negative the shift is done to the right (towards the least significant bits, then it is a division by $2^{-\mathtt{n2}}$).

Using `lshift_int` you may extract a bit field from an integer representation, as in:

```
#let extract n i = lshift_int (land_int (n, lshift_int (#1,i)), -i);;
Value extract is <fun> : int -> int -> int
```

```
#map (extract #24) (interval #0 #15);;
[0; 0; 0; 1; 1; 0; 0; 0; 0; 0; 0; 0; 0; 0; 0; 0] : int list

#map (extract (#-24)) (interval #0 #15);;
[0; 0; 0; 1; 0; 1; 1; 1; 1; 1; 1; 1; 1; 1; 1; 1] : int list
```

lnot_int: A bit of `lnot_int (n)` has value 0 iff the corresponding bit of `n` has value 1, else 1.

2.5 Miscellaneous functions on type int

num_bits_int†: int → int
num_decimal_digits_int†: int → int
power_int : int × int → int
power_int_positive_int†: int × int → big_int

`num_bits_int i` (resp. `num_decimal_digits_int i`) is the number of digits of the binary (resp. decimal) representation of `i`.

`power_int (i, n)` (resp. `power_int_positive_int (i, n)`) is the `int` (resp. `big_int`) representation of i^n, with n any positive or null `int`.

In fact, as long as the result is an integer `power_int` doesn't fail.

Example

```
#num_bits_int #20;;
5 : int

#num_decimal_digits_int #20;;
2 : int

#power_int (#2, length_of_int);;
-32768 : int

#power_int (#2, succ_int length_of_int);;
0 : int

#power_int_positive_int (#2, length_of_int);;
#(32768) : big_int

#power_int_positive_int (#2, succ_int length_of_int);;
#(65536) : big_int

#power_int (#-1, #-2);;
1 : int
```

Chapitre 3

Floating point numbers

Elements of type `float` are floating point representations of decimal numbers, in a range depending on the machine which runs the CAML system but generally included in $[-2^{128}+1; 2^{128}-1]$ and in 10-radix representation, absolute value of exponent is generally less than 38. There exists beyond these numbers a value *Infinity* and many incorrect values (e.g. *NaN* standing for "Not a number"). For example, for the computer on which this document is preprocessed the biggest floating point number is almost $2^{128}-1 \approx 3.4028235677e38$. So examples with 3.403e38 imply *Infinity*. Reading floating point numbers greater than $2^{1024}-1 \approx 1.797693134862315e308$, or lesser than 10^{-340} but not null, produce an error and the message depends on the computer.

3.1 Arithmetic operations for floating point numbers

3.1.1 Incrementing and decrementing

Incrementing and decrementing numbers

pred_float : float → float
succ_float : float → float

`pred_float n` (resp. `succ_float n`) is a primitive equivalent to `n-1` (resp `n+1`).

Example

```
#pred_float #1.0;;
0.0 : float

#pred_float #0.0;;
-1.0 : float

#succ_float #1.0;;
2.0 : float

#succ_float #0.0;;
1.0 : float
```

Incrementing and decrementing counters

incr_float : float ref → float
decr_float : float ref → float

When one has created a reference containing a number (whose type is `float`), it is convenient to increment and decrement its value, using these two (primitive) functions semantically equivalent to:

```
let incr_float i = i:= succ_float !i and decr_float i = i:= pred_float !i;;
```

3.1.2 Arithmetic operations for floating point numbers

add_float : float × float → float
minus_float : float → float
sub_float : float × float → float
mult_float : float × float → float
div_float : float × float → float

`add_float`, `sub_float`, `mult_float`, `div_float` perform respectively the addition, subtraction, multiplication and division on floating point numbers.

`minus_float x` is the opposite of `x`.

3.2 Comparisons between floating point numbers

eq_float : float × float → bool
lt_float : float × float → bool
le_float : float × float → bool
gt_float : float × float → bool
ge_float : float × float → bool
abs_float : float → float
max_float : float → float → float
min_float : float → float → float

`eq_float` is the semantic and syntactic equality on floating point numbers.

`lt_float`, `le_float`, `gt_float`, `ge_float` are arithmetic comparisons corresponding respectively to `<`, `<=`, `>`, `>=`.

`abs_float n` is the absolute value of the number `n`.

`max_float f1 f2` and `min_float f1 f2` are respectively the greater and the smaller of the numbers `f1` and `f2`.

Example

```
#eq_float (#3.402e38, #3.403e38);;
false : bool

#eq_float (#3.403e38, #3.404e38);;
true : bool
```

```
#lt_float (#3.402e38, #3.403e38);;
true : bool

#lt_float (#3.403e38, #3.404e38);;
false : bool

#gt_float (#3.403e38, #3.404e38);;
false : bool

#abs_float #-3.0;;
3.0 : float

#max_float #1.0 #2.0;;
2.0 : float

#min_float #1.0 #2.0;;
1.0 : float
```

3.3 Floating point transcendental operations

sin : float → float
cos : float → float
asin : float → float
acos : float → float
atan : float → float
exp : float → float
log : float → float
log10 : float → float
sqrt : float → float
power_float : float × float → float

`sin` and `cos` are the classical trigonometric functions. As an example one can create a function `tan` using the mathematical definition:
Example

```
#let tan x = div_float (sin x, cos x);;
Value tan is <fun> : float -> float

#tan 1.0;;
1.557408 : float

#atan it;;
0.9999999 : float

#let pi_over_2 = div_float (3.14159265,2.0);;
Value pi_over_2 is 1.570796 : float

#(* Computation with floating point numbers approximates *)
#tan pi_over_2;;
```

```
13245400.0 : float

#(* But sometimes consistently ! *)
#atan it;;
1.570796 : float
```

`asin`, `acos` and `atan` are the reciprocal functions of these trigonometric functions.

`log` and `exp` are the Neperian logarithm and the exponential function. `log10` is the decimal logarithm.

`sqrt` is the square root function and `power_float (x, y)` yields x^y.

3.4 Coercion functions on floating point numbers

3.4.1 Reading floating point numbers

sys_float_of_string : int × string × int × int → float
float_of_string : string → float

CAML is able to read floating point numbers entered directly, following the lexical convention:

```
FLOAT ::= '#' {'-' | '+'} Digit+ '.' Digit+ {'e' {'-' | '+'} Digit+}
```

Example

```
##1.0;;
1.0 : float

##1.0e2;;
100.0 : float

##1.0e-1;;
0.09999999 : float

##1.2e+2;;
120.0 : float

##-1.2e+2;;
-120.0 : float

##+1.2e+2;;
120.0 : float
```

Using standard arithmetic, the sharp character is optional (see chapter 8 for more information).

`sys_float_of_string (base, s, off, len)` maps the substring $s_{off,len}$ in base `base` ($2 \leq$ `base` ≤ 16) onto a floating point number.

`float_of_string s` is semantically equivalent to:

```
let float_of_string s =
  sys_float_of_string (#10, s, #0, length_string s)
;;
```

Example

```
#sys_float_of_string (#10, "123e-42", #0, #6);;
0.0123 : float

#float_of_string "123e-4";;
0.0123 : float
```

Printing floating point numbers

string_of_float : float → string
string_for_read_of_float : float → string
display_float : float → unit
print_float : float → unit
print_float_for_read : float → unit

`string_of_float` and `string_for_read_of_float` map elements of type `float` onto corresponding elements of type `string`. `string_for_read_of_float` adds the sharp character `#` before the number according to the lexical convention for numbers of type `float`.

`display_float` prints in a trivial way (i.e. without formatting its argument) floating point numbers.

`print_float` is the formatting function for floating point numbers.

`print_float_for_read` prints floating point numbers such that it is readable by the CAML system.

On the one hand, `display_float` and `print_float` print floating point numbers without printing sharp character `#`. On the other hand, `print_float_for_read` prints floating point numbers with sharp character `#`.

For more details on the differences between `display`- and `print`- and `print_*_for_read`-like functions see chapter "Formatting Functions" in the reference manual.

Example

```
#string_of_float #123.45;;
"123.45" : string

#string_for_read_of_float #123.45;;
"#123.45" : string

#print_float #1.0;;
1.0() : unit

#print_float_for_read #1.0;;
#1.0() : unit
```

Chapitre 4

Natural numbers

Elements of type `nat` are quasi arbitrarily large unsigned numbers. All operations on this type are performed with in place update.

4.1 Natural numbers representation

- Numbers of type nat are base B decompositions of natural numbers :

$$N = \sum_{i=0}^{i<n} N_i \, B^i.$$

length_of_digit†**:** int
set_length_of_digit†**:** int → unit

In fact, $B = 2^{length\_of\_digit}$ and in the current implementation, we have by default:

$$length\_of\_digit = 32.$$

This value can be modified using the function `set_length_of_digit`.

- We note here the subnat of

$$nat = \sum_{i=0}^{i<n} nat_i \, B^i$$

beginning at the off-th digit with length len digits

$$nat_{off,len} = \sum_{i=0}^{i<len} nat_{off+i} \, B^i.$$

The following conditions must be verified :

$$\begin{array}{rcl} off & \geq & 0 \\ len & \geq & 1 \\ off + len & \leq & n. \end{array}$$

These conditions are supposed to be verified and are tested only in cautious arithmetic mode (see chapter 8) with the following instruction:

```
#arith cautious;;
```

If not respected it can cause a segment violation.

- In a straightforward manner, the i-th digit of

$$N = \sum_{i=0}^{i<n} N_i \, B^i$$

is N_i. The following condition must be :

$$0 \leq i < n.$$

Beware:

→ Notice that all functions on type `nat` are accessible only in open arithmetic mode. So they are all followed by a †. For more information on the open arithmetic mode see chapter 8.

4.2 Creation of natural numbers

create_nat†: int → nat
make_nat†: int → nat
copy_nat†: nat × int × int → nat

`create_nat len` and `make_nat len` create a new natural number of *len* digits ($len \geq 1$). `create_nat` is the primitive function, it is not supposed to initialize the created number, on the other hand, `make_nat` is supposed to initialize the created number to zero.

`make_nat` is equivalent to the following CAML definition:

```
let make_nat len =
  let res = create_nat (len) in
    set_to_zero_nat (res, #0, len);
res
;;
```

Beware:

→ In fact in this version `create_nat` and `make_nat` are identical and initialize the created number.

`copy_nat (nat, off, len)` is a copy of the value of subnat $nat_{off,len}$.

4.3 Assignment of natural numbers

set_digit_nat$^{\dagger *}$**:** nat × int × int → unit
blit_nat$^{\dagger *}$**:** nat × int × nat × int × int → unit
set_to_zero_nat$^{\dagger *}$**:** nat × int × int → unit

`set_digit_nat (nat, off, digit)` assigns nat_{off} to the value of $digit$ of type `int`.

`blit_nat (nat1, off1, nat2, off2, len2)` assigns subnat $nat2_{off2,len2}$ to the subnat $nat1_{off1,len2}$.

The following conditions must be verified :

$$\begin{aligned} off1 &\geq 0 \\ off2 &\geq 0 \\ len2 &> 0 \\ off1 + len2 &\leq size(nat1) \\ off2 + len2 &\leq size(nat2) \end{aligned}$$

where $size(nat)$ designates the length of nat.

Notice that $nat1$ and $nat2$ can be the same.

`set_to_zero_nat (nat, off, len)` sets subnat $nat_{off,len}$ to zero.

Example

```
#let nat1 = create_nat (#2)
#and nat2 = create_nat (#2);;
Value nat1 is #<0> : nat
Value nat2 is #<0> : nat

#set_to_zero_nat (nat1, #0, #2);
#set_digit_nat (nat2, #0, #1);
#set_digit_nat (nat2, #1, #1);
#debug_print_nat nat1;;
|00000000|00000000|() : unit

#debug_print_nat nat2;;
|00000001|00000001|() : unit

#blit_nat (nat1, #0, nat2, #1, #1);
#debug_print_nat nat1;;
|00000000|00000001|() : unit

#debug_print_nat nat2;;
|00000001|00000001|() : unit

#blit_nat (nat1, #1, nat1, #0, #1);
#debug_print_nat nat1;;
|00000001|00000001|() : unit

#debug_print_nat nat2;;
|00000001|00000001|() : unit
```

4.4 Lengths of a natural number

length_nat†: nat $\to$ int
num_digits_nat$^{\dagger *}$: nat $\times$ int $\times$ int $\to$ int

`length_nat nat` is the presupposed length of nat and `num_digits_nat (nat, off, len)` is the real length of subnat $nat_{off,len}$.
Example

```
#let nat = create_nat (#2) in
#  set_digit_nat (nat, #0, #1);
#  set_digit_nat (nat, #1, #0);
#  (length_nat nat, num_digits_nat (nat, #0, #2))
#;;
(2,1) : int * int
```

4.5 Miscellaneous informations on natural numbers

is_digit_int†: nat $\times$ int $\to$ bool
is_digit_normalized$^{\dagger *}$: nat $\times$ int $\to$ bool
is_digit_zero$^{\dagger *}$: nat $\times$ int $\to$ bool
is_digit_odd†: nat $\times$ int $\to$ bool
is_zero_nat†: nat $\times$ int $\times$ int $\to$ bool
num_leading_zero_bits_in_digit$^{\dagger *}$: nat $\times$ int $\to$ int
is_nat_int†: nat $\times$ int $\times$ int $\to$ bool
nth_digit_nat$^{\dagger *}$: nat $\times$ int $\to$ int

`is_digit_int (nat, off)` tests if nat_{off} can be coerced to an `int`, i.e. this digit has less than $length\_of\_int$ bits, i.e. is smaller than $biggest\_int$.

`is_digit_normalized (nat, off)` returns true if

$$nat_{off} \geq \frac{B}{2}.$$

`is_digit_zero (nat, off)` returns true if $nat_{off} = 0$.

`is_digit_odd (nat, off)` returns true if nat_{off} is odd.

`is_zero_nat (nat, off, len)` tests if subnat $nat_{off,len}$ is null.

`num_leading_zero_bits_in_digit (nat, off)` returns the number of most significant bits consecutively null of nat_{off}.

`is_nat_int (nat, off, len)` tests if $nat_{off,len}$ can be coerced in an `int`.

`nth_digit_nat (nat, off)` returns nat_{off}.
Example

```
#let nat = create_nat #4;;
Value nat is #<0> : nat

#for i = #0 to #3 do set_digit_nat (nat, i, i) done;;
```

```
() : unit

#debug_print_nat nat;;
|00000003|00000002|00000001|00000000|() : unit

#is_digit_int (nat, #0);;
true : bool

#is_nat_int (nat, #0, #2);;
false : bool

#nth_digit_nat (nat, #1);;
1 : int

#is_digit_normalized (nat, #1);;
false : bool

#is_digit_zero (nat, #0);;
true : bool

#is_digit_zero (nat, #1);;
false : bool

#is_digit_odd (nat, #1);;
true : bool

#is_digit_odd (nat, #2);;
false : bool

#is_zero_nat (nat, #0, #1);;
true : bool

#is_zero_nat (nat, #0, #4);;
false : bool

#num_leading_zero_bits_in_digit (nat, #0);;
32 : int

#num_leading_zero_bits_in_digit (nat, #1);;
31 : int
```

Notice that `num_leading_zero_bits_in_digit` can be used to define most of these preceding functions as follows:

```
let is_digit_normalized (nat, off) =
  num_leading_zero_bits_in_digit (nat, off) == 0
;;

let is_digit_zero (nat, off) =
  num_leading_zero_bits_in_digit (nat, off) == length_of_digit
;;
```

```
let is_zero_nat (nat, off, len) =
  num_digits_nat (nat, off, len) == 1 &&
  num_leading_zero_bits_in_digit (nat, off) >= length_of_digit
;;

let is_digit_int (nat, off) =
  num_leading_zero_bits_in_digit (nat, off) >= length_of_digit - length_of_int
;;

let is_nat_int (nat, off, len) =
  num_digits_nat (nat, off, len) == 1 &&
  num_leading_zero_bits_in_digit (nat, off) >= length_of_digit - length_of_int
;;
```

In practice these functions are not so defined and their existence gives less interest to `num_leading_zero_bits_in_digit` since all the significant information of this function is present more pleasantly in the other functions.

4.6 Arithmetic operations on natural numbers

4.6.1 Addition operators on natural numbers

incr_nat$^{\dagger *}$**:** nat × int × int × int → int
add_nat$^{\dagger *}$**:** nat × int × int × nat × int × int × int → int

Incrementing natural numbers

`incr_nat (nat, off, len, c_in)` adds in place the carry in *c_in* (*c_in* = 0 or *c_in* = 1) to the subnat $nat_{off,len}$ and returns the carry out *c_out*.

More formally speaking `incr_nat` performs the following operation :

$$\begin{aligned} & nat_{off,len} + c\_in = \\ & \quad \left(\sum_{i=0}^{i<len} nat_{off+i}\, B^i \right) + c\_in = \\ & \quad \left(\sum_{i=0}^{i<len} nat'_{off+i}\, B^i \right) + c\_out\, B^{len}. \end{aligned}$$

`incr_nat` replaces each digit nat_{off+i} with its equivalent digit nat'_{off+i} and returns the carry out *c_out*, with value 0 or 1.

$c\_out = 1$ if and only if $nat_{off,len} = B^{len} - 1$ and $c\_in = 1$. `len` may be null, so the carry out equals the carry in.

Example

```
#let nat = make_nat #4;;
Value nat is #<0> : nat
```

```
#complement_nat (nat, #0, #4);;
() : unit

#debug_print_nat nat;;
|FFFFFFFF|FFFFFFFF|FFFFFFFF|FFFFFFFF|() : unit

#incr_nat (nat, #2, #2, #1);;
1 : int

#debug_print_nat nat;;
|00000000|00000000|FFFFFFFF|FFFFFFFF|() : unit

#incr_nat (nat, #0, #3, #1);;
0 : int

#debug_print_nat nat;;
|00000000|00000001|00000000|00000000|() : unit

#incr_nat (nat, #0, #3, #1);;
0 : int

#debug_print_nat nat;;
|00000000|00000001|00000000|00000001|() : unit

#incr_nat (nat, #0, #3, #1);;
0 : int

#debug_print_nat nat;;
|00000000|00000001|00000000|00000002|() : unit
```

Addition on natural numbers

`add_nat (nat1, off1, len1, nat2, off2, len2, c_in)` adds subnats $nat1_{\texttt{off1},\ \texttt{len1}}$, $nat2_{\texttt{off2},\ \texttt{len2}}$ and the carry in c_in, places result in subnat $nat1_{off1,len1}$, propagates this carry on subnat $nat1_{off1+len2,\,len1-len2}$ and returns the carry out c_out.

More formally speaking `add_nat` performs the following operation :

$$
\begin{aligned}
&nat1_{off1,len1} + nat2_{off2,len2} + \text{c\_in} = \\
&\quad \left(\sum_{i=0}^{i<len1} nat1_{off1+i}\, B^i\right) + \left(\sum_{i=0}^{i<len2} nat2_{off2+i}\, B^i\right) + c\_in = \\
&\quad \left(\sum_{i=0}^{i<len1} nat'_{off1+i}\, B^i\right) + c\_out\, B^{len1}.
\end{aligned}
$$

`add_nat` replaces each digit $nat1_{off1+i}$ with its equivalent digit nat'_{off1+i} and returns the carry out c_out, with value 0 or 1. This implies $0 \leq c\_in \leq 1$ and $len1 \geq len2$.

$nat1$ and $nat2$ can be physically the same one under the condition $off1 \leq off2$.

If $len2 = 0$, `add_nat (nat1, off1, len1, nat2, off2, len2, c_in)` is equivalent to `incr_nat (nat1, off1, len1, c_in)`.

Example

```
#let nat1 = nat_of_int #2;;
Value nat1 is #<2> : nat

#add_nat (nat1, #0, #1, nat_of_int #3, #0, #1, #0);;
0 : int

#debug_print_nat nat1;;
|00000005|() : unit

#add_nat (nat1, #0, #1, nat_of_int #3, #0, #1, #1);;
0 : int

#debug_print_nat nat1;;
|00000009|() : unit

#let nat1 = nat_of_int #3;;
Value nat1 is #<3> : nat

#complement_nat (nat1, #0, #1);;
() : unit

#add_nat (nat1, #0, #1, nat_of_int #3, #0, #1, #1);;
1 : int

#debug_print_nat nat1;;
|00000000|() : unit

#let nat1 = create_nat #3;;
Value nat1 is #<0> : nat

#for i = #0 to #1 do set_digit_nat (nat1, i, i) done;
#set_digit_nat (nat1, #2, #0);
#complement_nat (nat1, #0, #2);
#debug_print_nat nat1;;
|00000000|FFFFFFFE|FFFFFFFF|() : unit

#add_nat (nat1, #0, #3, nat1, #1, #1, #0);;
0 : int

#debug_print_nat nat1;;
|00000000|FFFFFFFF|FFFFFFFD|() : unit

#let nat2 = copy_nat (nat1, #0, #3);;
Value nat2 is #<18446744073709551613> : nat

#add_nat (nat1, #0, #2, nat1, #1, #1, #0);;
1 : int

#debug_print_nat nat1;;
|00000000|00000000|FFFFFFFC|() : unit
```

```
#add_nat (nat2, #0, #3, nat2, #1, #1, #0);;
0 : int

#debug_print_nat nat2;;
|00000001|00000000|FFFFFFFC|() : unit
```

4.6.2 Subtraction operators on natural numbers

complement_nat$^{\dagger *}$**:** nat × int × int → unit
decr_nat$^{\dagger *}$**:** nat × int × int × int → int
sub_nat$^{\dagger *}$**:** nat × int × int × nat × int × int × int → int

Complement of natural numbers

`complement_nat (nat, off, len)` replaces each digit of the subnat $nat_{off,len}$ with its B-complement, corresponding to logical inversion of each bit of each concerned digit.

More formally speaking `complement_nat` performs the following operation :

$$\overline{nat_{off,len}} = \overline{\sum_{i=0}^{i<len} nat_{off+i}\, B^i} = \sum_{i=0}^{i<len} \overline{nat_{off+i}}\, B^i = \sum_{i=0}^{i<len} (B - nat_{off+i} - 1)\ B^i.$$

No side effect is performed if $len = 0$.

Example

```
#let nat = create_nat #4;;
Value nat is #<0> : nat

#for i = #0 to #3 do
#  set_digit_nat (nat, i, i)
#done;
#debug_print_nat nat;;
|00000003|00000002|00000001|00000000|() : unit

#complement_nat (nat, #1, #2);
#debug_print_nat nat;;
|00000003|FFFFFFFD|FFFFFFFE|00000000|() : unit
```

Decrementing natural numbers

`decr_nat (nat, off, len, b_in)` subtracts in place the borrow in b_in ($b\_in = 0$ or $b\_in = 1$) to the subnat $nat_{off,len}$ and returns the borrow out b_out.

More formally speaking `decr_nat` performs the following operation :

$$\begin{aligned} nat_{off,len} + B^{len} + b\_in - 1 &= \\ \left(\sum_{i=0}^{i<len} nat_{off+i}\, B^i\right) + B^{len} + b\_in - 1 &= \end{aligned}$$

$$\left(\sum_{i=0}^{i<len} nat'_{off+i}\, B^i\right) + b\_out\, B^{len}.$$

`decr_nat` replaces each digit nat_{off+i} with its equivalent digit nat'_{off+i} and returns the borrow out b_out, with value 0 or 1.

$b\_out = 0$ if and only if $nat_{off,len} = 0$ and $b\_in = 0$. len may be null, so the borrow out equals the borrow in.

Example

```
#let nat = make_nat #4;;
Value nat is #<0> : nat

#set_digit_nat (nat, #3, #1);;
() : unit

#debug_print_nat nat;;
|00000001|00000000|00000000|00000000|() : unit

#decr_nat (nat, #0, #4, #1);;
1 : int

#debug_print_nat nat;;
|00000001|00000000|00000000|00000000|() : unit

#decr_nat (nat, #0, #4, #0);;
1 : int

#debug_print_nat nat;;
|00000000|FFFFFFFF|FFFFFFFF|FFFFFFFF|() : unit

#decr_nat (nat, #0, #4, #0);;
1 : int

#debug_print_nat nat;;
|00000000|FFFFFFFF|FFFFFFFF|FFFFFFFE|() : unit

#decr_nat (nat, #0, #4, #1);;
1 : int

#debug_print_nat nat;;
|00000000|FFFFFFFF|FFFFFFFF|FFFFFFFE|() : unit
```

Subtraction on natural numbers

`sub_nat (nat1, off1, len1, nat2, off2,len2, b_in)` subtracts $nat2_{off2,len2}$ and the borrow in b_in to $nat1_{off1,len1}$, places result in subnat $nat1_{off1,len1}$, propagates the borrow of this subtraction on subnat $nat1_{off1+len2,len1-len2}$ and returns the borrow out b_out.

More formally speaking `sub_nat` performs the following operation :

$$nat1_{off1,len1} - nat2_{off2,len2} + B^{len1} + b\_in - 1 =$$

$$\left(\sum_{i=0}^{i<len1} nat1_{off1+i}\, B^i\right) - \left(\sum_{i=0}^{i<len2} nat2_{off2+i}\, B^i\right) + B^{len1} + b\_in - 1 =$$
$$\left(\sum_{i=0}^{i<len1} nat'_{off1+i}\, B^i\right) + b\_out\, B^{len1}.$$

`sub_nat` replaces each digit $nat1_{off1+i}$ with its equivalent digit nat'_{off1+i} and returns the borrow out b_out, with value 0 or 1. This implies $0 \leq b\_in \leq 1$ and $len1 \geq len2$.

$nat1$ and $nat2$ can be physically the same one under the condition $off1 \leq off2$.

If $len2 = 0$, `sub_nat (nat1, off1, len1, nat2, off2, len2, b_in)` is equivalent to `decr_nat (nat1, off1, len1, b_in)`.

Example

```
#let nat = make_nat #4;;
Value nat is #<0> : nat

#complement_nat (nat, #3, #1);;
() : unit

#debug_print_nat nat;;
|FFFFFFFF|00000000|00000000|00000000|() : unit

#sub_nat (nat, #0, #1, nat, #1, #1, #0);;
0 : int

#debug_print_nat nat;;
|FFFFFFFF|00000000|00000000|FFFFFFFF|() : unit

#sub_nat (nat, #0, #2, nat, #2, #1, #0);;
1 : int

#debug_print_nat nat;;
|FFFFFFFF|00000000|00000000|FFFFFFFE|() : unit

#sub_nat (nat, #2, #1, nat, #3, #1, #0);;
0 : int

#debug_print_nat nat;;
|FFFFFFFF|00000000|00000000|FFFFFFFE|() : unit

#sub_nat (nat, #0, #3, nat, #0, #1, #0);;
0 : int

#debug_print_nat nat;;
|FFFFFFFF|FFFFFFFF|FFFFFFFF|FFFFFFFF|() : unit
```

4.6.3 Multiplication operators on natural numbers

mult_digit_nat : nat × int × int × nat × int × int × nat × int → int
mult_nat : nat × int × int × nat × int × int × nat × int × int → int
shift_left_nat : nat × int × int × nat × int × int → unit
square_nat : nat × int × int × nat × int × int → int

Multiplication of a natural number by a digit

`mult_digit_nat (nat1, off1, len1, nat2, off2, len2, nat3, off3)` multiplies subnat $nat2_{off2,len2}$ by digit $nat3_{off3}$, adds the result to $nat1_{off1,len1}$ and returns the carry out c_out.

More formally speaking `mult_digit_nat` performs the following operation :

$$\begin{aligned} & nat1_{off1,len1} + nat3_{off3} \times nat2_{off2,len2} = \\ & \left(\sum_{i=0}^{i<len1} nat1_{off1+i}\, B^i\right) + nat3_{off3} \left(\sum_{i=0}^{i<len2} nat2_{off2+i}\, B^i\right) = \\ & \left(\sum_{i=0}^{i<len1} nat'_{off1+i}\, B^i\right) + c\_out\, B^{len1}. \end{aligned}$$

`mult_digit_nat` replaces each digit $nat1_{off1+i}$ with its equivalent digit nat'_{off1+i} and returns the carry out c_out, with value 0 or 1. This implies $len1 > len2$.

If $len2 = 0$ and $c\_out = 0$ and no side effect is performed. $nat1$ and $nat2$ can be physically the same one under the condition $off1 \leq off2$. The digit $nat3_{off3}$ may be any digit of $nat1$ or $nat2$.
Beware:

→ For speed, the cases $nat3_{off3} = 0$ and $nat3_{off3} = 1$ are explicitly tested.

Example

```
#let nat1 = make_nat #5
#and nat2 = create_nat #4;;
Value nat1 is #<0> : nat
Value nat2 is #<0> : nat

#set_digit_nat (nat1, #1, #2);;
() : unit

#for i = #0 to #3 do set_digit_nat (nat2, i, i) done;;
() : unit

#debug_print_nat nat1;;
|00000000|00000000|00000000|00000002|00000000|() : unit

#debug_print_nat nat2;;
|00000003|00000002|00000001|00000000|() : unit

#mult_digit_nat (nat1, #0, #5, nat2, #0, #4, nat_of_int #15, #0);;
0 : int
```

```
#debug_print_nat nat1;;
|00000000|0000002D|0000001E|00000011|00000000|() : unit

#mult_digit_nat (nat1, #0, #5, nat2, #0, #4, nat2, #2);;
0 : int

#debug_print_nat nat1;;
|00000000|00000033|00000022|00000013|00000000|() : unit
```

Multiplication on natural numbers

`mult_nat (nat1, off1, len1, nat2, off2, len2, nat3, off3, len3)` multiplies subnats $nat2_{off2,len2}$ and $nat3_{off3,len3}$, adds the result to $nat1_{off1,len1}$ and returns the carry out c_out.

More formally speaking `mult_nat` performs the following operation :

$$\begin{aligned} & nat1_{off1,len1} + nat2_{off2,len2} \times nat3_{off3,len3} = \\ & \quad \left(\sum_{i=0}^{i<len1} nat1_{off1+i}\, B^i\right) + \left(\sum_{i=0}^{i<len2} nat2_{off2+i}\, B^i\right) \left(\sum_{i=0}^{i<len3} nat3_{off3+i}\, B^i\right) = \\ & \quad \left(\sum_{i=0}^{i<len1} nat'_{off1+i}\, B^i\right) + c\_out\, B^{len1}. \end{aligned}$$

`mult_nat` replaces each digit $nat1_{off1+i}$ with its equivalent digit nat'_{off1+i} and returns the carry out c_out, with value 0 or 1. This implies $len1 \geq len2 + len3$. If $len2 = 0$ and $c\_out = 0$ no side effect is performed.

Beware:

→ $nat2$ and $nat3$ have quite similar roles, but for speed reasons it is preferable that $len2 \geq len3$,

→ No overlap is possible between $nat1_{off1,len1}$ and $nat2_{off2,len2}$ or $nat3_{off3,len3}$.

Example

```
#let nat1 = create_nat #4
#and nat2 = create_nat #4;;
Value nat1 is #<0> : nat
Value nat2 is #<0> : nat

#for i = #0 to #3 do
#  set_digit_nat (nat2, i, i);
#  set_digit_nat (nat1, i, add_int (i, #4))
#done;
#debug_print_nat nat2;;
|00000003|00000002|00000001|00000000|() : unit

#debug_print_nat nat1;;
|00000007|00000006|00000005|00000004|() : unit

#mult_nat (nat1, #0, #4, nat2, #0, #2, nat2, #2, #2);;
0 : int
```

```
#debug_print_nat nat1;;
|00000007|00000009|00000007|00000004|() : unit

#complement_nat (nat2, #0, #4);;
() : unit

#debug_print_nat nat2;;
|FFFFFFFC|FFFFFFFD|FFFFFFFE|FFFFFFFF|() : unit

#mult_nat (nat1, #0, #4, nat2, #0, #2, nat2, #2, #2);;
1 : int

#debug_print_nat nat1;;
|00000003|00000008|0000000D|00000007|() : unit
```

Shifting left natural numbers

`shift_left_nat (nat1, off1, len1, nat2, off2, shift)` shifts left any digit of subnat $nat1_{off1,len1}$. The $shift$ bits out of any digit replace the $shift$ free bits of the next digit. $shift$ zeros replace the $shift$ free bits of the first digit and the least significant bits of digit $nat2_{off2}$ get back the $shift$ bits out of the last digit. So `shift_left_nat` multiplies $nat1_{off1,len1}$ by 2^{shift}.

More formally speaking `shift_left_nat` performs the following operation :

$$2^{shift}\, nat1_{off1,len1} = \\ 2^{shift}\left(\sum_{i=0}^{i<len1} nat1_{off1+i}\, B^i\right) = \\ \left(\sum_{i=0}^{i<len1} nat1'_{off1+i}\, B^i\right) + nat2'_{off2}\, B^{len1}.$$

`shift_left_nat` replaces each digit $nat1_{off1+i}$ with its equivalent digit $nat1'_{off1+i}$ and $nat2_{off2}$ with $nat2'_{off2}$. This implies $0 \leq shift < length\_of\_digit$.

Beware:

→ For speed, the case $shift = 0$ is explicitly tested.

Example

```
#let nat = create_nat #4;;
Value nat is #<0> : nat

#for i = #0 to #3 do set_digit_nat (nat, i, i) done;;
() : unit

#debug_print_nat nat;;
|00000003|00000002|00000001|00000000|() : unit

#shift_left_nat (nat, #1, #3, nat, #0, #8);;
() : unit
```

```
#debug_print_nat nat;;
|00000300|00000200|00000100|00000000|() : unit

#shift_left_nat (nat, #1, #3, nat, #0, #16);;
() : unit

#debug_print_nat nat;;
|03000000|02000000|01000000|00000000|() : unit

#shift_left_nat (nat, #1, #3, nat, #0, #16);;
() : unit

#debug_print_nat nat;;
|00000200|00000100|00000000|00000300|() : unit
```

Squaring natural numbers

`square_nat (nat1, off1, len1, nat2, off2, len2)` squared subnat $nat2_{off2,len2}$, adds the result to 2 $nat1_{off1,len1}$ and returns the carry out `c_out`.

More formally speaking `square_nat` performs the following operation :

$$2\ nat1_{off1,len1} + \left(nat2_{off2,len2}{}^2\right) = \left(\sum_{i=0}^{i<len1} nat1_{off1+i}\, B^i\right) + \left(\sum_{i=0}^{i<len2} nat2_{off2+i}\, B^i\right)^2 = \left(\sum_{i=0}^{i<len1} nat'_{off1+i}\, B^i\right) + c\_out\, B^{len1}.$$

`square_nat` replaces each digit $nat1_{off1+i}$ with its equivalent digit nat'_{off1+i} and returns the carry out c_out, with value 0 or 1. This implies $len1 \geq 2 \times len2$. If $len2$ is null, $c\_out = 0$ and no side effect is performed. No overlap is possible between $nat1_{off1,len1}$ and $nat2_{off2,len2}$.

Example

```
#let nat2 = create_nat #2
#and nat1 = make_nat #4;;
Value nat2 is #<0> : nat
Value nat1 is #<0> : nat

#set_digit_nat (nat2, #0, #1);
#set_digit_nat (nat2, #1, #2);
#debug_print_nat nat2;;
|00000002|00000001|() : unit

#square_nat (nat1, #0, #4, nat2, #0, #2);;
0 : int

#debug_print_nat nat1;;
|00000000|00000004|00000004|00000001|() : unit
```

4.6.4 Division operators on natural numbers

div_digit_nat$^{\dagger *}$**:** nat × int × nat × int × nat × int × int × nat × int → int
div_nat†**:** nat × int × int × nat × int × int → unit
shift_right_nat$^{\dagger *}$**:** nat × int × int × nat × int × int → unit
gcd_int_nat†**:** int × nat × int × int → int
gcd_nat†**:** nat × int × int × nat × int × int → int

Division of a natural number by a digit

`div_digit_nat (quo, offq, rem, offr, nat1, off1, len1, nat2, off2)` divides subnat $nat1_{off1,len1}$ by digit $nat2_{off2}$, places the quotient in subnat $quo_{offq,len1-1}$ and the remainder in digit rem_{offr}.

More formally speaking `div_digit_nat` satisfies the following equation :

$$\sum_{i=0}^{i<len1} nat1_{off1+i}\, B^i = \left(\sum_{i=0}^{i<len1-1} quo_{offq+i}\, B^i \right) \times nat2_{off2} + rem_{offr}$$

with $0 \leq rem_{offr} < nat2_{off2}$.

`div_digit_nat` implies $len1 > 1$, $nat1_{off1+len1-1} < nat2_{off2}$ and $length\_nat(quo) \geq off1 + len1 - 1$.

Example

```
#let nat = create_nat #4;;
Value nat is #<0> : nat

#for i = #0 to #3 do set_digit_nat (nat, i, i) done; debug_print_nat nat;;
|00000003|00000002|00000001|00000000|() : unit

#div_digit_nat (nat, #0, nat, #2, nat, #0, #3, nat, #3);;
3 : int

#debug_print_nat nat;;
|00000003|00000000|AAAAAAAB|00000000|() : unit

#div_digit_nat (nat, #0, nat, #2, nat, #0, #3, nat, #3);;
3 : int

#debug_print_nat nat;;
|00000003|00000000|38E38E39|00000000|() : unit

#div_digit_nat (nat, #0, nat, #2, nat, #0, #3, nat, #3);;
3 : int

#debug_print_nat nat;;
|00000003|00000002|12F684BD|AAAAAAAA|() : unit
```

```
#div_digit_nat (nat, #0, nat, #2, nat, #1, #2, nat, #0);;
0 : int

#debug_print_nat nat;;
|00000003|12F684BF|12F684BD|00000003|() : unit
```

Division of natural numbers

`div_nat (nat1, off1, len1, nat2, off2, len2)` divides subnat $nat1_{off1,len1}$ by subnat $nat2_{off2,len2}$, places the quotient in the subnat $nat1_{off1+len2,len1-len2}$ and the remainder in subnat $nat1_{\texttt{off1},\ \texttt{len2}}$.

More formally speaking `div_nat` verifies the following equation :

$$\sum_{i=0}^{i<len1} nat1_{off1+i}\, B^i =$$
$$\left(\sum_{i=0}^{i<len1-len2} nat'_{off1+len2+i}\, B^i\right) \times \left(\sum_{i=0}^{i<len2} nat2_{off2+i}\, B^i\right) + \left(\sum_{i=0}^{i<len2} nat'_{off+i}\, B^i\right)$$

with

$$0 \leq \sum_{i=0}^{i<len2} nat'_{off1+i}\, B^i < \sum_{i=0}^{i<len2} nat2_{off1+i}.$$

`div_nat` replaces each digit $nat1_{off1+i}$ with its equivalent digit nat'_{off1+i}. This implies $len1 \geq len2$ and $nat1_{off1+len1-1} < nat2_{off2+len2-1}$.

Example

```
#let nat = create_nat #5;;
Value nat is #<0> : nat

#for i = #0 to #4 do set_digit_nat (nat, i, i) done; debug_print_nat nat;;
|00000004|00000003|00000002|00000001|00000000|() : unit

#div_nat (nat, #0, #3, nat, #3, #2);;
() : unit

#debug_print_nat nat;;
|00000004|00000003|7FFFFFFF|00000003|80000003|() : unit
```

Shifting right natural numbers

`shift_right_nat (nat1, off1, len1, nat2, off2, shift)` shifts right each digit of $nat1_{off1,len1}$. The $shift$ bits out of any digit replace the $shift$ free bits of the previous digit. $shift$ zeros replace the $shift$ free bits of the last digit and the most significant bits of digit $nat2_{off2}$ get back the $shift$ bits out of the first digit. So `shift_right_nat` divides $nat1_{off1,len1}$ by 2^{shift}.

More formally speaking `shift_right_nat` verifies the following equation :

$$\sum_{i=0}^{i<len1} nat1_{off1+i}\, B^i = 2^{shift}\left(nat2'_{off2}\, B^{-1} + \sum_{i=0}^{i<len1} \texttt{nat1}'_{off1+i}\, B^i\right).$$

`shift_right_nat` replaces each digit $nat1_{off1+i}$ with its equivalent digit $\mathtt{nat1}'_{off1+i}$ and $nat2_{off2}$ with $\mathtt{nat2}'_{off2}$. This implies $0 \leq shift < length\_of\_digit$. For speed, the case $shift = 0$ is explicitly tested.
Example

```
#let nat = create_nat #4;;
Value nat is #<0> : nat

#for i = #0 to #3 do set_digit_nat (nat, i, i) done; debug_print_nat nat;;
|00000003|00000002|00000001|00000000|() : unit

#shift_right_nat (nat, #1, #3, nat, #0, #8);;
() : unit

#debug_print_nat nat;;
|00000000|03000000|02000000|01000000|() : unit

#shift_right_nat (nat, #1, #3, nat, #0, #16);;
() : unit

#debug_print_nat nat;;
|00000000|00000300|00000200|00000000|() : unit

#shift_right_nat (nat, #1, #3, nat, #0, #16);;
() : unit

#debug_print_nat nat;;
|00000000|00000000|03000000|02000000|() : unit
```

Greatest common divisor between an `int` and a `nat`

`gcd_int_nat (i, nat, off, len)` computes the gcd of “small” int i and subnat $nat_{off,len}$ and puts it in digit nat_{off}. If $i = 0$ then the resulting value of the function is 1 and digit nat_{off} is unchanged, else the resulting value of the function is 0 and the gcd is placed in digit nat_{off}.
Example

```
#let nat = create_nat #2;;
Value nat is #<0> : nat

#set_digit_nat (nat, #0, #3);
#set_digit_nat (nat, #1, #2);
#debug_print_nat nat;;
|00000002|00000003|() : unit

#gcd_int_nat (#10, nat, #0, #2);;
0 : int

#debug_print_nat nat;;
|00000002|00000005|() : unit
```

Greatest common divisor between natural numbers

`gcd_nat (nat1, off1, len1, nat2, off2, len2)` places the gcd of subnats $nat1_{off1,len1}$ and $nat2_{off2,len2}$ in $nat1_{off1,len}$ where len is the resulting value of `gcd_nat`. $len2$ can be smaller than $len1$ unless subnat $nat1_{off1,len1}$ is null.
Example

```
#let nat = create_nat #3;;
Value nat is #<0> : nat

#for i = #0 to #2 do set_digit_nat (nat, i, i) done;
#debug_print_nat nat;;
|00000002|00000001|00000000|() : unit

#let len = gcd_nat (nat, #0, #3, nat, #0, #2);;
Value len is 2 : int

#debug_print_nat (copy_nat (nat, #0, len));;
|00000001|00000000|() : unit
```

4.7 Comparisons on natural numbers

compare_digits_nat : nat × int × nat × int → int
compare_nat†**:** nat × int × int × nat × int × int → int
eq_nat†**:** nat × int × int × nat × int × int → bool
le_nat†**:** nat × int × int × nat × int × int → bool
ge_nat†**:** nat × int × int × nat × int × int → bool
lt_nat†**:** nat × int × int × nat × int × int → bool
gt_nat†**:** nat × int × int × nat × int × int → bool

Comparison between digits

`compare_digits_nat (nat1, off1, nat2, off2)` returns -1 if digit $nat1_{off1}$ is less than digit $nat2_{off2}$, 0 if they are equal and 1 otherwise.
Example

```
#let nat = create_nat #4;;
Value nat is #<0> : nat

#for i = #0 to #3 do set_digit_nat (nat, i, i) done; debug_print_nat nat;;
|00000003|00000002|00000001|00000000|() : unit

#compare_digits_nat (nat, #0, nat, #0);;
0 : int

#compare_digits_nat (nat, #0, nat, #1);;
-1 : int
```

```
#compare_digits_nat (nat, #1, nat, #0);;
1 : int

#complement_nat (nat, #0, #2);;
() : unit

#debug_print_nat nat;;
|00000003|00000002|FFFFFFFE|FFFFFFFF|() : unit

#compare_digits_nat (nat, #0, nat, #0);;
0 : int

#compare_digits_nat (nat, #0, nat, #1);;
1 : int

#compare_digits_nat (nat, #1, nat, #0);;
-1 : int
```

Comparison between natural numbers

compare_nat (nat1, off1, len1, nat2, off2, len2) returns -1 if subnat $nat1_{off1,len1}$ is less than subnat $nat2_{off2,len2}$, 0 if they are equal and 1 otherwise.
Beware:

→ For speed, if $len1 \neq len2$ then $compare\_nat(nat1, off1, len1, nat2, off2, len2) = compare\_int(len1, len2)$. So if one of the indicated length is greater than the real length of the corresponding subnat, the result of this comparison may be wrong.

Example

```
#let nat = create_nat #4;;
Value nat is #<0> : nat

#for i = #0 to #2 do set_digit_nat (nat, i, i) done; debug_print_nat nat;;
|00000000|00000002|00000001|00000000|() : unit

#compare_nat (nat, #1, #2, nat, #0, #2);;
1 : int

#compare_nat (nat, #2, #2, nat, #2, #1);;
1 : int

#compare_nat (nat, #3, #2, nat, #3, #1);;
1 : int
```

eq_nat, **le_nat**, **ge_nat**, **lt_nat** and **gt_nat** can be defined as follows :

```
let eq_nat args = eq_int (compare_nat args, #0)
and le_nat args = le_int (compare_nat args, #0)
and ge_nat args = ge_int (compare_nat args, #0)
and lt_nat args = lt_int (compare_nat args, #0)
and gt_nat args = gt_int (compare_nat args, #0)
;;
```

4.8 Coercion functions on natural numbers

4.8.1 Coercion between types `int` and `nat`

sys_int_of_nat : nat × int × int → int
int_of_nat†: nat → int
nat_of_int : int → nat

`sys_int_of_nat (nat, off, len)` converts subnat $nat_{off,len}$ into an `int` when it is possible.

`int_of_nat` converts its argument into an `int` when it is possible. This function is semantically equivalent to:

```
let int_of_nat nat = sys_int_of_nat (nat, #0, length_nat nat);;
```

Example

```
#let nat = create_nat #2;;
Value nat is #<0> : nat

#set_digit_nat (nat, #0, #1024);;
() : unit

#complement_nat (nat, #1, #1);;
() : unit

#debug_print_nat nat;;
|FFFFFFFF|00000400|() : unit

#sys_int_of_nat (nat, #0, #1);;
1024 : int

#sys_int_of_nat (nat, #1, #1);;

Evaluation Failed: failure "sys_int_of_nat"

#sys_int_of_nat (nat, #0, #2);;

Evaluation Failed: failure "sys_int_of_nat"

#int_of_nat (make_nat #2);;
0 : int
```

`nat_of_int` converts its argument into a nat unless it is negative.

Example

```
#nat_of_int #0;;
#<0> : nat

#nat_of_int #1;;
#<1> : nat

#nat_of_int #-1;;
```

```
Evaluation Failed: failure "nat_of_int"
```

4.8.2 Coercion between types `float` and `nat`

sys_float_of_nat†**:** nat × int × int → float
float_of_nat **:** nat → float
nat_of_float **:** float → nat

`sys_float_of_nat (nat, off, len)` converts subnat $nat_{off,len}$ into a floating point number.

`float_of_nat` converts its argument into a floating point number. This function is equivalent to:

```
let float_of_nat nat = sys_float_of_nat (nat, #0, length_nat nat);;
```

`nat_of_float` converts its floating point argument into a natural number when it is a positive integer.

Example

```
#let nat = create_nat #2;;
Value nat is #<0> : nat

#set_digit_nat (nat, #1, #1);;
() : unit

#sys_float_of_nat (nat, #0, #1);;
0.0 : float

#sys_float_of_nat (nat, #1, #1);;
1.0 : float

#float_of_nat nat;;
4294967000.0 : float

#nat_of_float #1.0;;
#<1> : nat

#nat_of_float #1.0e10;;
#<9999999000> : nat

#nat_of_float #1.5;;

Evaluation Failed: failure "sys_big_int_of_string"
```

4.8.3 Coercion between types `string` and `nat`

Reading natural numbers

sys_nat_of_string : int × string × int × int → nat
nat_of_string : string → nat

It is not possible to read directly natural numbers, but coercion from strings exists.

`sys_nat_of_string (base, s, off, len)` maps the substring $s_{off,len}$ in base `base` onto a natural number.

A string must verify the following convention for recognition as a natural number:

```
Separator ::= ’ ’ | ’\f’ | ’\n’ | |\r’ | ’\t’ | ’\\’
Pseudodigit ::= Digit | Separator
NAT ::= Separator* Digit Pseudodigit*
```

`nat_of_string s` is equivalent to:

```
let nat_of_string s =
  sys_nat_of_string (#10, s, #0, length_string s)
;;
```

Example

```
#sys_nat_of_string (#16, "FF", #0, #2);;
#<255> : nat

#nat_of_string "123";;
#<123> : nat
```

Printing natural numbers

base_digit_of_char†**:** char × int → int
sys_string_of_digit†**:** nat × int → string
string_of_digit†**:** nat → string
sys_string_list_of_nat†**:** int × nat × int × int → string list
sys_string_of_nat : int × string × nat × int × int × string → string
string_of_nat†**:** nat → string
string_for_read_of_nat†**:** nat → string
sys_print_nat†**:** int × string × nat × int × int × string → unit
print_nat†**:** nat → unit
print_nat_for_read†**:** nat → unit
debug_string_vect_nat†**:** nat → string vect
debug_string_nat†**:** nat → string
debug_print_nat†**:** nat → unit

`base_digit_of_char (c, base)` verifies `c` is a correct `base`-digit (as regards to usual conventions only uppercase letters are authorized in case of `base` greater than 10) and give its decimal value.

`sys_string_of_digit (nat, off)` map digit nat_{off} onto the corresponding string in 10-radix representation.

`string_of_digit nat` map first digit nat_0 onto the corresponding string in 10-radix representation.

`sys_string_list_of_nat (base, nat, off, len)` maps `base`-radix representation of subnat $nat_{off,len}$ onto a list of strings, the first string corresponds to the beginning of the number, and so on. Usually one string is enough for representing nats but for example the longest `nat` needs 32 strings for printing its 2-radix representation.

`sys_string_list_of_nat` is the most general printing function. It takes more time than the other printing functions but it can print any natural number even when all other printing functions fail. So the CAML system when printing values of type `nat`, `big_int`, `ratio` or `num` uses this function.

`sys_string_of_nat (base, before, nat, off, len, after)` writes into the resulting string first the string `before`, then the representation of subnat $nat_{off,len}$ in base `base`, and last the string `after`. This function is optimized in base 10.

`string_of_nat nat` is semantically (but not practically for efficiency reasons) equivalent to:

```
let string_of_nat nat =
  sys_string_of_nat (#10, "", nat, #0, length_nat nat, "")
;;
```

`string_for_read_of_nat nat` is analogous to `string_of_nat nat` except that the number is written between `#<` and `>`, according to conventions to write natural numbers.

This function is equivalent to:

```
let string_for_read_of_nat nat =
  sys_string_of_nat (#10, "#<", nat, #0, length_nat nat, ">")
;;
```

Beware:

→ This convention is not properly speaking a lexical convention since no natural number can be read directly, even though between `#<` and `>`.

`sys_print_of_nat (base, before, nat, off, len, after)` print the successive strings of `sys_string_list_of_nat (base, nat, off, len)`, between string `before` and `after`.

`print_nat` and `print_nat_for_read nat` are equivalent to:

```
let print_nat nat =
  sys_print_nat (#10, "", nat, #0,
                 num_digits_nat (nat, #0, length_nat nat), "")
;;

let print_nat_for_read nat =
  sys_print_nat (#10, "#<", nat, #0,
                 num_digits_nat (nat, #0, length_nat nat), ">")
;;
```

`debug_string_vect_nat nat` and `debug_string_nat` map nat onto its 16-radix representation, without normalizing the number according to its real length. If the result cannot fit in only one string, `debug_string_nat nat` fails but `debug_string_vect_nat nat` is a vector of more than one string, the first string correspond to the end of the number, and so on.

`debug_print_nat nat` prints the successive strings of `debug_string_vect_nat nat`.

Example

```
#base_digit_of_char (`9`, #10);;
9 : int

#base_digit_of_char (`A`, #10);;

Evaluation Failed: failure "base_digit_of_char"

#base_digit_of_char (`A`, #16);;
10 : int

#let nat = create_nat #2;;
Value nat is #<0> : nat

#set_digit_nat (nat, #0, #1);;
() : unit

#set_digit_nat (nat, #1, #2);;
() : unit

#string_of_digit nat;;
"1" : string

#sys_string_of_digit (nat, #1);;
"2" : string

#(* A good example for this function would have a really long result.
#   The resulting string should have more than biggest_int digits so
#   takes more than six pages, too long for doing it here but it works well *)
#sys_string_list_of_nat (#10, nat, #0, #1);;
#sys_string_of_nat (#16, "nat 16-radix representation = ", nat, #0, #2, "");;
["1"] : string list

#sys_string_of_nat (#2, "nat 2-radix representation = ", nat, #0, #2, "");;
"nat 2-radix representation = 1000000000000000000000000000000001" : string

#sys_string_of_nat (#8, "nat 8-radix representation = ", nat, #0, #2, "");;
"nat 8-radix representation = 100000000001" : string

#sys_string_of_nat (#10, "nat 10-radix representation = ", nat, #0, #2, "");;
"nat 10-radix representation = 8589934593" : string

#sys_string_of_nat (#10, "nat first digit 10-radix representation = ",
#                   nat, #0, #1, "");;
"nat first digit 10-radix representation = 1" : string

#string_of_nat nat;;
"8589934593" : string

#string_for_read_of_nat nat;;
"#<8589934593>" : string
```

```
#sys_print_nat (#16, "nat 16-radix representation = ", nat, #0, #2, "");;
nat 16-radix representation = 200000001() : unit

#print_nat nat;;
8589934593() : unit

#print_nat_for_read nat;;
#<8589934593>() : unit

#(* Same remark as for sys_string_list_of_nat *)
#debug_string_vect_nat nat;;
[|"|00000002|00000001|"|] : string vect

#debug_string_nat nat;;
"|00000002|00000001|" : string

#debug_print_nat nat;;
|00000002|00000001|() : unit
```

Special printing functions

sys_latex_print_nat†**:** int $\times$ string $\times$ nat $\times$ int $\times$ int $\times$ string $\rightarrow$ unit
latex_print_nat†**:** nat $\rightarrow$ unit
latex_print_for_read_nat†**:** nat $\rightarrow$ unit
sys_print_beautiful_nat†**:** int $\times$ string $\times$ nat $\times$ int $\times$ int $\times$ string $\rightarrow$ unit
print_beautiful_nat†**:** nat $\rightarrow$ unit

These special printing functions for natural numbers can be found into two libraries:

format_latex_numbers.ml: functions that split numbers into small lines so LaTeX is able to handle them correctly,

format_numbers.ml: functions to present the `nat` as a numerical table according to common conventions.

`sys_latex_print_nat (base, before, nat, off, len, after)` prints `base` representation of subnat $nat_{off,len}$ between `before` and `after`, cutting line each `latex_margin` characters.

`latex_print_nat` and `latex_print_for_read_nat` are equivalent to:

```
let latex_print_nat nat =
  sys_latex_print_nat (#10, "", nat, #0,
                       num_digits_nat (nat, #0, length_nat nat), "")
;;

let latex_print_for_read_nat nat =
  sys_latex_print_nat (#10, "#<", nat, #0,
                       num_digits_nat (nat, #0, length_nat nat), ">")
;;
```

As an example this manual uses `latex_print_for_read_nat` as the printer for natural numbers.

`sys_print_beautiful_nat (base, before, nat, off, len, after)` prints `base`-radix representation of subnat $nat_{off,len}$ between `before` and `after` as a numerical table according to classical conventions.

`print_beautiful_nat` is equivalent to:

```
let print_beautiful_nat nat =
  sys_print_beautiful_nat (#10, "", nat, #0,
                     num_digits_nat (nat, #0, length_nat nat), "")
;;
```

You can load the libraries with:

```
load_lib_file "format_latex_numbers";;
load_lib_file "format_numbers";;
```

Example

```
#let nat = make_nat #50;;
Value nat is #<0> : nat

#set_digit_nat (nat, #49, #1);;
() : unit

#print_beautiful_nat nat;;
10352 21304  67682 24942  50182 38987  58125 56497  26918 22829
15323 02783  39025 97280  29067 09060  12854 18454  96784 42287
66593 04257  74796 15304  02560 49867  14394 71495  42742 79767
67721 14810  59401 01347  02920 37191  58120 99936  58212 56286
17621 84379  22016 11343  67961 01316  26312 73818  64212 94556

73571 30891  00244 69518  17226 86663  79357 25012  27715 49253
01910 77163  01297 77340  98941 24750  20844 05834  13946 43737
89454 47786  92139 36390  56949 58460  67318 10387  73284 47182
89469 13683  23793 71211  71506 21872  25034 33123  21446 06272
33274 17368  32441 05465  856
() : unit
```

4.9 Logical operations on natural numbers

land_digit_nat : nat × int × nat × int → unit
lor_digit_nat : nat × int × nat × int → unit
lxor_digit_nat : nat × int × nat × int → unit

`land_digit_nat (nat1, off1, nat2, off2)` performs "logical and" on digits $nat1_{off1}$ and $nat1_{off2}$, the resulting digit is put in $nat1_{off1}$.

Example

```
#let nat = create_nat #4;;
Value nat is #<0> : nat

#for i = #0 to #3 do set_digit_nat (nat, i, i) done; debug_print_nat nat;;
```

```
|00000003|00000002|00000001|00000000|() : unit

#land_digit_nat (nat, #3, nat, #1); debug_print_nat nat;;
|00000001|00000002|00000001|00000000|() : unit

#land_digit_nat (nat, #0, nat, #2); debug_print_nat nat;;
|00000001|00000002|00000001|00000000|() : unit
```

`lor_digit_nat (nat1, off1, nat2, off2)` performs "logical or" on digits $nat1_{off1}$ and $nat1_{off2}$, the resulting digit is put in $nat1_{off1}$.
Example

```
#let nat = create_nat #4;;
Value nat is #<0> : nat

#for i = #0 to #3 do set_digit_nat (nat, i, i) done; debug_print_nat nat;;
|00000003|00000002|00000001|00000000|() : unit

#lor_digit_nat (nat, #3, nat, #1); debug_print_nat nat;;
|00000003|00000002|00000001|00000000|() : unit

#lor_digit_nat (nat, #0, nat, #2); debug_print_nat nat;;
|00000003|00000002|00000001|00000002|() : unit
```

`lxor_digit_nat (nat1, off1, nat2, off2)` performs "logical exclusive or" on digits $nat1_{off1}$ and $nat1_{off2}$, the resulting digit is put in $nat1_{off1}$.
Example

```
#let nat = create_nat #4;;
Value nat is #<0> : nat

#for i = #0 to #3 do set_digit_nat (nat, i, i) done; debug_print_nat nat;;
|00000003|00000002|00000001|00000000|() : unit

#lxor_digit_nat (nat, #3, nat, #1); debug_print_nat nat;;
|00000002|00000002|00000001|00000000|() : unit

#lxor_digit_nat (nat, #0, nat, #2); debug_print_nat nat;;
|00000002|00000002|00000001|00000002|() : unit
```

4.10 Miscellaneous power functions

sqrt_nat†**:** nat $\times$ int $\times$ int $\rightarrow$ nat
power_base_int†**:** int $\times$ int $\rightarrow$ nat
power_base_nat†**:** int $\times$ nat $\times$ int $\times$ int $\rightarrow$ nat

`sqrt_nat (nat, off, len)` computes the integer part of the square root of subnat $nat_{off,len}$.

`power_base_int (i, p)` computes i^p, where p is a positive `int`, and the resulting value is exact and a natural number.

`power_base_nat (base, nat, off, len)` computes `base` to $nat_{off,len}$ power.

Example

```
#let nat = nat_of_int #1729;;
Value nat is #<1729> : nat

#sqrt_nat (nat, #0, #1);;
#<41> : nat

#power_base_int (#10, #24);;
#<1000000000000000000000000> : nat

#power_base_nat (#10, nat_of_int #24, #0, #1);;
#<2003764205206896640> : nat
```

Chapitre 5

Big integers

Elements of type `big_int` are arbitrarily large signed numbers. All operations on this type are performed with copy update.

We have the following definition of type `big_int`:

```
type big_int = { Sign : int; Abs_Value : nat };;
```

In fact the sign is -1, 0 or 1 according to the classical conventions on signs as follows with generical writing:

```
let sign (x) =
  when x<0 -> -1
     | x=0 -> 0
     |  _  -> 1
;;
```

You cannot create a `big_int` with a sign different from -1, 0 and 1. The sign is supposed to correspond to the real sign of the number and particularly that the equivalence

$$sign(x) = 0 \Longleftrightarrow x = 0$$

is always respected. It is advisable not to introduce a `big_int` violating this convention by directly typing in such a record.

5.1 Creation of a `big_int`

create_big_int†: int × nat → big_int

We present here only the primitive function to create a `big_int`. You can of course create a `big_int` with coercion functions or using lexical convention.

`create_big_int (sign, nat)` verifies the sign condition and creates the corresponding `big_int`.

Example

```
#create_big_int (#1, nat_of_int #1);;
#(1) : big_int
```

```
#create_big_int (#0, nat_of_int #1);;

Evaluation Failed: failure "create_big_int"

#create_big_int (#-1, nat_of_int #1);;
#(-1) : big_int

#create_big_int (#0, nat_of_int #0);;
#(0) : big_int

#create_big_int (#1, nat_of_int #0);;

Evaluation Failed: failure "create_big_int"
```

5.2 Arithmetic operations for `big_ints`

5.2.1 Incrementing `big_ints`

pred_big_int : big_int → big_int
succ_big_int : big_int → big_int

`pred_big_int n` (resp. `succ_big_int n`) is a primitive equivalent to `n-1` (resp `n+1`).

Example

```
#succ_big_int (big_int_of_int biggest_int);;
#(32768) : big_int

#succ_big_int (big_int_of_int monster_int);;
#(-32767) : big_int

#pred_big_int (big_int_of_int least_int);;
#(-32768) : big_int

#pred_big_int #(1.2e3);;
#(1199) : big_int

#succ_big_int #(1234567890);;
#(1234567891) : big_int
```

5.2.2 Arithmetic operations for `big_ints`

add_int_big_int†: int × big_int → big_int
add_big_int : big_int × big_int → big_int
minus_big_int : big_int → big_int
sub_big_int : big_int × big_int → big_int
mult_int_big_int†: int × big_int → big_int
mult_big_int : big_int × big_int → big_int
square_big_int†: big_int → big_int

quomod_big_int†**:** big_int × big_int → big_int × big_int
quo_big_int : big_int × big_int → big_int
div_big_int : big_int × big_int → big_int
mod_big_int : big_int × big_int → big_int
gcd_big_int†**:** big_int × big_int → big_int

`add_big_int`, `sub_big_int`, `mult_big_int` perform respectively the addition, subtraction and multiplication on `big_int`s.

`add_int_big_int (i, bi)` and `mult_int_big_int (i, bi)` perform respectively the addition and multiplication between the `int` i and the `big_int` bi and yield a `big_int`.

`minus_big_int bi` is the opposite of `bi`.

`square_big_int bi` computes the square of the `big_int bi` faster than multiplication in the manner of `square_nat`.

`quomod_big_int (bi1, bi2)` is the pair of the quotient and the remainder of the Euclidian division of `bi1` by `bi2`.

These two results may be consulted separately with `quo_big_int` (or `div_big_int` which is exactly the same function) and `mod_big_int` respectively.

`gcd_big_int (bi1, bi2)` computes the gcd of the two `big_int`s `bi1` and `bi2`. This gcd is always positive or null.

Example

```
#minus_big_int (big_int_of_int biggest_int);;
#(-32767) : big_int

#minus_big_int (big_int_of_int least_int);;
#(32767) : big_int

#minus_big_int (big_int_of_int monster_int);;
#(32768) : big_int

#mult_int_big_int (#2, big_int_of_int biggest_int);;
#(65534) : big_int

#quomod_big_int (#(123), #(3));;
(#(41),#(0)) : big_int * big_int

#quomod_big_int (#(123), #(-3));;
(#(-41),#(0)) : big_int * big_int
```

5.3 Comparisons between `big_ints`

eq_big_int†**:** big_int × big_int → bool
compare_big_int†**:** big_int × big_int → int
lt_big_int : big_int × big_int → bool
le_big_int : big_int × big_int → bool
gt_big_int : big_int × big_int → bool
ge_big_int : big_int × big_int → bool

sign_big_int†**:** big_int → int
abs_big_int : big_int → big_int
max_big_int : big_int × big_int → big_int
min_big_int : big_int × big_int → big_int
zero_big_int†**:** big_int

`eq_big_int`, `lt_big_int`, `le_big_int`, `gt_big_int`, `ge_big_int` are arithmetic comparisons corresponding respectively to `=`, `<`, `<=`, `>`, `>=`.

The primitive function for these functions is `compare_big_int`, that is semantically equivalent to:

```
let compare_big_int (bigint1, bigint2) =
 when (bigint1 = bigint2) -> #0
    | (bigint1 < bigint2) -> #-1
    |          _          -> #1;;
```

`sign_big_int` is completely equivalent to:

```
let sign_big_int bi = bi.Sign
;;
```

`abs_big_int bi` is the absolute value of the `big_int` `bi`.

`max_big_int (bi1, bi2)` and `min_big_int (bi1, bi2)` are respectively the greater and the smaller of the `big_ints` `bi1` and `bi2`.

`zero_big_int` is the null `big_int`.

Example

```
#eq_big_int (succ_big_int (big_int_of_int biggest_int),
#            big_int_of_int monster_int);;
false : bool

#gt_big_int (succ_big_int (big_int_of_int biggest_int),
#            big_int_of_int biggest_int);;
true : bool

#sign_big_int (succ_big_int (big_int_of_int biggest_int));;
1 : int

#sign_big_int (big_int_of_int monster_int);;
-1 : int

#abs_big_int (big_int_of_int monster_int);;
#(32768) : big_int
```

5.4 Coercion functions on big_ints

5.4.1 Coercion between ints and big_ints

is_int_big_int†: big_int → bool
int_of_big_int†: big_int → int
big_int_of_int†: int → big_int

`is_int_big_int bi` yields true if and only if the `big_int bi` fits in an `int`.
`int_of_big_int bi` is the `int` representing the `big_int bi` if `bi` fits in an `int`.
`big_int_of_int` maps an `int` onto the corresponding `big_int`.

Example

```
#is_int_big_int (big_int_of_int biggest_int);;
true : bool

#is_int_big_int (big_int_of_int monster_int);;
true : bool

#is_int_big_int (abs_big_int (big_int_of_int monster_int));;
false : bool

#is_int_big_int (succ_big_int (big_int_of_int biggest_int));;
false : bool

#int_of_big_int (big_int_of_int biggest_int);;
32767 : int

#int_of_big_int (succ_big_int (big_int_of_int biggest_int));;

Evaluation Failed: failure "int_of_big_int"

#big_int_of_int monster_int;;
#(-32768) : big_int
```

5.4.2 Coercion between floating point numbers and big_int

float_of_big_int†: big_int → float
big_int_of_float†: float → big_int

`float_of_big_int` and `big_int_of_float` map elements of type `big_int` into elements of type `float`.

Example

```
#float_of_big_int #(1.23456789e8);;
123456800.0 : float

#big_int_of_float it;;
#(123456800) : big_int
```

5.4.3 Coercion between nats and big_ints

big_int_of_nat†: nat → big_int
nat_of_big_int†: big_int → nat

`big_int_of_nat` maps naturally a `nat` onto the corresponding `big_int`.
`nat_of_big_int bi` maps `big_int bi` onto the corresponding `nat` if `bi` is positive.
`is_big_int_nat` can be written as :

```
#let is_big_int_nat bi = ge_int (sign_big_int bi, #0);;
Value is_big_int_nat is <fun> : big_int -> bool
```

Example

```
#big_int_of_nat (nat_of_string "123456789");;
#(123456789) : big_int

#nat_of_big_int #(1.2e3);;
#<1200> : nat

#nat_of_big_int #(-1.2e3);;

Evaluation Failed: failure "nat_of_big_int"

#is_big_int_nat #(1.2e3);;
true : bool

#is_big_int_nat #(-1.2e3);;
false : bool
```

5.4.4 Coercion between strings and `big_int`

Reading `big_ints`

simple_big_int_of_string†: int × string × int × int → big_int
decimal_of_string†: int × string × int × int → string × int
sys_big_int_of_string†: int × string × int × int → big_int
big_int_of_string†: string → big_int

CAML is able to read `big_int` entered directly, according to the following lexical convention:

```
Separator ::= ’ ’ | ’\f’ | ’\n’ | |\r’ | ’\t’ | ’\\’
Pseudodigit ::= Digit | Separator
Decimal ::= {’-’| ’+’} Separator* Digit Pseudodigit*
            {’.’ Separator* Digit Pseudodigit*} {’e’ {’-’ | ’+’} INT}
BIG_INT ::= ’#(’ Decimal ’)’
```

Example

```
##(12);;
#(12) : big_int
```

```
##(12.0);;
#(12) : big_int

##(1.2e1);;
#(12) : big_int

##(120e-1);;
#(12) : big_int
```

`simple_big_int_of_string` recognizes simple expressions for `big_int`, i.e. without mantissa and without floating point.

`decimal_of_string` is the fundamental function for recognizing numerical expressions with floating point and mantissa. `decimal_of_string (base, s, off, len)` yields the pair of a string representing a simple `big_int` and an `int` representing the corresponding mantissa. Roughly speaking, if $s_{off,len}$ is of the form `s1.s2es3` then the result is `(s1^s2, s3-length_string s2)`.

`sys_big_int_of_string (base, s, off, len)` maps the substring $s_{off,len}$ in base `base` onto a `big_int`.

`big_int_of_string s` is equivalent to:

```
let big_int_of_string s =
  sys_big_int_of_string (#10, s, #0, length_string s)
;;
```

Example

```
#simple_big_int_of_string (#10, "1200", #0, #4);;
#(1200) : big_int

#simple_big_int_of_string (#10, "1.2e3", #0, #5);;

Evaluation Failed: failure "base_digit_of_char"

#decimal_of_string (#10, "1200", #0, #4);;
("1200",0) : string * int

#decimal_of_string (#10, "1.2e3", #0, #5);;
("12",2) : string * int

#decimal_of_string (#10, "1.2e-1", #0, #6);;
("12",-2) : string * int

#sys_big_int_of_string (#16, "FF", #0, #2);;
#(255) : big_int

#big_int_of_string "1.2e3";;
#(1200) : big_int
```

Printing big_ints

sys_string_of_big_int : int × string × big_int × string → string
string_of_big_int†: big_int → string
string_for_read_of_big_int†: big_int → string
sys_print_big_int†: int × string × big_int × string → unit
print_big_int†: big_int → unit
print_big_int_for_read : big_int → unit

`sys_string_of_big_int (base, before, bi, after)` writes into the resulting string first the string `before`, then the representation of the `big_int bi` in the base `base`, and finally the string `after`.

This function is equivalent to:

```
let sys_sys_string_of_big_int (base, before, bi, after) =
 sys_string_of_nat (base,
                   (if eq_int (bi.Sign, #-1)
                       then (before ^ "-") else before),
                   bi.Abs_Value, #0, num_digits_big_int bi, after)
;;
```

`string_of_big_int bi` is equivalent to:

```
let string_of_big_int bi =
  sys_string_of_big_int (#10, "", bi, "")
;;
```

`string_for_read_of_big_int bi` is analogous to `string_of_big_int bi` except that the number is written between `#(` and `)`, according to conventions to write `big_int`s.

It is equivalent to:

```
let string_for_read_of_big_int bi =
 sys_string_of_big_int (#10, "#(", bi, ")")
;;
```

`sys_print_of_big_int (base, before, bi, after)` prints the `base`-radix representation of the `big_int bi` between strings `before` and `after`.

`print_big_int` and `print_big_int_for_read nat` are equivalent to:

```
let print_big_int bi = sys_print_big_int (#10, "", bi, "");;

let print_big_int_for_read bi = sys_print_big_int (#10, "#(", bi, ")");;
```

Example

```
#sys_string_of_big_int
#  (#16, "big_int 16-radix representation = ", #(1.2e3), "");;
"big_int 16-radix representation = 4B0" : string

#sys_string_of_big_int (#2, "big_int 2-radix representation = ", #(1.2e3), "");;
"big_int 2-radix representation = 10010110000" : string

#sys_string_of_big_int (#8, "big_int 8-radix representation = ", #(1.2e3), "");;
```

```
"big_int 8-radix representation = 2260" : string

#sys_string_of_big_int
#  (#10, "big_int 10-radix representation = ", #(1.2e3), "");;
"big_int 10-radix representation = 1200" : string

#string_of_big_int #(1.2e3);;
"1200" : string

#string_for_read_of_big_int #(1.2e3);;
"#(1200)" : string

#sys_print_big_int (#16, "big_int 16-radix representation = ", #(1.2e3), "");;
big_int 16-radix representation = 4B0() : unit

#print_big_int #(1.2e3);;
1200() : unit

#print_big_int_for_read #(1.2e3);;
#(1200)() : unit
```

Special printing functions

sys_latex_print_big_int†**:** int $\times$ string $\times$ big_int $\times$ string $\rightarrow$ unit
latex_print_big_int†**:** big_int $\rightarrow$ unit
latex_print_for_read_big_int†**:** big_int $\rightarrow$ unit
sys_print_beautiful_big_int†**:** α $\times$ string $\times$ big_int $\times$ string $\rightarrow$ unit
print_beautiful_big_int†**:** big_int $\rightarrow$ unit

These special printing functions for `big_int`s cna be found into two libraries:

format_latex_numbers.ml: functions that split numbers into small lines so LaTeX is able to handle them correctly,

format_numbers.ml: functions to present the `big_int` as a numerical table according to common conventions.

`sys_latex_print_big_int (base, before, bi, after)` prints `base` representation of the `big_int` `bi` between `before` and `after`, cutting line each `latex_margin` characters.
`latex_print_big_int` and `latex_print_for_read_big_int` are equivalent to:

```
let latex_print_big_int bi =
  sys_latex_print_big_int (#10, "", bi, "")

and latex_print_for_read_big_int bi =
  sys_latex_print_big_int (#10, "#(", bi, ")")
;;
```

As an example this manual uses `latex_print_for_read_big_int` as the printer for `big_int`s.

`sys_print_beautiful_big_int (base, before, bi, after)` prints `base`-radix representation of the `big_int bi` between `before` and `after` as a numerical table according to classical conventions.

`print_beautiful_big_int` is equivalent to:

```
let print_beautiful_big_int bi =
  sys_print_beautiful_big_int (#10, "", bi, "")
;;
```

You can load the libraries with:

```
load_lib_file "format_latex_numbers";;
load_lib_file "format_numbers";;
```

Example

```
#print_beautiful_big_int #(1e312);;
10000 00000  00000 00000  00000 00000  00000 00000  00000 00000
00000 00000  00000 00000  00000 00000  00000 00000  00000 00000
00000 00000  00000 00000  00000 00000  00000 00000  00000 00000
00000 00000  00000 00000  00000 00000  00000 00000  00000 00000
00000 00000  00000 00000  00000 00000  00000 00000  00000 00000

00000 00000  00000 00000  00000 00000  00000 00000  00000 00000
00000 00000  000
() : unit
```

5.5 Miscellaneous information functions on `big_ints`

nth_digit_big_int†**:** big_int × big_int → int
leading_digit_big_int†**:** big_int → int
num_digits_big_int†**:** big_int → int

`nth_digit_big_int (bi, n)` returns the n-th 2^{32}-digit of the absolute value of bi.

`leading_digit_big_int bi` sis the leading digit of the 2^{32}-radix representation of the `big_int bi`.

`num_digits_big_int bi` is the length of the 2^{32}-radix representation of the `big_int bi`.

5.6 Miscellaneous power functions

sqrt_big_int†**:** big_int → big_int
power_int_positive_int†**:** int × int → big_int
power_int_positive_big_int†**:** int × big_int → big_int
base_power_big_int†**:** int × int × big_int → big_int
power_big_int_positive_int†**:** big_int × int → big_int
power_big_int_positive_big_int†**:** big_int × big_int → big_int

`sqrt_big_int bi` computes the integer part of the square root of the `big_int bi`.

`power_int_positive_int (i1, i2)` yields the `big_int` resulting from the computation of $i1^{i2}$ where $i1$ and $i2$ are `int`s and $i2 \geq 0$.

`power_int_positive_big_int (i, bi)` yields the `big_int` resulting from the computation of i^{bi} where bi is a positive `big_int`.

`base_power_big_int (base, n, bi)` computes $bi \times base^n$ as far as it is an integer.

`power_big_int_positive_int (bi, i)` yields the `big_int` resulting from the computation of bi^i where i is a positive `int`.

`power_big_int_positive_big_int(bi1, bi2)` yields the `big_int` resulting from the computation of $bi1^{bi2}$ where $bi2$ is a positive `big_int`.

Example

```
#sqrt_big_int #(1e10);;
#(100000) : big_int

#power_int_positive_int (#10, #11);;
#(100000000000) : big_int

#power_int_positive_big_int (#10, #(11));;
#(1215752192) : big_int

#base_power_big_int (#10, #2, #(12));;
#(1200) : big_int

#base_power_big_int (#10, #-2, #(1.2e3));;

Evaluation Failed: failure "base_power_big_int"

#base_power_big_int (#10, #-1, #(12));;

Evaluation Failed: failure "base_power_big_int"

#power_big_int_positive_int (#(10), #11);;
#(100000000000) : big_int

#power_big_int_positive_big_int (#(10), #(11));;
#(100000000) : big_int
```

Chapitre 6

Rational numbers of type ratio

Elements of type `ratio` are arbitrary large rational numbers. They are internally represented with a couple of `big_int`s, the second one being a positive integer. *infinity* (`1/0` or `-1/0`) and *undefined* (`0/0`) exist as natural values of type `ratio` on request.

We have the following definition of type `ratio`:

```
type ratio = { Numerator : big_int; Denominator : big_int; Normalized : bool };;
```

The denominator is supposed to be positive and the sign of an element of type `ratio` is supposed to be the sign of its numerator. It is advisable not to introduce a value of type `ratio` violating this convention by typing in such a record directly.

6.1 Normalization of `ratio`s

normalize_ratio†**:** ratio → ratio
get_normalize_ratio†**:** unit → bool
set_normalize_ratio†**:** bool → bool
cautious_normalize_ratio†**:** ratio → ratio
get_normalize_ratio_when_printing†**:** unit → bool
set_normalize_ratio_when_printing†**:** bool → bool
cautious_normalize_ratio_when_printing†**:** ratio → ratio
is_normalized_ratio†**:** ratio → bool

Normalization is adaptable. Two flags allow you to choose when you want to normalize rational numbers: after each computation, only before printing, or never. You can set or consult them with the `(g|s)et_normalize_ratio{_when_printing}` functions. For more information see chapter 8.

To force the normalization of a particular rational number, use the normalization primitive `normalize_ratio`.

On the other hand, if you want the normalization to be performed only if one of these flags is turned on, use `cautious_normalize_ratio` or `cautious_normalize_ratio_when_printing`.

When the `Normalized` label of a `ratio` is set to `true`, normalization is not performed anymore on this number. So the gcd of the numerator and denominator of a rational number is computed at the most once. `is_normalized_ratio r` tells you if the rational number `r` is already normalized.

6.2 Creation of a `ratio`

create_ratio†: big_int × big_int → ratio
create_normalized_ratio†: big_int × big_int → ratio

We present here only the primitive functions to create a number of type `ratio`: you can of course create a `ratio` with coercion functions or using lexical convention.

`create_ratio (num, den)` creates the `ratio` with numerator (resp. denominator) equal to `num` (resp. `den`), set up the sign of the rational number, verifies if its denominator is not null if necessary, normalizes this rational if normalization during computing is requested and sets `Normalized` label with the value of this normalization flag.

`create_normalized_ratio (num, den)` creates the `ratio` with numerator (resp. denominator) equal to `num` (resp. `den`), reports the eventual sign of `den` on the numerator of the created `ratio`, verifies if denominator is not null when null denominators are prohibited and sets `Normalized` label with *true*. It is interesting to use this creation function when you are sure a rational number to be already normalized, since this `ratio` will be no more normalized, but off course you must be right.

By default null denominators are prohibited. For more information see chapter 8.

Example

```
#create_ratio (#(2), #(3));;
#[2/3] : ratio

#create_ratio (#(2), #(-3));;
#[-2/3] : ratio

#create_ratio (#(4), #(6));;
#[2/3] : ratio

#create_ratio (#(1), #(0));;

Evaluation Failed: failure "create_ratio infinite or undefined rational number"

#create_normalized_ratio (#(4), #(-6));;
#[-4/6] : ratio
```

6.3 Miscellaneous information functions on `ratios`

numerator_ratio†: ratio → big_int
denominator_ratio†: ratio → big_int
null_denominator†: ratio → bool
verify_null_denominator†: ratio → bool

`numerator_ratio` and `denominator_ratio` consult the corresponding label of the record.

`null_denominator r` and `verify_null_denominator` test if the denominator of the `ratio` is null. `verify_null_denominator` fails when a null denominator is encountered and null denominators are prohibited.

Example

```
#null_denominator {Numerator=#(1);Denominator=#(0);Normalized=false};;
true : bool

#verify_null_denominator {Numerator=#(1);Denominator=#(0);Normalized=false};;

Evaluation Failed: failure "infinite or undefined rational number"
```

6.4 Arithmetic operations for ratios

add_int_ratio†**:** int × ratio → ratio
add_big_int_ratio†**:** big_int × ratio → ratio
add_ratio : ratio × ratio → ratio
minus_ratio : ratio → ratio
sub_ratio : ratio × ratio → ratio
mult_int_ratio†**:** int × ratio → ratio
mult_big_int_ratio†**:** big_int × ratio → ratio
mult_ratio : ratio × ratio → ratio
square_ratio†**:** ratio → ratio
inverse_ratio†**:** ratio → ratio
div_int_ratio†**:** int × ratio → ratio
div_big_int_ratio†**:** big_int × ratio → ratio
div_ratio_int†**:** ratio × int → ratio
div_ratio_big_int†**:** ratio × big_int → ratio
div_ratio : ratio × ratio → ratio

`add_ratio` adds two rational numbers.

`add_int_ratio` (resp. `add_big_int_ratio`) performs the addition of an `int` (resp. `big_int`) to a rational number. These functions are semantically equivalent to:

```
let add_int_ratio (i, r) = add_ratio (ratio_of_int i, r)
and add_big_int_ratio (bi, r) = add_ratio (ratio_of_big_int bi, r)
;;
```

`minus_ratio r` is the opposite of the rational number r.

`sub_ratio` is the subtraction between rational numbers.

`mult_ratio` multiplies two rational numbers.

`mult_int_ratio` (resp. `mult_big_int_ratio`) performs the multiplication of an `int` (resp. `big_int`) to a rational number. These functions are semantically equivalent to:

```
let mult_int_ratio (i, r) = mult_ratio (ratio_of_int i, r)
and mult_big_int_ratio (bi, r) = mult_ratio (ratio_of_big_int bi, r)
;;
```

`square_ratio r` computes the square of the rational number r. It is equivalent to:

```
let square_ratio r =
  { Numerator = square_big_int r.Numerator;
    Denominator = square_big_int r.Denominator;
    Normalized = r.Normalized }
;;
```

`inverse_ratio` is equivalent to:

```
let inverse_ratio r =
  { Numerator = r.Denominator;
    Denominator = r.Numerator;
    Normalized = r.Normalized}
;;
```

`div_ratio` performs the division on rational numbers.

`div_int_ratio` (resp. `div_big_int_ratio`) performs the division of an `int` (resp. `big_int`) by a rational number. These functions are semantically equivalent to:

```
let div_int_ratio (i, r) = div_ratio (ratio_of_int i, r)
and div_big_int_ratio (bi, r) = div_ratio (ratio_of_big_int bi, r)
;;
```

`div_ratio_int` (resp. `div_ratio_big_int`) performs the division of a rational number by an `int` (resp. `big_int`). These functions are semantically equivalent to:

```
let div_ratio_int (r, i) = div_ratio (r, ratio_of_int i)
and div_ratio_big_int (r, bi) = div_ratio (r, ratio_of_big_int bi)
;;
```

Example

```
#add_ratio (#[1/2], #[1/2]);;
#[1/1] : ratio

#sub_ratio (#[1/2], #[1/2]);;
#[0/1] : ratio

#inverse_ratio #[1/2];;
#[2/1] : ratio

#mult_int_ratio (#2, #[1/2]);;
#[1/1] : ratio

#square_ratio #[1/3];;
#[1/9] : ratio

#div_int_ratio (#2, #[2]);;
#[1/1] : ratio
```

6.5 Rounding `ratios`

is_integer_ratio†: ratio → bool
integer_ratio†: ratio → big_int
floor_ratio†: ratio → big_int
round_ratio†: ratio → big_int
ceiling_ratio†: ratio → big_int

`is_integer_ratio` normalizes its argument, in order to determine if it is an integer. It is equivalent to:

```
let is_integer_ratio r =
 eq_big_int ((normalize_ratio r).Denominator, unit_big_int)
;;
```

`integer_ratio` is the odd function of truncature for rational numbers.

`floor_ratio` (resp. `round_ratio`, `ceiling_ratio`) are the mathematical functions for rounding to the lower (resp. nearest, higher) integer.

Example

```
#is_integer_ratio #[1.2e3/1200];;
true : bool

#integer_ratio #[1/3];;
#(0) : big_int

#floor_ratio #[1/3];;
#(0) : big_int

#round_ratio #[1/3];;
#(0) : big_int

#ceiling_ratio #[1/3];;
#(1) : big_int

#integer_ratio #[1/2];;
#(0) : big_int

#floor_ratio #[1/2];;
#(0) : big_int

#round_ratio #[1/2];;
#(1) : big_int

#ceiling_ratio #[1/2];;
#(1) : big_int

#integer_ratio #[-1/3];;
#(0) : big_int

#floor_ratio #[-1/3];;
#(-1) : big_int

#round_ratio #[-1/3];;
#(0) : big_int

#ceiling_ratio #[-1/3];;
#(0) : big_int

#integer_ratio #[-1/2];;
#(0) : big_int
```

```
#floor_ratio #[-1/2];;
#(-1) : big_int

#round_ratio #[-1/2];;
#(-1) : big_int

#ceiling_ratio #[-1/2];;
#(0) : big_int
```

6.6 Comparisons between and with `ratio`s

eq_big_int_ratio†**:** big_int × ratio → bool
compare_big_int_ratio†**:** big_int × ratio → int
lt_big_int_ratio†**:** big_int × ratio → bool
le_big_int_ratio†**:** big_int × ratio → bool
gt_big_int_ratio†**:** big_int × ratio → bool
ge_big_int_ratio†**:** big_int × ratio → bool
eq_ratio†**:** ratio × ratio → bool
compare_ratio†**:** ratio × ratio → int
lt_ratio : ratio × ratio → bool
le_ratio : ratio × ratio → bool
gt_ratio : ratio × ratio → bool
ge_ratio : ratio × ratio → bool
sign_ratio†**:** ratio → int
report_sign_ratio†**:** ratio × big_int → big_int
abs_ratio : ratio → ratio
min_ratio : ratio × ratio → ratio
max_ratio : ratio × ratio → ratio

`lt_ratio`, `le_ratio`, `gt_ratio`, `ge_ratio` (resp. `lt_big_int_ratio`, `le_big_int_ratio`, `gt_big_int_ratio`, `ge_big_int_ratio`) are arithmetic comparisons corresponding respectively to `<`, `<=`, `>`, `>=` between elements of type `ratio` (resp. element of type `big_int` and `ratio`).

The primitive function for these functions is `compare_ratio` (resp. `compare_big_int_ratio`), that is semantically equivalent to:

```
let compare_ratio (r1, r2) =
 when (r1 = r2) -> #0
    | (r1 < r2) -> #-1
    |     _     -> #1

and compare_big_int_ratio (bi, r) =
 when (bi = r) -> #0
    | (bi < r) -> #-1
    |     _     -> #1
;;
```

`sign_ratio` is equivalent to:

```
let sign_ratio r = sign_big_int r.Numerator ;;
```

`report_sign_ratio (bi, r)` transfers the sign of the `big_int bi` on the rational number `r`.

`abs_ratio r` is the absolute value of the rational number `r`.

`max_ratio (r1, r2)` and `min_ratio (r1, r2)` are respectively the greater one and the smaller one of rational numbers `r1` and `r2`.

Example

```
#compare_ratio (#[1.2e3],#[12e2]);;
0 : int

#compare_big_int_ratio (#(1.2e3),#[1200/1]);;
0 : int

#compare_big_int_ratio (#(1.2e3),#[12e2]);;
0 : int
```

6.7 Coercion functions on ratios

6.7.1 Coercion between ints and ratios

int_of_ratio†: ratio → int
ratio_of_int†: int → ratio

`int_of_ratio r` is the `int` representing the rational number `r` if `r` fits in an `int`, otherwise `int_of_ratio` fails.

`ratio_of_int` maps naturally an `int` onto the corresponding `ratio`.

Example

```
#int_of_ratio #[1.2e3];;
1200 : int

#int_of_ratio (add_int_ratio (#1, ratio_of_int biggest_int));;

Evaluation Failed: failure "int_of_ratio"

#int_of_ratio (add_int_ratio (#-1, ratio_of_int least_int));;
-32768 : int

#ratio_of_int #1;;
#[1/1] : ratio
```

6.7.2 Coercion between nats and ratios

nat_of_ratio†: ratio → nat
ratio_of_nat†: nat → ratio

`nat_of_ratio r` maps rational number `r` onto the corresponding `nat`, if `r` is an integer and positive.

`ratio_of_nat` maps naturally a `nat` onto the corresponding rational number.

Example

```
#nat_of_ratio #[1.2e3];;
#<1200> : nat

#nat_of_ratio #[-1.2e3];;

Evaluation Failed: failure "nat_of_ratio"

#nat_of_ratio #[1.2];;

Evaluation Failed: failure "nat_of_ratio"

#nat_of_ratio #[1/2];;

Evaluation Failed: failure "nat_of_ratio"
```

6.7.3 Coercion between `big_int`s and `ratio`s

big_int_of_ratio†**:** ratio → big_int
ratio_of_big_int†**:** big_int → ratio

`big_int_of_ratio r` maps rational number `r` onto the corresponding `big_int`, if `r` is an integer.
`ratio_of_big_int` maps naturally a `big_int` onto the corresponding rational number.

Example

```
#big_int_of_ratio #[1.2];;

Evaluation Failed: failure "big_int_of_ratio"
```

6.7.4 Coercion between floating point numbers and `ratio`s

float_of_ratio†**:** ratio → float
ratio_of_float†**:** float → ratio

`float_of_ratio` and `ratio_of_float` map elements of type `ratio` into elements of type `float`.

Example

```
#float_of_ratio #[1.23456789e8];;
123456800.0 : float

#ratio_of_float it;;
#[123456800/1] : ratio
```

6.7.5 Coercion between strings and `ratio`s

Reading `ratio`s

sys_ratio_of_string[†]**:** int × string × int × int → ratio
ratio_of_string[†]**:** string → ratio

CAML is able to read numbers of type `ratio` entered directly, according to the following lexical convention:

```
Separator ::= ’ ’ | ’\f’ | ’\n’ | |\r’ | ’\t’ | ’\\’
Pseudodigit ::= Digit | Separator
Decimal ::= {’-’| ’+’} Separator* Digit Pseudodigit*
            {’.’ Separator* Digit Pseudodigit*} {’e’ {’-’ | ’+’} INT}
RATIO ::= ’#[’ Decimal {’/’ Decimal} ’]’
```

Example

```
##[12];;
#[12/1] : ratio

##[12.0];;
#[12/1] : ratio

##[1.2e1];;
#[12/1] : ratio

##[120e-1];;
#[12/1] : ratio

##[12/1];;
#[12/1] : ratio

##[60/5];;
#[12/1] : ratio

##[1.2/1e-1];;
#[12/1] : ratio

##[120/10];;
#[12/1] : ratio
```

`sys_ratio_of_string (base, s, off, len)` maps the substring $s_{off,len}$ in base `base` onto a element of type `ratio`.

`ratio_of_string s` is equivalent to:

```
let ratio_of_string s =
  sys_ratio_of_string (#10, s, #0, length_string s)
;;
```

Example

```
#sys_ratio_of_string (#16, "FF", #0, #2);;
#[255/1] : ratio

#ratio_of_string "1.2e3/5e1";;
#[24/1] : ratio
```

Printing `ratios`

sys_string_of_ratio†**:** int × string × ratio × string → string
string_of_ratio†**:** ratio → string
string_for_read_of_ratio†**:** ratio → string
sys_print_ratio†**:** int × string × ratio × string → unit
print_ratio†**:** ratio → unit
print_ratio_for_read†**:** ratio → unit

`sys_string_of_ratio (base, before, r, after)` returns the concatenation of the string `before`, the string representation of rational number `r` in base `base`, and finally the string `after`. In fact if the approximation when printing rational numbers is turned on, the result is the floating point approximation to the floating point precision of the rational number. This is not implemented for `base` $\neq 10$. For more information see section 6.9 and chapter 8.

`string_of_ratio r` is equivalent to:

```
let string_of_ratio r =
  sys_string_of_ratio (#10, "", r, "")
;;
```

`string_for_read_of_ratio r` is analogous to `string_of_ratio r` except that the number is written between `#[` and `]`, according to conventions to write "big" numbers.

It is equivalent to:

```
let string_for_read_of_ratio r =
 sys_string_of_ratio (#10, "#[", r, "]")
;;
```

`sys_print_of_ratio (base, before, r, after)` prints the `base`-radix representation of the rational number `r` between strings `before` and `after`. It is equivalent to:

```
let sys_print_ratio (base, before, r, after) =
 print_string (sys_string_of_ratio (base, before, r, after))
;;
```

`print_ratio` and `print_ratio_for_read` are equivalent to:

```
let print_ratio r = sys_print_ratio (#10, "", r, "");;

let print_ratio_for_read r = sys_print_ratio (#10, "#[", r, "]");;
```

Example

```
#sys_string_of_ratio
#  (#16, "ratio 16-radix representation = ", #[1.2e3/1.1e2], "");;
"ratio 16-radix representation = 78/B" : string

#sys_string_of_ratio
#  (#2, "ratio 2-radix representation = ", #[1.2e3/1.1e2], "");;
"ratio 2-radix representation = 1111000/1011" : string

#sys_string_of_ratio
```

```
#  (#8, "ratio 8-radix representation = ", #[1.2e3/1.1e2], "");;
"ratio 8-radix representation = 170/13" : string

#sys_string_of_ratio
#  (#10, "ratio 10-radix representation = ", #[1.2e3/1.1e2], "");;
"ratio 10-radix representation = 120/11" : string

#string_of_ratio #[1.2e3/1.1e2];;
"120/11" : string

#string_for_read_of_ratio #[1.2e3/1.1e2];;
"#[120/11]" : string

#sys_print_ratio (#16, "ratio 16-radix representation = ", #[1.2e3/1.1e2], "");;
ratio 16-radix representation = 78/B() : unit

#print_ratio #[1.2e3/1.1e2];;
120/11() : unit

#print_ratio_for_read #[1.2e3/1.1e2];;
#[120/11]() : unit
```

Special printing functions

sys_latex_print_ratio†: int × string × ratio × string → unit
latex_print_ratio†: ratio → unit
latex_print_for_read_ratio†: ratio → unit
sys_print_beautiful_ratio†: int × string × ratio × string → unit
print_beautiful_ratio†: ratio → unit

These special printing functions for rational numbers can be found into two libraries:

format_latex_numbers.ml: functions that split numbers into small lines so LaTeX is able to handle them correctly,

format_numbers.ml: functions to present the rational number as a numerical table according to common conventions.

`sys_latex_print_ratio (base, before, r, after)` prints `base` representation of the rational number `r` between `before` and `after`, cutting line each `latex_margin` characters.

`latex_print_ratio` and `latex_print_for_read_ratio` are equivalent to:

```
let latex_print_ratio r = sys_latex_print_ratio (#10, "", r, "")

and latex_print_for_read_ratio r = sys_latex_print_ratio (#10, "#[", r, "]")
;;
```

As an example this manual uses `latex_print_for_read_ratio` as the printer for rational numbers.

`sys_print_beautiful_ratio (base, before, r, after)` prints `base`-radix representation of the rational number `r` between `before` and `after` as a numerical table according to classical conventions.

`print_beautiful_ratio` is equivalent to:

```
let print_beautiful_ratio r =
  sys_print_beautiful_ratio (#10, "", r, "")
;;
```

You can load the libraries with:

```
load_lib_file "format_latex_numbers";;
load_lib_file "format_numbers";;
```

Example

```
#print_beautiful_ratio #[1.2e3/1.1e2];;
120/11() : unit

#print_beautiful_ratio #[12345e105/1.1e2];;
12345 00000  00000 00000  00000 00000  00000 00000  00000 00000
00000 00000  00000 00000  00000 00000  00000 00000  00000 00000
00000 0000
/11() : unit

#print_beautiful_ratio #[12345e1/1.1e201];;
2469/
22000 00000  00000 00000  00000 00000  00000 00000  00000 00000
00000 00000  00000 00000  00000 00000  00000 00000  00000 00000
00000 00000  00000 00000  00000 00000  00000 00000  00000 00000
00000 00000  00000 00000  00000 00000  00000 00000  00000 00000
() : unit
```

6.8 Miscellaneous power functions

power_ratio_positive_int†: ratio $\times$ int $\rightarrow$ ratio
power_ratio_positive_big_int†: ratio $\times$ big_int $\rightarrow$ ratio

`power_ratio_positive_int (r, i)` yields the rational number resulting from the computation of r^i where i is a positive `int`.

`power_ratio_positive_big_int (r, bi)` yields the rational number resulting from the computation of r^{bi} where r is a positive rational number.

Example

```
#power_ratio_positive_int (#[1.09],#10);;
#[23673636745921172340 1/100000000000000000000] : ratio

#power_ratio_positive_big_int (#[2], #(32));;
#[4294967296/1] : ratio
```

6.9 Floating point approximations of rational numbers

msd_ratio†: ratio → int
round_futur_last_digit†: string × int × int → bool
approx_ratio_fix†: int × ratio → string
approx_ratio_exp†: int × ratio → string
float_of_rational_string†: ratio → string

`msd_ratio r` is the position of the leading digit of the decimal expansion of a strictly positive rational number. If

$$r = \sum_{k=-p}^{k=N} r_k 10^k$$

then we have `msd_ratio r = N`.

`round_futur_last_digit (s, off, len)` rounds off in place the last but one digit-character of the substring $s_{off,len}$ and yields true if and only if it causes an overflow, i.e. if s contains an integer of the form $10^n - 1$. In this case after transformation s will contain only 0 characters and the resulting value of the function is true.

`approx_ratio_fix (n, r)` approaches the rational number `r` with a `n` 10-digits fix point representation. This is an odd function and the last digit is round off. In an informal manner, the printing format is:

```
integer_part . decimal_part_with_n_digits
```

`approx_ratio_exp (n, r)` approaches the rational number `r` with a `n` 10-digits floating point representation. This is an odd function and the last digit is round off. In an informal manner, the printing format is:

```
(+|-) (0. n_first_digits e msd) | (1. n_zeros e (msd+1))
```

`float_of_rational_string` is the same function with a default precision value according to the floating point precision flag. For more information see chapter 8.

Example

```
#msd_ratio #[1.2e3/1.1e2];;
1 : int

#let s = "+109090909090909";;
Value s is "+109090909090909" : string

#round_futur_last_digit (s, #0, #16);;
false : bool

#s;;
"+109090909090919" : string

#approx_ratio_fix (#12, #[1.2e3/1.1e2]);;
"+10.909090909091" : string

#approx_ratio_fix (#-1, #[1.2e3/1.1e2]);;
"+1" : string
```

```
#approx_ratio_exp (#12, #[1.2e3/1.1e2]);;
"+0.109090909091e2" : string

#approx_ratio_exp (#2, #[1.2e3/1.1e2]);;
"+0.11e2" : string

#float_of_rational_string #[1.2e3/1.1e2];;
"0.109090909091e2" : string
```

Chapitre 7

Rational numbers of type num

Elements of type `num` are arbitrarily large rational numbers, with optimization of the representation. We have the following definition of type `num`:

```
type num = Int of int | Big_int of big_int | Ratio of ratio
;;
```

7.1 Normalization of nums

normalize_num†: num → num
cautious_normalize_num_when_printing†: num → num

`normalize_num n` optimizes the representation of `n`. A rational will be normalized and if possible coerced to an integer. A "big" integer will be mapped onto a "small" one if possible.

`cautious_normalize_num_when_printing n` normalizes `n` in the above sense when the normalization when printing flag is turned on. For more information see chapter 8.

Beware:

→ It is supposed that "big" integers do not fit in a "small" one. You can introduce a "small" integer with constructor `Big_int`, this will not cause any error but it will be less efficient.

7.2 Miscellaneous information functions on nums

numerator_num†: num → num
denominator_num†: num → num

`numerator_num` and `denominator_num` are respectively the numerator and the denominator of the normalized representation of its argument.
Example

```
#numerator_num #{4/6};;
#{2} : num

#denominator_num #{4/6};;
#{3} : num
```

7.3 Arithmetic operations for nums

7.3.1 Incrementing and decrementing

Incrementing and decrementing numbers

pred_num : num → num
succ_num : num → num

`pred_num n` (resp. `succ_num n`) is a primitive equivalent to `n-1`(resp `n+1`).

Example

```
#pred_num #{4/6};;
#{-1/3} : num

#succ_num #{4/6};;
#{5/3} : num
```

Incrementing and decrementing counters

incr_num : num ref → num
decr_num : num ref → num

When one has created a reference containing a number (whose type is `num`), it is convenient to increment and decrement its value, using these two (primitive) functions semantically equivalent to:

```
let incr_num i = i:= succ_num !i and decr_num i = i:= pred_num !i;;
```

7.3.2 Arithmetic operations for nums

add_num : num × num → num
minus_num : num → num
sub_num : num × num → num
mult_num : num × num → num
square_num†**:** num → num
div_num : num × num → num
quo_num : num × num → num
mod_num : num × num → num

`add_num`, `minus_num r`, `sub_num` performs respectively the addition, opposite and subtraction on nums.

`mult_num` and `square_num` perform multiplication and square on nums.

`div_num`, `quo_num`, `mod_num` performs respectively the division and Euclidian division quotient and remainder on nums.

Example

```
#add_num (#{1/3}, #{2/3});;
#{1/1} : num

#sub_num (#{1/2}, #{1/2});;
#{0/1} : num

#square_num #{1/3};;
#{1/9} : num

#div_num (#{1/3}, #{1/5});;
#{5/3} : num

#quo_num (#{1/3}, #{1/5});;
#{1} : num

#mod_num (#{1/3}, #{1/5});;
#{2/15} : num
```

7.4 Rounding nums

is_integer_num†**:** num → bool
integer_num†**:** num → num
floor_num†**:** num → num
round_num†**:** num → num
ceiling_num†**:** num → num

`is_integer_num` is equivalent to:

```
let is_integer_num = function
  Ratio r   -> is_integer_ratio r
|     _     -> true
;;
```

`integer_num` is the odd function of truncature for rational numbers.

`floor_num` (resp. `round_num`, `ceiling_num`) are the mathematical functions for rounding to the lower (resp. nearest, higher) integer.

Example

```
#is_integer_num #{1.2e3/1200};;
true : bool

#integer_num #{1/3};;
#{0} : num

#floor_num #{1/3};;
#{0} : num

#round_num #{1/3};;
#{0} : num
```

```
#ceiling_num #{1/3};;
#{1} : num

#integer_num #{1/2};;
#{0} : num

#floor_num #{1/2};;
#{0} : num

#round_num #{1/2};;
#{1} : num

#ceiling_num #{1/2};;
#{1} : num

#integer_num #{-1/3};;
#{0} : num

#floor_num #{-1/3};;
#{-1} : num

#round_num #{-1/3};;
#{0} : num

#ceiling_num #{-1/3};;
#{0} : num

#integer_num #{-1/2};;
#{0} : num

#floor_num #{-1/2};;
#{-1} : num

#round_num #{-1/2};;
#{-1} : num

#ceiling_num #{-1/2};;
#{0} : num
```

7.5 Comparisons between nums

eq_num : num $\times$ num $\rightarrow$ bool
compare_num†: num $\times$ num $\rightarrow$ int
lt_num : num $\times$ num $\rightarrow$ bool
le_num : num $\times$ num $\rightarrow$ bool
gt_num : num $\times$ num $\rightarrow$ bool
ge_num : num $\times$ num $\rightarrow$ bool
sign_num†: num $\rightarrow$ int
abs_num : num $\rightarrow$ num

min_num : num → num → num
max_num : num → num → num

`lt_num`, `le_num`, `gt_num`, `ge_num` are arithmetic comparisons corresponding respectively to `<`, `<=`, `>`, `>=` between elements of type `num`.

The primitive function for these functions is `compare_num`, that is semantically equivalent to:

```
let compare_num (r1, r2) =
 when (r1 = r2) -> #0
    | (r1 < r2) -> #-1
    |     _     -> #1
;;
```

`sign_num` is equivalent to:

```
let sign_num = function
  Int i -> sign_int i
| Big_int bi -> sign_big_int bi
| Ratio r -> sign_ratio r
;;
```

`abs_num n` is the absolute value of `n`.

`max_num (n1, n2)` and `min_num (n1, n2)` are respectively the greater one and the smaller one of rational numbers `n1` and `n2`.

Example

```
#sign_num (succ_num (num_of_int biggest_int));;
1 : int

#sign_num (num_of_int monster_int);;
-1 : int

#abs_num (succ_num (num_of_int biggest_int));;
#{32768} : num

#abs_num (num_of_int monster_int);;
#{32768} : num
```

7.6 Coercion functions on nums

7.6.1 Coercion between ints and nums

int_of_num : num → int
num_of_int : int → num
is_int : num → bool

`int_of_num n` is the `int` representing the rational number `n` if `n` fits in an `int`.

`num_of_int` maps an `int` onto the corresponding `num`.

`is_int` is equivalent to:

```
let is_int = function
  Int _ -> true
|     _  -> false
;;
```

Beware:

→ No normalization is performed in this function, so if applied to an `int` with not optimized constructor, the result is false. This is possible only if you type in a "small" integer with a label `Big_int` or `Ratio`.

Example

```
#int_of_num #{1.2e3};;
1200 : int

#int_of_num (succ_num (num_of_int biggest_int));;

Evaluation Failed: failure "int_of_big_int"

#int_of_num (pred_num (num_of_int least_int));;
-32768 : int

#num_of_int #1;;
#{1} : num
```

7.6.2 Coercion between floating point numbers and `nums`

float_of_num : num → float
num_of_float : float → num

`float_of_num` and `num_of_float` map elements of type `num` into elements of type `float`.

Example

```
#float_of_num #{1.23456789e8};;
123456800.0 : float

#num_of_float it;;
#{123456800} : num
```

7.6.3 Coercion between `nats` and `nums`

nat_of_num†: num → nat
num_of_nat†: nat → num

`nat_of_num n` maps rational number `n` onto the corresponding `nat` if `n` is an integer and positive. `num_of_nat` maps a `nat` onto the corresponding rational number.

Example

```
#nat_of_num #{1.2e3};;
#<1200> : nat

#nat_of_num #{-1.2e3};;

Evaluation Failed: failure "nat_of_int"

#nat_of_num #{1.2};;

Evaluation Failed: failure "nat_of_ratio"

#nat_of_num #{1/2};;

Evaluation Failed: failure "nat_of_ratio"
```

7.6.4 Coercion between `big_int`s and `nums`

big_int_of_num†: num → big_int
num_of_big_int†: big_int → num

`big_int_of_num n` maps rational number `n` onto the corresponding `big_int` if `n` is an integer else fail.

`num_of_big_int` maps a integer of type `big_int` onto the corresponding rational number. If this "big" integer fits in a "small" one, this representation is adopted.

Example

```
#big_int_of_num #{1.2};;

Evaluation Failed: failure "big_int_of_ratio"
```

7.6.5 Coercion between `ratio`s and `nums`

ratio_of_num†: num → ratio
num_of_ratio†: ratio → num

`ratio_of_num` and `num_of_ratio` map a rational number onto its optimized representation.

Example

```
#num_of_ratio #[1];;
#{1} : num

#ratio_of_num it;;
#[1/1] : ratio
```

7.6.6 Coercion between strings and `nums`

Reading `nums`

sys_num_of_string†**:** int × string × int × int → num
num_of_string : string → num

CAML is able to read numbers of type `num` entered directly, according to the following lexical convention:

```
Separator ::= ’ ’ | ’\f’ | ’\n’ | |\r’ | ’\t’ | ’\\’
Pseudodigit ::= Digit | Separator
Decimal ::= {’-’| ’+’} Separator* Digit Pseudodigit*
            {’.’ Separator* Digit Pseudodigit*} {’e’ {’-’ | ’+’} INT}
NUM ::= ’#{’ Decimal {’/’ Decimal} ’}’
```

Example

```
##{12};;
#{12} : num

##{12.0};;
#{12} : num

##{1.2e1};;
#{12} : num

##{120e-1};;
#{12} : num

##{12/1};;
#{12} : num

##{60/5};;
#{12} : num

##{1.2/1e-1};;
#{12} : num

##{120/10};;
#{12} : num
```

Using non standard arithmetic, the `#{` and `}` characters are optional (see chapter 8 for more information).

`sys_num_of_string (base, s, off, len)` maps the substring $s_{off,len}$ in base `base` onto a element of type `num`.

`num_of_string s` is equivalent to:

```
let num_of_string s =
  sys_num_of_string (#10, s, #0, length_string s)
;;
```

Example

```
#sys_num_of_string (#16, "FF", #0, #2);;
#{255} : num

#num_of_string "1.2e3/5e1";;
#{24} : num
```

Printing nums

sys_string_of_num†**:** int × string × num × string → string
string_of_num : num → string
string_for_read_of_num : num → string
display_num : num → unit
echo_num : num → unit
sys_print_num†**:** int × string × num × string → unit
print_num : num → unit
print_num_for_read : num → unit

`sys_string_of_num (base, before, r, after)` writes into the resulting string first the string `before`, then the representation of rational number `n` in base `base`, and last the string `after`. In fact if the approximation when printing rational numbers is turned on, the result is the floating point approximation to the floating point precision of the rational number. This feature is not implemented with `base` $\neq 10$. For more information see section 6.9 and chapter 8.

`string_of_num n` is semantically equivalent to:

```
let string_of_num n =
  sys_string_of_num (#10, "", n, "")
;;
```

`string_for_read_of_num n` is analogous to `string_of_num n` except that the number is written between `#{` and `}`, according to conventions to write "big" numbers.

It is equivalent to:

```
let string_for_read_of_num r =
 sys_string_of_num (#10, "#{", r, "}")
;;
```

`display_num` and `echo_num` are equivalent to:

```
let display_num n = display_string (string_of_num n)

and echo_num n = echo_string (string_of_num n)
;;
```

For more information on the differences between `display`, `echo`, `print`-* like functions see the corresponding chapter of the reference manual.

`sys_print_of_num (base, before, r, after)` prints the `base`-radix representation of the rational number `n` between strings `before` and `after`. It is equivalent to:

```
let sys_print_num (base, before, r, after) =
 print_string (sys_string_of_num (base, before, r, after))
;;
```

`print_num` and `print_num_for_read` are equivalent to:

```
let print_num n = sys_print_num (#10, "", n, "");;

let print_num_for_read n = sys_print_num (#10, "#{", n, "}");;
```

Example

```
#sys_string_of_num
#  (#16, "num 16-radix representation = ", #{1.2e3/1.1e2}, "");;
"num 16-radix representation = 78/B" : string

#sys_string_of_num
#  (#2, "num 2-radix representation = ", #{1.2e3/1.1e2}, "");;
"num 2-radix representation = 1111000/1011" : string

#sys_string_of_num
#  (#8, "num 8-radix representation = ", #{1.2e3/1.1e2}, "");;
"num 8-radix representation = 170/13" : string

#sys_string_of_num
#  (#10, "num 10-radix representation = ", #{1.2e3/1.1e2}, "");;
"num 10-radix representation = 120/11" : string

#string_of_num #{1.2e3/1.1e2};;
"120/11" : string

#string_for_read_of_num #{1.2e3/1.1e2};;
"#{120/11}" : string

#sys_print_num (#16, "num 16-radix representation = ", #{1.2e3/1.1e2}, "");;
num 16-radix representation = 78/B() : unit

#print_num #{1.2e3/1.1e2};;
120/11() : unit

#print_num_for_read #{1.2e3/1.1e2};;
#{120/11}() : unit
```

Special printing functions

sys_latex_print_num†: int × string × num × string → unit
latex_print_num†: num → unit
latex_print_for_read_num†: num → unit
sys_print_beautiful_num†: int × string × num × string → unit
print_beautiful_num†: num → unit

These special printing functions for `nums` can be found into two libraries:

format_latex_numbers.ml: functions that split numbers into small lines so LaTeX is able to handle them correctly, format_numbers.ml: functions to present the rational number as a numerical table according to common conventions.

`sys_latex_print_num (base, before, r, after)` prints `base` representation of the rational number `n` between `before` and `after`, cutting line each `latex_margin` characters.

`latex_print_num` and `latex_print_for_read_num` are equivalent to:

```
let latex_print_num r = sys_latex_print_num (#10, "", r, "")

and latex_print_for_read_num r = sys_latex_print_num (#10, "#{", r, "}")
;;
```

As an example this manual uses `latex_print_for_read_num` as the printer for rational numbers.

`sys_print_beautiful_num (base, before, r, after)` prints `base`-radix representation of the rational number `n` between `before` and `after` as a numerical table according to classical conventions.

`print_beautiful_num` is equivalent to:

```
let print_beautiful_num r =
  sys_print_beautiful_num (#10, "", r, "")
;;
```

You can load the libraries with:

```
load_lib_file "format_latex_numbers";;
load_lib_file "format_numbers";;
```

Example

```
#print_beautiful_num #{1e562};;
10000 00000  00000 00000  00000 00000  00000 00000  00000 00000
00000 00000  00000 00000  00000 00000  00000 00000  00000 00000
00000 00000  00000 00000  00000 00000  00000 00000  00000 00000
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00000 00000  00000 00000  00000 00000  00000 00000  00000 00000

00000 00000  00000 00000  00000 00000  00000 00000  00000 00000
00000 00000  000
() : unit

#print_beautiful_num #{1.2e3/1.1e2};;
120/11() : unit

#print_beautiful_num #{12345e105/1.1e2};;
12345 00000  00000 00000  00000 00000  00000 00000  00000 00000
00000 00000  00000 00000  00000 00000  00000 00000  00000 00000
```

```
00000 0000
/11() : unit

#print_beautiful_num #{12345e1/1.1e201};;
2469/
22000 00000  00000 00000  00000 00000  00000 00000  00000 00000
00000 00000  00000 00000  00000 00000  00000 00000  00000 00000
00000 00000  00000 00000  00000 00000  00000 00000  00000 00000
00000 00000  00000 00000  00000 00000  00000 00000  00000 00000
() : unit
```

7.7 Miscellaneous power functions

power_num_int†**:** num × int → num
power_num_big_int†**:** num × big_int → num
power_num : num × num → num

`power_num_int (n, i)` and `power_num_big_int (n, bi)` yield the rational number resulting respectively from the computation of n^i and n^{bi}.

`power_num (n1, n2)` yields the rational number resulting from the computation of $n1^{n2}$ where n2 is an integer, possibly a "big" one.

Example

```
#power_num_int (#{1.09},#10);;
#{236736367459211723401/100000000000000000000} : num

#power_num_big_int (#{2}, #(32));;
#{0} : num

#power_num_big_int (#{2}, #(-32));;
#{1/0} : num

#power_num (#{2/3},#{3});;
#{8/27} : num
```

7.8 Floating point approximations of nums

approx_num_fix : int × num → string
approx_num_exp : int × num → string
float_num†**:** num → num

`approx_num_fix (n, r)` approaches the rational number **n** with a **n** 10-digits fix point representation. This is an odd function and the last digit is round off. In an informal manner, the printing format is:

```
integer_part . decimal_part_with_n_digits
```

`approx_num_exp (n, r)` approaches the rational number `n` with a `n` 10-digits floating point representation. This is an odd function and the last digit is round off. In an informal manner, the printing format is:

```
(+|-) (0. n_first_digits e msd) | (1. n_zeros e (msd+1))
```

`float_num` is equivalent to:

```
let float_num n = num_of_float (float_of_num n)
;;
```

Example

```
#approx_num_fix (#12, #{1.2e3/1.1e2});;
"+10.909090909091" : string

#approx_num_fix (#-1, #{1.2e3/1.1e2});;
"+1" : string

#approx_num_fix (#10, #{355/113});;
"+3.1415929204" : string

#approx_num_exp (#12, #{1.2e3/1.1e2});;
"+0.109090909091e2" : string

#approx_num_exp (#2, #{1.2e3/1.1e2});;
"+0.11e2" : string

#approx_num_exp (#10, #{355/113});;
"+0.3141592920e1" : string

#float_num #{1.2e3/1.1e2};;
#{1090909/100000} : num
```

Chapitre 8

Numerical directives

8.1 Standard arithmetic

`#standard arith ...`

There are 2 arithmetical modes in CAML, identified by the value of the directive `#standard arith`:

- using the standard arithmetic, only small integers and floating point numbers are recognized by default, floating point arithmetic is an approximate arithmetic and small integer arithmetic is an exact ring arithmetic,
- using the non-standard arithmetic, all numbers are recognized by default of type `num`. Rational operations are performed exactly. Transcendental operations are not defined on type `num`: they can only be applied on floating point representations of numbers, using explicit coercions.

Of course, any number can be read as an entity of (almost) any type, using the lexical convention associated to this type (no lexical convention for type `nat`).

By default, the mode is the standard one, but this can be changed using the `#standard arith` directive.

```
#standard arith false;;
```

tells the compiler to use mathematical arithmetic instead of the standard one. Of course,

```
#standard arith true;;
```

undoes this.

For more information, see chapter "Directives and pragmas" in the reference manual.

8.2 Fast arithmetic

`#fast arith ...`

The `#fast arith true` directive tells the compiler that from now on, it has to omit tests as far as possible. When accessing vectors or strings tests for range are ellipsed.

By default this flag is set to false.

8.3 Open arithmetic

`#open arith`

This directive provides more arithmetical functions. These additional functions are indicated in this manual by a † after the name of the function.

Its action is irreversible so it is not the arithmetical mode by default and calling this directive several times adds no more functions. This manual is printed in this arithmetic mode.

This directive is called by:

```
#open arith;;
```

8.4 Debugging nat arithmetic

`#arith cautious`

This directive transforms the arithmetic primitives on nats into cautious primitives. In this mode, every arithmetic function on nats verifies the correction of its arguments according to the conditions indicated in chapter 4.

Its action is irreversible so these verifications are not performed by default and calling this directive several times adds no more verifications.

8.5 Arithmetic overloading

```
overload ... with ...
#arith ...
-
```

get_arith_overloading : unit → arith list
prefix + : int × int → int
prefix * : int × int → int
prefix ** : int × int → int
prefix - : int × int → int
prefix / : int × int → int
prefix <= : int × int → bool
prefix < : int × int → bool
prefix >= : int × int → bool
prefix > : int × int → bool
float : int → float
pred : int → int
succ : int → int
incr : int ref → int
decr : int ref → int
prefix quo : int × int → int
prefix mod : int × int → int
max : int → int → int
min : int → int → int
abs : int → int

prefix power : int × int → int

Most of the operations on numbers are appliable to almost every kind of numbers, e.g. basic arithmetic operations like `+`, `*`, `-` and `/`. Some others are specialized for one precise kind of numbers as is the primitive operation `succ_int : int -> int`.

Roughly speaking, the overloading lets the user not worry about all these subtleties about numbers, especially in the non-standard arithmetic mode. On numerical overloading see section 8.5.

Symbols can be overloaded with the `overload` construction as in:

```
overload f with g;;
```

where f is the overloaded symbol and g is the function which is called when $g\ (x)$ is well-typed in the context of the application of f to argument x.

By default no numerical overloading is performed. The preceding functions are defined on type `int` in standard arithmetic, else `num`.

`#arith arithmetic_type` loads a set of predefined usual overloading declarations for type `arithmetic_type`. The possible types are: int, float, big_int, ratio, num.

`#arith overloading` loads a set of predefined usual overloading declarations for types int, num and float.

`get_arith_overloading ()` indicates for which numerical types the predefined overloading mechanism is active.

8.6 Normalization of rational numbers during computation

get_normalize_ratio†**:** unit → bool
set_normalize_ratio†**:** bool → bool

`get_normalize_ratio` (resp. `set_normalize_ratio`) consults (resp. modifies) the value of the flag which controls the automatic normalization of rational numbers during computation.

By default, this flag is set to false. This choice will be justified in appendix B.

8.7 Normalization of rational numbers when printing

get_normalize_ratio_when_printing†**:** unit → bool
set_normalize_ratio_when_printing†**:** bool → bool

`get_normalize_ratio_when_printing` (resp. `set_normalize_ratio_when_printing`) consults (resp. modifies) the status of normalization of rational numbers process during printing.

By default, this flag is set to true.

8.8 Floating point approximation when printing rational numbers

get_approx_printing : unit → bool
set_approx_printing : bool → bool
get_floating_precision : unit → int
set_floating_precision : int → int

`get_approx_printing` (resp. `set_approx_printing`) consults (resp. modifies) the flag of rational printing mode. If turned on, each rational number will be printed as its floating point approximation according to the current default floating point precision flag, consultable (resp. modifiable) with `get_floating_precision` (resp. `set_floating_precision`).

By default, this flag is set to false.

Example

```
#get_approx_printing ();;
false : bool

#set_approx_printing true;;
true : bool

##[2/3];;
#[2/3] : ratio

#get_floating_precision ();;
12 : int

#set_floating_precision #2;;
2 : int

##{1/4};;
#{1/4} : num
```

8.9 Error when a rational denominator is null

get_error_when_null_denominator : unit → bool
set_error_when_null_denominator : bool → bool

`get_error_when_null_denominator` (resp. `set_error_when_null_denominator`) consults (resp. modifies) the flag which indicates if null denominators causes run-time errors or not.

By default, this flag is set to true.

8.10 Arith_status or how to remember (or use) all this

arith_status : unit → unit

`arith_status ()` prints the current status of each arithmetic flag and recall how to manage it.

Example

```
#arith_status();;
```

```
Standard arithmetic --> ON
     (returned by arith_is_standard ())
     (modifiable with #standard arith <your choice>)

Fast arithmetic --> OFF
     (returned by arith_is_fast ())
     (modifiable with #fast arith <your choice>)

Open arithmetic --> OFF
     (returned by arith_is_open ())
     (modifiable with #open arith)

Cautious arithmetic --> OFF
     (returned by arith_is_cautious ())
     (modifiable with #arith cautious)

Arithmetic overloading --> no overloading on numbers
     (returned by get_arith_overloading ())
     (modifiable with #arith overloading or #arith <the numerical type to be added>)

Normalization during computation --> OFF
     (returned by get_normalize_ratio ())
     (modifiable with set_normalize_ratio <your choice>)

Normalization when printing --> ON
     (returned by get_normalize_ratio_when_printing ())
     (modifiable with set_normalize_ratio_when_printing <your choice>)

Floating point approximation when printing rational numbers --> ON
     (returned by get_approx_printing ())
     (modifiable with set_approx_printing <your choice>)
  Default precision = 2
     (returned by get_floating_precision ())
     (modifiable with set_floating_precision <your choice>)

Error when a rational denominator is null --> ON
     (returned by get_error_when_null_denominator ())
     (modifiable with set_error_when_null_denominator <your choice>)
() : unit
```

Annexe A

Numerical computations using CAML: π as a case-study

A.1 Introduction

Here is one of Ramanujan's formulas for π, one of the fastest known formulas:

$$\frac{1}{\pi} = \sum_{n=0}^{\infty} (-1)^n \frac{12(6n)!}{(n!)^3(3n)!} \frac{13591409 + 545140134n}{(640320^3)^{n+\frac{1}{2}}} = \sum_{n=0}^{\infty} u_n = \lim_{n\to\infty} S_n$$

with

$$S_n = \sum_{i=0}^{n} u_n.$$

David and Gregory Chudnovsky have calculated over 2 Billion digits with this formula, setting the current record (see [1] for the 1 Billion record).

A.2 First approach

The implementation of this formula raises several problems which are, in increasing order of difficulty:

1. computing $n!$,
2. computing $\sqrt{640320}$ with the necessary precision,
3. determining the number of terms in the series necessary for a given precision.

A.2.1 Factorial function

The first problem is solved with the definition of `fact` as follows:

```
#let rec fact n = if n = 0 then 1 else n * fact (n-1)
#;;
Value fact is <fun> : int -> int
```

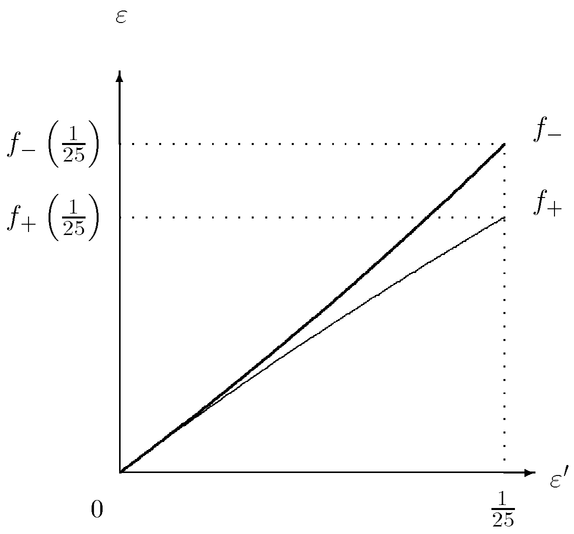

Figure A.1 : Graph of $\varepsilon = f(\varepsilon')$.

A.2.2 Square root with a given precision

A square root function is available on numbers of type `big_int`:

```
sqrt_big_int : big_int -> big_int
```

We will use this function to define:

```
#let sqrt640320 digits =
#  let pow = 10**digits in
#    num_of_big_int (sqrt_big_int (big_int_of_num (640320*pow*pow)))/pow
#;;
Value sqrt640320 is <fun> : num -> num
```

A.2.3 Required number of terms for a given precision

Theoretical stop

Let ε (resp. ε') be the error when evaluating π (resp. $\frac{1}{\pi}$) via $\frac{1}{S_n}$ (resp. S_n). Let us try to bound ε' above by a simple function of ε. We have:

$$\varepsilon = \left|\pi - \frac{1}{S_n}\right| = \frac{\pi}{|S_n|}\left|S_n - \frac{1}{\pi}\right| = \frac{\pi\varepsilon'}{|S_n|} = \begin{cases} \dfrac{\pi^2\varepsilon'}{1+\pi\varepsilon'} & \text{if } S_n \geq \frac{1}{\pi} \\[2ex] \dfrac{\pi^2\varepsilon'}{1-\pi\varepsilon'} & \text{otherwise} \end{cases}$$

Let f_+ and f_- be each of these functions with the obvious sign convention. Figure A.1 represents the graph of these two functions on the interval $[0, \frac{1}{25}]$.

Let f be either f_+ or f_- according to the related position of S_n and $\frac{1}{\pi}$. We see that f is a strictly increasing and continuous function on the interval $[0, \frac{1}{25}]$, so f is invertible and its reciprocal function f^{-1} is strictly increasing and we have:

$$0 \leq \varepsilon \leq \alpha \Longleftrightarrow 0 \leq \varepsilon' \leq f^{-1}(\alpha).$$

The reciprocal functions are the following:

$$\begin{aligned} f_+^{-1}(\alpha) &= \frac{\alpha}{\pi\,(\pi-\alpha)} \\ f_-^{-1}(\alpha) &= \frac{\alpha}{\pi\,(\pi+\alpha)}. \end{aligned}$$

For our purpose, it is sufficient to have:

$$f^{-1}(\alpha) \geq \frac{\alpha}{\pi\,(\pi+\frac{1}{25})} \geq \frac{\alpha}{10} = g^{-1}(\alpha)$$

and the preceding equivalence becomes:

$$0 \leq \varepsilon' \leq g^{-1}(\alpha) \leq f^{-1}(\alpha) \Longleftrightarrow 0 \leq \varepsilon \leq f(g^{-1}(\alpha)) \leq \alpha.$$

Hence a sufficient condition on ε' for ε to be smaller than α for example, is:

$$\varepsilon' \leq \frac{\alpha}{10}.$$

So we know when the required precision is obtained since this series converges according to the alternating series criterion :

$$\varepsilon' = \left|\frac{1}{\pi} - S_n\right| < |u_{n+1}|.$$

And quite finally,

$$|u_{n+1}| \leq \frac{\alpha}{10} \Longrightarrow \left|\pi - \frac{1}{S_n}\right| < \alpha$$

and we need to compute $\frac{1}{\pi}$ to a precision of one more digit than the required precision on π.

Practical stop

In fact, the computed term $\texttt{term}_n$ is different of u_n, since

$$\texttt{term}_n = \frac{a_n}{\sqrt{640320} - \varepsilon''},$$

with $\varepsilon'' > 0$ since `sqrt_big_int` (so `sqrt640320`) approximate the integer square root by the integer below it.

Hence, $|\texttt{term}_n|$ is greater than $|u_n|$ and

$$|\texttt{term}_n| \leq \frac{\alpha}{10} \Longrightarrow \left|\pi - \frac{1}{S_n}\right| < \alpha.$$

So it is sufficient to test if the computed term `term` is small enough.

A.2.4 The program

```
##standard arith false;;
Pragma false : bool

##open arith;;
Pragma () : unit

#let rec fact n = if n = 0 then 1 else n * fact (n-1)
#;;
Value fact is <fun> : num -> num

#let sqrt640320 digits =
#  let pow = 10**digits in
#    num_of_big_int (sqrt_big_int (big_int_of_num (640320*pow*pow)))/pow
#;;
Value sqrt640320 is <fun> : num -> num

#(* The rational approximation *)
#let approx_pi digits =
#  let s = ref 0
#  and term = ref (12*13591409/(640320 * (sqrt640320 (digits+1))))
#  and n = ref 1 in
#    while abs(!term) > 10**-(digits+1) do
#       s := !s + !term;
#       term := (-1)**!n * 12 * fact (6*!n) * (13591409 + 545140134*!n) /
#               ((fact !n)**3 * fact (3*!n) * (640320**(3*!n+1)) *
#                (sqrt640320 (digits+1)));
#       n := !n+1
#       done;
#   1/!s
#;;
Value approx_pi is <fun> : num -> num

#(* For floating point representation with all digits exacts *)
#let show_pi digits =
#  approx_num_fix (int_of_num digits, approx_pi digits)
#;;
Value show_pi is <fun> : num -> string
```

A.2.5 Running the program

Let us use this program for calculating 500 digits of π:

```
#(* Set the timer *)
#timers true;;
() : unit

#approx_pi 500;;
#{109533578894262971661206045498256604078365325503362726370262155560667384279186651900779927
  570310190946061842438218939937169222525519183014235012866529280008563672816062819257319798
  971669125103765337473687536585524549496603999150548618506290892909133189481696463521578380
```

```
  1448722947133006665514688857590021466464576103715153519039079266143540441676609829643728799
  2405429171924879141922655736744303766263070229790518231549222338371729386608278032379285888
  9604998388663823187583042735478992758858387056278193805805085302213644443801951437991707022
  0887676678657743258009565723141535231244982166512370599707745681757636785337917726771537455
  4053577695470509730187718139398591047753271004141807010496687441272841398144541782133198266
  6449959080579083451657644731165108153115983315760319918898939736536760859744232773055636644
  564116488796814040993607972020766441302977772413041475000372275498188
 /
  3486562103113610258441135839462719300426264703287105224551713832590454655855472689715737299
  6744257130209617326566828308846699041772024398851828570216008728779236385643424375394550255
  8464785521831208899565954646403511470284716044081809229103617116794462306646379350308037188
  9952155309257838950471936639634622393735987016915253324420202131548580304960498293810694844
  7126453954656292671150521437348397090253060600352610443540592215221333026157970354534875433
  7422563114581945414343677722262003454282393476183677094128117456370859041868814579690451099
  7608488703365707984721204153741478699909169500737237057918979842504678806540433932752226244
  4447643426003991766290457421409256610584935183437163812157364020472523338514957528096098966
  0972353675890449166907165464576871973899103057917848809445552820979910050876355425702679399
  174906121649837703323993802310976075631288040312938392162322998046875} :
 num
Evaluation has needed: Runtime: 17.34s GC: 2.90s
```

Now let us try with 1000 digits:

```
#approx_pi 1000;;

Evaluation Failed: failure "create_nat: number too long"
Evaluation has needed: Runtime: 95.96s GC: 10.35s
```

Two remarks:

- the running time is long and the computation of 1000 digits fails because the rational numbers used have too big numerator and denominator,
- this program is very redundant; some calculations are done several times.

It's time to optimize this algorithm a bit.

A.3 Second approach

A.3.1 The algorithm

In this section, we will factorize as far as possible the calculation using Horner techniques. We can deduce a factorial from the preceding iteration with less computation and optimize the computation of the power in the denominator.

Let us introduce some notations that will be used in the program too:

$$
\begin{aligned}
\texttt{sign}_n &= (-1)^n \\
\texttt{lin}_n &= 13591409 + 545140134n \\
\texttt{six\_n\_fact}_n &= (6n)! \\
\texttt{three\_n\_fact}_n &= (3n)!
\end{aligned}
$$

$$
\begin{aligned}
\texttt{fact\_n\_pow3}_{\,n} &= (n!)^3 \\
\texttt{pow\_n}_{\,n} &= 640320^{3n+1}
\end{aligned}
$$

We will use the following transformation:

$$
\begin{aligned}
\texttt{sign}_{\,n+1} &= -\,\texttt{sign}_{\,n} \\
\texttt{lin}_{\,n+1} &= 545140134 + \texttt{lin}_{\,n} \\
\texttt{six\_n\_fact}_{\,n+1} &= (6n+6)(6n+5)(6n+4)(6n+3)(6n+2)(6n+1)\ \texttt{six\_n\_fact}_{\,n} \\
\texttt{three\_n\_fact}_{\,n+1} &= (3n+3)(3n+2)(3n+1)\ \texttt{three\_n\_fact}_{\,n} \\
\texttt{fact\_n\_pow3}_{\,n+1} &= (n+1)^3\ \texttt{fact\_n\_pow3}_{\,n} \\
\texttt{pow\_n}_{\,n+1} &= (640320^3)\ \texttt{pow\_n}_{\,n}
\end{aligned}
$$

with :

$$
\begin{aligned}
\texttt{sign}_{\,0} &= 1 \\
\texttt{lin}_{\,0} &= 13591409 \\
\texttt{six\_n\_fact}_{\,0} &= 1 \\
\texttt{three\_n\_fact}_{\,0} &= 1 \\
\texttt{fact\_n\_pow3}_{\,0} &= 1 \\
\texttt{pow\_n}_{\,0} &= 640320
\end{aligned}
$$

and we compute only once the square root and the tenth power for the test.

A.3.2 Stop

With S_n the partial sum of the series without the square root and R_n the approximation of this square root, we try to approach $\pi = \frac{\sqrt{640320}}{S_\infty}$ with $\frac{R_n}{S_n}$. We have :

$$
\begin{aligned}
S_n &= S_\infty - \varepsilon' \\
R_n &= \sqrt{640320} - \varepsilon'',
\end{aligned}
$$

where ε'' is a positive number and ε' either positive or negative.

Let us try to bound $|\varepsilon'|$ and ε'' above by a simple function of ε, the error when evaluating π. We have:

$$\pi - \frac{R_n}{S_n} = \frac{\sqrt{640320}}{S_n + \varepsilon'} - \frac{\sqrt{640320} - \varepsilon''}{S_n} = \frac{-\sqrt{640320}\,\varepsilon' + S_n\varepsilon'' + \varepsilon'\varepsilon''}{S_n(S_n + \varepsilon')}.$$

If we simplify this expression eliminating the references to S_n, we obtain

$$\varepsilon = \left|\pi - \frac{R_n}{S_n}\right| = \left|\frac{-\pi\varepsilon' + \varepsilon''}{\frac{\sqrt{640320}}{\pi} - \varepsilon'}\right| \le \frac{\varepsilon'' + \pi|\varepsilon'|}{\frac{\sqrt{640320}}{\pi} - |\varepsilon'|} = f_{\varepsilon''}(|\varepsilon'|).$$

Thus

$$f_{\varepsilon''}(|\varepsilon'|) \le \alpha \Longrightarrow \varepsilon \le \alpha.$$

$f_{\varepsilon''}$ is a continuous and strictly increasing function so we have :

$$|\varepsilon'| \leq f_{\varepsilon''}^{-1}(\alpha) \Longrightarrow f_{\varepsilon''}(|\varepsilon'|) \leq \alpha.$$

The reciprocal function $f_{\varepsilon''}^{-1}$ is

$$f_{\varepsilon''}^{-1}(\alpha) = \frac{\frac{\sqrt{640320}}{\pi}\,\alpha - \varepsilon''}{\pi + \alpha}.$$

Under the condition

$$\varepsilon'' \leq 100\,\alpha,$$

and $\alpha \leq 1$, we have

$$f_{\varepsilon''}^{-1}(\alpha) \geq \frac{\frac{\sqrt{640320}}{\pi} - 100}{\pi + 1}\,\alpha > 10\,\alpha.$$

Hence a sufficient condition on ε'' and $|\varepsilon'|$ for ε to be smaller than α for example, is:

$$\begin{aligned} \varepsilon'' &\leq 100\,\alpha \\ |\varepsilon'| &\leq 10\,\alpha. \end{aligned}$$

In terms of digits, we need to compute the partial sum and the square root to a precision of respectively one and two decimal digits less than the required precision on π.

A.3.3 The program

```
#let sqrt640320 digits =
#  let pow = 10**digits in
#    (num_of_big_int (sqrt_big_int (big_int_of_num (640320*pow*pow))), pow)
#;;
Value sqrt640320 is <fun> : num -> num * num

#let approx_pi digits =
#  let s = ref 0
#  and sign = ref 1
#  and six_n_fact = ref 1
#  and three_n_fact = ref 1
#  and fact_n_pow3 = ref 1
#  and lin = ref (13591409)
#  and pow3 = 640320**3 in
#  let pow_n = ref 640320
#  and term = ref (12*13591409/640320)
#  and (sqrt, pow) = (sqrt640320 (digits-2))
#  and n = ref 0 in
#    let pow_test = pow*10 in
#    while abs(!term)*pow_test > 1 do
#       s := !s + !term;
#       lin := 545140134 + !lin;
#       sign := - !sign;
#       six_n_fact := (6*!n+6)*(6*!n+5)*(6*!n+4)*(6*!n+3)*(6*!n+2)*(6*!n+1)
#                     *!six_n_fact;
#       three_n_fact := (3*!n+3)*(3*!n+2)*(3*!n+1)*!three_n_fact;
```

```
#       pow_n := pow3*!pow_n;
#       term := (12 * !sign * !six_n_fact * !lin) /
#               (!fact_n_pow3 * !three_n_fact * !pow_n);
#       fact_n_pow3 := (!n+1)**3*!fact_n_pow3;
#       n := !n+1
#       done;
#  sqrt/(!s*pow)
#;;
Value approx_pi is <fun> : num -> num
```

A.3.4 Running the program

```
#echo_values false;;
 : unit

#approx_pi 500;;
 : num
Evaluation has needed: Runtime: 5.53s GC: 1.91s

#approx_pi 1000;;
 : num
Evaluation has needed: Runtime: 61.11s GC: 7.64s
```

Now let us try with 2000 digits:

```
#approx_pi 2000;;

Evaluation Failed: failure "create_nat: number too long"
Evaluation has needed: Runtime: 82.65s GC: 13.12s
```

Computing 500 digits of π is now faster, computing 1000 digits does not create too long numbers anymore but computing 2000 digits still create too long integers. So we must improve the algorithm once again.

A.4 Third approach

A.4.1 The algorithm

We remark that:

$$\begin{aligned}(n+1)^3 &= (3n^2+3n+1)+n^3 \\ 3\,(n+1)^2+3\,(n+1)+1 &= (6n+6)+(3n^2+3n+1) \\ 6\,(n+1)+6 &= 6+(6n+6)\end{aligned}$$

Let $\mathtt{T}_n$, $\mathtt{pown3}_n$, $\mathtt{binom}_n$ and $\mathtt{sixn}_n$ be the following quantities:

$$\begin{aligned}\mathtt{T}_n &= \frac{(-1)^n\ 12\ (6n)!}{(n!)^3(3n)!\ (640320)^{3n+1}} \\ \mathtt{lin}_n &= 13591409+545140134n\end{aligned}$$

$$
\begin{aligned}
\texttt{pown3}_{\,n} &= n^3 \\
\texttt{binom}_{\,n} &= 3n^2+3n+1 \\
\texttt{sixn}_{\,n} &= 6n
\end{aligned}
$$

We have :

$$
\begin{aligned}
\texttt{T}_{\,n+1} &= \frac{-8\ \times\ (\texttt{sixn}_{\,n}+5)\,(\texttt{sixn}_{\,n}+3)\,(\texttt{sixn}_{\,n}+1)}{\texttt{pown3}_{\,n+1}\ \times\ \texttt{pow3}}\,\texttt{T}_{\,n} \\
\texttt{lin}_{\,n+1} &= 545140134+\texttt{lin}_{\,n} \\
\texttt{pown3}_{\,n+1} &= \texttt{binom}_{\,n}+\texttt{pown3}_{\,n} \\
\texttt{binom}_{\,n+1} &= \texttt{sixn}_{\,n+1}+\texttt{binom}_{\,n} \\
\texttt{sixn}_{\,n+1} &= 6+\texttt{sixn}_{\,n}
\end{aligned}
$$

with

$$
\begin{aligned}
\texttt{T}_{\,0} &= \frac{12}{640320} \\
\texttt{lin}_{\,0} &= 13591409 \\
\texttt{pown3}_{\,0} &= 0 \\
\texttt{binom}_{\,0} &= 1 \\
\texttt{sixn}_{\,0} &= 0.
\end{aligned}
$$

A.4.2 The program

```
#let approx_pi digits =
#  let s = ref 0
#  and sixn = ref 0
#  and T = ref (12/640320)
#  and binom = ref 1                        (* 3n^2+3n+1 *)
#  and pown3 = ref 0                        (* n^3 *)
#  and lin = ref (13591409)                 (* 13591409+545140134n *)
#  and pow3 = 640320**3
#  and (sqrt, pow) = sqrt640320 (digits - 2)
#  and term = ref (12*13591409/640320) in
#  let pow_test = pow*10 in
#    while abs(!term)*pow_test > 1 do
#       s := !s + !term;
#       pown3 := !binom + !pown3;
#       T := -8 * (!sixn+1) * (!sixn+3) * (!sixn+5) / (!pown3 * pow3) * !T;
#       lin := 545140134 + !lin;
#       term := !T * !lin;
#       sixn := !sixn + 6;
#       binom := !sixn + !binom
#       done;
#  sqrt/(!s*pow)
#;;
Value approx_pi : num -> num
```

A.4.3 Running the program

```
#approx_pi 500;;
 : num
Evaluation has needed: Runtime: 4.20s GC: 1.91s

#approx_pi 1000;;
 : num
Evaluation has needed: Runtime: 40.59s GC: 6.61s
```

Now let us try it with 2000 digits:

```
#approx_pi 2000;;

Evaluation Failed: failure "create_nat: number too long"
Evaluation has needed: Runtime: 81.30s GC: 15.05s
```

The runtime decreases but there are still too many allocation of numbers. So the next step is computation with no allocation of numbers of type `nat`. We will prepare this step using first signed integers.

A.5 Fourth approach

A.5.1 The algorithm

Let us note that each partial sum is naturally a quotient of two integers $\mathtt{N}_n$ and $\mathtt{D}_n$ with :

$$
\begin{aligned}
\mathtt{D}_n &= 640320^{3n+1}\ (n!)^3 \\
\mathtt{prod}_n &= (-1)^n\ \frac{12\ (6n)!}{(3n)!} \\
\mathtt{sum}_n &= 13591409 + 545140134n \\
\mathtt{N}_n &= 640320^{3n}\ n!^3\ \sum_{i=0}^{i=n}(-1)^i\ \frac{12(6i)!}{(3i)!}\ \frac{13591409 + 545140134i}{640320^{3i}\ i!^3} \\
\mathtt{pown3}_n &= n^3 \\
\mathtt{binom}_n &= 3n^2 + 3n + 1 \\
\mathtt{sixn}_n &= 6n.
\end{aligned}
$$

So we have recursively

$$
\begin{aligned}
\mathtt{D}_{n+1} &= 640320^3\ (n+1)^3\ \mathtt{D}_n \\
\mathtt{prod}_{n+1} &= -8\ (6n+1)\ (6n+3)\ (6n+5)\ \mathtt{prod}_n \\
\mathtt{sum}_{n+1} &= 545140134 + \mathtt{sum}_n \\
\mathtt{N}_{n+1} &= 640320^3\ (n+1)^3\ \mathtt{N}_n + \mathtt{prod}_{n+1}\ \mathtt{sum}_{n+1} \\
\mathtt{pown3}_{n+1} &= \mathtt{binom}_n + \mathtt{pown3}_n \\
\mathtt{binom}_{n+1} &= \mathtt{sixn}_{n+1} + \mathtt{binom}_n \\
\mathtt{sixn}_{n+1} &= 6 + \mathtt{sixn}_n
\end{aligned}
$$

and

$$\begin{aligned}
\mathtt{D}_0 &= 640320\\
\mathtt{prod}_0 &= 12\\
\mathtt{sum}_0 &= 13591409\\
\mathtt{N}_0 &= 12\times 13591409\\
\mathtt{pown3}_0 &= 0\\
\mathtt{binom}_0 &= 1\\
\mathtt{sixn}_0 &= 0.
\end{aligned}$$

We stop the computation when

$$\left|\frac{\mathtt{prod}_n\ \mathtt{sum}_n}{\mathtt{D}_n}\right| \le 100\ \alpha.$$

We have $\alpha = 10^{-digits}$ so we test

$$\left|\mathtt{prod}_n\ \mathtt{sum}_n\ 10^{digits-1}\right| \le \mathtt{D}_n.$$

A.5.2 The program

```
#let approx_pi digits =
#  let prod = ref 12
#  and sum = ref 13591409
#  and D = ref 640320
#  and N = ref (12*13591409)
#  and sixn = ref 0
#  and binom = ref 1                      (* 3n^2+3n+1 *)
#  and pown3 = ref 0                        (* n^3 *)
#  and (sqrt, pow) = sqrt640320 (digits-2)
#  and pow3 = 640320**3 in
#  let pow_test = pow*10 in
#    while abs(!prod*!sum*pow_test) > !D do
#       prod := -8*(!sixn+1)*(!sixn+3)*(!sixn+5)*!prod;
#       sum := 545140134+!sum;
#       pown3 := !binom + !pown3;
#       D := !pown3*pow3*!D;
#       N := !pown3*pow3*!N+!prod*!sum;
#       sixn := !sixn+6;
#       binom := !sixn + !binom
#       done;
#  sqrt*!D/(!N*pow)
#;;
Value approx_pi : num -> num
```

A.5.3 Running the program

```
#approx_pi 500;;
 : num
Evaluation has needed: Runtime: 0.17s
```

Now let us try with 10000 digits:

```
#approx_pi 10000;;
 : num
Evaluation has needed: Runtime: 260.32s GC: 48.96s
```

Computing 500 digits of π is now quite fast, but computing 10000 digits takes too much time and too much GC. It uses less room in the heap but the amount of GC is still enormous. So we will now translate the preceding algorithm with `big_int` functions as an intermediary step for type `nat`.

A.5.4 `big_int` version

We increment `sixn` during the computation of $\mathtt{T}_{n+1}$ and we integrate the alternate sign in it. We can consider the denominator as

$$\mathtt{D}_{n+1} = (640320\ (n+1))^3\ \mathtt{D}_n.$$

Let us remind the definition of `D`, `prod`, `sum`, `N`, `incr_den`, `binom`, `monom` and `sixn` to us.

$$\begin{aligned}
\mathtt{D}_n &= 640320^{3n+1}\ (n!)^3 \\
\mathtt{prod}_n &= (-1)^n\ \frac{12\ (6n)!}{(3n)!} \\
\mathtt{sum}_n &= 13591409 + 545140134n \\
\mathtt{N}_n &= 640320^{3n}\ n!^3\ \sum_{i=0}^{i=n} (-1)^i \frac{12(6i)!}{(3i)!} \frac{13591409 + 545140134i}{640320^{3i}\ i!^3} \\
\mathtt{incr_den}_n &= 640320^3\ n^3 \\
\mathtt{binom}_n &= 640320^3\ (3n^2 + 3n + 1) \\
\mathtt{monom}_n &= 640320^3\ (6n) \\
\mathtt{sixn}_n &= 6n + 1.
\end{aligned}$$

So we have recursively

$$\begin{aligned}
\mathtt{D}_{n+1} &= 640320^3\ (n+1)^3\ \mathtt{D}_n \\
\mathtt{prod}_{n+1} &= -8\ (6n+1)\ (6n+3)\ (6n+5)\ \mathtt{prod}_n \\
\mathtt{sum}_{n+1} &= 545140134 + \mathtt{sum}_n \\
\mathtt{N}_{n+1} &= 640320^3\ (n+1)^3\ \mathtt{N}_n + \mathtt{prod}_{n+1}\ \mathtt{sum}_{n+1} \\
\mathtt{incr_den}_{n+1} &= \mathtt{binom}_n + \mathtt{incr_den}_n \\
\mathtt{binom}_{n+1} &= \mathtt{monom}_{n+1} + \mathtt{binom}_n \\
\mathtt{monom}_{n+1} &= \mathtt{monom}_n + \mathtt{six_pow3} \\
\mathtt{sixn}_{n+1} &= 6 + \mathtt{sixn}_n
\end{aligned}$$

and

$$\mathtt{D}_0 = 640320$$

$$\begin{aligned}
\texttt{prod}_0 &= 12\\
\texttt{sum}_0 &= 13591409\\
\texttt{N}_0 &= 12 \times 13591409\\
\texttt{incr_den}_0 &= 0\\
\texttt{binom}_0 &= 262537412640768000\\
\texttt{monom}_0 &= 0\\
\texttt{sixn}_0 &= 1.
\end{aligned}$$

We simplify the stop test, we determine an upper bound `sizeB` on the size of the 2^{32}-radix representation of this number and then we test if the product `prod` $\times$ `sum` shifted by `sizeB` digits is less than `D`. For our purpose, it is sufficient to have:

$$size(\texttt{prod}) + size(\texttt{sum}) + \texttt{sizeB} < size(\texttt{D}).$$

```
##standard arith true;;
Pragma  : bool

##open arith;;
Pragma  : unit

#let test (prod, sum, D, sizeB) =
#  (num_digits_big_int prod)+(num_digits_big_int sum)+sizeB >
#  (num_digits_big_int D)
#;;
Value test : big_int * big_int * big_int * int -> bool

#let sqrt640320 digits =
#  let pow = power_int_positive_int (#10, digits) in
#    create_ratio (sqrt_big_int (mult_big_int (#(640320), square_big_int pow)),
#                  pow)
#;;
Value sqrt640320 : int -> ratio

#let approx_pi digits =
#  let prod = ref #(12)
#  and sum = ref #(13591409)
#  and sixn = ref #1
#  and N = ref #(163096908)
#  and D = ref #(640320)
#  and incr_den = ref #(0)                 (* (640320n)^3 *)
#  and binom = ref #(262537412640768000)   (* incr_den = incr_den + binom *)
#  and monom = ref #(0)                    (* binom = binom + monom *)
#  and six_pow3 = #(1575224475844608000)   (* monom = monom + six_pow3,
#                                             six_pow3=640320^3 * 6 *)
#  and d = max_int #0 (pred_int digits) in
#  let sizeB =                             (* 10^d <= (2^32)^sizeB *)
#    int_of_big_int (ceiling_ratio (mult_int_ratio (d, #[3.322/32]))) in
#    while test (!prod, !sum, !D, sizeB) do
#      prod := mult_int_big_int (#-8, !prod);
```

```
#       prod := mult_int_big_int (!sixn, !prod);
#       sixn := add_int (!sixn, #2);
#       prod := mult_int_big_int (!sixn, !prod);
#       sixn := add_int (!sixn, #2);
#       prod := mult_int_big_int (!sixn, !prod);
#       sixn := add_int (!sixn, #2);
#       sum := add_big_int (!sum, #(545140134));
#       incr_den := add_big_int (!incr_den, !binom);
#       monom := add_big_int (!monom, six_pow3);
#       binom := add_big_int (!binom, !monom);
#       N := add_big_int (mult_big_int (!prod, !sum),
#                         mult_big_int (!incr_den, !N));
#       D := mult_big_int (!incr_den, !D)
#    done;
#    let r = (sqrt640320 (pred_int d)) in
#      create_ratio (mult_big_int (numerator_ratio r, !D),
#                    mult_big_int (denominator_ratio r, !N))
#;;
Value approx_pi : int -> ratio
```

A.5.5 Running the program

```
#approx_pi #10000;;
 : ratio
Evaluation has needed: Runtime: 60.61s GC: 34.18s
```

Now let us try with 20000 digits:

```
#approx_pi #20000;;
 : ratio
Evaluation has needed: Runtime: 179.98s GC: 171.16s
```

The runtime is quite better, but the GC-time is almost equal to the runtime for 20000 digits, so we will now improve the GC-time with type `nat`.

A.6 Final version: type `nat`

We need to know an upper bound of the size of the value of each variable. These sizes are increasing functions of the number of iterations so we will now compute an upper bound for this number and for the ultimate size of each variable.

A.6.1 Number of iterations

The series

$$\frac{S_n}{\sqrt{640320}} = \frac{\sum_{n=0}^{\infty}(-1)^n\, a_n}{\sqrt{640320}} = \sum_{n=0}^{\infty}(-1)^n \frac{12(6n)!}{(n!)^3(3n)!}\frac{13591409 + 545140134n}{640320^{3n+\frac{3}{2}}}$$

converges according to the alternated series criterion so we have:

$$\varepsilon' = \left| \frac{\sqrt{640320}}{\pi} - S_n \right| < a_{n+1},$$

and it is sufficient that an upper bound of a_n is less than the precision needed. We try to find a simple upper bound of a_n.

According to Stirling's formula:

$$\left(\frac{n}{e}\right)^n \sqrt{2\pi n} < n! < \left(\frac{n}{e}\right)^n \sqrt{2\pi n}\ \ e^{\frac{1}{12n}}$$

we have

$$\begin{aligned} (6n)! &< \left(\frac{6n}{e}\right)^{6n} \sqrt{12\pi n}\ \ e^{\frac{1}{72n}} \\ n!^3 &> \left(\frac{n}{e}\right)^{3n} \left(\sqrt{2\pi n}\right)^3 \\ (3n)! &> \left(\frac{3n}{e}\right)^{3n} \sqrt{6\pi n} \end{aligned}$$

and

$$\frac{(6n)!}{n!^3(3n)!} < \frac{12^{3n}\ e^{\frac{1}{72n}}}{2\ (\sqrt{\pi n})^3}.$$

So we have

$$a_n < \frac{1}{2\pi^{\frac{3}{2}}} \left(\frac{12}{640320}\right)^{3n+1} \frac{e^{\frac{1}{72n}}\ (13591409 + 545140134n)}{n^{\frac{3}{2}}}.$$

Hence

$$a_n < \frac{e^{\frac{1}{72}}\ 558731543}{106720\ \pi^{\frac{3}{2}}}\ \frac{1}{n^{\frac{1}{2}}\ (151931373056000)^n}$$

if we bound above $(13591409 + 545140134n)$ by $(13591409 + 545140134) \times n$ and $e^{\frac{1}{72n}}$ by $e^{\frac{1}{72}}$. Let λ and x respectively be

$$\lambda = \frac{e^{\frac{1}{72}}\ 558731543}{106720\ \pi^{\frac{3}{2}}}$$

and $x = 151931373056000$, the preceding inequality becomes:

$$a_n < \lambda\ \frac{1}{n^{\frac{1}{2}}\ x^n}.$$

So a sufficient condition for $u_n < \alpha$ is

$$n\ x^{2n} > \frac{\lambda^2}{\alpha^2}$$

and a fortiori

$$x^{2n} > \frac{\lambda^2}{\alpha^2},$$

that is to say

$$n \geq \left\lceil \frac{\ln_B(\lambda) - \ln_B(\alpha)}{\ln_B(x)} \right\rceil .$$

With $B = 2^{32}$ we have

$$n = \lceil 0.21 + 0.68 \times d \rceil$$

where d is the number of precision digits in base B for $\frac{\sqrt{640320}}{\pi}$ so $d = digits - 2$ where $digits$ is the number of precision digits in base B for π and

$$n = \lceil -1.14 + 0.68 \times digits \rceil \text{ with } B = 2^{32}.$$

We can notice that the successive upper bounds we have determine were quite good since this final upper bound for n is quasi-optimal.

A.6.2 Size of the objects with base 2^{32}

We consider as a reasonable value of n, any integer smaller than 3000 according to the maximum number of digits. So $\ln_B(n)$ is bounded above by $\ln_B(3000)$ from now on.

Let us remark that 640320^3 and $\frac{(6n)!}{(3n)!}$ $(n \geq 1)$ have at least 24 as common factor. If $n = 1$, 24 divides $\frac{(6n)!}{(3n)!} = 6 \times 5 \times 4$, and if $n \geq 2$,

$$\binom{6n}{3n} = \frac{(6n)!}{(3n)!\,(3n)!}$$

is an integer so

$$\frac{(6n)!}{(3n)!} = (3n)! \binom{6n}{3n}$$

is a multiple of $(3n)!$ so a multiple of $2 \times 3 \times 4 = 24$. So we can reduce the size and the gcd of `N` (hence `prod` too) and `D` if we simplify them by a factor 24 each step.

Recall the definition of `D`, `prod`, `sum`, `N`, `incr_den`, `binom`, `monom` and `sixn` to us.

$$\begin{aligned}
\texttt{D}_n &= \left(\frac{640320}{12}\right) \times \left(\frac{640320^3}{24}\right)^n (n!)^3 \\
\texttt{prod}_n &= \frac{(6n)!}{24^n\,(3n)!} \\
\texttt{sum}_n &= 13591409 + 545140134n \\
\texttt{N}_n &= 640320^{3n}\, n!^3 \sum_{i=0}^{i=n} (-1)^i \frac{12(6i)!}{(3i)!} \frac{13591409 + 545140134i}{640320^{3i}\, i!^3} \\
\texttt{incr_den}_n &= \frac{640320^3}{24} n^3 \\
\texttt{binom}_n &= \frac{640320^3}{24} (3n^2 + 3n + 1) \\
\texttt{monom}_n &= \frac{640320^3}{24} (6n) \\
\texttt{sixn}_n &= 6n \\
\texttt{twon}_n &= 2n + 1.
\end{aligned}$$

So we have recursively

$$
\begin{array}{rcll}
\mathtt{D}_{\,n+1} & = & \mathtt{incr\_den}_{\,n+1}; \mathtt{D}_{\,n} & \\
\mathtt{prod}_{\,n+1} & = & (6n+1)\ (6n+5)\ \mathtt{twon}_{\,n}\ \mathtt{prod}_{\,n} & \\
\mathtt{sum}_{\,n+1} & = & 545140134 + \mathtt{sum}_{\,n} & \\
\mathtt{N}_{\,n+1} & = & \mathtt{incr\_den}_{\,n+1}\ \mathtt{N}_{\,n} + \mathtt{prod}_{\,n+1}\ \mathtt{sum}_{\,n+1} & \text{if } n+1 \text{ is even} \\
 & = & \mathtt{incr\_den}_{\,n+1}\ \mathtt{N}_{\,n} - \mathtt{prod}_{\,n+1}\ \mathtt{sum}_{\,n+1} & \text{otherwise} \\
\mathtt{incr\_den}_{\,n+1} & = & \mathtt{binom}_{\,n} + \mathtt{incr\_den}_{\,n} & \\
\mathtt{binom}_{\,n+1} & = & \mathtt{monom}_{\,n+1} + \mathtt{binom}_{\,n} & \\
\mathtt{monom}_{\,n+1} & = & \mathtt{monom}_{\,n} + \mathtt{six\_pow3} & \\
\mathtt{sixn}_{\,n+1} & = & 6 + \mathtt{sixn}_{\,n} & \\
\mathtt{twon}_{\,n+1} & = & 2 + \mathtt{twon}_{\,n} &
\end{array}
$$

and

$$
\begin{array}{rcl}
\mathtt{D}_{\,0} & = & 53360 \\
\mathtt{prod}_{\,0} & = & 1 \\
\mathtt{sum}_{\,0} & = & 13591409 \\
\mathtt{N}_{\,0} & = & 13591409 \\
\mathtt{incr\_den}_{\,0} & = & 0 \\
\mathtt{binom}_{\,0} & = & 10939058860032000 \\
\mathtt{monom}_{\,0} & = & 0 \\
\mathtt{sixn}_{\,0} & = & 0.
\end{array}
$$

The ultimate value of `sum` $= 13591409 + 545140134n$ takes 1 digit if $n \leq 7$ and 2 for all other reasonable values of n.
`twon` and `sixn` take only a small `int` for the reasonable values of n, but they are used as a digit.

We have :

$$\mathtt{prod}_n = \frac{(6n)!}{24^n\ (3n)!} < \sqrt{2}\ \left(\frac{72n^3}{e}\right)^n\ e^{\frac{1}{72n}}$$

so

$$\ln_B(\mathtt{prod}_n) < \ln_B\left(\frac{72 \times 3000^3}{e}\right)\ n + \ln_B\left(\sqrt{2}\ e^{\frac{1}{72}}\right) < \lceil 1.23061n + 0.01625117 \rceil.$$

We have :

$$\mathtt{D}_n = 10939058860032000^n\ \times\ 53360 \times n!^3 < 10939058860032000^n\ \times\ 53360 \times \left(\frac{n}{e}\right)^{3n}\ \left(\sqrt{2\pi n}\right)^3\ e^{\frac{1}{4n}}$$

so

$$\ln_B(\mathtt{D}_n) < \ln_B\left(\frac{10939058860032000\ \times\ 3000^3}{e^3}\right)\ n + \ln_B\left(53360\ \left(\sqrt{2 \times 3000\ \pi}\right)^3\ e^{\frac{1}{4}}\right) < \lceil 2.613n + 1.168 \rceil.$$

`monom`, `binom` and `incr_den` take less than 3 digits.

`N` takes $\lceil 2.613n + 1.418 \rceil$ digits.

In fact, the final product with the square root is performed in `N` and `D` so the reserved size for these variables is increased by the number of digits.

Let us notice that the precision of $\texttt{digits} - 2$ for the square root was for $B = 10$ and we need to take `digits` of precision for $B = 2^{32}$.

A.6.3 The complete program

```
##standard arith true;;
Pragma  : bool

##fast arith true;;
Pragma  : unit

##open arith;;
Pragma  : unit

#let sqrt640320 digits =
#  let len = mult_int (#2, digits) in
#  let real_len = succ_int len in
#  let nat = create_nat real_len in
#    set_digit_nat (nat, len, #10005);
#    shift_left_nat (nat, #0, real_len, create_nat #1, #0, #6);
#    (* now nat = 10005*2^6*2^(32*digits) *)
#    sqrt_nat (nat, #0, real_len)
#;;
Value sqrt640320 : int -> nat

#let approx_pi digits =
#  (* Number of digits *)
#  let iter = int_of_big_int (
#               ceiling_ratio (
#                 add_ratio (#[-1.14],
#                            mult_int_ratio (digits, #[0.68])))) in
#  (* Sizes *)
#  let size_sum = if le_int (iter, #7) then #1 else #2
#  and max_size_prod = int_of_big_int (
#                        ceiling_ratio (
#                          add_ratio (#[0.01625117],
#                                     mult_int_ratio (iter, #[1.23061]))))
#  and common = mult_int_ratio (iter, #[2.613]) in
#  let max_sizeD = succ_int (add_int (digits, int_of_big_int (
#                      ceiling_ratio (add_ratio (#[1.168], common)))))
#  and max_sizeN = add_int (digits, int_of_big_int (
#                      ceiling_ratio (add_ratio (#[1.418], common)))) in
#  (* Creations *)
#  let prod = make_nat max_size_prod
#  and sum = make_nat size_sum
#  and N = ref (make_nat max_sizeN)
#  and D = ref (make_nat max_sizeD)
```

```
#  and incr_den = make_nat #3
#  and binom = make_nat #3
#  and monom = make_nat #3
#  and six_pow3 = nat_of_string "65634353160192000"
#  and sixn = make_nat #1
#  and twon = make_nat #1
#  and incr_sum = nat_of_string "545140134"
#  and aux1 = ref (make_nat max_sizeN) (* aux1 and aux2 are auxiliary *)
#  and aux2 = ref (make_nat max_sizeN) (* variables for computing N and D *)
#  and size_aux1 = ref #1
#  and size_aux2 = ref #1
#  and size_prod = ref #1
#  and new_size_prod = ref #1
#  and sizeN = ref #1
#  and sizeD = ref #1
#  in
#    (* Initialisation *)
#    set_digit_nat (prod, #0, #1);
#    blit_nat (sum, #0, nat_of_string "13591409", #0, #1);
#    blit_nat (!N, #0, nat_of_string "13591409", #0, #1);
#    blit_nat (!D, #0, nat_of_string "53360", #0, #1);
#    blit_nat (incr_den, #0, nat_of_string "10939058860032000", #0, #2);
#    blit_nat (binom, #0, nat_of_string "10939058860032000", #0, #2);
#    (* Treatment *)
#    for i = 1 to iter do
#        (* Treatment of prod *)
#        new_size_prod := min_int max_size_prod (succ_int !size_prod);
#        mult_digit_nat (prod, #0, !new_size_prod,
#                        prod, #0, !new_size_prod, sixn, #0);
#        incr_nat (sixn, #0, #1, #1);
#        incr_nat (sixn, #0, #1, #1);
#        incr_nat (sixn, #0, #1, #1);
#        incr_nat (sixn, #0, #1, #1);
#        mult_digit_nat (prod, #0, !new_size_prod,
#                        prod, #0, !new_size_prod, sixn, #0);
#        incr_nat (sixn, #0, #1, #1);
#        incr_nat (sixn, #0, #1, #1);
#        mult_digit_nat (prod, #0,
#                        (min_int max_size_prod (succ_int !new_size_prod)),
#                        prod, #0, !new_size_prod, twon, #0);
#        incr_nat (twon, #0, #1, #1);
#        incr_nat (twon, #0, #1, #1);
#        size_prod :=
#          num_digits_nat (prod, #0,
#                          (min_int max_size_prod (succ_int !new_size_prod)));
#        (* Treatment of sum *)
#        add_nat (sum, #0, size_sum, incr_sum, #0, #1, #0);
#        (* Treatment of N *)
#       mult_nat (!aux1, #0, add_int (!size_prod, size_sum),
#                 prod, #0, !size_prod, sum, #0, size_sum);
#       size_aux1 := num_digits_nat (!aux1, #0, add_int (!size_prod, size_sum));
```

```
#       (exchange N aux2);
#       (exchange sizeN size_aux2);
#       mult_nat (!N, #0, add_int (!size_aux2, #3),
#                 !aux2, #0, !size_aux2, incr_den, #0, #3);
#       sizeN := num_digits_nat (!N, #0, add_int (!size_aux2, #3));
#       (if (i mod 2 = 0)
#           then add_nat (!N, #0, succ_int (max_int !sizeN !size_aux1),
#                         !aux1, #0, !size_aux1, #0)
#           else sub_nat (!N, #0, !sizeN, !aux1, #0, !size_aux1, #1));
#       sizeN := num_digits_nat (!N, #0, succ_int (max_int !sizeN !size_aux1));
#       (* Treatment of D *)
#       set_to_zero_nat (!aux1, #0, !size_aux1);
#       size_aux1 := 1;
#       set_to_zero_nat (!aux2, #0, !size_aux2);
#       size_aux2 := 1;
#       (exchange D aux2);
#       (exchange sizeD size_aux2);
#       mult_nat (!D, #0, add_int (!size_aux2, #3),
#                 !aux2, #0, !size_aux2, incr_den, #0, #3);
#       sizeD := num_digits_nat (!D, #0, add_int (!size_aux2, #3));
#       set_to_zero_nat (!aux2, #0, !size_aux2);
#       size_aux2 := 1;
#       (* Treatment of incr_den *)
#       add_nat (monom, #0, #3, six_pow3, #0, #2, #0);
#       add_nat (binom, #0, #3, monom, #0, #3, #0);
#       add_nat (incr_den, #0, #3,binom, #0, #3, #0)
#   done;
#   let sq = sqrt640320 digits in
#     create_ratio (
#       ((exchange D aux2);
#        (exchange sizeD size_aux2);
#        mult_nat (!D, #0, min_int (add_int (!size_aux2, (succ_int digits)))
#                                  max_sizeD,
#                  !aux2, #0, !size_aux2, sq, #0, succ_int digits);
#        big_int_of_nat !D),
#        (blit_nat (!N, digits, !N, #0, !sizeN);
#        big_int_of_nat !N))
#;;
Value approx_pi : int -> ratio

#let num_of_digits digits =
#  int_of_big_int (ceiling_ratio (mult_int_ratio (digits, #[0.1038103])))
#;;
Value num_of_digits : int -> int

#let approx_pi10 digits = approx_pi (num_of_digits digits)
#;;
Value approx_pi10 : int -> ratio

##use (lib_directory ^ "format_numbers");;
Directive  : unit
```

```
#let show_pi digits =
#  let s = approx_ratio_fix (digits, approx_pi10 digits) in
#  let len = sub_int (length_string s, #3) in
#  let v = vector #1 of (sub_string s #3 len) in
#  let res = beautiful_string_vect_of_string_vect
#             (v, #0, #0, big_int_of_int len) in
#  print_string "+3.";print_newline();
#  for i = #0 to pred_int (vect_length res) do
#    print_string res.(i)
#  done
#;;
Value show_pi : int -> unit
```

A.6.4 Running the program

```
#timers true;;
 : unit

#echo_values false;;
 : unit

#approx_pi10 10000;;
 : ratio
Evaluation has needed: Runtime: 36.68s

#approx_pi10 20000;;
 : ratio
Evaluation has needed: Runtime: 155.06s GC: 0.95s
```

And now a beautiful printing of 2000 digits (contained quite exactly in one page):

```
#show_pi 2000;;
+3.
14159 26535  89793 23846  26433 83279  50288 41971  69399 37510
58209 74944  59230 78164  06286 20899  86280 34825  34211 70679
82148 08651  32823 06647  09384 46095  50582 23172  53594 08128
48111 74502  84102 70193  85211 05559  64462 29489  54930 38196
44288 10975  66593 34461  28475 64823  37867 83165  27120 19091

45648 56692  34603 48610  45432 66482  13393 60726  02491 41273
72458 70066  06315 58817  48815 20920  96282 92540  91715 36436
78925 90360  01133 05305  48820 46652  13841 46951  94151 16094
33057 27036  57595 91953  09218 61173  81932 61179  31051 18548
07446 23799  62749 56735  18857 52724  89122 79381  83011 94912

98336 73362  44065 66430  86021 39494  63952 24737  19070 21798
60943 70277  05392 17176  29317 67523  84674 81846  76694 05132
00056 81271  45263 56082  77857 71342  75778 96091  73637 17872
14684 40901  22495 34301  46549 58537  10507 92279  68925 89235
42019 95611  21290 21960  86403 44181  59813 62977  47713 09960

51870 72113  49999 99837  29780 49951  05973 17328  16096 31859
50244 59455  34690 83026  42522 30825  33446 85035  26193 11881
71010 00313  78387 52886  58753 32083  81420 61717  76691 47303
59825 34904  28755 46873  11595 62863  88235 37875  93751 95778
18577 80532  17122 68066  13001 92787  66111 95909  21642 01989

38095 25720  10654 85863  27886 59361  53381 82796  82303 01952
03530 18529  68995 77362  25994 13891  24972 17752  83479 13151
55748 57242  45415 06959  50829 53311  68617 27855  88907 50983
81754 63746  49393 19255  06040 09277  01671 13900  98488 24012
85836 16035  63707 66010  47101 81942  95559 61989  46767 83744

94482 55379  77472 68471  04047 53464  62080 46684  25906 94912
93313 67702  89891 52104  75216 20569  66024 05803  81501 93511
25338 24300  35587 64024  74964 73263  91419 92726  04269 92279
67823 54781  63600 93417  21641 21992  45863 15030  28618 29745
55706 74983  85054 94588  58692 69956  90927 21079  75093 02955

32116 53449  87202 75596  02364 80665  49911 98818  34797 75356
63698 07426  54252 78625  51818 41757  46728 90977  77279 38000
81647 06001  61452 49192  17321 72147  72350 14144  19735 68548
16136 11573  52552 13347  57418 49468  43852 33239  07394 14333
45477 62416  86251 89835  69485 56209  92192 22184  27255 02542

56887 67179  04946 01653  46680 49886  27232 79178  60857 84383
82796 79766  81454 10095  38837 86360  95068 00642  25125 20511
73929 84896  08412 84886  26945 60424  19652 85022  21066 11863
06744 27862  20391 94945  04712 37137  86960 95636  43719 17287
46776 46575  73962 41389  08658 32645  99581 33904  78027 59010
 : unit
Evaluation has needed: Runtime: 1.96s
```

Annexe B

To normalize or not to normalize rational numbers?

In this appendix, we try to answer the two following questions:

- when is it necessary to normalize,
- is it on average a good strategy always to normalize or not to normalize.

B.1 On average not to normalize is the better strategy

Let p (resp. p_n) be the probability for two integers a and b, to be mutually prime (resp. to have a gcd equal to n). Plainly, $\gcd(a,b) = n$ if and only if $a \bmod n = 0$ and $b \bmod n = 0$ and

$$\gcd\left(\frac{a}{n}, \frac{b}{n}\right) = 1.$$

Thus

$$p_n = \frac{1}{n} \times \frac{1}{n} \times p.$$

Hence we have :

$$1 = \sum_{n=1}^{n=\infty} \frac{p}{n^2} = p \times \sum_{n=1}^{n=\infty} \frac{1}{n^2} = p\,\frac{\pi^2}{6}$$

thus

$$p = \frac{6}{\pi^2}$$

and

$$p_n = \frac{6}{\pi^2\, n^2}.$$

Numerically speaking p is greater than 60% and the probability that two integers have gcd less than or equal to 10 is greater than 94%.

So 60% of normalization of rational numbers are absolutely without effect and 94% will need a long time and simplify the rational number by a very small factor compared to the size of the rational number.

This viewpoint guides our choice of the default position for the "normalize ratio when computing" flag (see section 8.6 of this manual).

But now we will see an example where it is much faster to normalize after each step of the computation than not to normalize.

B.2 Exception

B.2.1 Exposition

Here is the origin of this sequence. Jean-Michel Muller in [4] illustrates the problem of round off errors with it. This sequence converges slowly to 6 but with non exact rational arithmetic it seems to converges fastly to 100. In fact, for computing one term, it is much faster to normalize the term after each step, since successive gcd of no normalized numerator and denominator are much bigger than the normalized numerator and denominator.

And now here is the definition of this sequence:

$$\begin{aligned} a_0 &= \frac{11}{2} \\ a_1 &= \frac{61}{11} \\ a_{n+1} &= 111 - \frac{1130 - \frac{3000}{a_{n-1}}}{a_n} \end{aligned}$$

In a closed form,

$$a_n = \frac{6^{n+1} + 5^{n+1}}{6^n + 5^n}.$$

B.2.2 Floating point version

To convince the reader of the different behaviour when using floating point arithmetic, here is

```
##open arith;;
Pragma  : unit

##arith float;;
Symbol prefix + overloaded with add_float : float * float -> float
Symbol prefix * overloaded with mult_float : float * float -> float
Symbol prefix ** overloaded with power_float : float * float -> float
Symbol prefix - overloaded with sub_float : float * float -> float
Symbol prefix / overloaded with div_float : float * float -> float
Symbol prefix <= overloaded with le_float : float * float -> bool
Symbol prefix < overloaded with lt_float : float * float -> bool
Symbol prefix >= overloaded with ge_float : float * float -> bool
Symbol prefix > overloaded with gt_float : float * float -> bool
Symbol - overloaded with minus_float : float -> float
Symbol pred overloaded with pred_float : float -> float
Symbol succ overloaded with succ_float : float -> float
Symbol incr overloaded with incr_float : float ref -> float
Symbol decr overloaded with decr_float : float ref -> float
```

```
Symbol max overloaded with max_float : float -> float -> float
Symbol min overloaded with min_float : float -> float -> float
Symbol abs overloaded with abs_float : float -> float
Symbol prefix power overloaded with power_float : float * float -> float

#let a n =
#  let rec arec = function
#    0 -> (11.0/2.0,61.0/11.0)
#  | n -> let (x, y) = arec (pred_int n) in
#           (y, (111.0 - (1130.0 - 3000.0/x)/y))
#in snd (arec (pred_int n))
#;;
Value a : int -> float

#a 20;;
 : float
```

B.2.3 The program

This sequence can be implemented as follows:

```
##open arith;;
Pragma  : unit

##arith ratio;;
Symbol prefix + overloaded with add_ratio : ratio * ratio -> ratio
Symbol prefix * overloaded with mult_ratio : ratio * ratio -> ratio
Symbol prefix - overloaded with sub_ratio : ratio * ratio -> ratio
Symbol prefix / overloaded with div_ratio : ratio * ratio -> ratio
Symbol prefix <= overloaded with le_ratio : ratio * ratio -> bool
Symbol prefix < overloaded with lt_ratio : ratio * ratio -> bool
Symbol prefix >= overloaded with ge_ratio : ratio * ratio -> bool
Symbol prefix > overloaded with gt_ratio : ratio * ratio -> bool
Symbol - overloaded with minus_ratio : ratio -> ratio
Symbol float overloaded with float_of_ratio : ratio -> float
Symbol max overloaded with max_ratio : ratio * ratio -> ratio
Symbol min overloaded with min_ratio : ratio * ratio -> ratio
Symbol abs overloaded with abs_ratio : ratio -> ratio

#let a n =
#  let rec arec = function
#    #0 -> (#[11/2],#[61/11])
#  | n -> let (x, y) = arec (pred_int n) in
#           (y, (#[111] - (#[1130] - #[3000]/x)/y))
#in snd (arec (pred_int n))
#;;
Value a : int -> ratio
```

and we compare the runtime for a_{20} with and without normalizing during computation:

```
#timers true;;
 : unit
```

```
#set_normalize_ratio true;;
 : bool

#a 20;;
 : ratio
Evaluation has needed: Runtime: 0.10s

#approx_ratio_fix (#3, it);;
 : string

#set_normalize_ratio false;;
 : bool

#a 20;;
 : ratio
Evaluation has needed: Runtime: 4.12s
```

and more outstandingly for a_{30} we obtain:

```
#set_normalize_ratio true;;
 : bool

#a 30;;
 : ratio
Evaluation has needed: Runtime: 0.17s GC: 1.00s

#approx_ratio_fix (#3, it);;
 : string

#set_normalize_ratio false;;
 : bool

#a 30;;

Evaluation Failed: failure "create_nat: number too long"
Evaluation has needed: Runtime: 29.01s GC: 1.89s
```

We see on this example that this is a very small rational number (its numerator and denominator have only three 2^{32}-digits). If we do not perform normalization during computation, it is represented by two integers with more than 8191 2^{32}-digits. This proves well how normalizing during computation may be important in exceptional cases.

So we recommend in practice to try both versions. If a rational computation seems particularly slow, maybe you should turn this flag on and try again.

Bibliographie

[1] D. V. Chudnovsky and G. V. Chudnovsky. The computation of classical constants. *Proc. Natl. Acad. Sci. USA 86* (November 1989), 8178–8182.

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[3] D. E. Knuth. *The Art of Computer Programming: Seminumerical Algorithms.* Addison-Wesley, 1981.

[4] J. M. Muller. *Arithmétique des ordinateurs.* Masson, 1989.

Concept index

S

Function index

Note the meaning of the two following symbols placed beyond the name of an arithmetic function :

- † this function is only available in open arithmetic mode,
- $*$ this function is one of the kernel part of the BigNum

A

I

M

N

Q

R

S