Topological Semantic for Hybrid Modal Logic and Spatial Logic

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Abstract

We presente spatial logics and spatio-temporal logics in the paradigm of hybrid modal logics. These hybrid logics have some particularities: they have a topological semantic, the topological spaces associated with them are T_0 -spaces and have a cellular structure and the nominals of the hybrid logics are sorted. From this particular case, we look at general semantic structures and the completeness of a sorted hybrid logic with pure axioms in a "sorted sense"!

1 Introduction

Digital topology is a tool to represente pictures on digital screens. Classical topology is not operational on discrete structures because only the discrete topology is Hausdorff topology on finite spaces and we give no information about space. The topological spaces used in this cas are Alexandroff T_0-spaces. The elements of these spaces are cells of various dimensions. We brievely present these topological spaces. A topological semantic for modal logic is the oldest semantic for modal logics. The work of Aiello Van Benthem [2] has recalled their interest. Topological semantic for hybrid logic seems to be new but the work made for Kripke relational one can be easily translated. Dimension of cells is a fondamental parameter so we think that a multi-sorted hybrid logic is convenient and we have to verify that the completeness theorems of [3] and can be generalized.

2 The topological structure

A topological set is a set E with a collection $\mathcal{O}(\mathcal{E})$ of subsets of E named the open sets of E such that E and $\emptyset$ are open sets, that the intersection of a finite number of open sets is an open set and that arbitrary union of open sets is an open set.

A subset of E is said to be closed if its complementary in E is open.
For $X \subset E$, the interior int(X) is the union of open sets included in X and the closure $\overline{X}$ is the intersection of closed sets that include X.
A neighboorhood of an element x is a subset of E that contains an open set which contains x.

It is presented a topological structure for finite or infinite spaces. There are not Hausdorff separated spaces but T_0-spaces.

Figure 1: Examples : a,b,c and counter-examples d,e,f of complexes

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2.1 T_0 Alexandroff spaces

Definition 2.1.1 (T_0-space) *A T_0-space(T_0S) is a topological space such as if x and x' are two distincts points , there exists an open set θ that contains exactly one of x and x'.*

Definition 2.1.2 (Alexandroff space) *A topological space E is said to be an Alexandroff space if for every element x of E, there exists an open set which contains x and is a subset of such another open set . It is called the smallest neighborhood of x and denoted by SN(x).*

Theorem 2.1.1 *A finite T_0S is Alexandroff.*

Proof : a finite intersection of open sets is an open set .

Theorem 2.1.2 (Partial orders and T_0S) *If E is a sparse T_0S, the relation R defined by: $R(x, y) \equiv y \in SN(x)$ is a partial order.*

Proof :

- antisymetry : if $x' \in SN(x)$ then $x\neg \in SN(x')$ and $SN(x') \subset SN(x)$ (definition of a T_0S).
- reflexivity : evident.
- transitivity : if R(x,y) and R(y,z) then $SN(y) \subset SN(x)$ and $SN(z) \subset SN(y)$ therefore $SN(z) \subset SN(x)$ and R(x,z).

For a partial order (E,R) one denotes $u \uparrow \{x \mid R(u, x)\}$. We say that a subset X of E is a filter if whenever it contains x, it contains $x \uparrow$.

Theorem 2.1.3 *If E,R is a partial order, we define a T_0 topology with the set of its filters as a basis for the open sets.*

Proof :

- The intersection of 2 filters is a filter and an arbitrary union of filters is a filter then we have defined a topology.
- An element x has a minimal neighbour

  Theorem 2.1.4 *: $x \uparrow$.*

  If R(x,y) and $x \neq y$ then $y \uparrow$ is an open set that doe's not contain x. If x and y are not comparable then $x \uparrow$ and $y \uparrow$ are 2 open sets that contain only one of the points.

  Exemple 2.1.1 (Tesselation) *We consider a tesselation of the euclidian plane and for each "paving-stone", we split it in cells :the interior and the boundary splitted in edges and corners. like in figure 1. There are cells for pixels and for inter-pixels.*

2.2 Dimension of a T_0-space

We have to give a formal definition of the dimension of a cell and then the dimension of the space is the max of the dimensions of its cells. In our spatial semantic, we want a point to have 0 for its dimension, a line to have 1 and and an area to have 2.2. In the relational perspective, a point is a minimal element and the dimension is the level of an element (the level of a point is the maximal length of this point to a minimal point). We give then the definition in the topological point de vue. We define the sets E_i of the elements of the topological space E whose dimension is i by induction:

- If for the element x , x is not in the minimal neighbourhood $SN(y)$ of an other point y, then $dim(x) = 0$. Such an element x is said to be minimal .
- If the sets $E_0, E_1, \ldots$E $_n$ are defined , we consider the the set $=X_n = E - (E_0, \vee E_1, \ldots \vee E_n)$. $E_{n+1} = \{x \mid$ x is not in the minimal neighbourhood $SN(y)$ of an other point $y \in X_n$. This notion is not defined for every T_O space, but we have the theorem:

  Theorem 2.2.1 *An Alexandroff T_0 space E is said SN bounded if there is an integer n such that $card(SN(x)) \leq n$ for all the elements of E. The dimension function is well defined for a SN-bounded Alexandroff T_0 space.*

  Proof.
  In every T_0 space, if $y \in SN(x)$ then $x \notin SN(y)$ and $SN(y) \subset SN(x)$. An element whose minimal neighbouhood is maximal is not contained in the minimal neighbourhood of an another element so the set E_0 is well defined and all E_i are also defined.
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3 Hybrid modal logic

We define a pure axiomatic for a hybrid sorted logic.

3.1 Axiomatic

The language $\mathcal{LHP}$ of propositional hybrid modal logic is composed of:

* A countable set of propositional variables $\Phi = \{p, q, \ldots\}$
* A finite number of sets of nominals: $\mathbf{N}_0, \mathbf{N}_1, \ldots, \mathbf{N}_p$. For each $k \in 0, \ldots, p$ we denote i_k an element of $\mathbf{N}_k$.
* Boolean connectives: $\neg, \wedge, \vee, \rightarrow$
* Modal operators $\Box, \Diamond$
* Axioms of hybrid modal logic as enumerated in **??**
* Pure formulas: the two first items are hybrid axioms for S4 which is the more general logic that have a topological semantic.
  1. PF1 $@_{\ i}\Diamond_i$
  2. PF2 $@_i\Diamond_j \wedge @_j\Diamond_k \rightarrow @_i\Diamond_k$.
  3. PF3 $i \rightarrow \Box(\Diamond i \rightarrow i)$
  4. PF4 $\Diamond i_p \rightarrow i_p$ where p is the index maximum for the sets of nominals.
  5. PF5 If $l \ \geq k$ then $\neg(@_{i_l}\Diamond i_k)$

3.2 Topological semantics

We define a model $\mathbf{M} = E, \mathcal{O}, \nu$ to be a topological set E and a function ν from E to the propositional variables (including nominals in the hybrid one) such that:

* $\nu(p) \subseteq E$.
* For any nominal i $\nu(i)$ is a singleton (the denotation of i) and if two nominals are of different sorts their images are distincts.

The function ν can be extended into the set of all formulas by the induction rules :

* $\nu(\neg\phi) = E - \nu(\phi)$
* $\nu(\phi \wedge \psi) = \nu(\phi) \cap \nu(\psi)$
* $\nu(\Box\phi) = int(\nu(\phi))$

* $\nu(\Diamond\phi) = \overline{\nu\phi}$

An equivalent definition is :

- If $p \in \Phi$, $M, x \models p$ iff $x \in \nu(p)$.
- $M, x \models \neg\phi$ iff not $M, x \models \phi$
- $\mathbf{M}, x \models (\phi \wedge \psi)$ iff $\mathbf{M}, x \models \phi)$ and $\mathbf{M}, x \models psi)$
- $\mathbf{M}, x \models (\Box\phi)$ iff exists a neighbourhood V of x where $\forall y \in V$ $\mathbf{M}, y \models \phi$
- $\Diamond$M, x $\models \Diamond\phi$ iff for every neibourhood V of x there exists $y \in V$ such that: $\mathbf{M}, y \models \phi$

Definition 3.2.2 ([Sorted named model) *A model* $\mathbf{M} = E, \mathcal{O}, \nu$ *is sorted named iff E is an Alexandroff* T_0 *-space of dimension p, if every point x of E is the denomination of some nominal V(x) and if dim(x)* $=$ $dim(V(x))$.

4 Soudness

The topological semantic of S4 was presented in McKinsey,Tartski's work (ref. The soudness here is prooved for Alexandroff T_0-spaces of finite dimension. and sorted named models.

Theorem 4.0.1 *If* $\mathcal{L}$ *is a hybrid sorted modal logic, a formula* $\phi \in \mathcal{L}$ *is valid in every sorted named model*

The pure formulas PF1 and PF2 of the definition are the translation of two properties of topological spaces. For Alexandroff spaces , they say respectively that; $x \in SN(x)$ and if $x \in SN(y)$ and $y \in Sn(z)$ then x $\in SN(z)$.
There are three specific axioms for this logic: we give the topological significance:

- PF3 (that is viewed as antisymetry for Kripke semantic) is characteristic of T_0 spaces: if $y \in SN(x)$ then $x \notin SN(y)$.
- PF4: the elements of maximal dimension are open sets.
- PF5: if $x \in N_k$ in $SN(\nu(x))$ there are only elements of dimension $> k$.

5 Completeness

For completeness, there is a theorem demonstrated in **??** for the general hybrid logic. We show how it can be generalized. This theorem is prooved with the help of lemmas 7.24, 7;25, 7.28 and 7.29.

– In lemma 7.24, if Γ is a maximal consistent set (MCS), the sets $Delta_i = \{\phi \mid @_k\phi \in \Gamma$ are MCS(s) and their properties stay true in a sorted logic.

– Lemma 7.25: extended Lindenbaum lemma) which adds nominals to the language to locate formulas like: $@_i\Diamond\phi$ and obtain a pasted MCS. In the sorted version we have to add new nominals of every sort in the extended language and formulas : $@_i\Diamond j_l$ if $i \in N_r$ and $l > r$.

– 7.27 and 7.28, Kripke models yielded by extended MCS are studied. We do analogous work with topological models (cf **??**): the elements of the topological space W_Γ are the MCS Δ_i yielded by a MCS completed and named as in the extended Lindenbaum lemma, and the open sets are arbitrary unions of sets of the form: for every $U_\phi = \{\Delta_i \mid \Diamond\phi \in \Delta_i$. This topological space is a Alexandroff T_0, lemma 7.27 become: if $u \in W_\Gamma$, there is a $v \in W_\Gamma$ such that: $v \in SN(u)$.

Theorem 5.0.2 (Completeness) *Let $\mathcal{LHP}$ be a sorted hybrid propositional logic with a dimension dim. Every set of consistent formulas of $\mathcal{LHP}$ is satisfiable in a countable named topological model.*

6 A spatio-temporal logic

This theory can be extended to a spatio-temporal one. We consider a bi-modal logic with a modality for the time and another for space.Time is described by a cellular complex of dimension1. Points are instants and lines are intervals of time. But there is a specific parameter for the time: the order relation between cells of time. The model is a topological ordered space. This structure of time allows to have a temporal logic with instants and intervals. But here all cells are separate and we don't have to manage the Allen's algebra.

7 Application: The logic of the tramway

A tramway in a town: there are geographic objects of several dimensions: crossroads of dimension 0, streets of dimension 1 and blocks of house of dimension 2. Some propositions like:

– There is a tramway station at this crossroads.

– This street is in the route of the tramway.

– This station is near an hospital.

– This block is near the route of the tramway.

And the schedule of the tramway is managed in a spatio-temporal logic. This is an example of the utility of sorted hybrid modal logics. Only in the propositional case , we have a great expressivity.

8 Conclusion and perspectives

There are several claims:

– Hybrid logics are good tools for applications.

– Sorted logic is often necessary.

– A cellular complex allows to manage precisisely the localization of events.

– Topological semantic can be get out of the museum!

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