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On the Pathwidth of Planar Graphs

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Abstract: Fomin and Thilikos in [5] conjectured that there is a constant c such that, for every 2-connected planar graph G, $\text{pw}(G^*) \leq 2\text{pw}(G)+c$ (the same question was asked simutaneously by Coudert, Huc and Sereni in [4]). By the results of Boedlander and Fomin [2] this holds for every outerplanar graph and actually is tight by Coudert, Huc and Sereni [4]. In [5], Fomin and Thilikos proved that there is a constant c such that the pathwidth of every 3-connected graph G satisfies: $\text{pw}(G^*) \leq 6\text{pw}(G) + c$. In this paper we improve this result by showing that the dual a 3-connected planar graph has pathwidth at most 3 times the pathwidth of the primal plus two. We prove also that the question can be answered positively for 4-connected planar graphs.

Key-words: planar graphs, pathwidth

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A Propos de la Pathwidth des Graphes Planaires

Résumé : Fomin et Thilikos[5], après avoir démontré que la pathwidth de tout graphes planaires 3-connexe est au plus 6 fois celle de son dual à une constante près, ont conjecturé que pour tout graphe planaire biconnexe G, $\mathrm{pw}(G^*) \leq 2\mathrm{pw}(G) + cte$. D'après Boedlander et Fomin [2] cela est vrai pour tout graphe outerplanaire. De plus cela est exact d'après Coudert, Huc and Sereni [4]. Dans cet article nous améliorons le résultat de Fomin et Thilikos en montrant que la pathwidth de tout graphe planaire 3-connexe est au plus 3 fois celle de son dual plus 2. Nous démontrons également que la conjecture est vrai pour tout graphe planaire dont le dual est 4-connexe.

Mots-clés : graphes planaires, pathwidth

1 Introduction

A *planar graph* is a graph that can be embedded in the plane without crossing edges. It is said to be *outerplanar* if it can be embedded in the plane without crossing edges and such that all its vertices are incident to the unbounded face. For any graph G, we denote by $V(G)$ its vertex set and by $E(G)$ its edge set. The *dual* of the planar graph G, denoted by G^*, is the graph obtained by putting one vertex for each face, and joining two vertices if and only if the corresponding faces are adjacent. Note that the dual of a planar graph can also be computed in linear time.

The notion of pathwidth was introduced by Robertson and Seymour [9]. A *path decomposition* of a graph $G = (V, E)$ is a set system $(X_1, \ldots, X_r)$ of V such that

1. $\bigcup_{i=1}^{r} X_i = V$;
2. $\forall xy \in E, \exists i \in \{1, \ldots, r\} : \{x, y\} \subset X_i$;
3. for all $1 \leq i_0 < i_1 < i_2 \leq r$, $X_{i_0} \cap X_{i_2} \subseteq X_{i_1}$.

The *width* of the path decomposition $(X_1, \ldots, X_r)$ is $\max_{1 \leq i \leq r} |X_i| - 1$. The *pathwidth* of G, denoted by $\mathrm{pw}(G)$, is the minimum width over its path decompositions.

Computing the pathwidth of graphs is an active research area, in which a lot of work has been done (Fer a survay see for instance [8]). Govindan et al. [6] gave an $\mathcal{O}(n \log(n))$ time algorithm for approximating the pathwidth of outerplanar graphs with a multiplicative factor of 3. For biconnected outerplanar graphs, Bodlaender and Fomin [2] improved upon this result by giving a linear time algorithm which approximates the pathwidth of biconnected outerplanar graphs with a multiplicative factor 2. To do so, they exhibit a relationship between the pathwidth of an outerplanar graph and the pathwidth of its dual. More precisely, the following holds.

Theorem 1 (Bodlaender and Fomin [2]) *Let G be a biconnected outerplanar graph without loops and multiple edges. Then $pw(G^*) \leq pw(G) \leq 2pw(G^*) + 2$.*

Since the weak dual of an outerplanar graph (which can be computed in linear time) is a tree and there exist linear time algorithms to compute the pathwidth of a tree [11], this yields the desired approximation.

Coudert, Huc and Sereni in [4] improved this result by proving the following theorem:

Theorem 2 (Coudert, Huc and Sereni [4]) *For every biconnected outerplanar graph G, we have $pw(G^*) \leq pw(G) \leq 2\ pw(G^*) - 1$ and all the values in the interval $[pw(G^*)\ ,\ 2\ pw(G^*) - 1]$ can be the pathwidth of G.*

Simultaneously Coudert, Huc and Sereni state the following question as an open problem in [4] and Fomin and Thilikos conjectured it in [5] :

Conjecture 1 ([5],[4]) *Is there a constant c such that, for every 2-connected planar graph G, $\frac{1}{2}pw(G^*) - c \leq pw(G) \leq 2pw(G^*) + c$?*

It is worth noting that this conjecture is motivated by the following result about the treewidth, conjectured by Robertson and Seymour [10] and proved by Lapoire [7] using algebraic methods (notice that Bouchitté, Mazoit and Todinca [3] gave a shorter and combinatorial proof of this result).

Theorem 3 ([7]) *For every planar graph G,* $\mathrm{tw}(G) \leq \mathrm{tw}(G^*) + 1$.

Fomin and Thilikos made an even stronger conjecture :

Conjecture 2 ([5]) *There is a constant c such that for every 2-connected planar graphs G of treewidth at least m, $pw(G^*) \leq \frac{m}{m-1} pw(G) + c$*

This conjecture does not hold. Indeed we can slightly modify the examples given in [4]. They are example of biconected outerplanar graphs G such that $\mathrm{pw}(G) = 2\mathrm{pw}(G^*) - 1$. We modify this family of graph by plugging a 3×3 grid on a face. This can be done without changing the pathwidth, whereas the treewidth increases from 2 to 3, so the equation is no longer satisfied.

The following theorem improves the previously known bound for 3-connected planar graphs.

Theorem 4 (Main theorem) *For every 3-connected planar graph G we have $pw(G) \leq 3\ pw(G^*) + 2$*

Actually our methods prove that the conjecture holds for every 4-connected planar graph.

2 Main Theorem

In this section we present the proof of our main theorem. We will use the following notations:

Given a graph $G = (V(G), E(G))$ of maximum degree $\Delta(G)$, we will note $V(G^*)$ either its face set or the vertex set of its dual and by f_G, e_G and n_G respectively the number of faces, edges and vertices of G. Given a set A, by $\mathcal{P}(A)$ we denote the family of all subsets of A.

Definition 1 *Let G and H be two graphs. A connected map from G to H is a map $\sigma : V(G) \to \mathcal{P}(V(H))$ from vertices of G to subsets of vertices of H satisfying the following two properties:*

1. *for every $v \in V(G)$, $\sigma(v)$ is connected.*
2. *for every adjacent vertices $v, w \in V(G)$, $\sigma(v) \cup \sigma(w)$ is also connected*

σ is of degree at most k if it also satisfies

- *$\forall w \in V(H)$ we have $|\sigma^{-1}(w) := \{v \in V(G) | w \in \sigma(v)\}| \leq k$*

Lemma 1 *Let G and H be two graphs. for any connected map σ of degree at most k from G to H, we have:*

- *$pw(G) \leq k\ pw(H) + k - 1$*
- *$tw(G) \leq k\ tw(H) + k - 1$*

Proof

- Given a path-decomposition of H of width ℓ, applying σ^{-1} on bags of our decomposition gives one of width $k.\ell + k - 1$ for G. This can be easily verified using the properties of σ listed above.
- Same proof gives the result for tree-width.

□

From now on we suppose G to be a 3 vertex connected planar graph. We aim to find a low degree connected map from G^* to G.

A *Face-To-Edge assignment* is a system of distinct representatives for faces of G. In other words, a Face-To-Edge assignment is a function τ such that we associate to a given face F of G an edge $\tau(F) = (v, w) \in E(F) \subset E(G)$, in such a way that two different faces are associated to different edges. Given a Face-To-Edge assignment, the map $\sigma : G^* \to \mathcal{P}(V(G))$ associates to every vertex F of G^* (face of G) the subset $V(F) \setminus V(\tau(F))$.

Proposition 1 *The so defined map σ is connected.*

Proof Two faces F_1, F_2 sharing an edge e can't be both associated to e since they are associated to different edges. Consequently $\sigma(F_1) \cup \sigma(F_2)$ is connected. $\square$

Given a Face-To-Edge assignment, let H be the subgraph of G consisting of non selected edges; i.e. $H = G \setminus \{\tau(F) | F \in V(G^*)\}$. Using Euler's Formula ($f_S + n_S = e_S + 2$) we know that H contains exactly $n-2$ edges. We have

Proposition 2 $\forall v \in V(G), \quad |\sigma^{-1}(v)| = deg_H(v)$

Proof A selected edge (an edge of $G \setminus H$) should be associated to one of the two faces containing it. Given a vertex v of G of degree d, it appears exactly in d faces. Suppose r edges incident to v are selected, so they should be associated to exactly r faces incident with v. v doesn't appear in $|\sigma(v)|$ of these faces, and appears in $|\sigma(v)|$ of other faces incident to v. So $|\sigma^{-1}(v)| = d - r = deg_H(v)$ $\square$

Corollary 1 *σ is of degree at most $\Delta(H)$.*

Remark that the average degree in H is always $leq 2$.

Definition 2 *We call $H \subset G$ a nice subgraph if it has $n-2$ edges and such that we can find a Face-To-Edge assignment τ with $\tau(F) \in E(G) \setminus E(H)$.*

Definition 3 *Given a graph G, we call adjacency graph, the bipartite graph A on vertex set ($V(G^*) \cup E(G \setminus H)$), with an edge between a vertex of $V(G^*)$ (i.e. a face F of G) and an edge of $G \setminus H$ if this edge belongs to F.*

Corollary 2 *Let H be a nice subgraph of G of max-degree Δ. Then we have $pw(G^*) \leq \Delta\ pw(G) + \Delta - 1$*

We will need the following theorem of Barnette [1]:

Theorem 5 *(Barnette) Every 3-connected planar graph has a spanning tree of max-degree 3.*

Corollary 3 *Every 3-connected planar graph G contains a nice subgraph of max-degree 3*

Proof By Barnette's theorem there exists a sub-forest H of G of max-degree 3 containing $n-2$ edges (i.e. is a spaning tree minus an edge). We want to prove that such a subgraph is nice by applying Hall's matching theorem to the adjacency graph A between faces of G and edges of $G \setminus H$. Given a set of faces $\{F_1, \ldots, F_i\}$ we should prove that in A the corresponding set has at least i neighbors. Considering the planar graph S obtained by taking the union of F_i, we have:

- $f_S \geq i + 1$ (because G^* is connected)
- $f_S + n_S = e_S + 2$ (Euler's formula)

We conclude $e_S - (n_S - 1) \geq i$. As H is a forest the number of edges of H incident with some vertex of this subgraph is at most $n_S - 1$. So the hypothesis of Hall's theorem is always satisfied. This proves that H is a nice graph. $\square$

As a corollary we have

Corollary 4 *There exists a connected map $\sigma : G^* \to G$ of degree at most 3.*

As a results from corrolary 2 and 4 we have our main theorem:

Theorem 6 *For every 3-connected planar graph G we have $pw(G) \leq 3\ pw(G^*) + 2$*

Furthermore our method proves the conjecture for planar graphs whose dual has an Hamiltonian path :

Theorem 7 *If G has a Hamiltonian path, we have $pw(G) \leq 2\ pw(G^*) + 1$*

Proof The Hamiltonian path gives a nice subgraph of G of max-degree 2. □

Corollary 5 *If G^* is 4-connected then we have $pw(G) \leq 2\ pw(G^*) + 1$*

Proof Thomassen proved in [**?**] that every 4-connected planar graph has a Hamiltonian cycle. Then by last theorem we have the result. □

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