Dimensionality of Space-Time as a Gauge-Invariance Parameter in Yang-Mills Calculations

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1. Introduction

Perturbative calculations of gauge invariant quantities necessarily proceed in a gauge noninvariant manner due to the gauge-fixing required in the lagrangian [1]. In order to verify the gauge-invariance of the final result and to check against possible errors, computations are usually repeated for different choices of the gauge-fixing or they are performed in a general class of gauges labelled by an arbitrary gauge-fixing parameter. In the latter case, one ascertains that the dependence on the gauge-parameter drops out for physical quantities. For Yang-Mills (YM) theories, the complicated tensor structure of the vertices makes calculations in a general gauge containing a gauge parameter extremely tedious. In this paper I describe how, for pure YM theories, one can perform calculations in any particular gauge with a convenient propagator (e.g. Feynman) and yet retain a nontrivial check on the gauge-invariance of the result. The inclusion of fermions will be separately discussed at various places.

The idea is based on the fact that pure quantum chromodynamics (QCD) in two dimensional space-time is a free theory. This is established by going to the axial ($A_1 = 0$) gauge whence all the commutators vanish. Since, by definition, gauge-*invariant* quantities are independent of the choice of gauge-fixing, all gauge-invariant quantities in pure QCD_2 must vanish. The strategy to use this fact for calculating physical quantities in some D_0 dimensional space-time ($D_0 = 4$ is popular but not necessary for the general discussion here) is as follows : perform the Lorentz algebra, i.e. the contraction of vertex and propagator indices, for an arbitrary D dimensions. If the quantity being calculated is truly gauge-invariant, then a *necessary* condition is that it vanish at $D = 2$. In this way, the dimensionality of space-time is used as a "gauge-invariance" parameter.

The nice thing about performing the Lorentz algebra in D dimensions is that it takes almost no more effort than in doing it for the physical D_0 dimensions. The benefit, as mentioned above, is that the D parameter used in a simple gauge provides one with an algebraically efficient way of checking gauge invariance. Of course, one may use the D parameter in conjunction with a conventional gauge parameter (α) to give additional checks and insight. The D parameter is a book-keeping device keeping track of the "relevant" $(D-2)$ pieces in a calculation while the α parameter prefaces the "irrelevant" pieces.

In gauge theories it has become common practice to use dimensional continuation to regulate ultraviolet (UV) divergences in loops [1]. In dimensional regularisation what is important is the analytic continuation of space-time in the measure of loop integrals. For this purpose the Lorentz algebra itself is sometimes performed in the physical spacetime. On the other hand, to use the dimensionality of space-time as a gauge-invariance parameter, it is precisely the analytically continued Lorentz algebra which is important. In this paper, both the Lorentz algebra and the measure for loop integrals is in some arbitrary D dimensions. Gauge invariance is checked by looking for factors of $(D-2)$ while UV divergences are extracted as usual by the $D \to D_0$ limit. Of course once one is satisfied with gauge-invariance, one can set $D = D_0$.

What about fermions? Clearly QCD_2 with fermions is a nontrivial theory. Fortunately, the contribution of fermions to amplitudes can be kept track of by using the usual trick of working with an arbitrary N_f copies of them. A gauge-invariant quantity must be separately gauge invariant in the $N_f = 0$ and $N_f \neq 0$ sectors. In the first sector, the calculations can be performed as described above using the D parameter to check gauge-invariance while the $N_f \neq 0$ sector can be analysed separately. Usually diagrams with one or more fermion lines are algebraically simpler to deal with than those with only gluon lines so the methodology described here is not without promise.

The idea outlined in the preceding paragraphs will be exemplified in this paper by studying the free energy, self-energy, electric mass, propagator poles and hard thermal loops in pure QCD with gauge group $SU(N_c)$ at finite temperature (T) for $D_0 = 4$. The temperature is assumed to be high enough so that perturbative calculations are sensible [2]. The measure for finite temperature loop integrals is

$$\int [dq] \equiv T \sum_{q_0} \int \frac{d^{(D-1)}q}{(2\pi)^{(D-1)}} \,, \tag{1}$$

where the sum is over discrete Matsubara frequencies [2], $q_0 = 2\pi nT$ for gauge bosons and ghosts, $n \in \mathcal{Z}$. Euclidean metric will be used and four- vectors will be denoted by uppercase, $Q^2 = q_0^2 + \vec{q}^2$ and $q \equiv |\vec{q}|$. For quantities which depend on the external momenta, an analytic continuation to Minkowski space is made as usual after the loop sums are done [2].

The gauges that will be frequently referred to are the strict Coulomb gauge $(\xi = 0)$, the α-covariant gauge with gauge-fixing term $(\partial_\mu A_\mu)^2/2(\alpha+1)$ and the Feynman gauge $(\alpha = 0)$. The Feynman rules, being standard [1, 2], will not be spelled out.

2. Free energy

The free-energy is physical quantity equal to the negative of the pressure and is directly obtainable by calculating bubble diagrams in perturbation theory [3]. Since it is physical, it must be gauge-invariant. In the Feynman gauge, the ideal gas pressure (P_0) for pure QCD is given by [2]

$$\frac{P_0 V}{T} = \ln\left\{\left[Det(-\partial^2\delta_{\mu\nu})\right]^{-\frac{1}{2}} . Det(-\partial^2)\right\} \tag{2}$$

$$= \left(\frac{D-2}{2}\right) \ln Det^{-1}(-\partial^2) \tag{3}$$

where V is the volume. The first determinant in (2) is the contribution of gluons while the second determinant is the ghost contribution. The expression in (3) resulted from doing the Lorentz algebra in D dimensions and it vanishes at $D = 2$ as expected since there are no physical gauge particles in QCD_2. Thus for the ideal gas pressure P_0, the $(D-2)$ factor required for gauge invariance has also the clear and well known interpretation in terms of contribution from physical states.

Consider next the order g^2 correction to the ideal gas pressure, P_2. Choosing the "Feynman-D-gauge" (i.e. Feynman gauge-fixing with D dimensional Lorentz algebra), one obtains after some elementary algebra,

$$P_2 = g^2 N_c (N_c^2 - 1) . \left[\int \frac{[dq]}{Q^2} \right]^2 . \left\{ -\frac{1}{2}(\frac{1}{2}) + \frac{1}{8}[2D(1-D)] + \frac{1}{12}[9(D-1)] \right\} \tag{4}$$

$$= -\left(\frac{D-2}{2} \right)^2 g^2 N_c (N_c^2 - 1) \left[\int \frac{[dq]}{Q^2} \right]^2 . \tag{5}$$

The terms within brackets in (4) come respectively from the two-loop vacuum diagrams with one, two and three gluon propagators. Shown explicitly in front of each contribution are the symmetry factors and the minus sign for the ghost loop. The net result in (5) vanishes for $D = 2$ as required for a gauge invariant quantity. If one had made errors (for example in the symmetry factors), these would likely have shown up in the nonvanishing of the net result at $D = 2$. A similar calculation in a conventional α-covariant gauge for the purposes of checking algebra is far more tedious, especially for the diagram with three gluon lines. The complexity of the algebra in an α-gauge in fact increases the sources of possible errors at intermediate steps. As a curiosity, it might interest the reader to note that nevertheless the result (5) can also be established in an $\alpha - D$-gauge *before* doing any explicit integrals, albeit with greater algebraic effort, the α dependence cancelling in the sum of diagrams as required.

The evaluation of integrals in (5) and its UV renormalisation for $D_0 = 4$ is standard [3] but has not been required for the discussion here. Note that since the $(D-2)$ factor prefaces everything in (5), both the UV part and the finite part propotional to T^4 are manifestly gauge-invariant. For full QCD, there will also be the gauge-invariant fermionic ($N_f \neq 0$) contribution which can be analysed separately.

3. Self-energy

The self-energy by itself is not a gauge-invariant quantity. However there exists a gauge-invariant piece of it which is easy to extract at low orders. This is the inverse screening length for static electric fields, also called the electric mass, m_{el}. If $\delta^{ab}\Pi_{\mu\nu}(k_0, \vec{k})$ is the gluon polarisation tensor at finite temperature, then at lowest order one may define

$$m_{el}^2 \equiv \Pi_{00}(0, \vec{k} \to 0) . \tag{6}$$

At one loop, the order $(gT)^2$ result for (6) is well known [2, 4]. In Ref.[5], working in an α-covariant gauge, Toimela found the amazing result that the next term of order $g^2|\vec{k}|T$ in the low momentum expansion of $\Pi_{00}(0,\vec{k})$ at one-loop was independent of the α parameter! It was conjectured [5] that this term might in fact be gauge *invariant*. I repeat here the analysis of [5] using the D-dimensional Lorentz algebra. In the α-covariant-D gauge, one finds for the sum of one-loop gluonic and ghost diagrams, the relevant object

$$\Pi_{00}(0,\vec{k}) = \frac{g^2 N_c}{2}[A_0(\vec{k}) + \alpha A_1(\vec{k}) + \alpha^2 A_2(\vec{k})] \;, \tag{7}$$

where

$$A_0(\vec{k}) = \int [dq] \frac{2(D-2)(2q_0^2 - Q^2) + 4k^2}{Q^2[q_0^2 + (\vec{q}-\vec{k})^2]} \;, \tag{8}$$

$$A_1(\vec{k}) = \int [dq] \frac{2[4(\vec{k}.\vec{q})^2 - 2k^2Q^2 + 2q_0^2k^2]}{Q^4[q_0^2 + (\vec{q}-\vec{k})^2]} \;, \tag{9}$$

$$A_2(\vec{k}) = \int [dq] \frac{q_0^2 k^4}{Q^4[q_0^2 + (\vec{q}-\vec{k})^2]^2} \;. \tag{10}$$

The only difference between (7-10) and the expression contained in [5, 6] is the presence of the factor $(D-2)$ in eq.(8). This factor is invisible in [5, 6] because they of course work with $D = D_0 = 4$. From the above expressions, one gets for the electric mass squared at order g^2 :

$$m_{el}^2 = 2g^2 N_c \left(\frac{D-2}{2}\right) \int [dq] \frac{(2q_0^2 - Q^2)}{Q^4} \;. \tag{11}$$

As expected, it is proportional to $(D-2)$. The expression (11) is easily obtained also in the strict Coulomb gauge. As for the free-energy example in the last section, eq.(11) was obtained without doing the explicit integrals which contain the detailed dynamical information. It should be apparent by now that the power of $(D-2)$ appearing for gauge-invariant quantities obtained directly from Feynman amplitudes is equal to the number of loops involved in the diagrams.

Now consider the order $|\vec{k}|T$ term in (7). As discussed in [5], this can only arise from the infrared region of the integrals. That is, it only arises from the $q_0 = 0$ part of the frequency sum (1) in (8,9). For the gauge-fixing dependent piece (9), the zero mode contains pieces exactly of order $|\vec{k}|T$ but the net contribution vanishes after the elementary integrals are done [5]. The zero mode in the α independent piece (8) contributes

$$T \int \frac{d^{(D-1)}q}{(2\pi)^{(D-1)}} \left[\frac{-2(D-2)}{(\vec{q}-\vec{k})^2} + \frac{4k^2}{q^2(\vec{q}-\vec{k})^2} \right] . \tag{12}$$

The first term in (12) vanishes by dimensional regularisation. The second gives a finite contribution proportional to $|\vec{k}|T$ as found in [5]. However this last piece has no $(D-2)$ factor associated with it and so it *cannot* be gauge-*invariant* eventhough it is α independent in the covariant gauge. Its gauge *non*invariance can be verified by an explicit calculation in the Coulomb gauge.

The dimensionally continued Lorentz algebra has enabled us to negate the conjecture of [5]. Though the analysis above was done in the $\alpha - D$-gauge to parallel that of [5], the conclusion would have followed with less effort already in the Feynman-D gauge. In the latter gauge, only eqn.(8) is involved and one could have concluded via eqn.(12) that the linear $|\vec{k}|T$ term is not gauge-invariant. Thus the D parameter works as well by itself, without any other gauge-fixing parameter. In this example, the D parameter has shown its full potential, for it allowed us to decide quickly against gauge-invariance because of the absence of the factor of $(D-2)$.

4. Propagator poles and Hard Thermal Loops

Just as at zero temperature, the physical poles of the propagator at non-zero temperature are gauge invariant [7]. For finite temperature QCD, the real part of the gauge propagator pole at zero external three momentum defines the induced thermal masses for the gluons and the leading ($\sim gT$) result is easily obtained at one-loop [2]. The imaginary part turns out to be of subleading order (g^2T) and a practical consistent calculation requires the Braaten-Pisarski resummation using effective propagators and vertices [8]. Both the real and imaginary parts of the poles, if calculated with the D dimensional algebra, must vanish at $D=2$ for $N_f=0$.

For QCD, there are an infinite number of bare one-loop diagrams which are as large as the tree amplitudes when the momentum entering the external legs is soft ($\sim gT$) and the internal loop momentum is hard ($\sim T$). These "hard thermal loops" (HTL) occur only at one-loop and have been extensively analysed by Braaten and Pisarski [8] and Frenkel and Taylor [9]. The HTL's exist for amplitudes when all the $N \geq 2$ external lines are gluons or when one pair is fermionic and the other $(N-2)$ are gluons. By explicit calculations in [8, 9], the HTL's were found to be the same in Coulomb, α-covariant and axial gauges and thus argued to be gauge-fixing independent. Subsequently Taylor and Wong constructed a gauge-invariant generating functional for the HTL's which has since been cast into myriad forms [10]. In some recent work, Blaizot and Iancu [11] have rederived the results of [8, 9, 10] by analysing the kinetic equations obtained through a self-consistent truncation of the Schwinger-Dyson equations for sources and fields at finite temperature.

¿From the expressions contained in [8] or [10, 11] one sees that $N_f=0$ sector of the N-gluon HTL is proportional to $(D-2)$ as required. However even the HTL's with external quark lines are seen to be proportional to $(D-2)$. As noted in the above papers, this is because the HTL's, which are the leading high temperature parts of the one loop diagrams, are essentially classical objects which receive contributions only from the $(D-2)$ physical transverse gluon degrees of freedom. That is, the $(D-2)$ factors for the HTL's say more than gauge invariance.

5. Conclusion

Dimensionally continued Lorentz algebra has been proposed and illustrated as an efficient and beneficial way to check gauge-invariance in pure YM theories. Gauge invariant quantities must be proportional to $(D-2)$. The converse is not necessarily true. For example, any quantity, even if gauge *variant*, when calculated in the axial gauge must be proportional to $(D-2)$ due to the free nature of pure QCD_2. Whether gauge-variant quantities proportional to $(D-2)$ exist in other gauges is left as an open question.

Fermions can be accomodated by using the number of flavours, N_f, as a parameter. The $N_f \neq 0$ part of any gauge-invariant quantity must be invariant by itself. At low orders in perturbation theory, one may even entertain the notion of calculating the $N_f = 0$ and $N_f \neq 0$ sectors with different gauge-fixing. For example, the pure glue part can be calculated in the Feynman-D gauge while the $N_f \neq 0$ part can be calculated in the α-gauge to check gauge-invariance. Whether such hybrid calculations are useful or practical should be decided on a case by case basis. Likewise, scalars can be coupled by taking N_s copies of them.

Finally some comment on the background field gauge [12]. This is one way of calculating in quantum field theory while keeping classical gauge invariance at every step. The gauge-invariance here is with respect to the background field B_μ which is introduced for this purpose and gives no information about the physical gauge-invariance of any quantity calculated. In particular, the quantum part of the action must still be gauge fixed. Thus even here one might use the D parameter without redundancy.

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