Remarks on Serre's modularity conjecture

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Abstract

In this article we give a proof of Serre's conjecture for the case of odd level and arbitrary weight. Our proof will not depend on any yet unproved generalization of Kisin's modularity lifting results to characteristic 2 (moreover, we will not consider at all characteristic 2 representations in any step of our proof).
The key tool in the proof is Kisin's recent modularity lifting result, which is combined with the methods and results of previous articles on Serre's conjecture by Khare, Wintenberger, and the author, and modularity results of Schoof for semistable abelian varieties of small conductor. Assuming GRH, infinitely many cases of even level will also be proved.

1 Introduction

Let $p > 2$ be a prime, and let $\bar{\rho}$ be an odd, irreducible, two-dimensional Galois representation with Serre's weight k and level N, with values in a finite field $\mathbb{F}$ of characteristic p. The "level", or "conductor", is defined as in [Se87] to be the prime-to-p part of the Artin conductor, see [Se87] for the definition of the weight.
We will be mainly interested in the case of representations of odd level, although some cases of even level will also be considered, but only cases where

ramification at 2 is semistable (in the sense of [Ri97]).
For such $\bar{\rho}$, and in particular for all representations of odd level, we will prove Serre's modularity conjecture (assuming GRH in the cases of even level), i.e., we will prove that $\bar{\rho}$ is modular (cf. [Se87]). As it is well-known, for a prime p given (by suitable twisting) it is enough to consider the case of $k \leq p+1$. In all steps of the proof, whenever a residual irreducible representation is considered, it will be tacitly assumed that such a twist is performed so that the weight satisfies this inequality.

In this article, as in previous articles proving special cases on Serre's conjecture, modularity is proved by "propagation", i.e., by applying the principle of "switching the residual characteristic" (originally applied in [Di04b] and [KW04] to prove the first cases of Serre's conjecture) to reduce the problem to some other case of the conjecture already solved. This "switching principle" follows from a combination of three main results:
- Existence of minimal lifts or lifts with prescribed properties ([Di04b], [KW04], [K05], [KW06])
- Existence of (strictly and strongly) compatible families ([Di04a]), and
- Modularity lifting results à la Wiles (Taylor-Wiles, Skinner-Wiles, Diamond, Savitt, Kisin).
At this point, the main "obstacle for propagation" is due to the technical conditions needed for the application of these modularity lifting results. However, in several cases, like the crystalline of small weight case ($k < p$, assuming that $p \neq 2k-3$, or the representation is semistable), or weight 2 semistable case, it is known that by combining different modularity lifting results the lifting is modular without imposing any condition on the residual representation, just modularity or reducibility (cf. [Di03] for the weight 2 case and [DM03], [K05] for higher weights).

By a suitable combination of "switchings", using modularity lifting results of Kisin, we will show how the proof of the general odd level case can be reduced to the proof of the level 3 case, a case that we will reduce in turn to some modularity results of Schoof for semistable abelian varieties of small conductor. Also, some cases of even level will be solved assuming GRH.

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2 Preliminary results

• Weight 2 lifts:

Let $\bar{\rho}$ be a residual representation of weight k, odd level N, and odd characteristic p. Assume that $k > 2$, $k < p$, and the image of $\bar{\rho}$ is non-solvable.

At several steps of the proof we will need to consider for such a $\bar{\rho}$ a p-adic "weight 2 minimal lift" ρ defined as in [KW06], theorem 5.1(2), and to introduce ρ in a strongly compatible family (in the sense of [Ki06a]) $\{\rho_\ell\}$.

Every representation in the family $\{\rho_\ell\}$ with ℓ odd is unramified outside N and p (and, of course, ℓ), it is Barsotti-Tate if $\ell \nmid pN$ and potentially semistable of weight 2 or potentially Barsotti-Tate for any odd ℓ, and it has inertial Weil-Deligne parameter at p equal to $(\omega_p^{k-2} \oplus 1, 0)$ where ω_p is the Teichmuller lift of the mod p cyclotomic character. In particular ρ_p is potentially Barsotti-Tate (and Barsotti-Tate over a subextension of the cyclotomic field).

Since the proof of Theorem 5.1 is not given in [KW06], let us explain how the existence of weight 2 minimal lifts is deduced. The existence of p-adic lifts of this type follows by the same strategy used in [Di04b] and [KW04] for the construction of minimal lifts. A key point is to use the potential modularity results of Taylor (cf. [Ta02] and [Ta01]) and base change to obtain a bound from above for the corresponding universal deformation ring given by a modular deformation ring (a ring that is known to be finite). For the case of minimal lifts of a semistable representation, this key result was obtained by the author and, independently, by Khare-Wintenberger. It follows from results of Boeckle (cf. [Bo03]) that this suffices for a proof after checking that the local conditions are such that the corresponding restricted local deformation rings have the right dimension, i.e., the local "defects" Δ_ℓ are 0 for every prime ℓ (cf. [Bo03]) so that $\dim R > 0$ holds for the global universal deformation ring. The existence of a strictly compatible family satisfying good local properties containing such lift follows as in [Di04a] (and due to results of T. Saito, the same argument also gives strong compatibility). In [Ki06a] the strategy of [Di04b] and [KW04] is explained in detail in a generality which is enough for the case we are considering. In fact, corollary (3.3.1) of [Ki06a] asserts the existence of a minimal potentially Barsotti-Tate deformation (of fixed determinant) of a suitable type τ (propositions (3.2.4) and (3.2.6) of loc. cit. are generalizations of Boeckle's result to this context). In our case the type we impose is $(\omega_p^{k-2} \oplus 1, 0)$ and it follows from results of Savitt and Kisin on a conjecture of Conrad, Diamond and Taylor and on the Breuil-Mezard conjecture (cf. [Sa05], theorem 6.21 and [Ki06a], lemma (3.2.1), which is proved in [Ki06b]) that for this type corollary (3.3.1) of [Ki06a] applies. Thus, the result of Boeckle and the upper bound deduced from potential modularity are enough to conclude, for an odd characteristic $p > k$, the existence of the weight 2 minimal lift (and the strongly compatible family containing it) as stated in [KW06], theorem 5.1(2).

• Raising the level and good-dihedral primes:

Because of the technical conditions needed to apply Kisin's modularity lifting results, in the general case we need to assume that the residual representations that we encounter through the proof have non-solvable image.
This is handled by a trick in [KW06] of adding some extra prime q to the level in order that all the representations that we encounter are "good dihedral at q" and thus have non-solvable image (cf. loc. cit., section 8.4 and section 6).

Remark: The possibility to introduce "extra ramification" follows from the existence of non-minimal p-adic lifts of certain type (non-minimal at q). Again, as in the case of minimal lifts (cf. [Di04b] and [KW04]) or weight 2 minimal lifts, the existence of such a lift (which mimics a result that is well-known for modular forms, namely, a case of "raising the level") follows by combining potential modularity (to obtain an upper bound for the corresponding universal deformation ring) with the results of Boeckle (to obtain a lower bound).

This way, after having "raised the level", we can assume that at each step we will encounter residual representations having a large prime q in the level such that $q^2 \mid N$ with:

$$\bar{\rho}_p|_{I_q} = \mathrm{diag}(\psi, \psi^q) \qquad (I)$$

where the character ψ has order t^α, $t \mid q+1$ an odd prime, with q and t sufficiently large.

This ramification at q will be preserved in all the steps of the proof if the primes q and t are chosen as in the definition of good dihedral prime (ibid, Def. 2.1).

As a matter of fact, we have to modify slightly the definition of good dihedral prime, because we want to work also in characteristics up to a certain bound B, a bound possibly larger than the weight and the primes in the level. Thus, to ensure that also in these characteristics the ramification at q is preserved we modify this definition as follows:

i) assume t is greater than all prime factors of N (except q), greater than k, and greater than B.

ii) assume that $q \equiv 1 \pmod 8$, and $q \equiv 1 \pmod r$ for every prime r up to the maximum of: the prime divisors of N different from q, k, and B.

Remark: this "modified" definition does not affect at all the proof of existence of such a q and t.

As in section 8.4 of [KW06], at a large characteristic t we add the extra ramification at q in order to reduce the proof to representations being good dihedral at q. In all steps of the proof we will work in characteristics which are sufficiently small with respect to q and t, namely smaller than a certain bound B previously given. Thus, we know that all residual representations that we encounter maintain the good dihedral prime q in their levels and thus have non-solvable image.

Conclusion: we can assume that we start with a representation which is good dihedral at a (very large) prime q.

3 Kisin's newest modularity lifting result

We will rely heavily on the modularity lifting result of Kisin in [Ki06c]. One of the technical conditions in this result (condition 4, a condition on the image of the restriction to the decomposition group at p of the residual representation) is only included because the available version of the preprint is just a preliminary version, but as the author explains it is not intrinsic and should be removed in a later version. So, we will use the "strong version" of Kisin's result, namely the main theorem (page 2 of Kisin's preprint) but without condition 4, since this new version of the proof should become soon available (see also the remark in the last section concerning the strong version of Kisin's result). This result will be used to ensure that modularity is preserved in the "moves", as long as the residual image is not solvable and the lifting is (locally at p) semistable over an abelian extension of $\mathbb{Q}_p$.

For the sake of completeness, in the last section (see section 7, Remarks 1, 3 and Final Remark) we will also present a variation of the proof that does not require this "strong version", thus we give a proof that uses the modularity lifting results in [Ki06c] just as they appeared in the available version of this preprint.

Now let us describe the moves that give the proof of Serre's conjecture for odd level.

4 Iterated killing ramification

The first step can roughly be described as follows: Killing ramification at primes in the level, one after the other (using minimal lifts to go from one to the other) we reduce to cases of "small level".

Since we need to ensure that modularity propagates well in all "moves" (i.e., every time that we switch the residual characteristic), we have to ensure that the conditions of "non-solvable residual image" and "liftings are semistable over abelian extension of $\mathbb{Q}_p$" needed to apply the strong version of Kisin's result are satisfied.
By adding extra ramification at a very large prime q in a suitable way (given by some character of very large order t), a trick of Khare and Wintenberger, as explained in a previous section as long as we work in characteristics smaller than a certain bound B all representations will be good-dihedral at q and will have non-solvable residual image.

Then, we have a residual representation such that the primes in the level (all odd) are:

$$p_1, p_2, \ldots\ldots p_r, q$$

(q the larger one, and larger than the weight k) in some characteristic p (also smaller than q). Choose r auxiliary primes $b_1, \ldots.b_r$ larger than all the p_i and than k (but smaller than the bound B) and let us transfer all ramification to these primes. The moves that we need here are the following:
-starting in characteristic p, take a minimal lift and move to characteristic b_1, and reduce mod b_1. Take a minimal weight 2 lift and move to characteristic p_1, and reduce mod p_1 (this is killing ramification at p_1). Then take a minimal lift an move to characteristic b_2, reduce mod b_2, take a minimal weight 2 lift, move to p_2, and so on.
Recall that modularity is preserved in all these moves due to results of Kisin (cf. [Ki04]) in the potentially Barsotti-Tate or weight 2 semistable case (where here potentially means over ANY extension), and for crystalline representations of small weight by results of Diamond-Flach-Guo and Kisin (cf. [DFG04] and [Ki05]).
We have transfered the ramification at the p_i to ramification at the b_i introduced in the weight 2 lifts, so now we only have to consider representation such that the primes in the level are:

$$b_1, \ldots.b_r, q$$

and since ramification at each b_i was introduced while taking the weight 2 lift, due to strong compatibility all compatible families that will appear in the rest of the proof will correspond to cases that are (the b_i-adic member, locally at b_i) semistable over an abelian extension.
Therefore, since we have reduced the problem to a case where the two technical conditions needed to apply (the strong version of) Kisin's result in [Ki06c] are satisfied, we now proceed with "iterated killing ramification", i.e., take a minimal lift and move to characteristic b_1, reduce mod b_1, take a minimal lift and move to characteristic b_2, reduce again, and so on. Modularity is preserved because of the results of Kisin, and we have reduced the proof of all cases of odd level to the case: characteristic p, weight k, level q^2, good dihedral at q, where k and the characteristic p are smaller that the bound B.

5 Reduction to the level 3 case

We need to prove modularity in the weight k, level q^2 case, good-dihedral at q (q a very large prime).
If we take a minimal lift and include it in a strongly compatible family, since 3 is not in the level the 3-adic representation in this family is crystalline. Since it is also good-dihedral at q, we can consider the residual mod 3 representation and it is enough to prove modularity for this residual representation (thanks to the strong version of Kisin's result). After twisting, we can assume that the weight of this mod 3 representation is either 2 or 4 (being an odd representation, whose determinant ramifies only at 3, the case of $k = 3$ is impossible). By considering a minimal lift (if $k = 2$) or a minimal weight 2 lift (if $k = 4$) we have reduced the proof to the case of weight 2 and level q^2 or $3q^2$, semistable at 3.

Remark: For the case $k = p + 1$ the minimal weight 2 lift corresponds to a representation which is weight 2 semistable at p (cf. [K05]).

Now we are ready to say good-bye to our good-dihedral prime q: with this weight 2 family of conductor q^2 or $3q^2$ we move to characteristic t (the order of the character describing ramification at q) and we consider the residual mod t representation.
This is a very important point, so a few remarks before going on:
-Remark 1: At this point we are loosing the good-dihedral prime q. This means that this mod t representation and all residual representations in the next steps, may not have non-solvable image. So from now on we can not apply any longer the modularity lifting result of Kisin. We will have to use (see, for example, Khare's proof of the level 1 case for a similar situation) other modularity lifting results. As already explained in the introduction (and articles [Di03], [DM03], [K05]) in several cases this is known to work well (even if we have no information on the residual image, it can even be reducible):
- semistable weight 2 lift
- potentially Barsotti-Tate lift which is Barsotti-Tate over the cyclotomic extension
- crystalline lift of weight k with $k < p$ or $k = p + 1$ (for $k = p + 1$ either the lift is ordinary and results of Skinner-Wiles apply or if not results of Kisin apply), assuming that the residual representation is semistable at all primes

different from p (or that $p \neq 2k-3$)
-Remark 2: When reducing mod t given the information on the ramification at q of the family, there are two possibilities (this is also noticed in [KW06]): the residual representation is either unramified or semistable (i.e., unipotent ramification) at q.

From now on, at each step the following remark applies: if the residual representation has solvable image (or reducible) then it is modular (or reducible) and also its lift is modular (because of the above remark 1). So the only case relevant is the case where the residual image is not solvable, which is the case we will consider.

So, if the mod t representation is unramified at q, it will have $k = 2$ (the t-adic lift was Barsotti-Tate) and $N = 3$ (semistable), a case of Serre's conjecture already solved. Thus, the case that remains is the case: $k = 2$, $N = 3q$, semistable at both primes (the case $k = 2$, $N = q$ has already been solved by Khare).
Take a minimal lift (which corresponds to an abelian variety with semistable reduction at 3 and q) and move to characteristic q and reduce mod q. It remains to solve the case $k = q + 1$, $N = 3$, semistable at 3, i.e., the "level 3 case".

6 Proof of the level 3 case

To conclude the proof, let us solve the level 3 case (semistable at 3). We apply Khare's weight reduction (as in [K05]) and the proof is reduced to the cases: $k = 2, 4, 6$, $N = 3$ (semistable at 3): the cases of higher weight follow from these by a sophisticated induction created by Khare. For case $k = 2$ this case is known, so we only have to consider the other two cases.

• Case $k = 6$, $N = 3$: Take a minimal lift and move to characteristic 5, and reduce mod 5. Since the case $(k, N) = (2, 3)$ is known, we assume we are in the case $(k, N) = (6, 3)$. Take a minimal weight 2 lift, it corresponds (as follows from the results of Taylor) to a semistable abelian variety with good reduction outside 3 and 5. By recent results of Schoof ("Modular curves and semistable abelian varieties over $\mathbb{Q}$", unpublished, results presented at an AMS meeting at San Francisco, see [Sc05] for a similar result in the case of one single small prime of bad semistable reduction) such an abelian variety is known to be modular. This concludes the proof in this case.

Case $k = 4$, $N = 3$: At this last step, we use a couple of Sophie-Germain primes: 3 and 7. Move to characteristic 7, take a minimal weight 2 lift and move to characteristic 3. Ramification at 7 may not be eliminated when reducing mod 3, but clearly the character ω_7^2, a character of order 3, trivializes over a finite field of characteristic 3. Therefore, the only possible ramification at 7 of this mod 3 representation is semistable (i.e., unipotent) ramification. Thus, we can have the following cases: $(k, N) = (2, 1), (2, 7), (4, 1), (4, 7)$ (semistable at 7). The only case unknown is the last one, so assume that you do have ramification at 7 and weight 4. Take a minimal weight 2 lift: it corresponds to a semistable abelian variety with good reduction outside 3 and 7. Again, Schoof has proved that any such variety is modular, and we are done.

Let us write the theorem we have proved, together with some well-known consequences (cf. [Se87] for the statement of Serre's conjecture and for the proof of the second consequence, and [Ri92] for the proof of the first consequence):

Theorem 6.1 *Serre's conjecture is true for any odd, two-dimensional, irreducible Galois representation whose Serre's level is odd.*
Every abelian variety defined over $\mathbb{Q}$ of GL_2 type having good reduction at 2 is modular.
Every rigid Calabi-Yau threefold defined over $\mathbb{Q}$ having good reduction at 2 is modular.

7 Concluding Remarks

Remark 1: On the Strong Version of Kisin's result:
The following remark of Kisin should be taken into consideration when applying the "strong version" of his modularity lifting result in [Ki06c] (the result without condition 4): Concerning this strong version, which is not written down yet, he just points out a problematic case:
"A case where one should be particularly careful is when $p = 3$ and the representation is an extension of 1 by ω or a twist of such a representation. It's possible that this case *might* have to be omitted, although it really should

work in this case also" (M. Kisin, personal communication, here ω denotes the mod p cyclotomic character).

Since precisely this case is needed at one step of the proof, taking into account the above remark of Kisin we propose an alternative method to avoid $p = 3$. Observe however that at the end of this section (see Remark 3 and Final Remark) we present a proof that does not require at all the "strong version" of Kisin's result (in any characteristic), so the cautious reader may omit the proof that we present here and move directly to Remark 3:
In the proof $p = 3$ is used to reduce to $k \leq 4$, but there is a way to circumvent the use of $p = 3$: We start with a residual representation of some weight k and level q^2 (q a very large prime, the representation good-dihedral at q). Instead of switching to $p = 3$, we switch to $p = 5$ and we reduce mod 5. Then the weight (up to suitable twist) is $k = 2, 4$, or 6, and due to the "strong version" of Kisin's theorem it is enough to prove modularity of this residual representation. The cases $k = 2, 4$ are handled as before (switching to $p = 3$), the only new case is $k = 6$.

So, what to do with the case "$k = 6$, level q^2"? We will reduce the proof of modularity here to weight $k = 2$ or 4. Observe that Khare's weight reduction can not be applied to reduce the weight k if $k = 6$, so this is a new type of weight reduction.
The moves that we have to do are the following: we start in $p = 5$, we will move to $p = 7$, then to $p = 3$ and back to $p = 7$ (this works because $(3, 7)$ is a Sophie Germain pair), and then to 5 again. More precisely: we start by taking a crystalline lift of weight 6 and move to $p = 7$, then reduce mod 7. Here we take a weight 2 lift, and move to 3 and reduce mod 3. In the worst case, the mod 3 representation will have semistable ramification at 7, so it gives a weight 2 level $7q^2$ representation. Take a minimal lift and move to $p = 7$, the mod 7 representation (in the worst case) will be of weight 8 and level q^2. Then take a minimal crystalline weight 8 lift and move to $p = 5$, and reduce mod 5. Applying Kisin's theorem (strong version) it is enough to check modularity of this mod 5 representation. For a 5-adic representation which is crystalline of Hodge-Tate weights $(0, 7)$, the Serre's weight of the corresponding mod 5 representation can be computed using results of Berger-Li-Zhu and Berger- Breuil (results for the case "crystalline of intermediate weights", cf. Theorem 3.2.1(3) and Comment 3.2.2 in [Be05]): there are several cases, but up to twist we always get $k = 2$ or 4, never $k = 6$. This

concludes the proof.

Remark 2: On even levels:
a) Assuming some extra base case, like $(k, N) = (2, 30)$, one can also prove Serre's conjecture in the case of level semistable at 2.
b) There is one case of even level that can be proved: the case of level 6 (semistable) and weight 2. The method is the following: move to characteristic 3, then the residual mod 3 representation has conductor 2 and weight 2 or 4. For these two cases, Serre's conjecture has been proved by Moon and Taguchi (cf. [MT03], they proved reducibility, of course) in characteristic 3, thus by "switching the residual characteristic" we know they hold in any odd characteristic. Then, applying modularity lifting results (modularity of the semistable weight 2 deformation) the proof is complete.
c) Assuming GRH, also the case of level 10 (semistable) and weight 2 is known: the minimal lift corresponds to a semistable abelian variety, which is modular by results of Calegari (assuming GRH, cf. [Ca04]). Therefore, assuming GRH, we can also prove the following cases of Serre's conjecture: level $2p$ (semistable), p any odd prime, weight 2. The method is the following: move to characteristic p, then the proof is reduced to prove the case: level 2, weight $k \geq 4$. Applying Khare's weight reduction, this can be solved for arbitrary weight assuming that some base cases are known: $k = 2, 4, 6$, $N = 2$. For $k = 2, 4$ this is known, thanks to the result of Moon and Taguchi. For $k = 6$, $N = 2$, we move to characteristic 5 and the residual mod 5 representation has a weight 2 lift corresponding to a semistable abelian variety of conductor (dividing) 10. Since assuming GRH such a variety is modular, we conclude the proof. As a corollary, it follows that any semistable abelian variety of GL_2 type with bad reduction only at 2 and an odd prime p is modular, assuming GRH.

Theorem 7.1 *Assume the Generalized Riemann Hypothesis. Then Serre's conjecture is true for any odd, two-dimensional, irreducible Galois representation of Serre's weight* 2 *and semistable level* $2p$ *for any odd prime* p *(and the case* $p = 3$ *holds unconditionally, i.e., independently of GRH) .*
Every semistable abelian variety defined over $\mathbb{Q}$ *of* GL_2 *type having bad reduction only at* 2 *and another prime* p *is modular.*

Remark 3: The Sophie Germain trick:
In the step of "iterated killing ramification" we have given a proof which

relies on the "strong version" of Kisin's modularity lifting result. There is an alternative approach which avoids this in this step, namely, one can perform two sets of "moves" (the first one using pseudo Sophie Germain primes) to reduce to a situation where condition (4) in Kisin's result is satisfied.
Let us describe this two sets of moves in more detail:
First we transfer the ramification to r primes which are "pseudo Sophie Germain", namely, primes b_i such that $(b_i - 1)/2$ is an odd prime or the product of two odd primes. It is known that there are infinitely many such primes, and also that the odd prime factors of $b_i - 1$ can be taken (both) arbitrarily large. We proceed to transfer ramification to these r primes, the ramification being introduced while taking weight 2 lifts (as in section 4), but immediately after introducing this ramification, we move to the odd characteristics (one or two) dividing b_i, always via weight 2 families, so that in these characteristics we kill part of the ramification at b_i: just to ease the notation assume that $(b_i - 1)/2 = a_i$ is prime, then in characteristic a_i the nebentypus $\omega_{b_i}^{k_i-2}$ which is a character of order a_i or $2a_i$ (depending on the parity of k_i) becomes a character of order at most 2. We conclude that (up to twist) the residual mod a_i representation has ramification at b_i of one of the following two types: semistable or given by a quadratic character. Using strong compatibility we conclude that after transferring ramification to the b_i we have reduced to a case where each compatible family that we will consider is (the b_i-adic member, locally at b_i) either semistable or quadratic-crystalline, i.e., it becomes crystalline when restricted to a quadratic extension. The primes b_i have been chosen sufficiently distant from each other.
The second set of moves is performed to transfer all semistable ramification to potentially crystalline ramification. For each b_i such that ramification at it is semistable, we choose a larger prime q_i such that $q_i - 1$ is divisible by $b_i - 1$. Then we start with a weight 2 family, we switch to characteristic b_i and reduce mod b_i (if this residual representation has weight 2 we have eliminated ramification at b_i, if not we continue) and take a minimal crystalline lift, which has weight $b_i + 1$. We move to characteristic q_i, reduce mod q_i, and take a weight 2 lift, thus ramification at q_i is given by the character $\omega_{q_i}^{b_i-1}$. Namely, we have transfered semistable ramification at b_i into potentially crystalline ramification at q_i, but most importantly, since the exponent $b_i - 1$ divides the order $q_i - 1$ of the character, ramification at q_i is crystalline over a PROPER subfield of the cyclotomic field, a subfield F_i such that the degree of the cyclotomic field over it grows with b_i (*).
At the end, both at those b_j where ramification was quadratic-crystalline

and at the q_i just considered, we conclude that we have reduced to a case where ramification is always potentially crystalline, over a PROPER subfield of the cyclotomic field, as in (*). Just to ease the notation, let us rename the primes b_j where ramification was quadratic crystalline also as q_j.
Not only the primes b_i in the first set of moves but also the primes q_i in the second will be assumed to be "sufficiently distant" from each other.
At this point, we are ready to perform iterated killing ramification to eliminate, one after the other, ramification at each of the primes in the level except, as in section 4, for the very large good-dihedral prime q. Just one warning: this must be done in increasing order. So we begin with a weight 2 family and we move to q_1, the smaller prime in the level, take the residual representation (and, as usual, twist to obtain minimal weight), then a minimal crystalline lift, move to q_2, and so on. Whenever we switch to a prime in the level with a family of weight higher than 2, we have to check that condition (4) in Kisin's result is satisfied. Since ramification at each q_i is potentially crystalline over a PROPER subfield of the cyclotomic field of index at least as large as b_i (as in (*)), and we switch to characteristic q_i with a family of weight at most q_{i-1}, we see that condition (4) is satisfied provided that $q_{i-1} << b_i$. This is why we insisted in taking the b_i and the q_i sufficiently distant from each other, in particular this means that we take them such that the above inequality is satisfied (if you want, choose first b_1, then $q_1 > b_1$, then $b_2 >> q_1$, then q_2, and so on). The good-dihedral prime q is the only prime that remains at the end in the level, then the bound B and the prime q should be chosen at the beginning sufficiently large so that all primes b_i and q_i in the above construction can be taken smaller than B, thus ensuring non-solvable residual image through the whole process.
This concludes the proof that iterated killing ramification can be done independently of the strong version of Kisin's result.

Final Remark:
Recall that "iterated killing ramification" (before starting we have added a good-dihedral very large prime q in the level) reduces the proof of all odd conductor cases of Serre's conjecture to the "level q^2, weight k case, good dihedral at q". If we apply here Khare's weight reduction to reduce to weight $k \leq 6$ and then Remark 1 above to reduce to $k = 2$ or 4 we see thanks to Remark 3 that the only step (in the whole proof) where we need the "strong version" of Kisin's result is a step contained in Remark 1: to conclude that residual modularity implies modularity for a 5-adic representation which is

crystalline of Hodge-Tate weights $(0, 7)$ and has residually non-solvable image (here the problem is that the lift might not satisfy the condition (4) in Kisin's result). So it is the step of reducing the proof of the weight 6 (and level q^2) case to cases of weight 2 or 4 the ONLY step where we still have assumed the strong version of Kisin's result.

Since this "strong version" of Kisin's result is not yet available in print, let us explain for the sake of completeness that there is another way of doing this weight reduction using available modularity lifting results (the results in [Ki04], [DFG04] and [Ki05], and those of Skinner-Wiles). The trick that we will apply is the same used with the pair of primes 3 and 2 in [KW06] to reduce the weight 4 case to the weight 2 case (but here we do not need 2-adic modularity lifting theorems since we do not work with $p = 2$): We start with a residual representation of weight 6, level q^2, which is good-dihedral at a large prime q. We switch to characteristic 5, reduce mod 5 and consider a weight 2 lift, corresponding to an abelian variety with semistable reduction at 5 and conductor $5 \cdot q^2$. Then we switch to characteristic 3, reduce mod 3 and here we observe that this mod 3 representation, since it is either unramified or has unipotent ramification at 5 (and in both cases we know that there is a lift with semistable ramification at 5, so in the unramified case the well-known necessary condition for raising the level at 5 is satisfied), and $3 \mid (5 + 1)$, admits a weight 2 lift where the ramification at 5 is no longer semistable but instead is given by a character of order 3. We obtain a lift of conductor $25 \cdot q^2$: what we have just constructed is a non-minimal lift (it is not minimal at 5) having the same kind of ramification at 5 and at the good-dihedral prime q.
We consider the strictly compatible family containing this 3-adic representation and we switch to characteristic 5. Using strict compatibility and the description of ramification at 5 (a character of order $3 \mid (5 + 1)$) we see that the residual mod 5 representation will have (after suitable twist) Serre's weight equal to 2 or 4, but never 6, because locally at 5 it is irreducible, corresponding to the case of fundamental characters of order 2.
Thus we conclude that there is a way of doing this last weight reduction which does not require the "strong version" of Kisin's result (moreover, it does not require the results of [Ki06c] at all). This completes the proof of Serre's conjecture in the case of odd conductor and arbitrary level.

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