Radiant and coradiant dualities

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Abstract

We associate a dual problem to a constrained optimization problem in which the objective is quasiconvex and either attains at 0 its global minimum or its global maximum. The attractive features of such a duality are that it does not require an additional parameter to set the dual and that the dual problem has a form which is similar to the one of the primal problem. We present conditions ensuring strong duality using separation properties. We relate our approach to the Lagrangian theory.

Key words Coradiant set. Coradiant function, dual problem. Even convexity. Lagrangian. Quasiconvexity. Radiant set. Radiant function.

1 Introduction

For several minimization problems over a feasible set F, the global minimizer of the objective function f over the whole space X containing F is known. For instance, if X is a normed vector space (n.v.s.) and if f is the norm of X, 0 is the global minimizer of f. Such a fact does not yield any information about the location of solutions of the constrained problem over F. However, this fact can be used under weaken convexity assumptions to get duality relationships which are so important (see [29] for instance for a recent study from the algorithmic viewpoint). The needs of relaxed convexity assumptions in several fields, in particular in mathematical economics, incite to push further the results obtained so far in this direction. Quasiconcavity is often considered as an admissible assumption when dealing with an utility function u because the preference sets which are its superlevel sets have a concrete content while u itself is usually out of reach.

In [25] P.T. Thach gets optimality conditions for constrained problems under even convexity and quasiconvexity assumptions. The concept of even convexity introduced by Fenchel ([4]) has been studied by several authors ([2], [9], [11], [19], [20]....). A recent comprehensive study has appeared in [6]. Here, we rather focus on more classical topological assumptions such as closedness and semicontinuity. Thus, our results rely on classical separation theorems and complete the ones in [25]. We take advantage of the viewpoint of abstract convexity, but our methods are close to the ones of familiar convex programming, albeit the functions we deal with are quasiconvex and not convex. Bringing the viewpoint of polarities enables to consider a whole range of possible dual problems; we give a short account of these possibilities. We also reveal the case of radiant functions and sets which is not treated in [25]. We endeavour to give a unified presentation and to relate these dualities to general duality schemes.

Recall that a subset C of X is said to be *radiant* if it is starshaped and convex, i.e. if it is convex and $0 \in C$ (thus, our definition differs from the one in [23] and [31] in which C is just starshaped). It is *evenly radiant* if it is the intersection of a family of open half-spaces containing 0. A subset C of X is said to be *coradiant* if it is convex and costarshaped (or starshaped at infinity) with $0 \notin C$, i.e. if it is convex, if $0 \notin C$ and if for all $x \in C$, $t > 1$ one has $tx \in C$. It is *evenly coradiant* if it is the intersection of a family of open half-spaces whose closures do not contain 0. The Hahn-Banach theorem shows that a closed coradiant (resp. radiant) subset is evenly convex. A function $f : X \to \overline{\mathbb{R}}$ is said to be (quasi)*radiant* if its sublevel sets are radiant, i.e. if for every $x \in X$ one has $f(tx) \leq f(x)$ for all $t \in [0, 1]$. Equivalently, f is radiant if its strict sublevel sets are radiant. The function f is said to be *coradiant* if its (strict) sublevel sets are coradiant, i.e. if $f(0) = +\infty$ and if for every $x \in X$ one has $f(tx) \leq f(x)$ for all $t \in [1, +\infty)$. It is *evenly coradiant* if its strict sublevel sets are evenly coradiant. Such functions are useful in mathematical economics (see [25, Section 4] and the references in [16]).

A remarkable fact about these classes of functions is that conjugates can be defined on the dual space X^* of X. For general quasiconvex functions, conjugacies are not as simple as they involve an extra parameter (see [9], [15], [20], [22] for instance).

2 An adapted framework

Given a function $f : X \to \mathbb{R}_\infty := \mathbb{R} \cup \{+\infty\}$ and a nonempty convex subset F of X, let us consider the constrained optimization problem

$$(P) \qquad \text{minimize } f(x) \qquad x \in F.$$

In the sequel, for $r \in \mathbb{R}$, we set $S_f^<(r) = \{x \in X : f(x) < r\}$ and $\alpha := \inf(P)$. Throughout we suppose f assumes at least one finite value on the feasible set F, so that $\alpha < +\infty$.

We introduce the dual problems

$$\begin{array}{llll} (D^{\bar{\wedge}}) & \text{maximize } -f^{\wedge}(x^*) & & x^* \in F^{\nabla}, \\ (D^{\underline{\vee}}) & \text{maximize } -f^{\vee}(x^*) & & x^* \in F^{\Delta} \end{array}$$

where

$$\begin{aligned} f^{\wedge}(x^*) &:= -\inf\{f(x) : x \in X,\ \langle x, x^* \rangle \geq 1\}, \qquad & F^{\nabla} &:= \{x^* \in X^* : \forall x \in F\ \langle x, x^* \rangle \geq 1\}, \\ f^{\vee}(x^*) &:= -\inf\{f(x) : x \in X,\ \langle x, x^* \rangle \leq 1\}, \qquad & F^{\Delta} &:= \{x^* \in X^* : \forall x \in F\ \langle x, x^* \rangle \leq 1\}. \end{aligned}$$

The conjugate $f^{\vee}$ is related to the conjugate f^R considered in [25] by the equality $f^{\vee}(x^*) = f^R(-x^*)$ for all $x^* \in X^*$; the conjugate $f^{\wedge}$ is not considered in [25].

In order to deal simultaneously with the two dual problems $(D^{\bar{\wedge}})$ and $(D^{\vee})$, we introduce the following notation. For $\divideontimes \in \{\wedge, \nabla\}$, we set $\varepsilon_{\divideontimes} := 1$, while for $\divideontimes \in \{\vee, \Delta\}$, we set $\varepsilon_{\divideontimes} := -1$. Then, we can gather the two dual problems into the single one

$$(D^{\divideontimes}) \qquad \text{maximize } -f^{\divideontimes}(x^*) \qquad x^* \in F^{\divideontimes},$$

where

$$F^{\circledast} := \{x^* \in X^* : \forall x \in F\ \langle x, \varepsilon_{\circledast} x^* \rangle \geq \varepsilon_{\circledast}\},$$
$$f^{\circledast}(x^*) := -\inf\{f(x) : x \in X,\ \langle x, \varepsilon_{\circledast} x^* \rangle \geq \varepsilon_{\circledast}\}.$$

Note that we can rewrite $(D^{\circledast})$ as the equivalent adjoint problem

$$(P^{\circledast}) \qquad \text{minimize } f^{\circledast}(x^*) \qquad x^* \in F^{\circledast}$$

which has a form similar to the one of (P). The adjoint problem of $(P^{\circledast})$ is

$$(P^{\circledast\circledast}) \qquad \text{minimize } f^{\circledast\circledast}(x^{**}) \qquad x^{**} \in F^{\circledast\circledast}.$$

Its restriction to $X \subset X^{**}$ coincides with (P) when $\circledast \in \{\vee, \Delta\}$, F is closed and radiant and f is evenly coradiant by [25, Thm 2.3] and the bipolar theorem. When $\circledast \in \{\wedge, \nabla\}$, F is evenly coradiant and f is lower semicontinuous and radiant, one can also show that the restriction of $(P^{\circledast\circledast})$ to $X \subset X^{**}$ coincides with (P).

We first observe that we have the weak duality inequality

$$-\inf(P^{\circledast}) = \max(D^{\circledast}) \leq \inf(P)$$

since for all $x^* \in F^{\circledast}$, we have $F \subset [\varepsilon_{\circledast} x^* \geq \varepsilon_{\circledast}] := \{x \in X : \langle \varepsilon_{\circledast} x^*, x \rangle \geq \varepsilon_{\circledast}\}$, hence $-f^{\circledast}(x^*) = \inf f([\varepsilon_{\circledast} x^* \geq \varepsilon_{\circledast}]) \leq \inf f(F)$.

Let us illustrate the preceding duality scheme with an example.

Example. Let X be some Euclidean space for a scalar product denoted by $(\cdot \mid \cdot)$. Let A be a symmetric linear positive definite operator on X and let $e \in X$ be an eigenvector of A with norm 1 corresponding to an eigenvalue λ. Consider problem (P) with f and F given by $f(x) := -\|x\|^{-2}$ for $x \in X \backslash \{0\}$, $f(0) = -\infty$ and

$$F := \{x \in X : (Ax \mid x) \geq (1 - (e \mid x))^2\}.$$

Then, identifying X^* with X, one has $f^{\wedge}(y) = \|y\|^2$ and $F^{\nabla} = \{e + z : z \in X, (A^{-1}z \mid z) \leq 1\}$. Thus F^{∇} is compact and the dual problem $(D^{\overline{\wedge}})$ has a solution. In fact, for $\overline{x} := (\sqrt{\lambda}+1)^{-1}e$, $\overline{y} = (\sqrt{\lambda}+1)e$ one has $\overline{x} \in F$, $f(\overline{x}) = -(\sqrt{\lambda}+1)^2$, $\overline{y} \in F^{\nabla}$, $-f^{\wedge}(\overline{y}) = -(\sqrt{\lambda}+1)^2$, so that $\overline{x}$ is a solution to (P) and $\overline{y}$ is a solution to $(D^{\overline{\wedge}})$.

3 Criteria for strong duality

Let us give conditions ensuring strong duality, i.e. that there is no duality gap and that $(D^{\circledast})$ has solutions when $\alpha := \inf(P)$ is finite. As in [25, Thm 3.2] for the case of $(D^{\underline{\vee}})$, a separation property entails such a result. We first prove a converse.

Proposition 1 *Suppose α is finite, there is no duality gap between (P) and $(D^{\circledast})$ and $(D^{\circledast})$ has a solution $\overline{x}^*$. Then the hyperplane $[\overline{x}^* = 1]$ separates F and $S_f^{<}(\alpha)$:*

$$\forall u \in F,\ x \in S_f^{<}(\alpha) \qquad \langle x, \varepsilon_{\circledast} \overline{x}^* \rangle < \varepsilon_{\circledast} \leq \langle u, \varepsilon_{\circledast} \overline{x}^* \rangle. \tag{1}$$

Proof. Since $\overline{x}^* \in F^{\divideontimes}$, we have $\varepsilon_{\divideontimes} \leq \langle u, \varepsilon_{\divideontimes}\overline{x}^* \rangle$ for all $u \in F$. Now $\alpha = -f^{\divideontimes}(\overline{x}^*)$ since there is no duality gap, so that $\alpha \leq f(x)$ for all $x \in X$ satisfying $\langle x, \varepsilon_{\divideontimes}\overline{x}^* \rangle \geq \varepsilon_{\divideontimes}$. Equivalently, for all $x \in S_f^<(\alpha)$ we have $\langle x, \varepsilon_{\divideontimes}\overline{x}^* \rangle < \varepsilon_{\divideontimes}$. Note that since F is nonempty, the two inequalities of the statement ensure that $\overline{x}^* \neq 0$. $\square$

Proposition 2 *Suppose that for some $x_0^* \in X^*$, $r > 0$ the hyperplane $[\varepsilon_{\divideontimes} x_0^* = \varepsilon_{\divideontimes} r]$ separates F and $S_f^<(\alpha)$ in the sense that*

$$\forall u \in F,\ x \in S_f^<(\alpha) \qquad \langle x, \varepsilon_{\divideontimes} x_0^* \rangle < \varepsilon_{\divideontimes} r \leq \langle u, \varepsilon_{\divideontimes} x_0^* \rangle. \tag{2}$$

Then there is no duality gap and $\overline{x}^* := r^{-1}x_0^*$ is a solution to $(D^{\divideontimes})$.

Proof. Let $\overline{x}^* := r^{-1}x_0^*$. By assumption (2), for all $u \in F$ we have $\langle u, \varepsilon_{\divideontimes}\overline{x}^* \rangle \geq \varepsilon_{\divideontimes}$, so that $\overline{x}^* \in F^{\divideontimes}$. Let us show that $f(x) \geq \alpha$ for every $x \in [\varepsilon_{\divideontimes}\overline{x}^* \geq \varepsilon_{\divideontimes}]$; that will ensure that $-f^{\divideontimes}(\overline{x}^*) \geq \alpha$ and that $\overline{x}^*$ is a solution to $(D^{\divideontimes})$. Suppose on the contrary that there exists some $x \in [\varepsilon_{\divideontimes}\overline{x}^* \geq \varepsilon_{\divideontimes}]$ such that $f(x) < \alpha$. Then $x \in S_f^<(\alpha)$, hence, by the first inequality in (2), $\langle x, \varepsilon_{\divideontimes}\overline{x}^* \rangle < \varepsilon_{\divideontimes}$, a contradiction. $\square$

Corollary 3 *Suppose f is radiant and upper semicontinuous and F is convex. If either $\alpha = -\infty$ or $\alpha > \inf f(X)$, then there is no duality gap and if α is finite $(D^{\overline{\wedge}})$ has a solution.*

Proof. The result is obvious when $\alpha = -\infty$. Thus we may suppose α is finite. Since $S_f^<(\alpha)$ is a nonempty open convex subset disjoint from F, the Hahn-Banach separation theorem ensures that one can find $x_0^* \in X^* \backslash \{0\}$, $r \in \mathbb{R}$ such that $S_f^<(\alpha) \subset [x_0^* < r]$ and $F \subset [x_0^* \geq r]$. As f is radiant, $S_f^<(\alpha)$ contains 0, so that we have $r > 0$. $\square$

A variant of this result can be given when X is finite dimensional. Here we say that f is *upper semicontinuous along rays* if its restriction to any line passing through 0 is upper semicontinuous.

Corollary 4 *Suppose X is finite dimensional, f is radiant and upper semicontinuous along rays and F is convex. If $\alpha > \inf f(X)$, then, there is no duality gap and if α is finite $(D^{\overline{\wedge}})$ has a solution.*

Proof. Again, we have $0 \in S_f^<(\alpha)$ and since f is upper semicontinuous along rays at 0, $S_f^<(\alpha)$ is absorbent. Now $0 \notin S_f^<(\alpha) - F$ which is convex. Since X is finite dimensional, one can find some $x_0^* \in X^* \backslash \{0\}$ such that $r := \sup x_0^*(S_f^<(\alpha)) \leq \inf x_0^*(F)$. Taking $x_0 \in X$ such that $x_0^*(x_0) > 0$ and using the fact that $S_f^<(\alpha)$ is absorbent, we get $r > 0$. Now, for all $x \in S_f^<(\alpha)$, since f is upper semicontinuous along rays, we have $f(tx) < \alpha$ for some $t > 1$. Thus $x_0^*(tx) \leq r$ and $x_0^*(x) < r$ and we can apply the proposition with $\varepsilon_{\divideontimes} = 1$. $\square$

Corollary 5 *Suppose f is coradiant and upper semicontinuous and F is convex and absorbent. Then there is no duality gap and if α is finite $(D^{\underline{\vee}})$ has a solution.*

Proof. Again, the result is obvious when $\alpha = -\infty$. Thus we suppose α is finite and we apply the Hahn-Banach separation theorem since $S_f^<(\alpha)$ is an open convex subset disjoint from F, so that one can find $x_0^* \in X^* \backslash \{0\}$, $r \in \mathbb{R}$ such that $S_f^<(\alpha) \subset [x_0^* > r]$ and $F \subset [x_0^* \leq r]$. As F is absorbent, we have $r > 0$ and we can apply the proposition with $\varepsilon_{\divideontimes} = -1$. $\square$

Combining the techniques of the proofs of the two preceding corollaries we get the following variant.

Corollary 6 *Suppose X is finite dimensional, f is coradiant, upper semicontinuous along rays and F is convex and absorbent. Then there is no duality gap and if α is finite $(D^{\vee})$ has a solution.*

4 Optimality conditions and dualities

Let us also show that a sufficient optimality condition ensures strong duality. For such a purpose, we introduce the normal cone to F at $\overline{x} \in F$ given by

$$N(F, \overline{x}) := \{\overline{x}^* \in X^* : \forall x \in F \ \langle \overline{x}^*, x - \overline{x} \rangle \leq 0\}$$

and the subdifferential $\partial^{\circledast}$ by

$$\partial^{\circledast} f(\overline{x}) := \{x^* \in X^* : \langle \overline{x}, \varepsilon_{\circledast} x^* \rangle \geq \varepsilon_{\circledast}, \ f(x) \geq f(\overline{x}) \ \ \forall x \in [\varepsilon_{\circledast} x^* \geq \varepsilon_{\circledast}]\}$$

which encompasses the subdifferentials $\partial^{\wedge}$ and $\partial^{\vee}$ defined in [15] along a general line introduced by Martínez-Legaz and Singer [10] (see also [25] for a related definition). Note that when $\langle \overline{x}, \varepsilon_{\circledast} x^* \rangle = 1$, one has $x^* \in \partial^{\circledast} f(\overline{x})$ if, and only if, $\varepsilon_{\circledast} x^*$ belongs to the Greenberg-Pierskalla subdifferential [5]. Denoting by $\partial_1^{\circledast} f(\overline{x})$ the set of $\overline{x}^* \in \partial^{\circledast} f(\overline{x})$ such that $\langle \overline{x}, \varepsilon_{\circledast} \overline{x}^* \rangle = 1$, one also observes that, when $\circledast = \vee$, so that $\varepsilon_{\circledast} = 1$, one has

$$\partial_1^{\circledast} f(\overline{x}) \subset \partial^{\circledast} f(\overline{x}) \subset [1, +\infty) \partial_1^{\circledast} f(\overline{x}).$$

The following result completes [25, Thm 3.3] as it also deals with the case of problem $(D^{\wedge})$; moreover, here $\partial_1^{\circledast} f(\overline{x})$ is replaced with the larger set $\partial^{\circledast} f(\overline{x})$. Since the Plastria subdifferential

$$\partial^{<} f(\overline{x}) := \{\overline{x}^* \in X^* : \forall x \in S_f(f(\overline{x})) \ f(x) - f(\overline{x}) \geq \langle x - \overline{x}, \overline{x}^* \rangle\}$$

is contained in $\varepsilon_{\circledast} \partial_1^{\circledast} f(\overline{x})$, this result also implies [7, Prop. 5].

Proposition 7 *Let $\overline{x} \in F$ and $\overline{x}^* \in X^*$ be such that $\varepsilon_{\circledast} \overline{x}^* \in \partial^{\circledast} f(\overline{x}) \cap (-N(F, \overline{x}))$. Then the hyperplane $\{x : \langle x, \varepsilon_{\circledast} \overline{x}^* \rangle = \varepsilon_{\circledast}\}$ separates F and $S_f^{<}(\alpha)$, $\overline{x}$ is a solution of (P), $\overline{x}^*$ is a solution of $(D^{\circledast})$ and there is no duality gap.*

Proof. Since $-\varepsilon_{\circledast} \overline{x}^* \in N(F, \overline{x})$, for all $x \in F$ we have $\langle x - \overline{x}, \varepsilon_{\circledast} \overline{x}^* \rangle \geq 0$, hence $\langle x, \varepsilon_{\circledast} \overline{x}^* \rangle \geq \langle \overline{x}, \varepsilon_{\circledast} \overline{x}^* \rangle \geq \varepsilon_{\circledast}$ by the first condition in the definition of $\partial^{\circledast} f(\overline{x})$. Since $F \subset [\varepsilon_{\circledast} \overline{x}^* \geq \varepsilon_{\circledast}]$ we have $\overline{x}^* \in F^{\circledast}$. The second condition yields $f(x) \geq f(\overline{x})$ for all $x \in [\varepsilon_{\circledast} \overline{x}^* \geq \varepsilon_{\circledast}]$, hence for all $x \in F$ and $\overline{x}$ is a solution of (P). Since $f(x) \geq f(\overline{x})$ for all $x \in [\varepsilon_{\circledast} \overline{x}^* \geq \varepsilon_{\circledast}]$, we get $-f^{\circledast}(\overline{x}^*) \geq f(\overline{x})$. It follows that $\overline{x}^*$ is a solution of $(D^{\circledast})$ and $\sup(D^{\circledast}) = \inf(P)$. □

This result shows the usefulness of having a solution of $(D^{\circledast})$: to solve (P) it suffices to find a minimizer $\overline{x}$ of f on the half-space $[\varepsilon_{\circledast} x^* \geq \varepsilon_{\circledast}]$ which is also a minimizer of $-\overline{x}^*$ on F. Both problems are simpler that the primal problem (P).

One may wonder about the relationships between this result and Proposition 1.

Proposition 8 *Suppose that for some $\overline{x}^* \in X^* \backslash \{0\}$ the hyperplane $[\varepsilon_{\circledast} x_0^* = \varepsilon_{\circledast} r]$ separates F and $S_f^{<}(\alpha)$ in the sense of relation (1). If some solution $\overline{x}$ of (P) is not a local minimizer of f, then $\overline{x}^*$ satisfies the optimality condition $\varepsilon_{\circledast} \overline{x}^* \in \partial^{\circledast} f(\overline{x}) \cap (-N(F, \overline{x}))$. Moreover one has $\langle \overline{x}, \varepsilon_{\circledast} \overline{x}^* \rangle = \varepsilon_{\circledast}$.*

Proof. Since $\overline{x} \in F$, we have $\langle \overline{x}, \varepsilon_{\circledast}\overline{x}^* \rangle \geq \varepsilon_{\circledast}$. Since $\overline{x}$ is not a local minimizer of f, $\overline{x}$ belongs to the closure of $S_f^<(\alpha)$, hence $\langle \overline{x}, \varepsilon_{\circledast}\overline{x}^* \rangle = \varepsilon_{\circledast}$. Since $\langle u, \varepsilon_{\circledast}\overline{x}^* \rangle \geq \varepsilon_{\circledast}$ for all $u \in F$, we get $-\varepsilon_{\circledast}\overline{x}^* \in N(F, \overline{x})$. It remains to show that $f(x) \geq f(\overline{x})$ for every $x \in [\varepsilon_{\circledast}\overline{x}^* \geq \varepsilon_{\circledast}]$ to ensure that $\varepsilon_{\circledast}\overline{x}^* \in \partial^{\circledast} f(\overline{x})$. Suppose there exists some $x \in [\varepsilon_{\circledast}\overline{x}^* \geq \varepsilon_{\circledast}]$ such that $f(x) < f(\overline{x})$. Then, since $f(\overline{x}) = \alpha$, we would have $x \in S_f^<(\alpha)$, hence $\langle x, \varepsilon_{\circledast}\overline{x}^* \rangle < \varepsilon_{\circledast}$, a contradiction. Thus $\overline{x}^* \in \partial^* f(\overline{x})$ and the conclusion follows from the preceding proposition. $\square$

Corollary 9 *Suppose f is radiant and upper semicontinuous and F is convex. Then, for every solution $\overline{x}$ of (P) which is not a local minimizer of f, there exists some solution $\overline{x}^*$ of $(D^{\overline{\wedge}})$ which satisfies the optimality conditions $\overline{x}^* \in \partial^{\wedge} f(\overline{x}) \cap (-N(F, \overline{x}))$, $\langle \overline{x}, \overline{x}^* \rangle = 1$ and there is no duality gap.*

Proof. This follows from the Hahn-Banach separation theorem since $S_f^<(\alpha)$ is an open convex subset containing 0 and disjoint from F. $\square$

A variant of this result can be given when X is finite dimensional.

Corollary 10 *Suppose X is finite dimensional, f is radiant and upper semicontinuous along rays at 0 and F is convex. Then, for every solution $\overline{x}$ of (P) which is not a local minimizer of f, there exists some solution $\overline{x}^*$ of $(D^{\overline{\wedge}})$ which satisfies the optimality conditions $\overline{x}^* \in \partial^{\wedge} f(\overline{x}) \cap (-N(F, \overline{x}))$, $\langle \overline{x}, \overline{x}^* \rangle = 1$.*

Proof. Again, we have $0 \in S_f^<(\alpha)$ and $f(0) < \alpha$ since $f(0) = \inf f(X) < f(\overline{x})$, as $\overline{x}$ is not a minimizer of f. Since f is upper semicontinuous along rays at 0, $S_f^<(\alpha)$ is absorbing. Now $0 \notin S_f^<(\alpha) - F$ which is convex. Since X is finite dimensional, one can find some $x_0^* \in X^* \backslash \{0\}$ such that $r := \sup x_0^*(S_f^<(\alpha)) \leq \inf x_0^*(F)$. Taking $x_0 \in X$ such that $x_0^*(x_0) > 0$ and using the fact that $S_f^<(\alpha)$ is absorbing, we get $r > 0$. Since $\overline{x}$ is not a local minimizer of f, $\overline{x}$ is in the closure of $S_f^<(\alpha)$, so that $r := \langle \overline{x}, x_0^* \rangle$. Let $\overline{x}^* := r^{-1}x_0^*$, so that $\langle \overline{x}, \overline{x}^* \rangle = 1$ and $-\overline{x}^* \in N(F, \overline{x})$. Since f is upper semicontinuous along rays, for every $x \in S_f^<(\alpha)$ we have $tx \in S_f^<(\alpha)$ for some $t > 1$, hence $\langle x, \overline{x}^* \rangle < 1$. Thus we can apply the proposition. $\square$

The assumption that X is finite dimensional can be eliminated, but the substituted assumption is more difficult to check. We keep the preceding notation and we say that a subset C of X is *evenly convex* if it is the intersection of a family of open half-spaces. Obviously, open or closed convex subsets of a normed vector space are evenly convex, but the class of evenly convex subsets is larger than the union of these two subclasses.

Corollary 11 *Suppose f is radiant and upper semicontinuous along rays and $S_f^<(\alpha) - F$ is evenly convex. Then, for every solution $\overline{x}$ of (P) which is not a local minimizer of f, there exists some solution $\overline{x}^*$ of $(D^{\overline{\wedge}})$ which satisfies the optimality conditions $\overline{x}^* \in \partial^{\wedge} f(\overline{x}) \cap (-N(F, \overline{x}))$, $\langle \overline{x}, \overline{x}^* \rangle = 1$.*

The proof is similar to the preceding one after using the fact that, since $0 \notin S_f^<(\alpha) - F$, one can find some $x_0^* \in X^*$ such that $\langle x - u, x_0^* \rangle < 0$ for all $x \in S_f^<(\alpha)$, $u \in F$. Since $0 \in S_f^<(\alpha)$, we have $r := \langle \overline{x}, x_0^* \rangle > 0$ and for $\overline{x}^* := r^{-1}x_0^*$ we get $\langle x, \overline{x}^* \rangle < \langle \overline{x}, \overline{x}^* \rangle = 1$ for all $x \in S_f^<(\alpha)$. $\square$

Now let us turn to the coradiant case.

Corollary 12 *Suppose f is coradiant and upper semicontinuous and F is convex. Then, for every solution $\overline{x}$ of (P) which is not a local minimizer of f, there exists some solution $\overline{x}^*$ of $(D^{\veebar})$ which satisfies the optimality conditions $-\overline{x}^* \in \partial^{\vee} f(\overline{x}) \cap (-N(F, \overline{x}))$, $\langle \overline{x}, \overline{x}^* \rangle = 1$ and there is no duality gap.*

Proof. Since F and $S_f^<(\alpha)$ are convex and disjoint, and $S_f^<(\alpha)$ is open, there exist $x_0^* \in X^*$ and $r \in \mathbb{R}$ such that

$$\forall u \in F,\ \forall x \in S_f^<(\alpha) \qquad \langle x, x_0^* \rangle > r \geq \langle u, x_0^* \rangle.$$

Taking $\varepsilon_{\circledast} = -1$ in Proposition 2 we get that relation (2) is satisfied, hence $\varepsilon_{\circledast}\overline{x}^* := r^{-1}\varepsilon_{\circledast}x_0^* \in \partial^{\vee} f(\overline{x}) \cap (-N(F, \overline{x}))$, $\langle \overline{x}, \varepsilon_{\circledast}\overline{x}^* \rangle = \varepsilon_{\circledast}$, so that $\overline{x}^*$ is a solution of $(D^{\veebar})$ and there is no duality gap. □

Corollary 13 *Suppose f is coradiant and upper semicontinuous along rays and F is convex and contains 0 in its interior. Then, for every solution $\overline{x}$ of (P) which is not a local minimizer of f, there exists some solution $\overline{x}^*$ of $(D^{\veebar})$ which satisfies the optimality conditions $-\overline{x}^* \in \partial^{\vee} f(\overline{x}) \cap (-N(F, \overline{x}))$, $\langle \overline{x}, \overline{x}^* \rangle = 1$ and there is no duality gap.*

Proof. Since int$F$ and $S_f^<(\alpha)$ are convex and disjoint, there exist $x_0^* \in X^*$ and $r \in \mathbb{R}$ such that

$$\forall u \in \mathrm{int}F,\ x \in S_f^<(\alpha) \qquad \langle x, x_0^* \rangle \geq r > \langle u, x_0^* \rangle.$$

Since $0 \in$int$F$, we have $r > 0$. Since F is contained in the closure of int$F$, for all $u \in F$, we have $\langle u, x_0^* \rangle \leq r$. Since $\overline{x}$ is not a local minimizer of f, $\overline{x}$ is in the closure of $S_f^<(\alpha)$. Thus $\langle \overline{x}, x_0^* \rangle = r$. Let $\overline{x}^* := r^{-1}x_0^*$. Then $\langle \overline{x}, \overline{x}^* \rangle = 1 \geq \langle u, \overline{x}^* \rangle$ for all $u \in F$, hence $\overline{x}^* \in N(F, \overline{x})$. Moreover, if $x \in S_f^<(\alpha)$, we have $\langle x, \overline{x}^* \rangle \geq 1$. Since f is upper semicontinuous along rays, we have $tx \in S_f^<(\alpha)$ for $t < 1$ close enough to 1; thus we have $\langle x, \overline{x}^* \rangle > 1$. Therefore $-x^* \in \partial^{\vee} f(\overline{x})$. □

5 Links with polarities

The conjugates we considered are particular instances of conjugates associated with a polarity. Recall that a *polarity* between two sets X, Y is a map $P : 2^X \to 2^Y$ between the power sets of X and Y which satisfies the relation

$$P(\bigcup_{i \in I} A_i) = \bigcap_{i \in I} P(A_i)$$

for every family $(A_i)_{i \in I}$ of subsets of X. We also denote $P(A)$ by A^P for $A \subset X$. The preceding relation yields, for any $A \subset X$

$$P(A) = \bigcap_{a \in A} P(\{a\}) = \{y \in Y : A \subset D(y)\},$$

where $D(y) := P^{-1}(y) := \{x \in X : y \in P(\{x\})\}$. Conversely, given a family $(D(y))_{y \in Y}$ of subsets of X, one gets a polarity by setting, for $A \subset X$, $P(A) = \{y \in Y : A \subset D(y)\}$. When

X and Y are topological vector spaces in duality, it is natural to take for family $(D(y))_{y \in Y}$ a family of half-spaces. In [13] we detected four families of such half-spaces of special interest. They give rise to four polar sets:

$$A^{\Delta} := \{y \in Y : A \subset [y \leq 1]\}, \qquad A^{\wedge} := \{y \in Y : A \subset [y < 1]\},$$
$$A^{\nabla} := \{y \in Y : A \subset [y \geq 1]\}, \qquad A^{\vee} := \{y \in Y : A \subset [y > 1]\};$$

here we change the notation for the first one, which is the usual polar set often denoted by A^0; we do that in order to remind that one passes from $A^{\wedge}$ to A^{Δ} by adding a bar to the symbol $<$, changing it into $\leq$.

We note the following observation which is an immediate consequence of the definitions.

Lemma 14 *For any subset A of X, the sets $A^{\wedge}$ and A^{Δ} are radiant; A^{Δ} is weak* closed and $A^{\wedge}$ is evenly convex. For any nonempty subset A of X, the sets $A^{\vee}$ and A^{∇} are coradiant; A^{∇} is weak* closed and $A^{\vee}$ is evenly convex.*

Now, for any function f on X, one can define a conjugate function f^P associated with a polarity P by setting:

$$f^P(y) := \sup\{-f(x) : x \in X \backslash D(y)\},$$

where $D(y) := P^{-1}(y)$. Taking for P one of the preceding four polarities, we get the following result in which

$$f^{\Delta}(y) := -\inf\{f(x) : x \in X, \ \langle x, y\rangle > 1\}, \qquad f^{\nabla}(y) := -\inf\{f(x) : x \in X, \ \langle x, y\rangle < 1\}.$$

Some properties of these conjugacies are presented in [13] and [14]. Here we note the following ones for our needs.

Lemma 15 *For any function f on X, the functions $f^{\wedge}$ and f^{Δ} are radiant and the functions $f^{\vee}$ and f^{∇} are coradiant. Moreover, f^{Δ} and f^{∇} are lower semicontinuous while $f^{\wedge}$ and $f^{\vee}$ are evenly quasiconvex. Furthermore, $f^{\wedge}(0) = f^{\Delta}(0) = -\infty$ and $f^{\vee}(0) = f^{\nabla}(0) = -\inf f(Y)$.*

Proof. These assertions are consequence of the preceding lemma and of the following relation, valid for every $r \in \mathbb{R}$, any function f and any polarity P :

$$\left[f^P \leq r\right] = [f < -r]^P .$$

This relation, established in [30], [16] follows from the equivalences:

$$\begin{aligned} \left(y \in \left[f^P \leq r\right]\right) &\Leftrightarrow (-r \leq f(z)\ \forall z \in X \backslash D(y)) \\ &\Leftrightarrow (-r > f(z) \Rightarrow z \in D(y)) \\ &\Leftrightarrow ([f < -r] \subset D(y)) \Leftrightarrow \left(y \in [f < -r]^P\right). \end{aligned}$$

□

6 Some variants

Since we have four different sorts of polar sets and four different sort of conjugate functions, it is tempting to study other combinations. Such a temptation is increased by the fact that the combinations we have selected above are mixed.

The inequalities $f^{\wedge} \geq f^{\Delta}$, $f^{\vee} \geq f^{\nabla}$ and the inclusions $F^{\wedge} \subset F^{\Delta}$, $F^{\vee} \subset F^{\nabla}$ entail the obvious relationships

$$\sup(D^{\wedge}_{\vee}) := \sup -f^{\wedge}(F^{\vee}) \leq \sup -f^{\wedge}(F^{\nabla}) =: \sup(D^{\bar{\wedge}}) \leq \sup(D^{\Delta}_{\nabla}) := \sup -f^{\Delta}(F^{\nabla}),$$
$$\sup(D^{\vee}_{\wedge}) := \sup -f^{\vee}(F^{\wedge}) \leq \sup -f^{\vee}(F^{\Delta}) =: \sup(D^{\underline{\vee}}) \leq \sup(D^{\nabla}_{\Delta}) := \sup -f^{\nabla}(F^{\Delta}),$$

Although the estimates provided by the dual problems $(D^{\wedge}_{\vee})$ and $(D^{\vee}_{\wedge})$ may be useful, the duality gaps between these problems and the primal one (P) are always larger than the duality gaps for the dual problems we have chosen. Thus, strong duality would be more difficult to get with the dual problems $(D^{\wedge}_{\vee})$ and $(D^{\vee}_{\wedge})$. The following examples show that the problems (D^{Δ}_{∇}) and (D^{∇}_{Δ}) do not satisfy the weak duality property in general and thus should be excluded, although they involve closed subsets and lower semicontinuous functions.

Example. Let $X := \mathbb{R}$, $F := (-\infty, 1]$, $f(r) := 1$ for $r \in (-\infty, 1)$, $f(r) = 0$ for $r \in [1, +\infty)$. Then $F^{\Delta} = [0, 1]$ and $-f^{\nabla}(1) = 1$. Thus $\sup(D^{\nabla}_{\Delta}) \geq 1 > 0 = \inf(P)$.

Example. Let $X := \mathbb{R}$, $F := [1, +\infty)$, $f(r) := 1$ for $r \in (1, +\infty)$, $f(r) = 0$ for $r \in (-\infty, 1]$. Then $F^{\nabla} = [1, +\infty)$ and $-f^{\Delta}(1) = 1$. Thus $\sup(D^{\Delta}_{\nabla}) \geq 1 > 0 = \inf(P)$.

These facts explain why we focus our attention on the dual problems $(D^{\bar{\wedge}})$ and $(D^{\underline{\vee}})$ rather than on (D^{Δ}_{∇}) and (D^{∇}_{Δ}) or $(D^{\wedge}_{\vee})$ and $(D^{\vee}_{\wedge})$. Under some semicontinuity assumptions, equalities hold in the inequalities $\sup(D^{\bar{\wedge}}) \leq \sup(D^{\Delta}_{\nabla})$ and $\sup(D^{\underline{\vee}}) \leq \sup(D^{\nabla}_{\Delta})$ in view of the following result.

Proposition 16 *If f is upper semicontinuous along rays, then $f^{\Delta} = f^{\wedge}$ and $f^{\nabla} = f^{\vee}$ so that $f^{\wedge}$ and $f^{\vee}$ are lower semicontinuous. Then, for all subsets F one has* $\sup(D^{\bar{\wedge}}) = \sup(D^{\Delta}_{\nabla})$ *and* $\sup(D^{\underline{\vee}}) = \sup(D^{\nabla}_{\nabla})$.

Proof. Since $f^{\wedge} \geq f^{\Delta}$, to prove that $f^{\Delta} = f^{\wedge}$, it suffices to show that for every $x^* \in X^*$ and every $r \in \mathbb{R}$ with $r \geq f^{\Delta}(x^*)$ we have $r \geq -f(x)$ for all $x \in [x^* \geq 1]$. The inequality $r \geq -f(x)$ holding when $x \in [x^* > 1]$, we may suppose $\langle x, x^* \rangle = 1$. Then, for all $t > 1$, we have $tx \in [x^* > 1]$, hence $-f(tx) \leq r$. Since f is upper semicontinuous along rays, we get $f(z) \geq \limsup_{t \to 1} f(tz) \geq -r$. The proof of the equality $f^{\nabla} = f^{\vee}$ is similar.

Proposition 17 *Let F be an arbitrary nonempty subset of X. If $f^{\wedge}$ is upper semicontinuous along rays, then* $\sup(D^{\bar{\wedge}}) = \sup(D^{\wedge}_{\vee})$. *If $f^{\vee}$ is upper semicontinuous along rays, then* $\sup(D^{\underline{\vee}}) = \sup(D^{\vee}_{\wedge})$.

Proof. It suffices to show that $\sup -f^{\wedge}(F^{\vee}) \geq \sup -f^{\wedge}(F^{\nabla})$. Given $r < \sup -f^{\wedge}(F^{\nabla})$ one can find some $x^* \in F^{\nabla}$ such that $r < -f^{\wedge}(x^*)$. For $t \in (1, +\infty)$ one has $tx^* \in F^{\vee}$ and since $f^{\wedge}$ is u.s.c. along rays, for t close enough to 1, one gets $-r > f^{\wedge}(tx^*)$. Thus $\sup -f^{\wedge}(F^{\wedge}) \geq -f^{\wedge}(tx^*) > r$ and, as r can be arbitrarily close to $\sup -f^{\wedge}(F^{\nabla})$, we get the expected inequality. The proof of the second assertion is similar, using the fact that $tF^{\Delta} \subset F^{\vee}$ for $t \in (0, 1)$. □

The following criteria are taken from [13, Prop. 4.8] in the case of $f^{\wedge}$; the proof for $f^{\vee}$ is similar.

Proposition 18 *If f is inf-compact for the weak topology, then $f^{\wedge}$ and $f^{\vee}$ are upper semi-continuous.*

If f is such that $\lim_{x \to 0} f(x) = -\infty$, then f^{Δ} is inf-compact for the weak topology on X^*.*

Corollary 19 *If f is such that $\lim_{x \to 0} f(x) = -\infty$, then (D^{Δ}_{∇}) has a solution. If moreover f is upper semicontinuous along rays, then $(D^{\bar{\wedge}})$ has a solution.*

Proof. The first assertion is consequence of the fact that the inf-compact function f^{Δ} attains its infimum over the weak* closed convex set F^{∇}. The second assertion follows from the equality $f^{\Delta} = f^{\wedge}$ when f is upper semicontinuous along rays. □

Now let us tackle the question of existence of solutions for the dual problem (D^{∇}_{Δ}). Since the sublevel sets of f^{∇} are coradiant, f^{∇} cannot be inf-compact unless it is identically $+\infty$.

Proposition 20 *Under each of the following assumptions the problem (D^{∇}_{Δ}) has a solution:*

(a) 0 is in the interior of F;

(b) X is finite dimensional and $\mathbb{R}_+ F - \mathbb{R}_+ S^{<}_{f}(\alpha)$ is dense in X.

If moreover f is upper semicontinuous along rays, then $(D^{\underline{\vee}})$ has a solution.

Proof. (a) The sublevel sets of the function f^{∇} being weak* closed convex, f^{∇} is weak* lower semicontinuous. Now, since F contains the closed ball B with center 0 and radius r for some $r > 0$, F^{Δ} is contained in B^{Δ} which is the closed ball with center 0 and radius $1/r$. Thus F^{Δ} is weak* compact and f^{∇} attains its infimum on F^{Δ}.

(b) Let us first observe that if A and B are two closed convex subsets of a finite dimensional Banach space X and if $0^+A \cap 0^+B = \{0\}$, then $A \cap B$ is bounded because $0^+(A \cap B) \subset 0^+A \cap 0^+B$; here, for a subset C of X, we denote by $0^+C := \{v \in X : V + C \subset C\}$ the recession cone of C. Now we have

$$0^+(F^{\Delta}) = F^0 := \{y \in Y : \forall x \in F \ \langle x, y \rangle \leq 0\},$$

as easily checked. Let $G := S^{<}_{f}(\alpha)$. Since $G^{\nabla} = [f^{\nabla} \leq -\alpha]$, we have

$$0^+([f^{\nabla} \leq -\alpha]) = 0^+(G^{\nabla}) = (-G)^0.$$

Since $\mathbb{R}_+ F - \mathbb{R}_+ G$ is dense in X, its polar cone is $\{0\}$, hence

$$0^+(F^{\Delta}) \cap 0^+([f^{\nabla} \leq -\alpha]) = F^0 \cap (-G)^0 = (\mathbb{R}_+ F - \mathbb{R}_+ G)^0 = \{0\}.$$

Thus $[f^{\nabla} \leq -\alpha] \cap F^{\Delta}$ is bounded, hence weak* compact and since f^{∇} is weak* lower semi-continuous, $-f^{\nabla}$ attains its infimum over F^{Δ} : (D^{∇}_{Δ}) has a solution. The last assertion is a consequence of the equality $f^{\nabla} = f^{\vee}$ when f is upper semicontinuous along rays. □

7 Mathematical programming problems

Now, let us try to answer to the natural question: does the preceding results enter a general theory of duality? In fact, we even consider a more general problem written under the form of a mathematical programming problem.

Suppose Z is another Banach space, $g : X \to Z$ is a map, C is a closed convex subset of W, and Y is the dual space of Z. Let $F := \{x \in X : g(x) \in C\}$. Then problem (P) turns into the problem

$$(M) \quad \text{minimize } f(x) : \ x \in X, \ g(x) \in C.$$

which can be rewritten as the minimization of $f(\cdot)+\iota_C(g(\cdot))$, where ι_C is the indicator function of C given by $\iota_C(z) := 0$ if $z \in C$, $\iota_C(z) = +\infty$ otherwise.

We can introduce the *perturbation* $P : X \times Z \to \mathbb{R}\cup\{+\infty\}$ given by

$$P(x, z) := f(x) + \iota_C(g(x) + z)$$

and its associated *performance function* p given by

$$p(z) := \inf\{f(x) : x \in X, \ g(x) + z \in C\}.$$

We observe that Y and Z can be coupled with the coupling function $c_\vee : Y \times Z \to \overline{\mathbb{R}}$ given by

$$c_\vee(y, z) = -\iota_{[y\leq 1]}(z).$$

Since $c_\vee(y, 0) = 0$ for all $y \in Y$, the perturbational dual problem of (M) is the problem

$$(D^\vee) \quad \text{maximize } -p^\vee(y) \quad y \in Y,$$

where $p^\vee$ is the conjugate of p for the coupling function $c_\vee$:

$$\begin{aligned} p^\vee(y) &:= -\inf\{p(z) - c_\vee(y, z) : z \in Z\} \\ &:= -\inf\{p(z) : z \in Z, \ \langle y, z\rangle \leq 1\}. \end{aligned}$$

However, this process cannot be applied with the coupling function $c_\wedge$ given by $c_\wedge(y, z) = -\iota_{[y\geq 1]}(z)$ nor with the coupling function c_Δ given by $c_\Delta(y, z) = -\iota_{[y>1]}(z)$. Thus, we take a direct Lagrangian approach rather than a perturbational approach.

We use the simple observation that since C is included in $C^{\Delta\Delta} := (C^\Delta)^\Delta$ (and in the three other bipolar sets of C), we have $\iota_C \geq \iota_{C^{\Delta\Delta}}$. Now

$$\iota_{C^{\Delta\Delta}} = \sup_{y\in C^\Delta} \iota_{[y\leq 1]},$$

as easily checked. It follows that we can introduce the sub-Lagrangian function $L_{\underline{\vee}}$ given by

$$\begin{aligned} L_{\underline{\vee}}(x, y) &:= f(x) + \iota_{[y\leq 1]}(g(x)) \quad && \text{for } y \in C^\Delta, \\ L_{\underline{\vee}}(x, y) &:= -\infty && \text{for } y \in Y\backslash C^\Delta. \end{aligned}$$

Here we use the terminology of [17] which means that

$$f(x) + \iota_C(g(x)) \geq \sup_{y \in Y} L_{\underline{\vee}}(x, y) \qquad \text{for all } x \in X,$$

$L_{\underline{\vee}}$ being called a Lagrangian when equality holds for all $x \in X$ (which is the case when C is closed and radiant). The Lagrangian dual function $d_{\underline{\vee}}$ is given by

$$d_{\underline{\vee}}(y) := \inf_{x \in X} L_{\underline{\vee}}(x, y).$$

In order to express it, we introduce the function $f_g : Z \to \overline{\mathbb{R}}$ given by

$$f_g(z) := \inf\{f(x) : x \in g^{-1}(z)\},$$

with the usual convention that $f_g(z) := +\infty$ when $g^{-1}(z)$ is empty. Then

$$d_{\underline{\vee}}(y) = \inf_{z \in Z}(f_g(z) + \iota_{[y \leq 1]}(z)) = -(f_g)^{\vee}(y) \quad \text{for } y \in C^{\Delta}, \quad d_{\underline{\vee}}(y) = -\infty \quad \text{for } y \in Y \backslash C^{\Delta}.$$

Thus, the dual problem can be written

$$(D^{\underline{\vee}}) \text{ maximize } -(f_g)^{\vee}(y) \quad \text{for } y \in C^{\Delta}.$$

When $Z = X$ and g is the identity mapping I_X, we recover the dual problem we have considered in section 2. A similar approach can be given for the dual problem $(D^{\overline{\wedge}})$. We also notice that using the inclusions $C \subset C^{\wedge\wedge}$ and $C \subset C^{\vee\vee}$ we can obtain new dual problems. These problems provide new estimates as weak duality holds; but strong duality results are not at hand.

When C is a closed convex cone, we have $C^{\Delta} = C^0 := \{y \in Y : \forall z \in C \ \langle y, z\rangle \leq 0\}$, the usual polar cone. Then, for $y \in C^{\Delta}$, we have

$$L_{\underline{\vee}}(x, y) := f(x) + \iota_{[y \leq 1]}(g(x)) \leq L_{<}(x, y) := f(x) + \iota_{[y \leq 0]}(g(x))$$

where $L_{<}$ is the surrogate Lagrangian considered in [7], [21]. Thus, if $\overline{y} \in C^{\Delta}$ is a multiplier for the Lagrangian $L_{\underline{\vee}}$, it is also a multiplier for the Lagrangian $L_{<}$.

The advantage of strong duality is reminded in the following statement which relies on [17, Prop. 1.2] and uses the fact that $L_{\underline{\vee}}(x, y) := f(x)$ when $\langle g(x), \overline{y}\rangle \leq 1$ and $L_{\underline{\vee}}(x, y) := +\infty$ otherwise.

Proposition 21 *Let $\overline{y}$ be a multiplier for the Lagrangian $L_{\underline{\vee}}$, i.e. a solution to the dual problem $(D^{\underline{\vee}})$ such that $(f_g)^{\vee}(\overline{y}) = \alpha$. Then $\overline{x}$ is a solution to (P) if, and only if, $\overline{x}$ is a solution to the simplified problem*

$$(Q_{\overline{y}}) \qquad \textit{minimize } f(x) \textit{ subject to the constraint } \langle g(x), \overline{y}\rangle \leq 1.$$

When g is a continuous linear map, in particular, when $Z = X$ and $g = I_X$, the feasible set of $(Q_{\overline{y}})$ is simply a half-space. Clearly, one has a similar result for the Lagrangian $L_{\overline{\wedge}}$.

References

[1] J.-P. Crouzeix, Continuity and differentiability properties of quasiconvex functions on $\mathbb{R}^n$, in "*Generalized concavity in optimization and economics*", S. Schaible and W.T. Ziemba, eds. Academic Press, New York, (1981), 109-129.

[2] A. Daniilidis and Martínez-Legaz, Characterizations of evenly convex sets and evenly quasiconvex functions, J. Math. Anal. Appl. 273 (2002), 58-56.

[3] W.E. Diewert, Applications of duality theory, in "*Frontiers of quantitative economics 2*", M.D. Intriligator and D.A. Kendrick, eds, North Holland, Amsterdam (1974), 106-171.

[4] W. Fenchel, A remark on convex sets and polarities, , Comm. Sém. Math. Univ. Lund (Medd. Lunds Univ. Math. Sem.), Tome supplémentaire (1952) 82-89.

[5] H.P. Greenberg and W.P. Pierskalla, Quasiconjugate function and surrogate duality, Cahiers du Centre d'Etude de Recherche Oper. 15 (1973), 437-448.

[6] V. Klee, E. Maluta and C. Zanco, Basic properties of evenly convex sets, J. Convex Anal. 14 (1) (2007), 137-148.

[7] N.T.H. Linh and J.-P. Penot, Optimality conditions for quasiconvex programs, SIAM J. Optim. 17 (2) (2006), 500-510.

[8] D.T. Luc, T.Q. Phong and M. Volle, A new duality approach to solving concave vector maximization problems, J. Global Optim. 36 (3) (2006), 401-423

[9] J.-E. Martínez-Legaz, A generalized concept of conjugation, in *Optimization, theory and algorithms*, J.-B. Hiriart-Urruty, W. Oettli, J. Stoer, (eds.), Dekker, New York (1983), 45-49.

[10] J.-E. Martínez-Legaz and I. Singer, Subdifferentials with respect to dualities, ZOR-Math. Methods Oper. Res. 42, pp. 109-125 (1995).

[11] U. Passy and E.Z. Prisman, U. Passy and E.Z. Prisman, Conjugacy in quasiconvex programming, Math. Programming 30 (1984), 121-146.

[12] U. Passy and E.Z. Prisman, A convexlike duality scheme for quasiconvex programs, Math. Programming 32 (1985), 278-300.

[13] J.-P. Penot, Duality for radiant and shady programs, Acta Math. Vietnamica 22 (2) (1997), 541-566.

[14] J.-P. Penot, Duality for anticonvex programs, J. Global Optim. 19 (2001), 163-182.

[15] J.-P. Penot, What is quasiconvex analysis?, Optimization 47 (2000), 35-110.

[16] J.-P. Penot, The bearing of duality on microeconomics, Advances in Math. Econ. 7 (2005), 113-139.

[17] J.-P. Penot, Unilateral analysis and duality, in GERAD, *Essays and Surveys in Global Optimization*, G. Savard et al eds, Springer, New York (2005), 1-37.

[18] J.-P. Penot, Critical duality, submitted to J. Global Optim.

[19] J.-P. Penot and M. Volle, Dualité de Fenchel et quasi-convexité, C.R. Acad. Sci. Paris Série I 304 (13) (1987), 371-374.

[20] J.-P. Penot and M. Volle, On quasi-convex duality, Math. Oper. Research 15 (4) (1990), 597-625.

[21] J.-P. Penot and M. Volle, Surrogate programming and multipliers in quasiconvex programming, SIAM J. Control and Optimization 42 (6) (2003), 1994-2003.

[22] J.-P. Penot and M. Volle, Duality methods for the study of Hamilton-Jacobi equations, ESAIM: Proceedings, 17 (2007).

[23] A.M. Rubinov, *Abstract Convexity and Global Optimization*, Kluwer, Dordrecht (2000).

[24] A.M. Rubinov, Radiant sets and their gauges, in *Quasidifferentiability and Related Topics*, V. Demyanov and A.M. Rubinov, eds., Kluwer, Dordrecht (2000).

[25] P.T. Thach, Diewert-Crouzeix conjugation for general quasiconvex duality and applications, J. Optim. Theory Appl. 86 (3) (1995), 719-743.

[26] P.T. Thach, Quasiconjugate of functions, duality relationships between quasiconvex minimization under a reverse convex convex constraint and quasiconvex maximization under a convex constraint and application, J. Math. Anal. Appl. 159 (1991) 299-322.

[27] P.T. Thach, Global optimality criterion and a duality with a zero gap in nonconvex optimization, SIAM J. Math. Anal. 24 (6) (1993), 1537-1556.

[28] P.T. Thach, A nonconvex duality with zero gap and applications, SIAM J. Optim. 4 (1) (1994), 44-64.

[29] Hoang Tuy, On solving nonconvex optimization problems by reducing the duality gap, J. Global Optim. 32 (2005), 349-365.

[30] M. Volle, Conjugaison par tranches, Annali Mat. Pura Appl. 139, pp. 279-312 (1985).

[31] A. Zaffaroni, Is every radiant function the sum of quasiconvex functions? Math. Methods Oper. Res. 59 (2004), 221-233.