Intermediate rank and property RD

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INTERMEDIATE RANK AND PROPERTY RD

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Abstract. We introduce concepts of intermediate rank for countable groups that "interpolate" between consecutive values of the classical (integer-valued) rank. Various classes of groups are proved to have intermediate rank behaviors. We are especially interested in interpolation between rank 1 and rank 2.

For instance, we construct groups "of rank $\frac{7}{4}$". Our setting is essentially that of non positively curved spaces, where concepts of intermediate rank include polynomial rank, local rank, and mesoscopic rank.

The resulting framework has interesting connections to operator algebras. We prove property RD in many cases where intermediate rank occurs. This gives a new family of groups satisfying the Baum-Connes conjecture. We prove that the reduced C^*-algebras of groups of rank $\frac{7}{4}$ have stable rank 1.

The paper is organized along the following thematic lines.

A) Rank interpolation from the viewpoint of property RD;
B) Triangle polyhedra and the classical rank;
C) Polynomial and exponential rank, growth rank and property RD;
D) Local rank, rank $\frac{7}{4}$, existence and classification results;
E) Triangle polyhedra and property RD;
F) Applications to the Baum-Connes conjecture;
G) C^*-algebraic rank, stable rank, real rank;
H) Mesoscopic rank. Mixed local rank.

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1. Introduction and statement of the results

1.1. **Definition of property RD.** Let Γ be a countable group endowed with a length ℓ. One says that Γ has Property RD with respect to ℓ if there is a polynomial P such that for any $r \in \mathbf{R}_+$ and $f, g \in \mathbf{C}\Gamma$ with $\mathrm{supp}(f) \subset B_r$ one has

$$\|f * g\|_2 \leq P(r)\|f\|_2\|g\|_2$$

where $B_r = \{x \in \Gamma,\ \ell(x) \leq r\}$ is the ball of radius r in Γ. For example groups of polynomial growth have property RD as the number of decompositions $xy = z$ for fixed $z \in \Gamma$ and $x, y \in \Gamma$ with $\ell(x) \leq r$, is polynomial in r. Property RD was introduced by Jolissaint in [39] after the work of Haagerup [33] on reduced C^*-algebras of free groups.

In the case of amenable groups property RD implies polynomial growth [39, 22, 62]. Thus $\mathrm{SL}_3(\mathbf{Z})$, for example, does not have property RD because it contains amenable subgroups which are not of polynomial growth (see [39], this is the only obstruction to property RD known so far).

Here are two fundamental examples of groups with property RD:

(1) free groups on finitely many generators have property RD (with respect to the usual word length), as was proved by Haagerup in [33],
(2) groups acting freely isometrically on Bruhat-Tits buildings of type $\tilde{A}_2$ (also called triangle buildings) have property RD with respect to the length induced from the 1-skeleton, as was proved by Ramagge, Robertson and Steger in [48].

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Haagerup's result was generalized by Jolissaint [39] and de la Harpe [34] to every hyperbolic groups in the sense of Gromov. The result of Ramagge, Robertson and Steger provided the first occurrences of property RD in "higher rank" situations. This was extended in [40], where Lafforgue proves that all cocompact lattices in $\mathrm{SL}_3(\mathbf{R})$ (and $\mathrm{SL}_3(\mathbf{C})$) have property RD.

We refer to Section 2 for details and further recent developments. We are interested here in interpolation between (1) and (2).

1.2. **Triangle polyhedra.** The *rank* of a non positively curved metric space X is an asymptotic invariant of X usually defined as the dimension of maximal flats in X. A *flat* in X is the image of an isometric embedding of an Euclidean space $\mathbf{R}^k$, $k \geq 2$. In [31, page 127] seven definitions of the rank are discussed. As mentioned there, they express the idea that X behaves hyperbolically in the dimensions above its rank.

Definition 1. We call *triangle polyhedron* a non positively curved (i.e. CAT(0)) simplicial complex X of dimension 2 without boundary and whose faces are **equilateral** triangles. A countable group is called a *triangle group* if it admits a proper and isometric action on a triangle polyhedron. Proper means that stabilizers are uniformly finite.

The role of triangle polyhedra below is to allow rank interpolation within a tractable geometrical framework. The word triangle refers to the Coxeter diagram of flats and follows the classical terminology for Tits buildings [52]. All flats in X are isometric to Euclidean planes $\mathbf{R}^2$ tessellated by equilateral triangles (i.e. they are of type $\tilde{A}_2$). Note that in the literature the terminology 'triangle groups' can also refer to a class (different from the above) of reflection groups of the plane (Euclidean or hyperbolic, see e.g. [35, V.38]).

Examples of triangle groups include $\tilde{A}_2$-groups, i.e. groups which act freely and simply transitively on triangle buildings (see [16] and [52] for a general reference on Tits buildings) as well as many hyperbolic groups (in particular all triangle groups that satisfy the 'girth > 6' local condition [31]). Note that by the *no flat criterion* (see [31, page 176]) the classical integer-valued rank (defined above) detects precisely hyperbolicity among triangle polyhedra. For clarity we adopt the following convention concerning the integer-valued rank.

A triangle polyhedron is said to have

- *rank 1* if it is hyperbolic,
- *rank 2* if it is symmetric, i.e., if it is a triangle building ([1, 4]).

For other triangle polyhedra the integer-valued rank is too coarse a rank invariant. We understand it to be non defined in these cases (other concepts shall be substituted to it).

Polyhedra in Definition 1 are not assumed to be locally finite a priori. Note that every countable group admits a triangle presentation (by adding generators to a given presentation that split relations into length 3 relations) but this presentation does not define a triangle polyhedron in general. Cohomological arguments implies that $\mathrm{SL}_3(\mathbf{Z})$ is not a triangle group (see also Theorem 5).

1.3. **Intermediate growth rank.** As we will see intermediate rank behaviors can be exhibited at the microscopic, mesoscopic, and macroscopic—or rather asymptotic—scale. We first discuss the latter along with the notion of *polynomial (growth) rank.*

Definition 2. A triangle polyhedron X is said to have *polynomial rank* if there exists a polynomial P such that for any simplicial geodesic segment γ in X, the number of flat equilateral triangles in X with base γ is bounded by $P(r)$, where r is the length of γ. One says that a triangle group has polynomial rank if it admits a proper and isometric action on a triangle polyhedron of polynomial rank.

In other words we restrict the *branching of flats* in X to be polynomial. This essentially captures spaces whose rank is "not too far from to 1". For instance triangle polyhedra which are hyperbolic or which have isolated flats are of polynomial rank. Triangle buildings, for which the branching of flats is exponential, are not.

In a similar way we define *subexponential rank* by replacing the above polynomial P by some given subexponential function. Triangle polyhedra which are not of subexponential rank are said to be of *exponential rank*. These notions are asymptotic in nature and, as in [31] (see page 127), can be detected at infinity.

Definition 2 is generalized to arbitrary countable groups endowed with a length (e.g. finitely generated groups with the word length) in Section 2. While being in the non amenable setting this generalization is strongly reminiscent of the theory of *growth of groups* (for which we refer to [35] and references therein) and will thus be called *polynomial growth rank*. It relies on tools that arise from the study of property RD, allowing to merely retain the polynomial growth of 'flats' rather than sharp flatness. Subexponential and exponential growth rank can be defined analogously. The proof that polynomial growth rank coincide with the above Definition 2 for triangle groups is given in Section 3 (Proposition 31). In the amenable case the growth rank gives back the classical notion of growth of groups (see Section 2.2). Regarding property RD the following holds.

Proposition 3. *Let Γ be a countable group endowed with a length ℓ. If Γ has polynomial growth rank with respect to ℓ then it has property RD with respect to ℓ.*

Examples of groups with polynomial growth rank are relatively hyperbolic groups with respect to a finite family of groups of polynomial growth. In this case property RD was already known by a theorem of Chatterji and Ruane [20]. Subexponential growth rank does not imply property RD in general, but it does imply some useful subexponential variations of it (see the end of Section 2.2).

1.4. **Polyhedra of rank $\frac{7}{4}$.** We now turn to constructions of triangle groups of intermediate rank. The groups exhibited in this Section will be called *groups of rank* $\frac{7}{4}$. As we will explain, the rank conditions that prevail in these constructions are *local*. Nonetheless, large rank tends to propagate to the asymptotic level and groups of rank $\frac{7}{4}$, at least for some of them (see Theorem 4), have exponential rank in the sense of the previous subsection (yet their rank is strictly lower than that of triangle buildings).

Recall that the link of a CAT(0) complex X at a point A is the set of directions at A in X, endowed with the angular metric (see e.g. [13, p. 103]). The tangent cone $\mathrm{Con}_A X$ of X at A is the CAT(0) cone over the link of A endowed with the angular metric. This is a CAT(0) space as well ([13, p. 190]). In this paper links are always assumed to be connected.

For a (geometrically finite) CAT(0) complex X the *local rank* at a point A represents the *proportion of flats in the tangent cone* $\mathrm{Con}_A X$ *of X at A*. We recall below what it means for a 2-dimensional CAT(0) complex to have local rank $\leq \frac{3}{2}$, local rank $\frac{3}{2}$, local rank 2, and we define local rank $\frac{7}{4}$.

Observe that in a triangle polyhedron links at vertices can be a priori any graph with girth 6 (this is equivalent to the CAT(0) condition, edges have length $\pi/3$) so that the proportion of tangent flats at a vertex corresponds to the proportion of 6-cycles in its link.

In [6, 7] the first author defined the local rank of a 2-dimensional CAT(0) complex X to be $\leq \frac{3}{2}$ if the following condition is satisfied.

- *Local rank $\leq \frac{3}{2}$.* For every vertex A of X, every segment (not necessarily simplicial) of length π in the link L at A is included in at most one cycle of length 2π in L.

Then the complex X is said to have *local rank* $\frac{3}{2}$ if it has rank $\leq \frac{3}{2}$ and if the following condition is satisfied (see [6]).

- *Local rank* $\geq \frac{3}{2}$. For every vertex A of X, every edge in the link L at A, as well as every pair of thick vertices at distance no greater than π, is included in at least one cycle of length 2π of L (where we say that a vertex is thick if its valency is at least 3).

It is shown in [7] that triangle polyhedra of rank $\leq \frac{3}{2}$ have isolated flats. In particular they have polynomial rank.

In the case of triangle buildings links corresponds to *projective planes*. They have local rank 2 in the sense that an incidence graph of a projective plane is a *spherical building*: compare [3, 4] and see also [1] for a semi-local definition of rank 2. A 2-dimensional CAT(0) complex X is said to have local rank 2 if the following condition is satisfied [4].

- *Local rank 2.* For every vertex A of X, every segment (not necessarily simplicial) of length π in L, is included in at least one cycle of length 2π.

For triangle polyhedra, well-known examples of link of local rank 2 (i.e., of spherical buildings) include the incidence graph of the Fano plane (see Figure 1.3 in [52]).

The only link which is both of rank $\leq \frac{3}{2}$ and of rank 2 is the circle of length 2π (cf. [4]). Thus the only 2-dimensional CAT(0) complex of local rank $\leq \frac{3}{2}$ and local rank 2 is $\mathbf{R}^2$. Note that this is a (thin, reducible) building, which indeed has low rank but still, large rank relatively to itself.

We say that a triangle polyhedron has *local rank* $\frac{7}{4}$, or merely *rank* $\frac{7}{4}$ when no confusion can arise, if its links at each vertex are isomorphic to the following graph, henceforth denoted $L_{\frac{7}{4}}$.

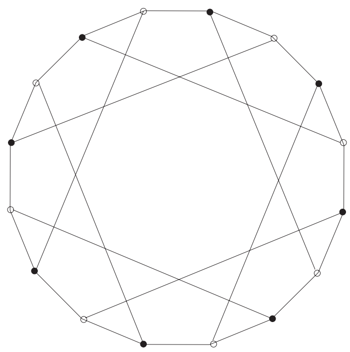

FIGURE 1. Rank $\frac{7}{4}$ for triangle polyhedra

This graph should be compared to the above-mentioned incidence graph of the Fano plane. It belongs to the family of so-called Generalized Pertersen graph (this is $GP(8,3)$). We refer to [24] (see in particular Fig. 3.3.c on page 22) for further informations. Justifications for its use in rank interpolation can be found in Proposition 32 of Section 4 below and the paragraph following it.

Our main results on the geometric structure of triangle polyhedra of rank $\frac{7}{4}$ are Theorem 35 and the results in Subsections 4.2 and 4.3, that we summarize as follows. By *complex of rank* $\frac{7}{4}$ we mean a compact CW-complex with triangle faces whose universal cover is a polyhedron of rank $\frac{7}{4}$ (see Definition 33).

Theorem 4. *There are precisely 12 orientable complexes of rank* $\frac{7}{4}$ *with one vertex. Their universal covers are triangle polyhedra of rank* $\frac{7}{4}$ *and they all have exponential (growth) rank. Moreover,*

(1) *three of them have an abelianization of non zero rank (which can be 1 or 2) and in particular they don't have the property T of Kazhdan,*
(2) *all of them contain copies of the free abelian group* $\mathbf{Z}^2$ *and satisfy the following additional property: for any copy of* $\mathbf{Z}^2$ *in* Γ*, there is a* $\gamma \in \Gamma$ *such that the pairwise intersection of the subgroups* $\gamma^n \mathbf{Z}^2 \gamma^{-n}$*,* $n \in \mathbf{Z}$*, is reduced to the identity.*

Tits showed in [57] that a triangle polyhedron all of whose links correspond to a projective plane are buildings (see also [4, 1]). It follows that they have exponential growth rank. In the case of general triangle polyhedra, the combination of germs of flats can be very intricate and, depending on their relative position, does not necessarily "integrate" to actual flats in X. We make this precise in Section 4.2 where we study the local flat structure imposed by the rank $\frac{7}{4}$ (this should be compared to the results in Section 6). The asymptotic rank assertions of Theorem 4 are derived in Section 4.3. We do not know whether a triangle polyhedron of rank $\frac{7}{4}$ always contains flats or not (see in particular Question 45).

In [29] Garland introduced another local invariant for non positively curved polyhedra of dimension 2 (and larger), called the *p-adic curvature*. Given a vertex A of a polyhedron X the p-adic curvature of X at A is defined to be the first non-zero eigenvalues λ_1 of the Laplacian on the link of A. Then he proved his famous vanishing cohomology results under the assumption $\lambda_1 > 1/2$, which eventually lead to the $\lambda_1 > 1/2$ criterion for property T [66, 45, 2]. The first eigenvalue of $L_{\frac{7}{4}}$ is

$$\lambda_1(L_{\frac{7}{4}}) = 0.42... < 1/2.$$

We do not know whether there are groups of rank $\frac{7}{4}$ which have property T (although Item (1) in Theorem 4 first seemed quite unexpected to us). Property T for triangle buildings was first established in [16] and the proof given there, based on (local) spherical analysis, fails to apply in our context because the automorphism group of $L_{7/4}$ is not sufficiently transitive (see Section 4.2).

1.5. **The Baum-Connes conjecture for triangle groups.** Our general criterion for proving property RD in the above framework is the following result.

Theorem 5. *Let* Γ *be a triangle group and let* ℓ *be the length on* Γ *induced by the 1-skeleton of a triangle polyhedron* X *on which* Γ *acts isometrically and properly. Then* Γ *has property RD with respect to* ℓ*.*

In particular the (twelve) groups of rank $\frac{7}{4}$ described in Theorem 4 have property RD. The classical scheme for establishing property RD consists in reducing the convolution product to partial convolutions over simpler triangles and we proceed exactly in the same way in the present paper (see Section 2 and the references therein). As in [48, 40] triangles will be reduced to flat equilateral triangles. Our contribution is in Section 3 and concerns the geometrical part of the proof.

Note that symmetric spaces tools (e.g. the retraction onto apartments that was useful for buildings in [48] or computations in SL_3 as in [40]) are not available in our context. According to a conjecture of Valette [62, page 66] property RD should hold for every groups properly isometrically and cocompactly on an affine building or a Riemannian symmetric space. If true, as Theorem 5 suggests, it might hold even more generally in situations where rank interpolations is available. Understanding to what extend Theorem 5 generalizes to groups acting on other type of (say, geometrically finite but not necessarily symmetric) CAT(0) simplicial complexes is an interesting open problem (compare this to Subsection 1.7).

Theorem 5 has the following consequence, which is a straightforward application of Lafforgue's Theorem [41].

Corollary 6. *Let Γ be a countable group admitting a proper, isometric, and cocompact action on a triangle polyhedron X. Then Γ satisfies the Baum-Connes conjecture, i.e. the Baum-Connes assembly map*

$$\mu_r : K_*^{\rm top}(\Gamma) \to K_*(C_r^*(\Gamma))$$

is an isomorphism.

See [10, 53, 62] for information on the Baum-Connes conjecture. We simply comment here that Lafforgue considered in [42] a strengthening of property T which holds for cocompact lattices in $\mathrm{SL}_3(\mathbf{Q}_p)$ (in particular $\tilde{A}_2$-groups) but fails for every hyperbolic groups. This version of property T can be seen as an obstruction for proving the Baum-Connes conjecture with coefficients using Banach KK-theory (see [42] where a proof of the Baum-Connes conjecture with coefficients for any hyperbolic groups is announced). It would be interesting in that respect to determine 'up to what rank' (necessarily < 2) Banach KK-theory techniques can be applied in the framework of triangle polyhedra to get Baum-Connes with coefficients. Note that the construction of Kasparov's element γ and of the homotopy between γ and 1 in (asymptotic versions of) $KK_\Gamma^{\rm ban}(\mathbf{C},\mathbf{C})$ are technically easier to perform in the context of triangle polyhedra than in the general case [41] of (strongly) bolic spaces, and coefficients appearing in the homotopy should be controllable to some extend (see [41] and Section 4 in [53]). The first-named author proved in [7] that groups of local rank $\leq \frac{3}{2}$ have the Haagerup property and thus they satisfy the Baum-Connes conjecture with coefficients by Higson-Kasparov's Theorem [37] (and $\gamma = 1$ in Kasparov's $KK_\Gamma(\mathbf{C},\mathbf{C})$ by [60]). The proof of strengthened property T for cocompact lattices of $\mathrm{SL}_3(\mathbf{Q}_p)$ in [42] relies on symmetric spaces tools and it is not clear at all that the same holds when the rank is (even slightly) lower, e.g. for some groups of rank $\frac{7}{4}$.

1.6. **C^*-algebraic rank.** Let A be a unital C^*-algebra. The stable rank $\mathrm{sr}(A)$ of A is an invariant of A taking values in $\{1,2,\ldots\}\cup\{\infty\}$ which was introduced by Rieffel [49]. In the commutative case $\mathrm{sr}(A)$ behaves as a dimension. Thus for a compact space X and $A = C(X)$ the C^*-algebra of complex-valued function on X one has

$$\mathrm{sr}(A) = \lfloor \dim X/2 \rfloor + 1.$$

In particular

$$\mathrm{sr}(C_r^*(\mathbf{Z}^2)) = 2$$

where $C_r^*(\mathbf{Z}^2) \simeq C(\mathbf{T}^2)$ is the C^*-algebra of the abelian free group $\mathbf{Z}^2$. We are interested here in the stable rank of reduced C^*-algebras of non amenable countable groups where, as opposed to the commutative case, an interpretation of $\mathrm{sr}(A)$ as a "dimension" of A is far less evident. In another direction, we mention that the case of nuclear (simple) algebras received much attention recently in connection to Elliott's classification program (see e.g. [59] and references). Villadsen [63] constructed for any integer n a simple, separable and unital AH-algebra of stable rank n.

We investigate here the relationships between the "asymptotic dimension of Γ" (especially from the intermediate rank point of view) and the stable rank of $C_r^*(\Gamma)$, in the case of triangle groups.

A unital C^*-algebra A has stable rank 1 if and only if the group $\mathrm{GL}(A)$ of invertible elements of A is norm dense in A. There are well-known structural consequences of the stable rank 1 condition (see [12]), especially concerning non stable K-theory properties of A. For instance the map $U(A)/U(A)_0 \to K_1(A)$ from the quotient of the unitary group of A by the connected component of the identity to the first K-theory group of A is an isomorphism.

In [26] Dykema, Haagerup and Rørdam proved that if Γ_1 and Γ_2 are two countable groups with $|\Gamma_1| \geq 2$ and $|\Gamma_2| \geq 3$ then

$$\mathrm{sr}(C_r^*(\Gamma_1 * \Gamma_2)) = 1.$$

In particular for the free groups F_n on $n \geq 2$ generators one has $\mathrm{sr}(C^*_r(F_n)) = 1$. In [27] Dykema and de la Harpe generalized these results and proved that if Γ is a torsion free non elementary hyperbolic group, or a cocompact lattice in a real, noncompact, simple, connected Lie group of real rank one with trivial center, one has

$$\mathrm{sr}(C^*_r(\Gamma)) = 1.$$

We also mention that the rank of a group and the stable rank of its reduced C^*-algebra are known to be related to each other in the realm of Lie groups. In [54] Sudo proved that for a connected noncompact real semisimple Lie group G the stable rank of $\mathrm{sr}(C^*_r(G))$ is 1 if the real rank of G is 1, while it is 2 if the real rank of G is ≥ 2. It is unknown if a similar dichotomy holds true for cocompact lattices in real Lie groups (see Problem 1.8 in [27]). The p-adic case is open as well, and in particular we don't know what the stable rank of the reduced C^*-algebra of $\tilde{A}_2$-groups is.

Theorem 7. *Let X be a complex of rank $\frac{7}{4}$ and let $\Gamma = \pi_1(X)$ be the fundamental group of X. Then the reduced C^*-algebra $C^*_r(\Gamma)$ of Γ has stable rank 1.*

The proof of this result occupies Section 5. We use a sufficient condition for stable rank 1 of Dykema and de la Harpe [27], which is recalled at the beginning of Section 5.

All previously known reduced group C^*-algebras with stable rank 1 were related to free products or hyperbolicity. In our case we know from Theorem 4 that there exist groups of rank $\frac{7}{4}$ with exponential rank and containing infinitely many subgroups isomorphic to $\mathbf{Z}^2$ (each of them further satisfying the conditions in item (2) of this Theorem). These groups are neither Gromov hyperbolic nor they are decomposable as non trivial free products. In many respect they are actually *closer to $\tilde{A}_2$-groups than to hyperbolic groups* (we remark that our proof of Theorem 7, however, definitely fails in the rank 2 case).

Another invariant of A, the real rank, was defined by Brown and Perdesen in [14]. It is denoted $\mathrm{rr}(A)$ and takes values in $\{0, 1, \ldots\} \cup \{\infty\}$. A unital C^*-algebra A has real rank 0 if and only if $\mathrm{GL}(A_{\mathrm{sa}})$ is dense in A_{sa}, where the subscript $_{\mathrm{sa}}$ denotes the self-adjoint subspace of A. In the commutative case one has $\mathrm{rr}(C(X)) = \dim X$, where X is a compact space, and in general the following relation holds for a C^*-algebra A (see [12]):

$$\mathrm{rr}(A) < 2\mathrm{sr}(A).$$

Thus the real rank of the reduced C^*-algebra of fundamental groups of compact complexes of rank $\frac{7}{4}$ is at most 1. Let us now show that it is 1.

Recall the following conjecture of Kaplansky and Kadison: for any torsion-free countable group Γ, the reduced C^*-algebra of Γ has no idempotent besides 0 and 1. As is well-known, this conjecture is a consequence of the surjectivity of the Baum-Connes assembly map μ_r (see [10, 62]). Thus, the absence of non trivial projection in the C^*-algebras considered in Theorem 7 is part of Corollary 6. On the other hand in a C^*-algebra of real rank 0 every self-adjoint element can be approximated by self-adjoint elements with finite spectrum, so in particular real rank 0 implies the existence of many non trivial projections [12].

Summarizing, the following is a straightforward consequence of Theorem 7, Corollary 6, and known facts.

Corollary 8. *Let X be a complex of rank $\frac{7}{4}$ and let $\Gamma = \pi_1(X)$ be the fundamental group of X. Then the reduced C^*-algebra $C^*_r(\Gamma)$ of Γ has real rank 1.*

It would be interesting to have a direct proof of this result.

1.7. **Mesoscopic rank, mixed local rank.** Let us now come to our last (and most refined) notion of intermediate rank. Interpolation occurs here from local to global, i.e., we aim at measuring the proportion of "flat pieces" of a space which are strictly in between the microscopic and the macroscopic scale.

Let X be a CAT(0) space of dimension 2 (without boundary) and A be a point of X. Consider the function $\varphi_A : \mathbf{R}_+ \to \mathbf{N}$ which associated to an $r \in \mathbf{R}_+$ the number of flat disks in X of center A *which are not included in a flat of* X. It can be seen as a way to measure the quantity of flats in X which are situated strictly in between local flats (i.e. flats of the tangent cones at A) and global ones (i.e. isometric copies of $\mathbf{R}^2$ in X containing A). We call φ_A the *mesoscopic rank profile* of X at A (or simply *mesoscopic profile* for short).

In the case of triangle polyhedra of extremal rank (rank 1 or rank 2) the mesoscopic profile trivializes as follows.

Proposition 9. *Let X be a triangle polyhedron. If X is hyperbolic (i.e. has rank 1) then its mesoscopic profile at every point is compactly supported. On the other hand X is a buildings (i.e. has rank 2) if and only if its mesoscopic profile vanishes identically at every point.*

Phenomenons start to appear for polyhedra of rank $\leq \frac{3}{2}$. The following graph, as one can show, is the mesoscopic profile at some vertex of the triangle polyhedron of rank $\frac{3}{2}$ constructed in [6].

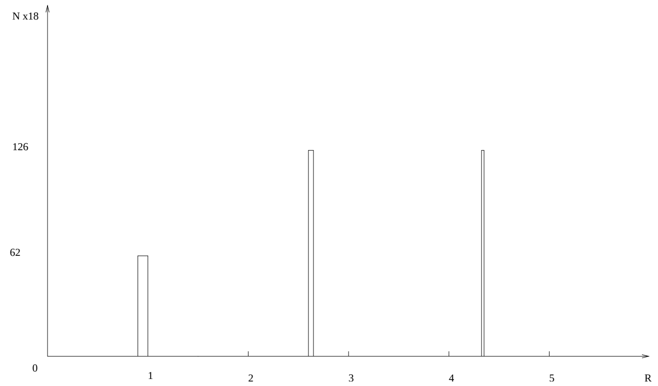

FIGURE 2. Mesoscopic profile of the rank $\frac{3}{2}$ polyhedron of [6]

Here the mesoscopic profile is is bounded (by $2268 = 126 \times 18$) and its support is an infinite union of disjoint intervals whose length tends to 0 at infinity.

Intermediate rank at the mesoscopic scale is defined as follows.

Definition 10. A CAT(0) space X of dimension 2 is said to be *of mesoscopic rank* at a point A if the support of φ_A contains a neighborhood of infinity.

The signification of Definition 10 is clear: in a space X of mesoscopic rank one can continuously rescale the radius of disks centered at A which are flat but not included in flats of X from some constant C up to ∞. It is trivial to work out examples of 2-dimensional CAT(0) spaces X without boundary which are of mesoscopic rank at *some* point A. What we aim to construct here are X for which the set of A satisfying this property is a (uniform) *lattice* in X. We say that a countable group is *of mesoscopic rank* if it admits a free and cocompact isometric action on a CAT(0) space X of dimension 2 which is of mesoscopic rank at some point.

(a) Surgery of exotic buildings. Recall that an affine building is said to be *exotic* if it is cocompact but not classical, i.e. not associated to an algebraic group over a local field (see e.g. [50, 51, 16, 5]). Fundamental work of Tits [57] led to the complete

classification of affine buildings of dimension ≥ 3: they all are classical. The situation is entirely different in dimension 2 (see [8, 9] and references). In Section 3 of [5], the first author constructed an exotic triangle buildings $\tilde{P}$ which is the universal cover of a compact complex P with two vertices (links at these vertices are trivalent, i.e. they are associated to the Fano plane). In fact the fundamental group $\pi_1(P)$ of P has finite index in the automorphisms group of $\tilde{P}$ ([5, Théorème 7]).

By surgery on the complex P, one can construct a compact complex $V_{\bowtie}$ with 8 vertices, whose universal cover is a CAT(0) space of dimension 2 without boundary (see Section 6 for details). This complex has *mixed local rank*: 2 of its vertex have rank 2 and the 6 others have rank $\frac{3}{2}$.

We call the fundamental group $\Gamma_{\bowtie} = \pi_1(V_{\bowtie})$ of $V_{\bowtie}$ a *group of friezes* and its universal cover $\tilde{V}_{\bowtie}$ a *complex of friezes* ('groupe de frizes' and 'complexe de frizes' in french).

(b) Polyhedra of rank $\frac{7}{4}$. In the classification of orientable complexes of rank $\frac{7}{4}$ with one vertex of Section 4 there is one, namely V_0^1 in the notations introduced there, that has quite a distinctive intermediate rank property: in a sense that will be make precise in Sections 4 and 6, its universal cover has *the "maximum" asymptotic rank within the range allowed by $L_{\frac{7}{4}}$*. As the local analysis in Section 4.2 will show, that this upper-bound is "attained" is quite remarkable. Its proof is the first step towards Item (b) of the following theorem.

Theorem 11. *The following groups are of mesoscopic rank.*

(a) *The group of friezes $\Gamma_{\bowtie}$ acting on the complex of friezes.*
(b) *The fundamental group of the complex V_0^1 (which is of rank $\frac{7}{4}$) acting on its universal cover $\tilde{V}_0^1$.*

Even more, these complexes have *exponential mesoscopic rank* in the sense that their mesoscopic profile converges exponentially to infinity at infinity. Recall that $\tilde{V}_0^1$ in (b) has all links isomorphic to $L_{\frac{7}{4}}$ and is *transitive on vertices*, showing that extremely homogeneous local data, that precludes in particular mixed the local rank and spaces built out of different shapes, may still create 'singular' flats disks in X at the mesoscopic scale (homogeneity of $L_{\frac{7}{4}}$ is studied in Subsection 4.2). This cannot happen for the *most* homogeneous local data (i.e. spherical buildings), as we already saw.

The group $\Gamma_{\bowtie}$ is a triangle group, as one can see after a suitable subdivision of the complex of friezes. Thus it satisfies property RD and the Baum-Connes conjecture. However, the proof of Theorem 5 is *very sensitive to the ambient geometry*, and establishing property RD for $\Gamma_{\bowtie}$ directly (without subdivising) would further increase the technical difficulties of Section 3. In fact it is while looking for a way to bypass this technicalities in the case of the *Wise group* that we first encountered mesoscopic rank phenomenon. We shall now briefly discuss this as a conclusion.

Write Γ_W for Wise's non Hopfian group, as constructed in [65]. Recall that Γ_W is the fundamental group of a compact complex whose universal cover a 2-dimensional CAT(0) space $\tilde{W}$. Then one can prove that:

(c) *Wise's group Γ_W is of mesoscopic rank.*

The proof is omitted here (it is quite similar to that of $\Gamma_{\bowtie}$). In [21] Section 5.2 the question is raised of whether Γ_W has property RD or not (see also 6.6 in [21], where Γ_W is proposed as a possible counter-example to property RD). As noted there, Γ_W does not acts on any cube complex (which implies property RD by Theorem 0.4 of [20]) nor it is relatively hyperbolic (it is actually of exponential growth rank).

Inspection of the proof Theorem 5 in the case of $\tilde{W}$ reveals that the situation is slightly worse than in the case of the non-subdivised $V_{\bowtie}$ but we believe, nevertheless, that Γ_W has property RD.

Theorem 11 is proved in Section 6. The proof of (b) is illustrated on Figure 3.

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Contents

2. Property RD and polynomial growth rank

Let Γ be a countable group. A triple $(x, y, z) \in \Gamma^3$ such that $xy = z$ is called a triangle in Γ. Given a set $\mathfrak{s}$ of triangles in Γ, finitely supported functions $f, g \in \mathbf{C}\Gamma$, and $z \in \Gamma$, define $f *_{\mathfrak{s}} g(z)$ by the expression

$$f *_{\mathfrak{s}} g(z) = \sum_{(x,y,z)\in\mathfrak{s}} f(x)g(y)$$

if a triangle of the form (x, y, z) belongs to $\mathfrak{s}$, and 0 otherwise. The convolution product over the family of all triangles in Γ is written $f * g$. Let ℓ be a length on Γ, i.e. a non negative function ℓ on Γ such that $\ell(e) = 0$, $\ell(x) = \ell(x^{-1})$ and $\ell(xy) \leq \ell(x) + \ell(y)$ for $x, y \in \Gamma$. Then $\mathfrak{s}$ is said to have property RD with respect to ℓ if one can find a polynomial P such that for any $r \in \mathbf{R}_+$ and $f, g \in \mathbf{C}\Gamma$ with $\mathrm{supp}(f) \subset B_r$ one has

$$\|f *_{\mathfrak{s}} g\|_2 \leq P(r)\|f\|_2\|g\|_2.$$

If $*_{\mathfrak{s}} = *$, i.e. if $\mathfrak{s}$ consists of all triangles in Γ, then the group Γ is said to have property RD with respect to ℓ (see [39]). For finitely generated groups this is independent of the choice of ℓ among word metrics associated to a finite generating sets, so we simply speak of property RD for Γ in that case. Note that it is sufficient to check the above inequality on non negative functions $f, g \in \mathbf{R}_+\Gamma$. A standard approach to prove property RD for Γ consists in reducing ℓ^2 estimates over $*$ to estimates over simpler partial convolutions $*_{\mathfrak{s}}$ (see [48, 40, 17, 56]). In this section we prove Lemma 15 and Lemma 24, which are the two main known tools for reducing convolution products, and we introduce a notion of polynomial growth rank for countable groups endowed with a length.

Remark 12. Our basic framework for this section will be that of a countable group endowed with a length. This has the advantage of simplifying the exposition without hiding the important issues and this is well adapted to groups acting freely and simply transitively on the vertex set of a triangle polyhedron (which is the case, for instance, of the groups of rank 7/4 constructed in Section 4). As shown in [48] the appropriate tools for generalizing these results to non necessarily simply transitive action are *transitive groupoids.* This is discussed in more details at the end of Subsection 2.3.

2.1. **Statement and proof of Lemma 15.** Let Γ be a countable group and ℓ be a length on Γ. A *3-path* from the identity e to a $z \in \Gamma$ is a triple $\gamma = (a_3, a_2, a_1)$ in Γ^3 such that $z = a_3 a_2 a_1$.

Definition 13. A Γ-indexed family of 3-paths in Γ, i.e. family $C = (C_z^r)_{z\in\Gamma,\ r\in\mathbf{N}^*}$ where C_z^r is a set of 3-paths from e to z in Γ for every $z \in \Gamma$ and $r \in \mathbf{R}^*$, is said to have *polynomial growth* if there exists a polynomial p_1 such that for any $r \in \mathbf{R}_+$ and any $z \in \Gamma$ one has $\# C_z^r \leq p_1(r)$.

Let $\mathfrak{s}$ and $\mathfrak{s}^-$ be two sets of triangles in Γ and $C = (C_z^r)_{z\in\Gamma,\ r\in\mathbf{R}_+}$ be a Γ-indexed set of 3-paths. For $(u, v, w) \in \mathfrak{s}^-$ and $r \in \mathbf{R}_+$ define $D^r_{(u,v,w)}$ to be the set of triple (a, b, c) in Γ^3 such that $(b^{-1}, u, a) \in C^r_{b^{-1}ua}$, $(c^{-1}, v, b) \in C^r_{c^{-1}vb}$ and $(c^{-1}, w, a) \in C^r_{c^{-1}wa}$.

Given $x \in \Gamma$ we often write $|x|$ for $\ell(x)$.

Definition 14. One says that $\mathfrak{s}^-$ *is a retract of* $\mathfrak{s}$ *along* C if there exists a polynomial p_2 such that for every $(x, y, z) \in \mathfrak{s}$ there exists $(u, v, w) \in \mathfrak{s}^-$ with $|u| \leq p_2(|x|)$ and $(a, b, c) \in D^{|x|}_{(u,v,w)}$ such that $b^{-1}ua = x$ and $c^{-1}wa = z$.

The idea of retracting to simpler sets of triangles originates in [48]. Definition 13 corresponds to Property H_δ and a part of Property $K_\delta a$ in [40], and Definition 14 is another part of Property $K_\delta a$ in [40] (see also [17] and Section 1.3 in [56]). Our assumptions here are actually slightly weaker (this was required to define polynomial growth rank below, in particular we do not to assume triangles in the retraction $\mathfrak{s}^-$ to be "balanced" at this stage, cf. Subsection 2.3).

The following lemma was first proved by Haagerup in [33, Lemma 1.4] in the case of finitely generated free groups, where the set $\mathfrak{s}$ of all triangles consists of *tripod triangles* (i.e. triangles which retract to $\mathfrak{s}^- = \{(e, e, e)\}$). It has then been extended in [39, 34, 48, 40]. The statement below corresponds to Proposition 2.3 and a part of Theorem 2.5 in [40] (compare [48]), and the proof given reproduces the arguments on pages 258 and 259 of this paper.

Lemma 15. *Let Γ be a group and ℓ be a length on Γ. Fix a Γ-indexed set of path C in Γ, a family $\mathfrak{s}$ of triangles in Γ and a retract $\mathfrak{s}^-$ of $\mathfrak{s}$ along C as in Definition 14. Assume that C has polynomial growth. Then for any $r \in \mathbf{R}_+$ and $f, g \in \mathbf{R}_+\Gamma$ with* $\operatorname{supp}(f) \subset B_r$ *there exist two functions $i, j \in \mathbf{R}_+\Gamma$ with* $\operatorname{supp}(i) \subset B_{p_2(r)}$ *such that*

$$\|f *_{\mathfrak{s}} g\|_2 \leq \sqrt{p_1(r)}\|i *_{\mathfrak{s}^-} j\|_2$$

and $\|i\|_2 \leq \sqrt{p_1(r)}\|f\|_2$, $\|j\|_2 \leq \sqrt{p_1(r)}\|g\|_2$ where p_1, p_2 are as above. Thus property RD holds for $\mathfrak{s}$ provided it does for $\mathfrak{s}^-$.

Proof. Let $f, g, h \in \mathbf{R}_+\Gamma$ with $\operatorname{supp}(f) \subset B_r$. As any triangle in $\mathfrak{s}$ can be retracted to $\mathfrak{s}^-$ one has

$$\begin{aligned}
\langle f *_{\mathfrak{s}} g \mid h\rangle &= \sum_{(x,y,z)\in\mathfrak{s}} f(x)g(y)h(z) \\
&\leq \sum_{(u,v,w)\in\mathfrak{s}^-,\ |u|\leq p_2(r)} \ \sum_{(a,b,c)\in D^r_{(u,v,w)}} f(b^{-1}ua)g(c^{-1}vb)h(c^{-1}wa) \\
&= \sum_{(u,v,w)\in\mathfrak{s}^-} \operatorname{Tr}(R_u S_v T_w)
\end{aligned}$$

where R_u is the operator on $\ell^2(\Gamma)$ defined for $|u| \leq p_2(r)$ by

$$\langle R_u \delta_a \mid \delta_b\rangle = f(b^{-1}ua)$$

if $(b^{-1}, u, a) \in C^r_{b^{-1}ua}$ and 0 elsewhere. The operators S_v, T_w are defined similarly for any v, w using g, h. Set

$$i(u) = \|R_u\|_2,\ j(v) = \|S_v\|_2 \text{ and } k(w) = \|T_w\|_2.$$

As $\mathrm{Tr}(R_u S_v T_w) \leq \|R_u\|_2 \|S_v\|_2 \|T_w\|_2$ one has

$$\langle f *_{\mathfrak{s}} g \mid h \rangle \leq \sum_{(u,v,w)\in \mathfrak{s}^-} i(u)j(v)k(w) = \langle i *_{\mathfrak{s}^-} j \mid k \rangle.$$

On the other hand

$$\|i\|_2^2 = \sum_{a,b,u|\ (b^{-1},u,a)\in C^r_{b^{-1}ua}} f(b^{-1}ua)^2 \leq \sum_x \# C^r_x f(x)^2 \leq p_1(r)\|f\|_2^2$$

and similarly $\|j\|_2^2 \leq p_1(r)\|g\|_2^2$ and $\|k\|_2^2 \leq p_1(r)\|h\|_2^2$. The lemma follows from the Cauchy-Schwarz inequality for $h = f *_{\mathfrak{s}} g$. □

2.2. **Polynomial growth rank.** Let Γ be a countable groups and ℓ be a length on Γ. Let $\delta \geq 0$. A 3-path (a_3, a_2, a_1) in Γ is said to be δ-geodesic if $|a_1| + |a_2| + |a_3| \leq |a_3 a_2 a_1| + \delta$.

Definition 16. One says that Γ has *polynomial growth rank* with respect to ℓ if there exists a $\delta \geq 0$, a family $C = (C^r_z)_{z\in\Gamma,\ r\in\mathbf{N}^*}$ of sets C^r_z of δ-geodesic 3-paths from e to z with polynomial growth (see Definition 13), a subset $\mathfrak{s}$ of triangle in Γ which is a retract along C of the family of all triangles in Γ, and a polynomial p_3 such that the for every z in Γ and every $r \in \mathbf{R}_+$ the number of triangles in $\mathfrak{s}$ of the form (x, y, z) with $|x| \leq r$ is no greater than $p_3(r)$.

Proposition 17. *Let Γ be countable group with length ℓ. If Γ has polynomial intermediate rank with respect to ℓ then it has property RD with respect to ℓ.*

Proof. Let $\mathfrak{s}$ be a family of triangles in Γ as given by Definition 16. By Proposition 15 there exists a polynomial p_1 such that for any $r \in \mathbf{R}_+$ and $f, g \in \mathbf{R}_+\Gamma$ with $\mathrm{supp}(f) \subset B_r$ there exist two functions $i, j \in \mathbf{R}_+\Gamma$ with $\mathrm{supp}(i) \subset B_{r+\delta}$ such that

$$\|f * g\|_2 \leq \sqrt{p_1(r)}\|i *_{\mathfrak{s}} j\|_2$$

and $\|i\|_2 \leq \sqrt{p_1(r)}\|f\|_2$, $\|j\|_2 \leq \sqrt{p_1(r)}\|g\|_2$. We then have

$$\begin{aligned}
\|i *_{\mathfrak{s}} j\|_2^2 &= \sum_{z\in\Gamma}\left(\sum_{(x,y,z)\in\mathfrak{s},\ |x|\leq r+\delta} i(x)j(y)\right)^2 \\
&\leq p_3(r+\delta)\sum_z \sum_{(x,y,z)\in\mathfrak{s},\ |x|\leq p_2(r)} i(x)^2 j(y)^2 \\
&= p_3(r+\delta)\sum_{x,\ |x|\leq p_2(r)} i(x)^2 \sum_{(x,y,z)\in\mathfrak{s},} j(x^{-1}z)^2 \\
&\leq p_3(r+\delta)\|i\|_2^2\|j\|_2^2.
\end{aligned}$$

where p_3 is a polynomial as in Definition 16. Hence,

$$\|f * g\|_2 \leq \sqrt{p_1(r)p_3(r+\delta)}\|i\|_2\|j\|_2 \leq p_1(r)^{\frac{3}{2}} p_3(r+\delta)^{\frac{1}{2}}\|f\|_2\|g\|_2$$

which proves the Proposition. □

Corollary 18. *An amenable group has polynomial growth rank if and only if it has polynomial growth* [35].

Proof. Recall that an amenable group has property RD if and only if it has polynomial growth [39]. So if Γ is amenable of polynomial growth rank then it is has polynomial growth by Proposition 17. Note that polynomial growth rank is stable under direct product but not semi-direct extensions. The converse is easily seen by choosing C^r_z to be reduced to $\{(e, z, e)\}$ for any $r \in \mathbf{N}^*$ and $\mathfrak{s}$ to be the set of all triangles in Γ. □

Examples of groups with property RD which doesn't have polynomial growth rank include groups acting freely and simply transitively on triangles buildings (i.e. $\tilde{A}_2$ groups). Examples of non amenable groups with polynomial growth rank are as follows.

Proposition 19. *Let Γ be a finitely generated group. If Γ is hyperbolic then it has polynomial intermediate rank (of degree 0). More generally, if Γ is hyperbolic relatively to finitely generated subgroups $\{\Lambda_1, \ldots \Lambda_n\}$ of polynomial growth (for the induced length function) have polynomial intermediate rank.*

Proof. In the case of hyperbolic groups, the required properties are satisfied if we choose C_z^r to be the set of δ-geodesic 3-paths from e to z of the form (b, e, a) with $|a| \leq r$, for some δ large enough and $\mathfrak{s}^- = \{(e, e, e)\}$. (See [34].)

So let Γ be a group which is hyperbolic relatively to $\{\Lambda_1, \ldots \Lambda_n\}$. In fact we assume more generally that Γ is (*)-relatively hyperbolic with respect to $\{\Lambda_1, \ldots \Lambda_n\}$ in the sense of Drutu and Sapir [25, Definition 2.8], for the length ℓ coming from some finite generating set of Γ, and argue exactly as in the beginning of the proof of Theorem 3.1 in [25]. Thus for $z \in \Gamma$ we fix a simplicial geodesic g_z from e to z and let C_z^r for $r \in \mathbf{R}_+$ be is the set of triples (b, h, a) which are called central decompositions in [25] (see Definition 3.3), where r is fixed to be equal to r_1 in their paper. Then one has that $C = (C_z^r)_{z,r}$ has polynomial growth (see Lemma 3.3 in [25]) and that all triangles of Γ retract along C to the family $\mathfrak{s}$ of all triangles in the subgroups Λ_i (see the begining of the proof of Theorem 3.1). Thus, if the subgroups Λ_i have polynomial growth with respect to ℓ then the number of triangles in $\mathfrak{s}$ with fixed basis z is polynomial. This proves the Proposition. $\square$

Note that if in the above proof the Λ_i are not of polynomial growth but have property RD, then we can still apply Lemma 15 to deduce that Γ has property RD as well, which is Theorem 1.1. of Drutu-Sapir's paper [25]. (Accordingly, not all relatively hyperbolic groups with property RD have polynomial intermediate rank.)

Definition 20. We say that a group Γ has *subexponential growth rank* with respect to a length ℓ if all conditions in Definition 16 are satisfied except perhaps for the polynomial growth assumption on p_1 and p_3, which we now allow to be subexponential, i.e. p_1 and p_2 are non negative functions on $\mathbf{R}_+$ such that $\lim_r p_1(r)^{1/r} = \lim_r p_3(r)^{1/r} = 1$.

Since for p_1, p_3 of subexponential growth the function $p_1(r)^{\frac{3}{2}} p_3(r+\delta)^{\frac{1}{2}}$ has subexponential growth as well, the proof of Proposition 17 shows that groups with subexponential growth rank satisfy a subexponential variation of property RD where in the definition P is replaced by a function, say κ, of subexponential growth. Amenable groups with subexponential property RD have subexponential growth. Indeed denoting χ_s the characteristic function of the ball of radius s, one has

$$|B_s| \leq \|\chi_s\|_r \leq \kappa(s)\|\chi_s\|_2 = \kappa(r)\sqrt{|B_s|}$$

where the first inequality follows from $|\sum_{x\in\Gamma} f(x)| \leq \|f\|_r$ for every $f \in \mathbf{C}\Gamma$ by weak containment of the trivial representation in the regular representation of Γ (we write $\|f\|_r$ for the norm of $f \in \mathbf{C}\Gamma$ acting by convolution on $\ell^2(\Gamma)$). Thus indeed Γ has subexponential growth (compare [39, 22, 62]) and in particular subexponential growth rank coincide with subexponential growth in the amenable setting. On the other hand arguing as in Proposition 19 we obtain that groups that are relatively hyperbolic with respect to groups of subexponential growth have subexponential growth rank. Taking free product $A * B$ of groups A, B of intermediate growth (e.g. the groups of Grigorchuk, see [35] and the references therein) shows that the class of groups with (optimal) subexponential growth rank κ vary when the growth of κ varies (relying upon examples by A. Erschler). Note also that for groups which are relatively hyperbolic with respect to $\{\Lambda_1, \ldots, \Lambda_n\}$, following [25], subexponential property RD is equivalent to subexponential property RD for the Λ_i. We do not know, however, the answer to the following 'intermediate growth rank problem'.

Question 21. Are there finitely generated groups admitting a proper and cocompact action on a triangle polyhedron X which have intermediate (i.e. subexponential but not polynomial) growth rank with respect to the length induced from the 1-skeleton of X ?

In fact $\tilde{A}_2$ groups, and other triangle groups constructed below, have exponential growth rank in the following sense.

Definition 22. A group Γ is said to have exponential growth rank with respect to a length ℓ if it is not of subexponential growth rank.

For future use (Section 5) we end this subsection with a discussion of the ℓ^2 spectral radius property (see [36, 27]) and related applications of property RD to random walks on groups [30]. In Section 5 of [30] Grigorchuk and Nagnibeda considered the *operator growth function of* Γ, defined as $F_r(z) = \sum_n a_n z^n$ with coefficients

$$a_n = \sum_{|x|=n} u_x$$

where u_x, $x \in \Gamma$, is the canonical unitary corresponding to Γ in $C_r^*(\Gamma)$ under the regular representation. The radius of convergence ρ_r of F_r satisfies

$$\frac{1}{\rho_r} = \limsup_{n\to\infty} \|a_n\|_r^{1/n} \leq \limsup_{n\to\infty} |S_n|^{1/n} = \frac{1}{\rho}$$

where ρ is the usual (inverse) exponential growth rate of Γ with respect to ℓ. Conjecture 2 in [30] states that Γ is amenable if and only if $\rho = \rho_r$. It is proved in [30] that $\rho = \rho_r = 1$ for amenable groups and that $\rho_r = \sqrt{\rho} < 1$ for non amenable hyperbolic groups (recall that Γ is amenable if and only if $\rho = 1$ by Kesten criterion). Valette noted that the proof given of [30] was only using property RD for Γ, and in fact that *radial property RD* was sufficient. This allows for instance to include every $\tilde{A}_n$-groups, for $n \geq 2$, in the above Conjecture 2 which have radial property RD thanks to the work of Valette [61] and Świątkowski [55].

In the same way *for a non amenable group* Γ *satisfying radial subexponential property RD with respect to* ℓ *one has* $\rho_r = \sqrt{\rho} < 1$, so Γ satisfies conjecture 2 in [30]. The proof is exactly as in [30] (see also [36]): by radial subexponential property RD we have

$$\|a_n\|_r \leq \kappa(n)\|a_n\|_2 = \kappa(n)\sqrt{|S_n|}$$

so

$$\frac{1}{\rho_r} \leq \limsup_{n\to\infty} \kappa(n)^{\frac{1}{n}} \sqrt{|S_n|}^{\frac{1}{n}} = \frac{1}{\sqrt{\rho}}.$$

As $\|a_n\|_2 \leq \|a_n\|_r$ always holds $\rho_r \leq \sqrt{\rho}$ as well. The same argument also shows the ℓ^2 spectral radius property for every element in the group algebra of Γ provided Γ has subexponential property RD, i.e. the spectral radius of every element $a \in \mathbf{C}\Gamma$ acting by convolution on $\ell^2(\Gamma)$ is equal to

$$\lim_{n\to\infty} \|a^{*n}\|_2^{1/n},$$

since the radius of the support of n-th convolution product a^{*n} is at most n times the radius of the support of a. Summarizing we have the following result (compare Proposition 8 in [36], Section 3 in [27], and Proposition 4 in [30]).

Proposition 23. *If* Γ *is a countable group with radial subexponential property RD, then* $\rho_r = \sqrt{\rho}$ *and thus* Γ *satisfies conjecture 2 in* [30]. *If moreover* Γ *has subexponential property RD (in particular if it has property RD), then it satisfies the* ℓ^2 *spectral radius property.*

As noted at the end of Section 3 of [27] finitely generated solvable groups provide examples of groups with the ℓ^2-spectral radius property which don't have property RD. All known examples of groups with the ℓ^2-spectral radius property seems, however, to have subexponential property RD.

2.3. **Statement and proof of Lemma 24.** We now recall the (crucial) analytical argument of Ramagge, Robertson and Steger [48, Lemma 3.2] for establishing property RD in the case of triangle buildings. For a family $\mathfrak{s}$ of triangles in Γ we call *dual of* $\mathfrak{s}$ the family $\mathfrak{s}^*$ of triangles of the form (x^{-1}, u, ux^{-1}) and (y, v^{-1}, vy^{-1}) with common basis $ux^{-1} = vy^{-1}$ whenever (x, y, z) and (u, v, z) are two triangle in $\mathfrak{s}$ with common basis z (cf. property $K_\delta b$ in [40] or Definition 1.30 in [56]). The proof below is contained in Lemma 3.2 of [48], see also the top of p. 260 in [40] or [56]. One says that a family $\mathfrak{s}$ of triangles is *balanced* if there is a polynomial p_4 such that for every $(x, y, z) \in \mathfrak{s}$ one has $\max\{|y|, |z|\} \leq p_4(|x|)$.

Lemma 24. *Let Γ be a countable group endowed with a length ℓ, and let $\mathfrak{s}$ be a balanced family of triangles in Γ. There exists a polynomial p_4 such that for $r \in \mathbf{R}_+$ and $f, g \in \mathbf{R}_+\Gamma$ with* $\operatorname{supp}(f) \subset B_r$ *one has*

$$\|f *_{\mathfrak{s}} g\|_2^2 \leq \|\check{f} *_{\mathfrak{s}^*} f\|_2 \|(g\chi_{B_{p_4(r)}}) *_{\mathfrak{s}^*} \check{g}\|_2$$

where $\check{h}(z) = h(z^{-1})$ for $h \in \mathbf{C}\Gamma$. So property RD holds for $\mathfrak{s}$ provided it does for $\mathfrak{s}^*$.

Proof. For $f, g \in \mathbf{R}_+\Gamma$ with $\operatorname{supp}(f) \subset B_r$ one has

$$\begin{aligned}
\|f *_{\mathfrak{s}} g\|_2^2 &= \sum_{z\in\Gamma} \sum_{(x,y,z)\in\mathfrak{s}} \sum_{(u,v,z)\in\mathfrak{s}} f(x)g(y)f(u)g(v) \\
&\leq \sum_{z'\in\Gamma} \sum_{x^{-1}uz'\in\mathfrak{s}^*} \sum_{yv^{-1}z'\in\mathfrak{s}^*,\ |y|\leq p_4(r)} f(x)g(y)f(u)g(v) \\
&= \sum_{z'\in\Gamma} (\check{f} *_{\mathfrak{s}^*} f)(z')((g\chi_{B_{p_4(r)}}) *_{\mathfrak{s}^*} \check{g})(z')
\end{aligned}$$

as $|y| \leq p_4(r)$ for $|x| \leq r$ as $\mathfrak{s}$ is balanced. The Lemma follows from the Cauchy-Schwarz inequality. ∎

Let us conclude this section by recalling the generalization of the above to *transitive* groupoids [48]. This notably allows to prove property RD for countable groups whose length is coming from a general free isometric actions on metric spaces (rather than vertex-transitive actions). So let Γ be a countable group acting freely on a metric space (X, d) and consider, following [48], the countable groupoid $G = X \times_\Gamma X$ of base $G^{(0)} = X/\Gamma$. Let ℓ be the length on G defined by $\ell([x, y]) = d(x, y)$ for $[x, y] \in G$ and $B_r = \{[x, y] \in G,\ \ell([x, y]) \leq r\}$. Then (G, ℓ) is said to have property RD if the usual convolution estimate (with respect to the groupoid law in G) is satisfied for $f, g \in \mathbf{R}_+G$ with $\operatorname{supp}(f) \subset B_r$. All definitions presented in this section (in particular retractions along Γ-indexed family of path of Subsection 2.1 and the above dualization procedure) extends to the case of (G, ℓ), and straightforward generalizations of Lemma 15 and Lemma 24 provide criteria for proving property RD for (G, ℓ). In turn property RD for (G, ℓ) is easily seen to imply property RD for Γ with respect to the length induced from d on one of its orbit in X (see e.g. [40, Prop. 2.1]).

Note however that an extension of the techniques presented in this section to other—non transitive, but say, r-discrete and locally compact—groupoids is an open problem in general, compare [43] and the last sections of [46].

3. Proof of Theorem 5

The proof relies on several preliminary lemmas. Throughout the section we let X be a fixed triangle polyhedron (Definition 1). A curve between two vertices A and B of X is said to be

- a *geodesic segment* if its length equals the CAT(0) distance between A and B. By the CAT(0) property there is a unique geodesic segment between any two points of X.
- a *simplicial geodesic segment* if it is simplicial, i.e. included in the 1-skeleton of X, and if its length coincide with the simplicial length between A and B in X, where the length of every edge in X is normalized to 1.

A geodesic segment is called *singular* if it is simplicial (up to parallelism this coincides with the usual definition in case X is symmetric [13, p. 322]).

Definition 25. Let γ be a geodesic segment between two vertices A and B of X. One calls *simplicial convex hull* of γ the reunion, denoted $\mathrm{Conv}(\gamma)$, of all triangles of X whose three vertices belong to simplicial geodesic segments from A to B in X.

By $\mathbf{R}^2$ we mean the Euclidean plane endowed with the tessellation by equilateral triangles. Isometries are assumed to preserve the simplicial structures. A *flat* in X is the image of an isometric embedding in X of the Euclidean plane $\mathbf{R}^2$. A *flat topological disk* in X is the image an isometric embedding in X of a topological disk of $\mathbf{R}^2$. In particular a *flat equilateral triangle* is the image of an isometric embedding of an equilateral triangle of $\mathbf{R}^2$.

Let D be an open topological disk in X with piecewise linear topological boundary of Δ. Let s be a point in Δ and L be the link of s in X. The disk D determine a path c in L from the two points of L corresponding the incoming and outgoing segments of Δ at s. The angle between these segments, i.e. the angular length of c in L, is called the *internal angle* of D at s and is denoted by θ_s.

Lemma 26. *Let γ be a geodesic segment between two vertices of X. Then there exist finite sets J and J° such that*

$$\mathrm{Conv}(\gamma) = \bigcup_{i\in J} G_i \cup \bigcup_{i\in J^\circ} S_i$$

where

1. *G_i, $i \in J$, is a closed flat topological disks of X which is, under an isometry with a closed disk of $\mathbf{R}^2$, a reunion of minimal galleries (see [52]) between two given vertices of $\mathbf{R}^2$,*
2. *S_i, $i \in J^\circ$, are singular geodesic segments included in γ,*
3. *$S_i \cap S_j$, $i, j \in J^\circ$, is empty, while $G_i \cap G_j$, $i, j \in J$, and $G_i \cap S_j$, $i \in J$, $j \in J^\circ$, are either empty or reduced to a vertex of γ.*

Proof. Let $A_0, A_1, \ldots A_n$ be the set of vertices of $\gamma = [A_0, A_n]$. Denote by I the set of integers $i \in [0 \ldots n-1]$ for which the segment $[A_i, A_{i+1}]$ is non singular and let I° be the complement of I in $[0 \ldots n-1]$. For each $i \in I$ let G_i^0 be the gallery from A_i to A_{i+1} in X, which can be defined in this context as the reunion the triangles of X whose interior intersects $]A_i, A_{i+1}[$. We call the set

$$n_\gamma = \bigcup_{i\in I} G_i^0 \cup \bigcup_{i\in I^\circ} [A_i, A_{i+1}]$$

the *nerve* of the simplicial convex hull of γ. Note that G_i^0, $i \in I$, is a flat disk satisfying property (1) of the lemma. Let J° the set of $i \in I^\circ$ such that $i - 1 \notin I^\circ$. For $j \in J^\circ$ we denote by S_j the reunion of segments $[A_i, A_{i+1}]$ for $i \in I^\circ$ such that $[j, j+1, \ldots i] \subset I^\circ$.

Call a vertex A_i, $i \in I$, regular if $i-1 \in I$ and if the distance in the link L_i of A_i in X between the two edges corresponding to n_γ, say e_i and f_i, equals $2\pi/3$. Let I_r be the set of $i \in I$ such that A_i is regular and let J be the complement of I_r in I.

For every regular vertex A_i, $i \in I_r$ choose two edges h_i^0 and h_i^1 of the link L_i such that the family $\{e_i, h_i^0, h_i^1, f_i\}$ forms a connected path in L_i (there might be two such paths) and denote by t_i^0 and t_i^1 the triangles in X containing A_i and having h_i^0 and h_i^1 respectively as basis (where we identified the link L_i with the simplicial sphere of radius 1 in X).

Let $j \in J$. Consider the largest integer $k < n$ such that for all integer $i < n$ with $j < i \leq k$ one has $i \in I$ and the vertex A_i is regular. Consider the set G_j^1 defined as

$$G_j^1 = \bigcup_{j<i\leq k} G_i^0 \;\cup\; \bigcup_{j<i\leq k} \{t_i^0 \cup t_i^1\}.$$

It is easy to see that G_i^1 is a flat disk which satisfies (1). Denote $B_j = A_{k+1}$ and fix, for every $j \in J$, an isometry φ_j between G_j^1 and a closed disk F_i of $\mathbf{R}^2$. Let also $\tilde{A}_j$ and $\tilde{B}_j$ be the points in $\mathbf{R}^2$ corresponding to A_j and B_j under this isometry and note that F_i is included in the simplicial convex closure E_j of $\tilde{A}_j$ and $\tilde{B}_j$ in $\mathbf{R}^2$ (which is a parallelogram).

Let $\mathcal{F}_j$ be the (finite) set of closed disk of $\mathbf{R}^2$ containing F_j and which are reunion of minimal gallery from $\tilde{A}_j$ to $\tilde{B}_j$ (so every disk in $\mathcal{F}$ is a subset of E_i). Consider the set $\mathcal{F}_j^0$ of elements of $\mathcal{F}_j$ which admit an isometric embedding in X which coincide with φ_j^{-1} on F_j. For each j choose a disk in $\mathcal{F}_j^0$ which has the maximal number of triangles and denote by G_j its corresponding embedding in X. We will show that the closed subset

$$C = \bigcup_{j\in J} G_j \cup \bigcup_{j\in J^\circ} S_j$$

of X coincide with the convex closure of γ.

Let $\mathcal{H}_0$ be the set of simplicial geodesics of C from A_0 to A_n and $\mathcal{H}_1$ be the set (which may be infinite a priori) of simplicial geodesics of X from A_0 to A_n which are not included in C. We will show that $\mathcal{H}_1$ is empty. Note that by construction every simplicial geodesic of C from A_0 to A_n is the reunion of a simplicial geodesic in the flat disk G_j, $j \in J$ and the CAT(0) geodesic S_j, $j \in J^\circ$. For every $g_0 \in \mathcal{H}_0$ and $g_1 \in \mathcal{H}_1$ there exists by the CAT(0) property a finite family $\mathcal{D} = \{D_0, \ldots, D_m\}$ of non empty topological disks of X with disjoint interiors, which are union of triangles, which are filling $g_0 \cup g_1$ in the sense that the subset $(g_0 \cup g_1) \bigcup (D_0 \cup \ldots \cup D_m)$ of X is contractile, and such that for every $i = 0 \ldots m$ the intersection of the topological boundary ∂D_i of D_i with $g_0 \cap g_1$ consists of two points. Moreover up to modifying g_0 among elements of $\mathcal{H}_0$ one can choose (for every $g_1 \in \mathcal{H}_1$) a g_0 such that the interior of $D_0 \cup \ldots \cup D_m$ is disjoint from the interior of C. Let $\mathcal{A}$ be the set of triples $(g_0, g_1, \mathcal{D})$ satisfying these conditions (so $\mathcal{A} \to \mathcal{H}_1$ which maps $(g_0, g_1, \mathcal{D})$ to g_1 is surjective).

Assume that $\mathcal{H}_1$ is non empty and pick a $(g_0, g_1, \mathcal{D})$ in $\mathcal{A}$ such that the number of triangles of $\mathcal{D}$ is minimal among all elements of $\mathcal{A}$. Let D be a disk in $\mathcal{D}$. By construction the topological boundary ∂D of D is included in $g_0 \cup g_1$.

Let s be a vertex of $\partial D \cap g_0$ which does not belong to $g_0 \cap g_1$. Let us show that the internal angle θ_s of D at s is at least π. By (1) $\theta_s \neq \pi/3$ so $\theta_s \geq 2\pi/3$. Assume that $\theta_s = 2\pi/3$ and denote by (x, s, t) and (t, s, y) the corresponding triangles in D, where $[x, s]$ and $[s, y]$ are two consecutive edges of g_0 because $s \notin g_0 \cap g_1$. By definition of S_j, $j \in J^\circ$, the point s does not belong to S_j (neither its interior nor its extremities) as this would contradict the fact that γ is geodesic and the definition of C. In particular $s \in G_j$ for some $j \in J$ which in turn implies that $s \in \partial G_j$. Indeed otherwise the path $[x, t] \cup [t, y]$ would create with two edges of C a cycle of length at most $\pi + 2\pi/3 < 2\pi$ in the link of s in X, contradicting the CAT(0) property. Furthermore one has $s \neq A_j$

and $s \neq B_j$ as otherwise (one at least of) these points would be regular. It follows that $[x, s] \cup [s, y]$ is included in the boundary of G_j. Now by (1) the internal angle of G_j at s is at most $4\pi/3$ so as X is CAT(0) this angle exactly equals $4\pi/3$. It follows that the disk $\tilde{G}_j = G_j \bigcup (x, s, t) \cup (t, s, y)$ belongs to $\mathcal{F}_j^0$, which contradicts the maximality of G_j. Hence $\theta_s \geq \pi$.

Let now s be a vertex of $\partial D \cap g_1$ which does not belong to $g_0 \cap g_1$ and let us show that the internal angle θ_s of D at s is at least π as well. As g_1 is a simplicial geodesic of X one has $\theta_s \geq 2\pi/3$, so we assume that $\theta_s = 2\pi/3$ and argue towards a contradiction. Denote by (x, s, t) and (t, s, y) the corresponding triangles in D so that $[x, s]$ and $[s, y]$ are two consecutive edges of g_1 because $s \notin g_0 \cap g_1$. Up to permuting x and y one can write $g_1 = h_0 \cup [x, s] \cup [s, y] \cup h_1$ where h_0 is a simplicial geodesic in X from A_0 to x and h_1 is a simplicial geodesic in X from y to A_n. Let $\tilde{g}_1 = h_0 \cup [x, t] \cup [t, y] \cup h_1$. As

$$\ell(\tilde{g}_1) \leq \ell(h_0) + \ell(h_1) + 2 = \ell(g_1)$$

the path $\tilde{g}_1$ is a simplicial geodesic from A_0 to A_n in X (where $\ell(g)$ denotes the simplicial length of g). Let $\tilde{D} = D \backslash \{(x, s, t) \cup (t, s, y)\}$ and let $\tilde{\mathcal{D}}$ be the reunion of $\tilde{D}$ and the disks in $\mathcal{D}$ which are distinct from D. Then $(g_0, \tilde{g}_1, \tilde{\mathcal{D}})$ is an element of $\mathcal{A}$ so by minimality of $(g_0, g_1, \mathcal{D})$ we get $\tilde{g}_1 \subset C$ and as the interiors of disks in $\mathcal{D}$ are disjoint from C it follows that $g_0 = \tilde{g}_1$. However this implies that the point $t \in g_0$ has an internal angle in D of $4\pi/3$, which contradicts what was established in the previous paragraph. So $\theta_s \geq \pi$.

It follows that the disk D has internal angles at every point $s \in \partial D$ at least π, except perhaps at the two points $\partial D \cap g_0 \cap g_1$. But this is a contradiction, no such a disk can exist in a CAT(0) space. Thus $\mathcal{H}_1$ is empty and it follows that

$$\mathrm{Conv}(\gamma) \subset C.$$

Now it easy to show that for any three vertices of a triangle in C there are geodesics of C from A_0 to A_n which contains these vertices (it is sufficient to prove the assertion for the flat G_j, which is easy). Moreover a geodesic of X from A_0 to A_n, say of length ℓ_0, is included in C by the above and so is a geodesic of C. Thus all geodesic of C have length ℓ_0. It follows that $\mathrm{Conv}(\gamma) = C$ and the lemma is proved. □

Lemma 27. *Let γ be a geodesic segment between two vertices A and B of X and let I be a vertex of* $\mathrm{Conv}(\gamma)$. *Then the simplicial convex closure of the geodesic segment $[A, I]$ is included in* $\mathrm{Conv}(\gamma)$ *and the reunion of any two simplicial geodesics in X from A to I and I to B respectively is a simplicial geodesic in X from A to B.*

Proof. By definition of the convex closure I belongs to a simplicial geodesic g of X from A to B, so $g = g_0 \cup g_1$ where g_0 and g_1 are simplicial geodesic of X from A to I and I to B respectively. Now for any geodesic g_0' and g_1' of X from A to I and I to B respectively we have $\ell(g_0' \cup g_1') \leq \ell(g_0) + \ell(g_1) = \ell(g)$ so $g_0' \cup g_1'$ is a simplicial geodesic of X from A to B and $g_0', g_1' \subset \mathrm{Conv}(\gamma)$. □

Recall that a subset D of X is said to be convex if for any two points $A, B \in D$ the geodesic segment $[A, B]$ from A to B is included in D (see [13]).

Lemma 28. *Let γ be a geodesic segment between two vertices of X. Then* $\mathrm{Conv}(\gamma)$ *is convex.*

Proof. Let A, B be two points of $\mathrm{Conv}(\gamma)$ and suppose that the geodesic segment $[A, B]$ is not included in $\mathrm{Conv}(\gamma)$. Denote by A' the closest point from A in $]A, B]$ which does not belong to the interior of $\mathrm{Conv}(\gamma)$, and denote by B' the closest point from A' in $]A', B]$ which belongs to $\mathrm{Conv}(\gamma)$. Note that both A' and B' are on the 1-skeleton of X so we can choose a simplicial geodesic path g from A' to B' inside $\mathrm{Conv}(\gamma)$. Let D be the unique disk in X with boundary $g \cup [A', B']$ so that, as in

Lemma 26, the internal angle of D at every point of g distinct from A' and B' is at least π. Since $[A', B']$ is a CAT(0)-geodesic the internal angle of D at every point of $]A', B'[$ is at least π as well so we get a contradiction. Thus $[A, B] \subset \mathrm{Conv}(\gamma)$. $\square$

Lemma 29. *Let A, B, C be three vertices of X and let Δ be the geodesic triangle of X with vertices A, B and C. Assume that the angle of Δ at A, B and C are non zero and denote by D the unique closed topological disc in X whose boundary is Δ. Let S_Δ be the reunion of the simplicial convex closures of the segments $[A, B]$, $[B, C]$ and $[A, C]$. Then the following assertions holds.*

1. *If $D \backslash S_\Delta$ is empty there is a point I in D and simplicial segments γ_{AI}, γ_{BI} and γ_{CI} from A to I, B to I and C to I respectively such that the simplicial paths $\gamma_{AI} \cup \gamma_{BI}$, $\gamma_{BI} \cup \gamma_{CI}$ and $\gamma_{AI} \cup \gamma_{CI}$ are simplicial geodesic segments of X from A to B, from B to C and from A to C respectively.*
2. *If $D \backslash S_\Delta$ is not empty then there exist a non empty flat equilateral triangle $T = (A', B', C')$ in D whose edges $\gamma_{A'B'}$, $\gamma_{B'C'}$ and $\gamma_{A'C'}$ are singular simplicial geodesic between three distinct vertices A', B' and C' of D, and three simplicial geodesic segments $\gamma_{AA'}, \gamma_{BB'}$ and $\gamma_{CC'}$ in X such that the simplicial paths $\gamma_{AA'} \cup \gamma_{A'B'} \cup \gamma_{B'B}$, $\gamma_{BB'} \cup \gamma_{B'C'} \cup \gamma_{C'C}$ and $\gamma_{AA'} \cup \gamma_{A'C'} \cup \gamma_{C'C}$ are simplicial geodesic segments in X from A to B, from B to C and from A to C respectively.*

Proof. Let S_{AB}, S_{BC} and S_{AC} be the simplicial convex closures of $[A, B]$, $[B, C]$ and $[A, C]$, so $S_\Delta = S_{AB} \cup S_{BC} \cup S_{AC}$, and let n_{AB}, n_{BC} and n_{AC} be their respective nerves. Assume first that $D \backslash S_\Delta$ is empty and let us distinguish the following two cases.

Suppose first that the disk D is included in the reunion of the simplicial convex closures of two of the edges of its boundary, say $D \subset S_{AB} \cup S_{AC}$. In particular $[B, C] \subset S_{AB} \cup S_{AC}$. As simplicial convex closures are unions of simplexes of X, the nerve n_{BC} of S_{BC} is included in $S_{AB} \cup S_{AC}$ and $n_{BC} \cap S_{AB} \cap S_{AC}$ is a non empty union of simplexes. Therefore we can find a vertex I in n_{BC} which belongs to $S_{AB} \cap S_{AC}$. Choose two simplicial geodesics γ_{BI} and γ_{CI} in (the boundary of) n_{BC} from B to I and C to I respectively. It follows from Lemma 26 that γ_{BI} and γ_{CI} are simplicial geodesic in X from B to I and C to I respectively. Consider a simplicial geodesic γ_{AI} in X from A to I. As I belongs to both S_{AB} and S_{AC} one has that $\gamma_{AI} \subset S_{AB} \cap S_{AC}$ and that $\gamma_{AI} \cup \gamma_{BI}$ and $\gamma_{BI} \cup \gamma_{CI}$ are simplicial geodesic from A to B and B to C respectively, as follows from Lemma 27. This shows that (1) holds in that case.

Suppose now that D is not included in the reunion of the simplicial convex closure of any two edges of its boundary. In particular the closure D' of $D \backslash (S_{AB} \cup S_{AC})$ has non empty interior. We claim that there is a point I in the interior of D which belongs to $\partial D' \cap S_{AB} \cap S_{AC}$. Indeed assume there is no such a point. Then every point of $P = \partial D' \backslash [B, C]$ is either in $S_{AB} \backslash S_{AC}$ or in $S_{AC} \backslash S_{AB}$. This implies that every connected component of P is entirely included in $S_{AB} \backslash S_{AC}$ or in $S_{AC} \backslash S_{AB}$ (as $P = (P \cap S_{AB}) \amalg (P \cap S_{AC})$ and both sets are closed in P). It follows that P is connected, as any disk whose boundary is included in a given convex closure, say S_{AB}, is actually entirely included in S_{AB}. So suppose for instance that P is included in $S_{AB} \backslash S_{AC}$. Then the closure of P is included in S_{AB} and so the geodesic $D' \cap [B, C]$ is included in S_{AB} as well by Lemma 28. Hence $\partial D' \subset S_{AB}$ which implies $D' \subset S_{AB}$ and contradicts the fact that D' is non empty. This proves the claim. So let $I \in \partial D' \cap S_{AB} \cap S_{AC}$ (in fact this point is unique). Since D' is non empty and $D \backslash S_\Delta$ is empty, one has $I \in S_{BC}$. Now choose simplicial geodesic γ_{AI}, γ_{BI} and γ_{CI} of X, from A to I, B to I and C to I respectively. Lemma 27 readily implies that the required conditions in (1) are satisfied. This conclude the proof of (1).

So for the remaining part of the proof, we assume that $D \backslash S_\Delta \neq \emptyset$.

Denote by $n_\Delta = n_{AB} \cup n_{BC} \cup n_{AC}$ the nerve of S_Δ. Note that every triangle t of n_{AB} is divided into two parts by the geodesic $[A, B]$ and that exactly one of this part has a non trivial intersection t^+ with the interior of D, and similarly for n_{BC} and n_{AC}. We write n_Δ^0 for the subset of n_Δ defined by

$$n_\Delta^0 = \bigcup_{s \subset n_\Delta} s \ \cup \ \bigcup_{t \subset n_\Delta} \overline{t^+}$$

where s runs over the singular segments of n_Δ and t over all its triangles, and $\overline{t^+}$ is the closure of t^+. Then the closure D_0 of $D \backslash n_\Delta$ is a closed topological disk included in D which is a non empty union of closed triangles and whose boundary ∂D_0 consists of the points in the boundary of n_Δ^0 which are in the interior of D, and the singular segments of n_Δ^0.

Let us construct by induction a finite decreasing sequence $D_0 \supset D_1 \supset \ldots \supset D_m$, $m \in \mathbf{N}$, of closed topological disks which are non empty unions of closed triangles such that the two following conditions, henceforth referred to as Property P_k, are satisfied for every non negative integer $k \leq m$:

(P_k^0) the pairwise intersection of the simplicial segments

$$\gamma_{AB}^k = \partial D_k \cap S_{AB}, \ \gamma_{BC}^k = \partial D_k \cap S_{BC} \text{ and } \gamma_{AC}^k = \partial D_k \cap S_{AC}$$

is reduced to a point, say,

$$\gamma_{AB}^k \cap \gamma_{AC}^k = A_k, \ \gamma_{AB}^k \cap \gamma_{BC}^k = B_k \text{ and } \gamma_{AC}^k \cap \gamma_{BC}^k = C_k,$$

(P_k^1) γ_{AB}^k, γ_{BC}^k and γ_{AC}^k are included into simplicial geodesic of X from A to B, from B to C and from C to A respectively.

and such that the internal angle of the disk D_m at every point $s \in \partial D_m$ distinct from A_m, B_m and C_m is at least π.

Note that Property P_0 is satisfied (P_0^1 is a consequence of Lemma 26). Assume the construction has been done up to some non negative integer k. If the internal angle of the disk D_k at every point $s \in \partial D_k$ distinct from A_k, B_k and C_k is $\geq \pi$ we set $m = k$ and stop the construction here. Otherwise there is an $s \in \partial D_k$ distinct from A_k, B_k and C_k whose internal angle θ_s in D_k is $\leq 2\pi/3$. Assume for instance that $s \in \gamma_{AB}^k$ (the case of γ_{BC}^k and γ_{AC}^k being similar). By Property P_k^1 there exists a simplicial geodesic segment g of X from A to B which contains γ_{AB}^k. In particular $\theta_s \neq \pi/3$, and so $\theta_s = 2\pi/3$. Denote by (x, s, t) and (y, s, t) the two corresponding triangles in D_k, where the two edges $[x, s]$ and $[s, y]$ are included in γ_{AB}^k. Up to permuting x and y we can write $g = g_1 \cup [x, s] \cup [s, y] \cup g_2$ so that $\ell(g) = \ell(g_1) + \ell(g_2) + 2$. Consider the simplicial curve $g' = g_1 \cup [x, t] \cup [t, y] \cup g_2$. Obviously $\ell(g') \leq \ell(g)$ so g' is a simplicial geodesic of X from A to B. There are two cases.

- Suppose that t belongs to the interior of D_k. Define

  $$D_{k+1} = D_k \backslash \{(x, s, t) \cup (y, s, t)\}$$

  and γ_{AB}^{k+1}, γ_{BC}^{k+1} and γ_{AC}^{k+1} to be the intersection with ∂D_{k+1} of g', of γ_{BC}^k and of γ_{AC}^k respectively. Then D_k is a topological disk which is a non empty reunion of triangles (as $D \backslash S_\Delta \neq \emptyset$) and property P_{k+1} is easily seen to be satisfied and we can iterate the construction.
- Suppose that t belongs to the boundary of D_k, say $t \in \gamma_{BC}^k$ for example. Denote by γ_{AB}^{k+1} the reunion of the portion of γ_{AB}^k between A_k and x and of the segment $[x, t]$, and by γ_{BC}^{k+1} the portion of γ_{BC}^k between t and C_k. Set $\gamma_{AC}^{k+1} = \gamma_{AC}^k$. Then the three paths γ_{AB}^{k+1}, γ_{BC}^{k+1} and γ_{AC}^{k+1} satisfies Property P_{k+1}^0 with $A_{k+1} = A_k$, $B_{k+1} = t$ and $C_{k+1} = C_k$, and they bound a disk D_{k+1} included in D_k. Since property P_{k+1}^1 is satisfied by construction, we only have to show that D_{k+1} is non empty. Assume towards a contradiction that it is

empty. Then t belongs to γ_{AC}^k and hence to the intersection $S_{AB} \cap S_{BC} \cap S_{AC}$. Choose simplicial geodesics of X from t to A, B and C, say to γ_{At}, γ_{Bt} and γ_{Ct} respectively. We have that $\gamma_{At} \cup \gamma_{Bt} \subset S_{AB}$ (Lemma 27) and so the disk $D_{AB} \subset D$ with boundaries $[A, B]$, γ_{At} and γ_{Bt} is included in S_{AB}. Arguing similarly for $[A, C]$ and $[B, C]$, we conclude that $D \backslash S_\Delta$ is empty, contrary to our standing assumption. Finally $D_{k+1} \neq \emptyset$, and this shows that we can iterate the construction in this case too.

Hence there exists a disk D_m satisfying Property P_m and whose internal angle of at every point $s \in \partial D_m$ distinct from A_m, B_m and C_m is at least π. Let us now prove that D_m defined above satisfies the conditions in Item (2) of the lemma. We need to show that D_m is a flat equilateral triangle.

Recall that for any open topological disk D in X with piecewise linear topological boundary Δ, one calls *geodesic curvature of* D at a point s of Δ is the number $\kappa_s = \pi - \theta_s$ where θ_s is the internal angle of D at s (note that κ_s is zero for all but a finite number of $s \in \Delta$). For a point x be a point in D, one calls *curvature of* D *at* x is defined as $\delta_x = 2\pi - \phi_x$, where ϕ_x the sum of the angles at x in D, i.e. the angular length of the circle in the link L_x of x in X defined by D (so the disk D is flat at x if and only if $\delta_x = 0$).

Denote by $T = \partial D_m$ the triangle with vertex A_m, B_m and C_m and simplicial edges γ_{AB}^m, γ_{BC}^m and γ_{AC}^m. By construction a point $s \in \partial D_m$ whose internal angle in D_m is $< \pi$, if it exists, is necessarily one of the three vertices A_m, B_m or C_m. This shows that the total geodesic curvature $\int_{\Delta_m} \kappa$ of the boundary Δ_m of D_m, that is the sum over all points $s \in \Delta_m$ distincts from A_m, B_m and C_m of κ_s, satisfies

$$\int_{\Delta_m} \kappa \leq 0.$$

Applying the Gauss-Bonnet Formula (for domains with piecewise linear boundary) to the disk D_m we get

$$(\pi - \theta_{A_m}) + (\pi - \theta_{B_m}) + (\pi - \theta_{C_m}) + \sum_{x \in D_m} \delta_x \;=\; 2\pi - \int_{\Delta_m} \kappa \geq 2\pi.$$

where θ_{A_m}, θ_{B_m} and θ_{C_m} are the internal angle of D_m at A_m, B_m and C_m respectively, and so

$$\theta_{A_m} + \theta_{B_m} + \theta_{C_m} - \sum_{x \in D_m} \delta_x \leq \pi.$$

Now the fact that D_m is simplicial (with non empty interior) implies that the values θ_{A_m}, θ_{B_m} and θ_{C_m} are at least $\pi/3$. It follows that $\theta_{A_m} = \theta_{B_m} = \theta_{C_m} = \pi/3$ and that the disk D_m is flat. Thus T is a flat equilateral triangles with simplicial edges. Furthermore Property P_m^1 implies that A_m belongs to $S_{AB} \cap S_{AC}$ and that $[A_m, B_m]$ is included in a simplicial geodesic in X from A to B. Fixing a simplicial geodesic γ_{AA_m} in X from A to A_m (which is included in $S_{AB} \cap S_{AC}$) and arguing similarly for B_m and C_m, we conclude that all conditions in (2) hold. This proves the lemma. □

We now prove Theorem 5.

Proof of Theorem 5. Let Γ be a group acting properly and isometrically on X and let $A_0 \in X$. Identify Γ to the orbit ΓA_0 and consider the length ℓ induced by the 1-skeleton of X. We prove that Γ has property RD with respect to ℓ by using Section 2 and the above lemmas.

Assume first the action of Γ to be free and transitive. For $z \in \Gamma$ and $r \in \mathbf{R}_+$ let U_z^r the set triples (a, u, b) with $z = bua$ such that the points aA_0, and uaA_0 belong to a simplicial geodesic from A_0 to zA_0 and such that the length of $[A_0, aA_0]$ and $[aA_0, uaA_0]$ is less than r. By Lemma 26 the family $U = (U_z^r)_{z \in \Gamma, r \in \mathbf{R}}$ has polynomial growth (in the sense of Definition 13). Let $\mathfrak{equ}$ be the family of equilateral triangles in Γ (i.e. the family of triangles (x, y, z) in Γ such that (A_0, xA_0, zA_0) is equilateral in X). Lemma 29 shows that $\mathfrak{equ}$ is a retract of the set of all triangles of Γ along U (see Definition 15, here $p_2(t) = t$). Indeed let A, B and C be three points of X and denote by ABC the corresponding geodesic triangle. Note that $[A, B] \cap [A, C]$ is either reduced to A or a geodesic segment of X. In the latter case we let A' be the extremity of this segment which is distinct from A. Otherwise set $A' = A$ and define similarly B' and C'. If $A' = B' = C'$ then ABC is a tripode triangle and it reduces to A' along (any choice of) simplicial geodesics from A' to A, B and C. If A', B' and C' do not coincide then we get a geodesic triangle $A'B'C'$ whose angles at A', B' and C' are non zero. Observe now that for any two vertices A', B' of X which are ordered $A \leq A' \leq B' \leq B$ on a given geodesic segment $[A, B]$ of X, the concatenation of any three simplicial geodesics from A to A', A' to B', and B' to B respectively is a simplicial geodesic from A to B. This together with Lemma 29 shows that $\mathfrak{equ}$ is a retract of the set of all triangles of Γ along U (the metric requirements on the retraction being readily satisfied from description of simplicial convex closure in Lemma 26). Hence by Lemma 15 property RD for Γ is equivalent to property RD for $\mathfrak{equ}$. Lemma 24 then shows that property RD for $\mathfrak{equ}$ is equivalent to property RD for the dual set $\mathfrak{equ}^*$ (see Section 2.3 and note that $\mathfrak{equ}$ is obviously balanced). As all triangles in $\mathfrak{equ}^*$ are tripods Lemma 15 applies.

As noted in [48] the same proof works for free isometric action provided we replace everywhere 'groups acting transitively' by 'transitive groupoids' (see Subsection 2.3). Next, as noted in [40, Section 3.1], the proof works also for proper isometric actions too provided we replace the space X by the disjoint union of all stabilizers Γ_A, $A \in X$, of the action of Γ on X, where each Γ_A is endowed with the complete graph structure and there is an edge between $\tilde{A} \in \Gamma_A$ and $\tilde{B} \in \Gamma_B$ if and only if there is an edge between A and B in X. Then Γ acts freely isometrically on $\tilde{X}$ (which is quasi-isometric to X) and the required polynomial growth conditions all are satisfied for $\tilde{X}$ because $\sup_{A \in X} \#\Gamma_A < \infty$, as is easily seen. $\square$

Note that we may reformulate Theorem 5 by saying that groups without property RD have no proper action on a triangle polyhedron. We don't know whether there exit groups that have "property T for triangle polyhedra targets", i.e. groups for which every isometric action on a triangle polyhedron has a bounded orbit (compare [32] and [64, 38, 47]).

Problem 30. Are there finitely generated infinite countable groups for which any action by isometry on a triangle polyhedron has a fixed point ?

We conjecture the answer to be positive.

Let us conclude this section with the fact that Definition 2 and Definition 16 coincide in case of triangle groups, which essentially follows from the above lemmas.

Proposition 31. *Let X be a triangle polyhedron. Then X has polynomial rank in the sense of Definition 2 if and only the space $(X^{(0)}, \ell)$ has polynomial growth rank in the sense of Definition 16, where ℓ is any length on the vertex set $X^{(0)}$ of X induced by the 1-skeleton of X from some base point $A_0 \in X^{(0)}$ (i.e. $\ell(A)$ is the simplicial distance between A_0 and $A \in X^{(0)}$).*

Proof. Assume that X doesn't have polynomial rank in the sense of definition 2 and let $(\gamma_{r_n})_{n \in \mathbf{N}}$ be a sequence of simplicial geodesic segments in X of length $r_n \in \mathbf{R}_+$ such that the growth of the number of flat equilateral triangle in X with base γ_{r_n} is faster than any given polynomial function of r_n. Then the edges of every equilateral triangle ABC of base γ_{r_n} are simplicial geodesic segment, so for any fixed $\delta > 0$, the set of δ-geodesics of ABC (for the simplicial structure) between the vertices A, B and C is in the δ-neighbourhood of the boundary of ABC. Hence letting T_n be the set of equilateral triangle of base γ_{r_n}, the set of retractions of triangles in T_n along any set of δ-geodesic as in Definition 16 contains as many triangles as T_n, up to a constant number depending only on δ. This shows that X does not have polynomial growth rank.

Conversely assume that X has polynomial rank in the sense of definition 2 and consider the retraction along the family C as in the proof of Theorem 5 (which consists of simplicial geodesic of X, i.e. $\delta = 0$). Then as shown in the lemmas above, retractions along C consist of flat simplicial equilateral triangles. In particular X has polynomial growth rank in the sense of Definition 16.

Note that this proof works as well for subexponential growth rank. □

4. Rank $\frac{7}{4}$

A triangle polyhedron is said to have *order* $q \in \mathbf{N}^*$ if each of its edges is contained into $q+1$ triangles. In this section $q = 2$. A graph is *ample* if the length of its smallest cycle is 6.

The following proposition is known in the sense that it is possible (e.g. using a computer) to classify all trivalent finite graphs of sufficiently small order. We give a direct proof below as it is the basis for local rank $\frac{7}{4}$ (we don't know of a reference where it is actually stated).

Proposition 32. *The smallest ample trivalent graph has 14 vertices. Moreover there exists a unique ample trivalent graph with*

(1) *14 vertices: this is the incidence graph L_2 of the Fano plane $P^2(\mathbf{F}_2)$.*
(2) *16 vertices: this is $L_{\frac{7}{4}}$ from the introduction.*
(3) *18 vertices: this is the graph $L_{\frac{3}{2}}$ of* [6].

(Recall that the number of vertices of a trivalent graph is $\frac{2}{3}\times$ the number of its edges and so is even.)

The graph L_2 is a spherical building. By [57] a triangle polyhedron of order 2 is an Euclidean building if and only if its link at each vertex is isomorphic to L_2, and there is a correspondence between local rank 2, i.e. links being spherical buildings, and global rank 2, i.e. being an affine building. We refer to [3, 4] for more details.

In [6] the first-named author considered the graph $L_{\frac{3}{2}}$ of item (3), constructed a compact complex P of dimension 2 with one vertex whose link is isometric to $L_{\frac{3}{2}}$ (see Section 1 in [6]), and proved the Haagerup property for its fundamental group. The universal cover $\tilde{P}$ of P has isolated flats and thus polynomial rank [7].

Proposition 32 shows that $L_{\frac{7}{4}}$ is in some sense the 'canonical' graph which occupies an intermediate position between rank $\frac{3}{2}$ to rank 2.

Definition 33. A triangle polyhedron is said to be of rank $\frac{7}{4}$ if the link at each of its vertices is $L_{7/4}$. A triangle group is said to be of rank $\frac{7}{4}$ if it admits a proper cocompact action on a polyhedron of rank $\frac{7}{4}$. We call *complex of rank* $\frac{7}{4}$ a compact CW-complex of dimension 2 with equilateral faces and whose universal cover is polyhedron of rank $\frac{7}{4}$ (for the usual piecewise linear metric). A complex of rank $\frac{7}{4}$ is said to be *orientable* if there is a coherent orientation of its faces.

Remark 34. To further interpolate the local rank between $\frac{3}{2}$, $\frac{7}{4}$, and 2, one can (for instance) increase the order q (that will not be studied in the present paper), or mix together various links at vertices whenever possible (see Section 6).

Proof of Proposition 32. Let L be an ample trivalent graph, n be the number of vertices and $m = \frac{3}{2}n$ be the number of edges. The universal cover T of L is the standard trivalent tree. Fix a vertex $*$ in T and let B_3 be the open ball of center $*$

and radius 3 in T. The ampleness condition shows that B_3 is included in a (connected) fundamental domain F of the action of the fundamental group $\Gamma = \pi_1(L)$ on T. Thus $m \geq 21$ and $n \geq 14$.

Assume $n = 14$. Then the 12 vertices in the boundary ∂B_3 of B_3 must be identified 3 by 3 without creating cycles of length < 6. One readily checks that there is indeed a possibility (namely L_2) which is unique up to isomorphism.

In the case $n = 16$ the ball B_3 contains all but 3 edges of F, so B_3/Γ has 15 vertices and $\partial B_3/\Gamma$ consists of 5 vertices: 2 of valence 3 (denoted A and B) and three of valence 2 (in B_3/Γ). Write $*'$ for the vertex of F which is not in B_3 and call branch a connected component of $B_3\backslash\{*\}$. Then any two vertices $*_A$ and $*_B$ projecting down to A and B respectively cannot be at distance 2 in a same branch (otherwise identifications of the two other extremities of this branch with $*'$ under Γ would create a small cycle). Thus each branch must contain (points in the orbits of) $*_A$ and $*_B$ at a distance equaling 4. But then the remaining identifications from a branch to the other are uniquely determined up to isomorphism. The only possibility is $L_{\frac{7}{4}}$.

Assume that $n = 18$. Then 6 edges are left and hence 2 vertices $*'$ and $*''$. In this case the 12 vertices of ∂B_3 are glued together so to have valency 2 in B_3/Γ and each of them is further glued to $*'$ or $*'$. In fact the 6 vertices of $\partial B_3/\Gamma$ must be in a same cycle of length 12 (in B_3/Γ) together with the 6 vertices of $\partial B_2/\Gamma$. Indeed it is not possible to have 2 cycles of length 6 as this would create a cycle of length 4 with one of the vertices of $\partial S_1/\Gamma$. Thus the graph B_3/Γ is indeed symmetric enough to have a unique way up to isomorphism to complete it into a trivalent ample graph with 18 vertices. The only possibility here is $L_{\frac{3}{2}}$ from [6]. $\square$

4.1. **Existence and classification results.** In this section we prove Theorem 35 stated below. More precisely, we prove all the assertions of this theorem except for the fact that there are *at most* 12 complexes of rank $\frac{7}{4}$ with one vertex, which is postponed until Subsection 4.4 (this is a computer assisted proof). In course of the proof we give an explicit description of all rank $\frac{7}{4}$ complexes appearing in item (1). The verification that these complexes are indeed of rank $\frac{7}{4}$ will be straightforward by checking (a), (b) and (c) below.

Theorem 35. *Let V be CW-complex of dimension 2 with triangular faces. Assume that*

(a) *V has 8 faces,*
(b) *each edge of V is incident to 3 faces,*
(c) *the link at each vertex of V is ample.*

Then V is a complex of rank $\frac{7}{4}$ with one vertex. In particular $\pi_1(V)$ is a triangle group of rank $\frac{7}{4}$. Moreover,

(1) *there are precisely 12 orientable complexes of rank $\frac{7}{4}$,*
(2) *the homology group $H_1(V, \mathbf{Z})$ of such a V can have rank 0 (i.e. can be torsion), rank 1 and rank 2.*

In particular there exist groups of rank $\frac{7}{4}$ which do not have Kazhdan's property T.

Proof. Let V be CW-complex of dimension 2 with triangular faces. Let us first show that (a), (b) and (c) implies that V is a complex of rank $\frac{7}{4}$ with one vertex. Let $\{s_1, \ldots s_n\}$ be the vertex set of V. By (b) and (c) the link L_{s_i} at each vertex s_i, $i = 1 \ldots n$, is an ample trivalent graph, so Proposition 32 implies that $|L_{s_i}^{(0)}| \geq 14$ for every $i = 1 \ldots n$, where $L_{s_i}^{(0)}$ is the vertex set of L_{s_i}. By (a) one has $|L_{s_i}^{(1)}| \leq 24$, denoting by $L_{s_i}^{(1)}$ the edge set of L_{s_i}. So Proposition 32 again implies $L_{s_i} = L_2$ or

$L_{s_i} = L_{\frac{7}{4}}$. This forces $n = 1$ which in turn implies $L_{s_1} = L_{\frac{7}{4}}$ by (a) again. Thus V is a complex of rank $\frac{7}{4}$ with one vertex.

To construct such a complex we thus have to understand how 8 triangles can be glued together on a base point $*$ so that the link L_* at $*$ is a trivalent ample graph. This should be compared to the case of triangle buildings as derived in [15] and [3].

So consider a bouquet B_8 of 8 oriented circles on which these triangles will be glued along their edges. Such a triangle t will be denoted $[x, y, z]$ where x, y, z are numbers in $\{1, \ldots, 8\} \cup \{1^-, \ldots, 8^-\}$ corresponding to the circles in B_8 on which its consecutive (for some fixed orientation of t) edges are attached. A minus sign occurs when the orientation of the circle in B_8 on which the edge of t is glued is opposite to that of t. Thus a complex of rank $\frac{7}{4}$ is entirely described by a list of 8 triples $[[x_1, y_1, z_1], \ldots [x_8, y_8, z_8]]$ with $x_i, y_i, z_i \in \{1, \ldots, 8\} \cup \{1^-, \ldots, 8^-\}$. Furthermore, condition (b) is equivalent to the fact that each number in $\{1, \ldots, 8\}$ appears exactly 3 times. This 8-list is called a presentation of the space V.

Assume that V is orientable. Then it has a presentation as above with $x_i, y_i, z_i \in \{1, \ldots, 8\}$. By the first part of Theorem 35 the only condition to check for V to be a complex of rank $\frac{7}{4}$ is that L_* is an ample graph. This can be further simplified by the following lemma.

Lemma 36. *If L_* has no cycle of length 2 and 4, then L_* is ample.*

Proof of Lemma 36. Let $[[x_1, y_1, z_1], \ldots [x_8, y_8, z_8]]$ be a presentation of V with x_i, y_i, $z_i \in \{1, \ldots, 8\}$. Fix some small ε_0 so that the link L_* coincide with the sphere of radius ε_0 and center $*$ in V. For every $i \in \{1, \ldots 8\}$ let $i^\flat$ and $i^\sharp$ be the two points (as order by the orientation in B_8) of the i-th circle of B_8 at distance ε_0 from $*$. The edges in L_* are then $[x_i^\flat, y_i^\sharp]$ $[y_i^\flat, z_i^\sharp]$ and $[z_i^\flat, x_i^\sharp]$ for $i = 1 \ldots 8$. In particular there is no cycle of length 3 or 5. $\square$

Thus, the classification of orientable complexes of rank $\frac{7}{4}$ can done in two steps:

(α) list all admissible presentations, i.e. presentations for which $x_i, y_i, z_i \in \{1, \ldots, 8\}$ and each number in $\{1, \ldots, 8\}$ appears exactly 3 times, and
(β) check in each case that the corresponding link has no cycle of length 2 and 4.

Some more details concerning the implementation of this procedure will be given in Section 4.4. After extensive computations we get the following list of the 12 complexes of rank $\frac{7}{4}$ announced in (1), together with their first homology groups and their fundamental groups. They are coming in 5 classes, according to the number of *adjacent identifications* in their link (equivalently the number of $[x, x, \cdot]$ in their presentation). The most symmetric ones (in particular V_0 all of whose identifications occur at distance 3 in L_*) can be found by hand from the link.

Type I (no adjacent identification). There are four orientable complexes in this class:

$$V_0 = [[1,2,6],[2,3,7],[3,4,8],[4,5,1],[5,6,2],[6,7,3],[7,8,4],[8,1,5]]$$
$$V_0^1 = [[1,2,3],[1,4,5],[1,6,4],[2,6,8],[2,8,5],[3,6,7],[3,7,5],[4,8,7]]$$
$$V_0^2 = [[1,2,3],[1,4,5],[1,6,7],[2,4,6],[2,8,5],[3,6,8],[3,7,5],[4,8,7]]$$
$$\check{V}_0^2 = [[1,2,3],[1,4,5],[1,6,7],[2,6,4],[2,8,5],[3,6,8],[3,7,5],[4,8,7]]$$

with respective first homology groups $H_1(\,\cdot\,, \mathbf{Z})$:

$$\mathbf{Z}/15\mathbf{Z}, \qquad \mathbf{Z}/3\mathbf{Z} \times \mathbf{Z}^2, \qquad (\mathbf{Z}/3\mathbf{Z})^3, \text{ and } \qquad (\mathbf{Z}/3\mathbf{Z})^3$$

Note that V_0 is the only polyhedron satisfying the additional following condition: for any $x = 1 \ldots 8$, there is $y = 1 \ldots 8$ such that $[x, y, \cdot]$ and $[y, x, \cdot]$ are faces. The fundamental group $\pi_1(V_0^1)$ is a group of rank $\frac{7}{4}$ with two generators s, t and two relations (where $t^s = s^{-1}ts$):

$$\pi_1(V_0) = \langle s, t \mid tst^2s^{-1}t = st^2st^2t^s,\ st^{-1}s^{-1}t^{-2}s = tst^2t^sst\rangle.$$

In the other cases the fundamental group has 3 generators and 3 relations:

$$\pi_1(V_0^1) = \langle u, v, w \mid u = vw^{-1}v^{-1}uw,\ uv = wvwuw,\ uvuw = wuvu\rangle.$$
$$\pi_1(V_0^2) = \langle u, v, w \mid uv = wvwuw,\ wu = uvwv^2,\ v = uvw^{-1}uwu\rangle.$$
$$\pi_1(\check{V}_0^2) = \langle u, v, w \mid uv = wvwuw,\ wu = vuvwv,\ wu^2v = vu^{-1}w\rangle.$$

Type II (one adjacent identification). A single orientable polyhedron in this class:

$$V_1 = [[1,1,2],[1,3,4],[2,5,6],[2,7,8],[3,5,7],[3,6,5],[4,6,8],[4,8,7]]$$

with $H_1(V_1, \mathbf{Z}) = \mathbf{Z}/3\mathbf{Z} \times \mathbf{Z}$, and

$$\pi_1(V_1) = \langle s, t \mid s^4ts^{-3}ts = ts^2t,\ t = s^2ts^{-1}t^{-1}s^2t^{-1}s^{-2}t^2s^2\rangle.$$

Type III (2 adjacent identifications). There are four orientable complexes in this class:

$$V_2^1 = [[1,1,3],[2,2,3],[1,4,5],[2,7,8],[3,5,7],[4,6,8],[4,7,6],[5,8,6]]$$
$$V_2^2 = [[1,1,3],[2,2,4],[3,7,4],[1,4,6],[2,5,3],[5,7,8],[5,8,6],[6,8,7]]$$
$$V_2^3 = [[1,1,3],[2,2,4],[1,5,2],[3,6,4],[3,7,6],[4,6,8],[5,7,8],[5,8,7]]$$
$$V_2^4 = [[1,1,3],[2,2,4],[1,5,2],[3,6,5],[3,7,8],[4,5,8],[4,6,7],[6,8,7]]$$

with respective first homology groups:

$$\mathbf{Z}/24\mathbf{Z}, \qquad (\mathbf{Z}/3\mathbf{Z})^2, \qquad \mathbf{Z}/3\mathbf{Z} \times \mathbf{Z}, \quad \text{and} \quad \mathbf{Z}/24\mathbf{Z}$$

and respective fundamental groups

$$\pi_1(V_2^1) = \langle s, t \mid st^2 = tst^2s^{-1}ts^2,\ ts^{-1}t^2 = s^3ts^2t^{-2}st^{-1}sts^2\rangle$$
$$\pi_1(V_2^2) = \langle s, t \mid s^2 = t^2s^4t^{-3}st,\ st = t^2s^{-2}t^{-4}s^3\rangle$$
$$\pi_1(V_2^3) = \langle s, t \mid s^2t^3s^3t^2 = t^2s^2,\ t^2 = s^2t^2s^2t^2s^{-2}ts\rangle$$
$$\pi_1(V_2^4) = \langle s, t \mid s^2t^2sts^2 = t^3st,\ s = t^3stst^{-1}s^2tst\rangle$$

Type IV (3 adjacent identifications). There is a single orientable complex in this class:

$$V_3 = [[1,1,4],[2,2,4],[3,3,5],[1,3,6],[2,5,7],[4,7,8],[5,8,6],[6,8,7]]$$

with $H_1(V_3, \mathbf{Z}) = \mathbf{Z}/6\mathbf{Z}$ and $\pi_1(V_3) = \langle s, t \mid st^3st = t^2sts^2,\ s^2 = t^2sts^{-2}ts^{-1}t^2st^3\rangle$.
Type V (4 adjacent identifications). Two orientable complexes in this class:

$$V_4^1 = [[1,1,5],[2,2,5],[3,3,6],[4,4,6],[1,3,8],[2,7,4],[5,8,7],[6,7,8]]$$
$$V_4^2 = [[1,1,5],[2,2,5],[3,3,6],[4,4,7],[1,3,8],[2,7,6],[4,8,6],[5,8,7]]$$

with $H_1(V_4^1, \mathbf{Z}) = (\mathbf{Z}/2\mathbf{Z})^2 \times \mathbf{Z}/12\mathbf{Z}$ and $H_1(V_4^2, \mathbf{Z}) = \mathbf{Z}/66\mathbf{Z}$, and

$$\pi_1(V_4^1) = \langle u, v, w \mid v^2 = u^2,\ w^2uw = uwu^2,\ w^2vuwvu^3wu^2 = e\rangle$$
$$\pi_1(V_4^2) = \langle s, t \mid t^2st^3ststs^2 = e,\ t^2s^{-2}t^{-1}s^{-1}t^2 = s^3ts^2\rangle$$

This, together with Subsection 4.4, concludes the proof of Theorem 35. □

Remark 37 (compare [28]). The first non zero eigenvalue of $L_{\frac{7}{4}}$ is

$$\lambda_1(L_{\frac{7}{4}}) = 1 - \frac{1}{\sqrt{3}} = 0.42... < \frac{1}{2}.$$

Indeed let σ_i, $i = 0, \dots, 7$ be the i-th matrix of cyclic permutation of the set of 8 elements ($\sigma_0 = \mathrm{Id}$). The random walk operator D on L in the canonical basis of vertices has the form

$$\frac{1}{3}\begin{pmatrix} 0 & A \\ A^t & 0 \end{pmatrix}$$

where A is the 8×8 matrix defined by $A = \mathrm{Id} + \sigma_2 + \sigma_7$. One has $AA^t = A^tA = 2\mathrm{Id} + P + \sigma_4$ where $P^2 = 8P$. Thus the eigenvalues of AA^t are 1, 3 and 9 and those of D are $\pm\frac{1}{3}, \pm\frac{1}{\sqrt{3}}$ and ± 1 (of order 3, 4 and 1 respectively).

Remark 38. Here is an example of a complex of rank $\frac{7}{4}$ which is not orientable:

$$\bar{V} = [[3, 1^-, 2], [3, 2^-, 4], [2, 6, 3^-], [5, 1^-, 6^-], [7, 4^-, 5], [8, 6^-, 7], [5, 8, 7], [1, 4, 8^-]]$$

It has a torsion-free $H_1(\bar{V}, \mathbf{Z}) = \mathbf{Z}$ and

$$\pi_1(\bar{V}) = \langle s, t \mid s^2t = t^2s^2t^2s^{-1}t^{-1}st^{-1}s,\ t = sts^{-1}ts^{-1}t^2s^{-1}t^{-2}st^{-1}s \rangle.$$

Remark 39. Some comments on the ℓ^2 invariants of polyhedra of rank $\frac{7}{4}$. Let V be a complex of rank $\frac{7}{4}$ with one vertex. Then an immediate computation gives

$$\chi(V) = 1$$

where $\chi(V)$ is the Euler characteristic of V. Thus, writing $\Gamma = \pi_1(V)$ and denoting by $\beta_0(\Gamma), \beta_1(\Gamma), \dots$ the ℓ^2-Betti numbers of Γ (see [31, 44]) we have that

$$\beta_2(\Gamma) = \beta_1(\Gamma) + 1 \geqslant 1,$$

while all other ℓ^2 Betti numbers vanish identically because Γ is infinite of dimension 2. Note then that the Atiyah conjecture [44], if true for for triangle groups, implies that $\beta_1(\Gamma) = 0$ and so $\beta_2(\Gamma) = 1$. Thus the ℓ^2 invariants of Γ are (presumably) identical to that of transitive triangle buildings (i.e. the rank 2 case), where one knows that $\beta_1 = 0$ because of property T.

4.2. **Homogeneity and structure of local flats.** We now study the local rank structure in rank $\frac{7}{4}$ polyhedra.

Endow $L_{\frac{7}{4}}$ with the uniform length (edges have length 1). The following concept is important to describe the local behavior of flats of rank $\frac{7}{4}$ polyhedra.

Definition 40. Let α, β be two vertices of $L_{\frac{7}{4}}$ at distance 3 in $L_{\frac{7}{4}}$. The couple (α, β) is said to be *of type* $\frac{3}{2}$ if there are exactly two distinct simplicial paths of length 3 in $L_{\frac{7}{4}}$ with extremities α and β. If not, then there are exactly three such paths, in which case we call (α, β) *of type 2*.

The bipartite structure of $L_{\frac{7}{4}}$ gives a partition of its vertex set into two sets of cardinal 8. We call the vertices of the first set (resp. the second set) of type 0 (resp. type 1). The following is straightforward.

Proposition 41. *Let α be a vertex of $L_{\frac{7}{4}}$. There are 5 vertices at distance 3 from α in $L_{\frac{7}{4}}$ (if α is of type $i = 0, 1$, these are the 5 vertices of $L_{\frac{7}{4}}$ of type $1 - i$ which are not adjacent to α). Three of them are vertices β such that the couple (α, β) is of type $\frac{3}{2}$. For the two others, (α, β) is of type 2. The diameter of $L_{\frac{7}{4}}$ is 4 and there is a unique vertex of $L_{\frac{7}{4}}$ at distance 4 from α.*

Let $G = \mathrm{Aut}(L_{\frac{7}{4}})$ be the automorphism group of $L_{\frac{7}{4}}$.

Proposition 42. *The group G is transitive on the tripods of $L_{\frac{7}{4}}$. The subgroup of G fixing pointwise a tripod in $L_{\frac{7}{4}}$ is trivial. Its stabilizer is isomorphic to $\mathbf{Z}/3\mathbf{Z} \rtimes \mathbf{Z}/2\mathbf{Z}$. In particular $|G| = 96$.*

It is interesting to compare this proposition to the proof of property T for (some) triangle buildings in [16]. One can see that it is the lack of transitivity of stabilizers of tripods which explains that their proof doesn't apply to the present situation (what we already know from Theorem 35).

Proof of Proposition 42. Let G_0 the subgroup of G fixing pointwise the tripod of $L_{\frac{7}{4}}$ at a vertex α. Then the vertex β of $L_{\frac{7}{4}}$ at distance 4 from α is fixed by G_0. Let S be the complement of the (open) tripod of α in $L_{\frac{7}{4}}$. This is a connected graph on which G_0 acts with at least 4 fixed points. From this one easily infers that the tripod of β is fixed by G_0. Thus G_0 is trivial. One can then check that the stabilizer of a tripod is isomorphic to $\mathbf{Z}/3\mathbf{Z} \rtimes \mathbf{Z}/2\mathbf{Z}$. Furthermore it readily seen on Figure 1 that G contains the group of dihedral symmetries (of order 16), which respects the type of vertices, as well as reflections which exchange the type of vertices. It follows that G is simply transitive on the tripods (and so $|G| = 6 \times 16 = 2^5$). $\square$

Let us describe in more details the graph $L_{\frac{7}{4}}$ from the local rank point of view. The signification of the following result is that, already at the local level in any polyhedron X of rank $\frac{7}{4}$, the link $L_{\frac{7}{4}}$ *imposes certain directions of (non-)branching* in X. This has to be compared to Section 6.

Proposition 43. *Let Π be a 6-cycle in $L_{\frac{7}{4}}$. Then one of the 3 couples (α, β) of points at distance 3 in Π is of type $\frac{3}{2}$, and the two others are of type 2. The group G is transitive on the 6-cycles of $L_{\frac{7}{4}}$ and has two orbits on the flags $f \subset \Pi$ where f is an edge of $L_{\frac{7}{4}}$ and Π a 6-cycle.*

Proof. On easily check on Figure 1 that there exists a 6-cycle Π_0 which satisfies the assertion of the Proposition, i.e. one of the 3 couples of points at distance 3 in Π, say (α_0, β_0) is of type $\frac{3}{2}$, and the two others are of type 2. In fact one can further assume that that there is a vertex δ_0 at distance 2 from α_0 on Π_0 such that, for the unique vertex β_0' at distance 1 from δ_0 which does not belong to Π_0, the couple (α_0, β_0') is of type 2.

Proposition 42 implies that G is transitive on the flags $A \subset \gamma$ where γ is a (simplicial) path of length 2 and A is an extremity of γ. Let Π be a 6-cycle, (α, β) be two vertex at distance 3 in Π, and let δ be a point at distance 2 from α on Π. Then there exists $g \in G$ such that $g(\alpha) = \alpha_0$ and $g(\delta) = \delta_0$. Then $g(\Pi)$ is a 6-cycle and we have either $g(\beta) = \beta_0$ or $g(\beta) = \beta_0'$.

Thus there are 3 possibilities for $g(\Pi)$ (all containing α_0 and δ_0). One readily checks that these three cycles satisfies the first assertion of the proposition. Then, choosing in $g(\Pi)$ the unique couple (α_1, β_1) of points at distance 3 of type $\frac{3}{2}$, and mapping the couple (α_1, δ_1), where δ_1 is at distance 2 from α_1 in $g(\Pi)$, to the couple (α_0, δ_0), one sees that there is $h \in G$ such that $hg(\Pi) = \Pi_0$.

Finally the fact that G has two orbits on the flags $f \subset \Pi$ where f is an edge of $L_{\frac{7}{4}}$ and Π a 6-cycle comes from the two possibilities for position of f relatively to the couple of rank $\frac{3}{2}$ in Π.

This proves the proposition. $\square$

4.3. **On the asymptotic structure of flats.** We now study how the local analysis of the previous subsection 'integrates' to polyhedra of rank $\frac{7}{4}$.

The proof of the following proposition relies on techniques developed in the proof of theorem 7 and is deferred to the end of Section 5.

Proposition 44. *Let V be a complex of rank $\frac{7}{4}$ and $\Gamma = \pi_1(V)$ be the fundamental group of V. Then for any copy of the free abelian group $\mathbf{Z}^2$ in Γ, there is a $\gamma \in \Gamma$*

such that the pairwise intersection of the subgroups $\gamma^n \mathbf{Z}^2 \gamma^{-n}$, $n \in \mathbf{Z}$, is reduced to the identity.

This proposition expresses the following dichotomy: either there is no copy of $\mathbf{Z}^2$ in Γ, or there is abundance of $\mathbf{Z}^2$ all of whose copies sit *'mixingly'* in Γ, the latter being a strong structural property of Γ. We don't know if there are always copies of $\mathbf{Z}^2$ in groups of rank $\frac{7}{4}$. In fact this is precisely what first led us to study polyhedra of rank $\frac{7}{4}$.

More precisely, one of our motivations for studying these polyhedra was the following well-known open problem in geometric group theory (see e.g. Question 1.1 in [11], that we formulate here in the non-positively curved case and dimension 2, as in the paragraph following Q 1.1. in [11]).

Question 45. Let Γ be a countable group admitting a non-positively curved finite $K(\Gamma, 1)$ of dimension 2. If the universal cover of $K(\Gamma, 1)$ contains a flat, does Γ contains $\mathbf{Z}^2$?

This question is especially intriguing for polyhedra of rank $\frac{7}{4}$ because of their local structure (as described in Subsection 4.2). This motivated our classification of complex of rank $\frac{7}{4}$ in Theorem 35, where our objective was to study the *most symmetric* cases, i.e., polyhedra that are transitive on vertices. Relying upon this classification we now clarify the issue of Question 45 in this particular case.

Proposition 46. *Let V be one of the 12 orientable complexes of rank $\frac{7}{4}$ of with one vertex (see Theorem 35) and $\Gamma = \pi_1(V)$ be the fundamental group of V. Then the universal cover $\tilde{V}$ has exponential rank. Moreover, Γ contains copies of $\mathbf{Z}^2$ and thus (non trivially) satisfies the conclusion of Proposition 44.*

Proof. We shall give full details for one of these complexes (we chose V_0^2) so as to present the arguments. The other cases can be derived similarly.

Recall that V_0^2 admits the presentation

$$V_0^2 = [[1,2,3],[1,4,5],[1,6,7],[2,4,6],[2,8,5],[3,6,8],[3,7,5],[4,8,7]].$$

Let $a_1, \dots, a_8$ be the generators $\Gamma_0^2 = \pi_1(V_0^2)$ of corresponding to $1, \dots, 8$. Then one readily checks that the two elements $x = a_1^3$ and $y = a_3 a_4$ commute in Γ_0^2. Hence they generate a subgroup Λ isomorphic to $\mathbf{Z}^2$. The fundamental domain P of the action of Λ on its corresponding flat in $\tilde{V}$ contains 12 triangles which are respectively (from left to right and bottom to top after a suitable embedding of P in $\mathbf{R}^2$):

$$\begin{gathered}[1,2,3],[2,8,5],[4,8,7],[1,6,7]\\ [1,4,5],[4,8,7],[3,6,8],[1,2,3]\\ [1,6,7],[3,6,8],[2,8,5],[1,4,5]\end{gathered}$$

Let us show that $\tilde{V}_0^2$ has exponential rank. For each vertex $A \in \tilde{V}_0^2$ consider the flat parallelogram P_A in $\tilde{V}_0^2$ associated to x and y and let $Y = \cup_{A \in \tilde{V}_0^2} P_A$. Then Y is a reunion of flats and from the description of P above, and the definition of exponential rank, it is not hard to check that it is enough to prove that the semi-group of Γ generated by the three elements

$$u = a_3 a_4, \ v = a_5 a_6, \text{ and } w = a_7 a_2,$$

has exponential growth. Write $u' = vwu$. One easily check on the presentation of V_0^2 that the for any vertex $A_0 \in \tilde{V}_0^2$ the three points A_0, uA_0 and $u'uA_0$ (resp. A_0, $u'A_0$ and $uu'A_0$) are on geodesic of $\tilde{V}_0^2$. But this implies that the semi-group generated by u and u' is free in Γ. Hence the (semi-)group of Γ generated by u, v and w has exponential growth. $\square$

4.4. **End of the proof of Theorem 4.** This subsection is devoted to the (computer assisted) proof that there are *at most 12* orientable compact complexes of rank $\frac{7}{4}$ with one vertex, as asserted in Theorem 35. For us the program below was primary used to obtain a representative list of examples of polyhedra of rank $\frac{7}{4}$, beyond the few ones we found by hand.

We only explain below what procedures are relevant to understand the source code and to check that its mathematical part was correctly implemented. Other procedures (e.g. displaying, sorting, memory management,...) are routine and omitted. Recall from the proof of Theorem 35 that polyhedra of rank $\frac{7}{4}$ can be represented by a list of numbers. At the mathematical level two procedures are important:

(a) the test that a list representing a polyhedron has the correct link, i.e., $L_{\frac{7}{4}}$ (see `test_link`),
(b) the iteration process that enumerates all possible list representing orientable polyhedra of rank $\frac{7}{4}$ with 1 vertex (see `case_1`).

Let us briefly explain how they are inserted in the body of the program (the code reproduced below is written in C++). We first need a class, called `polyhed`, which describes a generic orientable polyhedron of rank $\frac{7}{4}$ with 1 vertex:

```
class polyhed{ private: char index[8][3];
   public: polyhed(); polyhed(char [][3], char);~polyhed();char length;
      void add (char, char, char); void display(); void reorder();
      char compare(polyhed); bool edge(char, char ); bool two_path(char, char );
      bool test_link(char, char ,char ); };
```

The function `add` add the face (char, char, char) to the given polyhedron (represented by the 8×3 array `index`) and increments its length `length` of 1. The function `reorder` reorders `index` in a canonical way (i.e. removes the ambiguities arising in the process of coding of the actual polyhedra as an array of number). The function `compare` is a basic comparison of two polyhedra.

The function `edge` tests, given two numbers (k,l) between 1 and 8, whether $(i^\flat, k^\sharp)$ is already an edge of the link of the polyhedron P (see the proof of Theorem 35 for the notations $^{\flat}\,^{\sharp}$). Similarly `two_path` tests, given (k,l), whether there is a length 2 path in the link of P between $k^\flat$ and $l^\sharp$ or $k^\sharp$ and $l^\flat$. These functions are straightforward to implement. They are used in the last function `test_link`, which tests whether adding the face (i,j,k) to a polyhedron P (with `length`$<$ 8 faces) creates an admissible portion of the link:

```
bool polyhed::test_link(char i, char j, char k){ if (i==j and j==k) return true;
  if (edge(i, j) or edge(j, k) or edge(k, i)) return true;
  if (i==j){if (edge(k, k) or two_path(i, k)) return true;};
  if (j==k){if (edge(i, i) or two_path(i, j)) return true;};
  if (k==i){if (edge(j, j) or two_path(i, j)) return true;};
  if (edge(i, k) and edge(j, j)) return true;
  if (edge(j, i) and edge(k, k)) return true;
  if (edge(k, j) and edge(i, i)) return true;
  for (int a=0; a<length; a++){
        if (edge(index[a][0], j) and edge(i, index[a][1])) return true;
        if (edge(index[a][1], j) and edge(i, index[a][2])) return true;
        if (edge(index[a][2], j) and edge(i, index[a][0])) return true;
        if (edge(index[a][0], k) and edge(j, index[a][1])) return true;
        if (edge(index[a][1], k) and edge(j, index[a][2])) return true;
        if (edge(index[a][2], k) and edge(j, index[a][0])) return true;
        if (edge(index[a][0], i) and edge(k, index[a][1])) return true;
        if (edge(index[a][1], i) and edge(k, index[a][2])) return true;
```

```
    if (edge(index[a][2], i) and edge(k, index[a][0])) return true;}
  return false;}
```

Next one needs a class `list` to record the collected polyhedra. The function `add` in this class, given some polyhedron, reorders it in a 'canonical' order, compares it to the elements already present in the list, and adds a copy of it to the list if it was found to be new.

```
class list{ private: long int max; polyhed **K;
      public: List();~List(); void display(); void add (polyhed *); };
```

Finally we need to enumerate all possible 8-tuples representations of polyhedra. In fact doing this enumeration directly would have been too long, so we divided it into the following six subcases which take *a priori* into account (some of) the ambiguity in the representation of a polyhedron as a list. Thus, we claim that it is sufficient to enumerate all lists of the following form to exhaust all of the actual polyhedra (the proof is easy).

Case 1: $[(1,2,3),(4,4,5),(1,.,.),(2,.,.),(3,.,.),(1,.,.),(2,.,.),(3,.,.)]$
Case 2: $[(1,2,3),(4,5,6),(1,.,.),(2,.,.),(3,.,.),(1,.,.),(2,.,.),(3,.,.)]$
Case 3: $[(1,2,3),(1,3,2),(1,4,.),(2,.,.),(3,.,.),(.,.,.),(.,.,.),(.,.,.)]$
Case 4: $[(1,2,3),(1,3,4),(3,5,.),(1,.,.),(2,.,.),(2,.,.),(.,.,.),(.,.,.)]$
Case 5: $[(1,2,3),(1,3,4),(3,5,.),(2,1,.),(2,.,.),(.,.,.),(.,.,.),(.,.,.)]$
Case 6: $[(1,2,3),(1,3,4),(2,1,.),(3,2,.),(.,.,.),(.,.,.),(.,.,.),(.,.,.)]$

Here is the (recursive) procedure `case_1` which implements the first of the above cases. It takes as argument the list *K of all previously found polyhedra, the currently analyzed polyhedron (or rather its initial segment, of length l), the remaining possible edges to be added, encoded in Res (r is the length of Res).

```
void case_1(List *K, polyhed *t, char l, char Res [][2], char r){
  char X[r][2]; char x,u,v;
  for(int a=0; a<r; a++){ u=Res[a][0];
    if (Res[a][1]==1){ for(int i=0; i<a; i++){X[i][0]=Res[i][0];X[i][1]=Res[i][1];}
      for(int i=a+1; i<r; i++){X[i-1][0]=Res[i][0];X[i-1][1]=Res[i][1];} x=r-1;}
    else{ for(int i=0; i<r; i++){X[i][0]=Res[i][0];X[i][1]=Res[i][1];}
      x=r;X[a][1]--;}
    for(int b=0; b<x; b++){ char Y[x][2]; char y; v=X[b][0];
      if (X[b][1]==1){ for(int i=0;i<b;i++){Y[i][0]=X[i][0];Y[i][1]=X[i][1];}
          for(int i=b+1;i<x;i++){Y[i-1][0]=X[i][0];Y[i-1][1]=X[i][1];} y=x-1;}
      else{ for(int i=0;i<x;i++){Y[i][0]=X[i][0];Y[i][1]=X[i][1];} y=x;Y[b][1]--;}
      t→length=l;
      if (t→test_link(l%3+1,u,v)==false){ t→add(l%3+1,u,v);
          if (l<7){case_1(K,t,l+1,Y,y);} else {K→add(t);}}}}}
```

The function `case_1` is initialized in the `main` procedure as:

```
int main () {
  char a [2][3]={{1,2,3},{4,4,5}}; polyhed t(a,2); List L;
  char S[5][2]={{4,1},{5,2},{6,3},{7,3},{8,3}};
  case_1(&L,&t,2,S,5); L.display(); return 0; }
```

The functions implementing the five remaining cases are of a similar nature.

5. Proof of Theorem 7

Proof. Let V be a compact complex of rank $\frac{7}{4}$ (Definition 33). Let Γ be its fundamental group, X its universal cover and $\pi : X \to V$ the covering map. Fix a vertex A_0 of X and identify Γ with its orbit ΓA_0 in X. By Dykema and de la Harpe's Theorem 1.4 in [27], it is enough to show that Γ has the free semigroup property and the ℓ^2-spectral radius property. The latter follows from property RD (see Proposition 23) which itself follows from Theorem 5 so we aim to prove the free semi-group property in the present section. Recall [27] that a group Γ is said to have the *free semi-group property* if for every finite subset F of Γ there is $u \in \Gamma$ such that the set $uF = \{ua,\ a \in F\}$ is semi-free. A finite subset F of Γ is said to be *semi-free* if for every $n, m \in \mathbf{N}$, every $x_1, \dots x_n, y_1 \dots y_m \in F$, the equality $x_1 \dots x_n = y_1 \dots y_m$ implies that $m = n$ and $x_i = y_i$ for every $i = 1 \dots n$.

In order to prove the free semi-group property for Γ, which is achieved in Lemma 53, we need to introduce one more concept for rank $\frac{7}{4}$ polyhedra. Negative curvature is used extensively below via the Gauss-Bonnet formula.

Definition 47. We call *analytic geodesic* of X a singular CAT(0) geodesic (i.e. included in the 1-skeleton, see Section 3) whose angle at each vertex equals $4\pi/3$.

As V is compact the family of analytic geodesics of X projects under π to a finite set of closed geodesics of V. We call these closed geodesics the *rings* of V. Note that every analytic geodesic is periodic of period the length of its corresponding ring in V (because Γ acts freely on X with compact quotient V).

Example 48. One can show that the complex $\bar{V}$ of Remark 38 has a single ring $r = 841^{-}6537^{-}2$. Thus all analytic geodesics in the universal cover of $\bar{V}$ have period 8. Moreover the image of r in $H_1(\bar{V}, \mathbf{Z}) = \mathbf{Z}$ is equal to 8.

Lemma 49 (Analyticity)**.** *Let γ_1 and γ_2 be two analytic geodesics of X. Then exactly one of the following cases occurs:*

(1) *γ_1 and γ_2 are disjoint;*
(2) *the intersection of γ_1 and γ_2 is reduced to a point;*
(3) *$\gamma_1 = \gamma_2$.*

Proof. Assume that $I = \gamma_1 \cap \gamma_2$ contains at least two distinct points and let us prove that $\gamma_1 = \gamma_2$. One has $I = [A, B] \cap X$ for some points $A, B \in \bar{X}$, where $\bar{X}$ is the disjoint union of X and its boundary. Assume that $A \in X$. Then A is a vertex of X and as $B \neq A$ by assumption one of the edges of X containing A, say $[A, A']$, is included in I. However the link L_A (being isometric to $L_{\frac{7}{4}}$) contains a unique point A'' which is at (angular) distance $4\pi/3$ from A'. By definition of analyticity A'' belongs to both γ_1 and γ_2, which contradicts the definition of I. Thus $A \in \bar{X}\backslash X$ and similarly $B \in \bar{X}\backslash X$ so $\gamma_1 = \gamma_2$. □

Lemma 50. *Let γ be an analytic geodesic of period $t \in \Gamma$ in X. There exists an s in Γ such that for $A \in X$ the geodesic segment η from A to $tstA$ contains tA, stA, and is not included in an analytic geodesic.*

Proof. Let A_0 be a point of γ and g be an analytic geodesic of X such that $\gamma \cap g = \{A_0\}$. Let u be the period of g and write $B_0 = u^2 A_0$. Then the unique analytic geodesic γ' which contains $[B_0, tB_0]$ does not intersect γ. Indeed assume it does and denote $C_0 = \gamma \cap \gamma'$. Then by analyticity the angles of the geodesic triangle $\Delta = (A_0 B_0 C_0)$ at A_0, B_0, C_0 are at least $\pi/3$. In particular Δ bounds a topological disk D. With the notations of Lemma 29, the Gauss-Bonnet formula for D gives

$$\int_{\Delta} \kappa + (\pi - \theta_{A_0}) + (\pi - \theta_{B_0}) + (\pi - \theta_{C_0}) + \sum_{x \in D} \delta_x \ = \ 2\pi.$$

Recall that $\sum_{x \in D} \delta_x$ is the internal curvature of D, so $\sum_{x \in D} \delta_x \leq 0$, and $\int_{\Delta} \kappa$ is total geodesic curvature of D on its boundary, that is $\int_{\Delta} \kappa = \int_{]A_0, B_0[} \kappa + \int_{]B_0, C_0[} \kappa + \int_{]A_0, C_0[} \kappa$ where

$$\int_{]A_0, B_0[} \kappa = \sum_{s \in]A_0, B_0[} (\pi - \theta_s)$$

with θ_s is the internal angle of D at $s \in]A_0, B_0[$ (and similarly for the two other sides). Thus

$$\int_\Delta \kappa \leq \int_{]A_0,B_0[} \kappa = -\pi/3$$

by analyticity of $[A_0, B_0]$. However the inequality $(\pi - \theta_{A_0}) + (\pi - \theta_{B_0}) + (\pi - \theta_{C_0}) \leq 2\pi$ gives

$$\int_\Delta \kappa \geq 0$$

which is a contradiction.

Let $A_1 = t^{-3}A_0$ (resp. $B_1 = t^3 B_0$) and denote A_1' (resp. B_1') the point of $[A_1, A_0] \cap [A_1, B_1]$ (resp. $[B_0, B_1] \cap [A_1, B_1]$) such that $|A_1 - A_1'|$ (resp. $|B_1 - B_1'|$) is maximal. We now show that $A_1 \neq A_1'$ and $B_1 \neq B_1'$.

So assume toward a contradiction that $A_1 = A_1'$. Let $h = [A_1, A_0] \cup [A_0, B_0] \cup [B_0, B_1']$. By the above h is a piecewise geodesic path in X from A_1 to B_1' without self-intersection. Let B_1'' be the closest point from A_1 on h which belongs to $[A_1, B_1']$ and let h' be the part of h going from A_1 to B_1''. Consider the topological disk D' whose boundary is the piecewise geodesic closed curve $\Delta' = [A_1, B_1''] \cup h'$. Note that, the internal angles of D' at A_1 being non zero, the point B_1'' does not belong to γ and so $A_0 \in h'$. The point B_0 might or might not belong to h'; in both cases the following follows from the Gauss-Bonnet formula:

$$\begin{aligned}\int_{\Delta'} \kappa &\geq 2\pi - (\pi - \theta_{A_1}) - (\pi - \theta_{A_0}) - (\pi - \theta_{B_0}) - (\pi - \theta_{B_1''}) - \sum_{x \in D'} \delta_x \\ &> -(\pi - \theta_{A_0}) - (\pi - \theta_{B_0}) \geq -2\pi/3.\end{aligned}$$

Then analyticity of $[A_0, A_1]$ implies

$$\int_\Delta \kappa \leq \int_{]A_0,A_1[} \kappa = -\pi/3(|A_0 - A_1| - 1) \leq -2\pi/3$$

which is a contradiction.

Thus $A_1 \neq A_1'$ and similarly $B_1 \neq B_1'$. But this shows that the element $s = t^3u^2t^3$ of Γ satisfies that, for any $A \in X$, the geodesic segment η from A to $tstA$ contains both tA, stA. By construction, η is not included in an analytic geodesic. □

Let F be a finite subset of Γ and let $\alpha = \max_{a \in F} |a|$. Fix a analytic geodesic γ of period t in X and let s be as in Lemma 50. Let β be the length of s and denote

$$u = t^{4\alpha+7} s t^{9\alpha+\beta+19}.$$

Lemma 51. *Let $a \in F$ and $A \in X$. Then $t^{\alpha+3}aA$ is on the geodesic segment from A to uaA in X.*

Proof. Let $B = aA$, $C = uaA$ and consider the points A', B', C' of $[A, B] \cap [A, C]$, $[B, A] \cap [B, C]$ and $[C, A] \cap [C, B]$ respectively such that $|A - A'|$, $|B - B'|$, and $|C - C'|$ are maximal. By assumption we have $|A - B| \leq \alpha$ so $|B - B'| \leq \alpha$. Let D be the unique disk whose boundary is the geodesic triangle $\Delta = [A', B'] \cup [B', C'] \cup [C', A']$. If Δ is reduced to a point the lemma is clear so we assume this is not the case. If not we apply the Gauss-Bonnet formula to D:

$$\int_\Delta \kappa + (\pi - \theta_{A'}) + (\pi - \theta_{B'}) + (\pi - \theta_{C'}) + \sum_{x \in D} \delta_x \;=\; 2\pi.$$

which gives

$$\int_\Delta \kappa > 2\pi - 3\pi - \sum_{x \in D} \delta_x \geq -\pi$$

as $(\pi - \theta_{A'}) + (\pi - \theta_{B'}) + (\pi - \theta_{C'}) < 3\pi$. On the other hand

$$\int_\Delta \kappa \leq \int_{]B',C'[} \kappa = \sum_{s\in]B',C'[} (\pi - \theta_s)$$

As $|B - B'| \leq \alpha$ the point $C'' = t^{\alpha+4}B$ belongs to $]B', C[$ and $[B', C'']$ is an analytic geodesic of X. Write $]B', \tilde{C}[$ for the initial segment $]B', C'[\cap]B', C''[$. We have

$$\int_{]B',C'[} \kappa \leq \sum_{s\in]B',\tilde{C}[} (\pi - \theta_s) = -\frac{\pi}{3} \min\{|B' - C'| - 1, |B' - C''| - 1\}$$

because $]B', C'[$ is geodesic and $]B', \tilde{C}[$ is analytic. Therefore

$$-\frac{\pi}{3} \min\{|B' - C'| - 1, |B' - C''| - 1\} \geq \int_{]B',C'[} \kappa \geq \int_\Delta \kappa > -\pi$$

and so $\min\{|B' - C'| - 1, |B' - C''| - 1\} < 3$. As $|B' - C''| \geq 4$ it follows that $\tilde{C} = C'$ and $|B' - C'| \leq 3$ so we have $t^{\alpha+3}B \in [C', C]$. This proves the lemma. □

Lemma 52. *Let $(a_1, \ldots a_n)$ be a sequence of elements of F of length $n \in \mathbf{N}$. For $k = 0 \ldots n - 1$ define recursively points x_k in Γ (viewed as a subset of X) by*

$$x_{k+1} = ua_{k+1}x_k,$$

where $x_0 = e$. Let η_k be the geodesic segment from x_k to x_{k+1} in X. For any $k = 0 \ldots n-1$ there exist two points $x_k^+ < x_{k+1}^-$ on η_k such that $|x_k^+ - x_k| \leq (2\alpha+6)|t|$, $|x_{k+1}^- - x_{k+1}| \leq (2\alpha+6)|t|$, and such that $[x_k^+, x_{k+1}^-]$ are pairwise disjoint consecutive geodesic segment on the geodesic segment from x_0 to x_n.

Proof. For $i = 1 \ldots n-1$ let $\tilde{x}_i$ be the point of $\eta_{i-1} \cap \eta_i$ for which $|x_i - \tilde{x}_i|$ is maximal. Let us first show that $|x_i - \tilde{x}_i| \leq (2\alpha + 3)|t|$. By Lemma 51, the point $z = t^{\alpha+3}a_i x_i$ belongs to η_i and $|x_i - z| \leq (2\alpha+3)|t|$. Assume toward a contradiction that $z \in]x_i, \tilde{x}_i[$. Then the intersection of the segments $[z, x_i] \subset \eta_{i-1}$ and $[z, x_{i+1}] \subset \eta_i$ contains at least an edge which is readily seen to be in the analytic parts of η_{i-1} and η_i. From the definition of u and Lemma 49 we infer that $\eta_{i-1} \cap \eta_i \supset [x_i, z']$ where $z' = t^{-2\alpha-5}x_i$. Thus as $z = t^{\alpha+3}a_i x_i \in \eta_{i-1}$ we have

$$z \in [t^e z', t^{e+1}z']$$

for some index $2 \leq e \leq 2\alpha + 5$. However $z' \in \eta_i$ as well and as $z = t^{\alpha+3}a_i x_i \in \eta_i$ we also have

$$z' \in [t^f z, t^{f+1}z]$$

for some index $2 \leq f \leq 2\alpha + 5$. In particular $|tz - tz'| < |z - z'|$ which contradicts the fact that, being its period, t acts isometrically on γ. Thus $\tilde{x}_i \in [x_i, z]$ and $|x_i - \tilde{x}_i| \leq (2\alpha + 3)|t|$. Set $\tilde{x}_0 = x_0, \tilde{x}_n = x_n$ and let $\tilde{\eta}_i$ be the geodesic from $\tilde{x}_i$ to $\tilde{x}_{i+1}$, $i = 1 \ldots n-1$ (so $\tilde{\eta}_i \subset \eta_i$).

Let us prove that $\tilde{\eta}_i$ intersects $\tilde{\eta}_j$ if and only if they are consecutive (i.e. $i = j+1$ or $j = i+1$) for $i, j = 1 \ldots n-1$. Suppose on the contrary that there is a $i \in [0, n-2]$ and a $j > i+1$ such that $\tilde{\eta}_i \cap \tilde{\eta}_j \neq \emptyset$. We can further assume that j is the smallest index $j > i+1$ satisfying this condition. Let $z \in \tilde{\eta}_j$ be the closest point from $\tilde{x}_j$ which belongs to $\tilde{\eta}_i$. Denote $\tilde{\eta}_i' = [z, \tilde{x}_{i+1}]$ and $\tilde{\eta}_j' = [\tilde{x}_j, z]$. Then there is a topological disk D in X whose boundary is the piecewise geodesic simple curve

$$\Delta = \tilde{\eta}_i' \cup (\bigcup_{i<k<j} \tilde{\eta}_k) \cup \tilde{\eta}_j'.$$

The Gauss-Bonnet formula for D reads

$$\int_\Delta \kappa + (\pi - \theta_z) + \sum_{i<k\leq j} (\pi - \theta_{\tilde{x}_k}) + \sum_{x\in D} \delta_x \ = \ 2\pi.$$

As $(\pi-\theta_z)+\sum_{i<k\leq j}(\pi-\theta_{\tilde{x}_k})<(j-i+1)\pi$ and $\sum_{x\in D}\delta_x\leq 0$ we deduce that

$$\int_\Delta \kappa > 2\pi-(j-i+1)\pi=(i+1-j)\pi.$$

On the other hand

$$\int_\Delta \kappa = \sum_{i<k<j}\sum_{s\in\tilde{\eta}_k}(\pi-\theta_s)+\sum_{s\in\tilde{\eta}'_i}(\pi-\theta_s)+\sum_{s\in\tilde{\eta}'_j}+(\pi-\theta_s)\leq \sum_{i<k<j}\sum_{s\in\tilde{\eta}_k}(\pi-\theta_s).$$

So consider the geodesic $\tilde{\eta}_k\subset\eta_k$ for $k\in]i,j[$. As $|x_k-\tilde{x}_k|\leq(2\alpha+3)|t|$ and $|x_{k+1}-\tilde{x}_{k+1}|\leq(2\alpha+3)|t|$ we can find a segment S_k of length at least 3α in $\tilde{\eta}_j$ which is analytic. In particular

$$\sum_{s\in\tilde{\eta}_k}(\pi-\theta_s)\leq\sum_{s\in S_k}(\pi-\theta_s)\leq 3\alpha\cdot(-\pi/3)=-\alpha\pi$$

Thus we get

$$\int_\Delta \kappa\leq -\alpha(j-i-1)\pi$$

and so

$$(i+1-j)\pi<-\alpha(j-i-1)\pi$$

It follows that $\alpha<1$, which is a contradiction. This shows that $\tilde{\eta}_i$ intersects $\tilde{\eta}_j$ if and only if $i=j+1$ or $j=i+1$ for $i,j=1\ldots n-1$.

We now prove the lemma. Let $g=\cup_{k=0\ldots n-1}\tilde{\eta}_k$. By the above g is a piecewise geodesic curve in X from x_0 to x_n without self-intersection. We proceed by recurrence.

Set $x_0^+=x_0$ and denote by x_1^- the unique point of $\tilde{\eta}_0\cap[x_0,x_n]$ such that $]x_1^-,x_n[\cap\tilde{\eta}_0$ is empty. If $x_1^-=\tilde{x}_1$ and a neighbourhood of $\tilde{x}_1$ in $\tilde{\eta}_1$ is included in $[x_0,x_n]$ then we let $x_1^+=\tilde{x}_1$ and the conditions of the lemma are satisfied at x_1 (as $|x_1-\tilde{x}_1|\leq(2\alpha+3)|t|$). Otherwise let $x_1^+>x_1^-$ be the first point of g distinct from x_1^- which belongs to $[x_0,x_n]$. Note that $x_1^+\notin\tilde{\eta}_0$ as this would imply $x_1^-=x_1^+$. As g is a simple curve there is a non empty disk D whose boundary

$$\Delta=g_{[x_1^-,x_1^+]}\cup[x_1^-,x_1^+]$$

consists of the part $g_{[x_1^-,x_1^+]}$ of g which is in between x_1^- and x_1^+ and the geodesic segment $[x_1^-,x_1^+]\subset[x_0,x_1]$. Let $j\geq1$ such that $x_1^+\in\tilde{\eta}_j$. The Gauss-Bonnet formula for D shows that

$$\int_\Delta\kappa+(\pi-\theta_{x_1^-})+(\pi-\theta_{x_1^+})+\sum_{1\leq k\leq j}(\pi-\theta_{\tilde{x}_k})\geq 2\pi.$$

(where one could a priori have $x_1^-=x_1$ or $x_1^+=\tilde{x}_i$ for some $i>1$). Reiterating our argument above we get that $j=1$ and thus, $x_1^+\in\tilde{\eta}_1$. Hence the preceding formula implies

$$\int_\Delta\kappa>2\pi-3\pi=-\pi.$$

If $x_1^+ \in [x_1, t^{\alpha+3}ax_1]$ the condition $|x_1^+ - x_1| \leq (2\alpha+6)|t|$ of the lemma is satisfied. It follows that we can assume $x_1^+ \in [t^{\alpha+3}ax_1, x_2]$. Then

$$\begin{aligned}
\int_\Delta \kappa &= \sum_{s\in]x_1^-, \tilde{x}_1[} (\pi - \theta_s) + \sum_{s\in]\tilde{x}_1, x_1^+[} (\pi - \theta_s) + \sum_{s\in]x_1^-, x_1^+[} (\pi - \theta_s) \\
&\leq \sum_{s\in]x_1^-, \tilde{x}_1[} (\pi - \theta_s) + \sum_{s\in]\tilde{x}_1, x_1^+[} (\pi - \theta_s) \\
&\leq \sum_{s\in]x_1^-, \tilde{x}_1[} (\pi - \theta_s) + \sum_{s\in]t^3ax_1, x_1^+[} (\pi - \theta_s) \\
&\leq \min\{4\alpha|t|, |x_1^- - \tilde{x}_1|\} \cdot (-\pi/3) + \min\{4\alpha|t|, |x_1^+ - t^{\alpha+3}ax_1|\} \cdot (-\pi/3)
\end{aligned}$$

where the last inequality comes from the fact that the geodesics $[x_1^-, \tilde{x}_1]$ and $[t^{\alpha+3}ax_1, x_2]$ are analytic on a segment of length at least $4\alpha|t|$, starting from $\tilde{x}_1$ and $t^{\alpha+3}ax_1$ respectively. Thus

$$\min\{4\alpha|t|, |x_1^- - \tilde{x}_1|\} + \min\{4\alpha|t|, |x_1^+ - t^{\alpha+3}ax_1|\} \leq 3$$

As $4\alpha|t| > 3$ this implies $|x_1^- - \tilde{x}_1| \leq 3$ and $|x_1^+ - t^{\alpha+3}ax_1| \leq 3$. Thus $|x_1^- - x_1| \leq (2\alpha+6)|t|$ and $|x_1^+ - x_1| \leq (2\alpha+6)|t|$ as asserted.

The construction of x_k^- and x_k^+ (the latter provided $k < n$, where $x_n^- = x_n$) and the proof that $|x_k^- - x_k| \leq (2\alpha+6)|t|$ and $|x_k^+ - x_k| \leq (2\alpha+6)|t|$ can then be done exactly as for x_1^- and x_1^+ above so we will omit the details.

This concludes the proof of Lemma 52. □

Recall that $u = t^{4\alpha+7}st^{9\alpha+\beta+19}$ has been defined in the paragraph preceding Lemma 51.

Lemma 53. *The finite set uF is semi-free in Γ.*

Proof. Let $(a_1, \dots a_n)$ and $(b_1, \dots b_m)$ be a sequence of elements of F of length $n, m \in \mathbf{N}$ respectively. Assume that

$$ua_n \dots ua_1 = ub_m \dots ub_1$$

in Γ. Let $(x_0, \dots, x_n)$ and $(y_0, \dots, y_m)$ be the sequences of point of X associated to $(a_1, \dots a_n)$ and $(b_1, \dots b_m)$ respectively as in Lemma 52 (so $x_0 = y_0 = e$ and $x_n = y_m = ua_n \dots ua_1$ assuming $\Gamma \subset X$). Associated to $(x_0, \dots, x_n)$ (resp. $(y_0, \dots, y_m)$) we can find points $x_0^+, x_1^\pm \dots, x_{n-1}^\pm, x_n^-$ (resp. $y_0^+, y_1^\pm \dots, y_{n-1}^\pm, y_n^-$) on the geodesic $[x_0, x_n]$ which satisfy the conclusion of Lemma 52. Fix $k \in \{0 \dots n-1\}$. Since we have $|x_{k+1}^- - x_{k+1}| \leq (2\alpha+6)|t|$ and $|x_k^+ - x_k| \leq (2\alpha+6)|t|$ the point $w_k = t^{3\alpha+\beta+6}a_kx_k$ and the point $st^{6\alpha+13}w_k$ both belong to $[x_k^+, x_{k+1}^-]$. Note that $w_0 < w_1 < \dots < x_{n-1}$ on $[x_0, x_n]$. Similarly, define points w'_k on $[x_0, x_n]$, $k = 0 \dots m-1$, relative to the segments $[y_k^+, y_{k+1}^-]$ and satisfying analogous properties.

We claim that if w is a point on $[x_0, x_n]$ such that $st^{6\alpha+13}w \in [x_0, x_n]$ then $w = w_k$ for some $k = 0 \dots n-1$.

Let us first show that this implies the lemma. Indeed by the claim there is an increasing injection $j : [0, \dots, m-1] \to [0, \dots, n-1]$ such that $w'_k = w_{j(k)}$. Thus by symmetry we obtain that $n = m$ and $w'_k = w_k$ for $k = 0 \dots n-1$. On the other hand we have $x_{k+1} = t^{4\alpha+7}st^{6\alpha+13}w_k$ and so

$$y_{k+1} = t^{4\alpha+7}st^{6\alpha+13}w'_k = x_{k+1}.$$

Thus

$$ua_{k+1} = x_{k+1}x_k^{-1} = y_{k+1}y_k^{-1} = ub_{k+1}$$

for $k = 0 \dots n-1$. This shows that uF is semi-free.

Let us now prove the claim. Let w is a point on $[x_0, x_n]$ such that $w' = st^{6\alpha+13}w \in [x_0, x_n]$. We first show that $w \in [x_k^+, st^{6\alpha+13}w_k]$ for some index $k \in [0, n-1]$. If not then $w \in [st^{6\alpha+13}w_{k-1}, x_k^+]$ for some $k \in [1, n-1]$. As $|x_k - x_k^+| \leq (2\alpha + 6)|t|$ and $|st^{6\alpha+13}w_{k-1} - x_k| \leq (4\alpha + 7)|t|$ we have

$$|x_k^+ - st^{6\alpha+13}w_{k-1}| \leq 6\alpha|t| + 13|t|.$$

Lemma 51 and Lemma 52 then show that that the point $t^{6\alpha+13}w$ of $[x_0, x_n]$ is of the form $t^\ell a_k x_k$ for some index $\ell > \alpha + 3$, and as $|x_k - x_k^+| \leq (2\alpha + 6)|t|$ we also have

$$\ell \leq (6|t|\alpha + 13|t|) + (2\alpha|t| + 6|t|) = 8\alpha|t| + 19|t|.$$

As $8\alpha + 19 < 9\alpha + 19$ our choice of u (more precisely the exponent $9\alpha + \beta + 19$) shows that the points $t^{6\alpha+13}w$ and $st^{6\alpha+13}w$ are extremities of an analytic subsegment of $[x_0, x_n]$. But this contradicts Lemma 50 (i.e. the non analyticity of s).

Therefore $w \in [x_k^+, st^{6\alpha+13}w_k]$ for some index $k \in [0, n-1]$. Now the segment $[w, t^{6\alpha+13}w]$ is an analytic subsegment of $[x_0, x_n]$ and thus is disjoint from $[t^{6\alpha+13}w_k, st^{6\alpha+13}w_k]$ by construction of s. Thus $[w, t^{6\alpha+13}w] \subset [x_k^+, t^{6\alpha+13}w_k]$. However by definition of w_k the segment $[x_k^+, t^{6\alpha+13}w_k]$ is analytic as well, hence disjoint from $[t^{6\alpha+13}w, st^{6\alpha+13}w]$. This implies that $st^{6\alpha+13}w_k = st^{6\alpha+13}w$ and thus $w_k = w$, proving the claim.

This concludes the proof of Lemma 53. □

Theorem 7 follows from Lemma 53 and the paragraph preceding Definition 47. □

As announced in Section 4.3, we conclude this section with the proof of Proposition 44.

Proof of Proposition 44. Let V be a complex of rank $\frac{7}{4}$, $X = \tilde{V}$, and $\Gamma = \pi_1(V)$. Let Λ be a subgroup of Γ isomorphic to $\mathbf{Z}^2$. Fix an analytic geodesic γ in X of period $t \in \Gamma$ an set

$$u = t^{|s|+1} s t^{|s|+1}$$

where s is given by Lemma 50 and $|s|$ is its length.

Let us show that the pairwise intersection of the subgroups $u^n \Lambda u^{-n}$, $n \in \mathbf{Z}$, is reduced to the identity. Indeed assume that there is $n, m \in \mathbf{Z}$, and $\lambda_1, \lambda_2 \in \Lambda$ such that

$$u^n \lambda_1 u^{-n} = u^m \lambda_2 u^{-m}$$

and let us show that $n = m$ or $\lambda_1 = \lambda_2 = e$. We have

$$u^{n-m} \lambda_1 u^{m-n} = \lambda_2 \in \Lambda$$

so consider the geodesic parallelogram P in X with vertices A_0, $A_1 = \lambda_1 A_0$, $A_2 = u^{n-m}\lambda_1 A_0$ and $A_3 = u^{n-m}A_0$. We assume that $n \neq m$ and prove that $\lambda_1 = \lambda_2 = e$.

As the segment $[A_0, A_1]$ is flat by assumption, the point $A_0' \in [A_0, A_1] \cap [A_0, A_3]$ such that $|A_0 - A_0'|$ is maximal is at distance at most 1 from A_0 by definition of u. Similarly define (using flatness of $[A_0, A_1]$ and $[A_2, A_3]$) points A_1', A_2', A_3' in $[A_1, A_0] \cap [A_1, A_2]$, $[A_2, A_1] \cap [A_2, A_3]$, and $[A_3, A_2] \cap [A_3, A_0]$ respectively, sharing analogous properties. Using the Gauss-Bonnet formula one can prove that $A_0' = A_1'$ and $A_2' = A_3'$. We shall omit the details as they are similar to that appearing in the middle of the proof of Lemma 52. Then arguing as in the proof of Lemma 53 (using the element s in the definition of u) we get that $A_0 = A_1$ and $A_2 = A_3$. (We leave the details to the reader as well.) The action of Γ on X being free, we deduce that $\lambda_1 = \lambda_2 = e$. This proves the proposition. □

6. Mesoscopic rank

We first prove the statements in the first part of Subsection 1.7.

Proposition 54. *Let X be a locally compact triangle polyhedron.*

1. *If X is hyperbolic then the support of φ_A is a relatively compact subset of $\mathbf{R}$ for any vertex $A \in X$.*
2. *If X has local rank $\leq 3/2$, then for every vertex $A \in X$ the support of φ_A is included in a compact subset of $\mathbf{R}$, excepts perhaps for a reunion of semi-open intervals $I_1, I_2, I_3, \ldots$ of $\mathbf{R}$ of the form*
$$I_p = \left]\,(2p+1)\frac{\sqrt{3}}{2}, \sqrt{3p(p+1)+1}\,\right], \ p \in \mathbf{N},$$
where, observe, $|I_p| \to_p 0$. If moreover the order of X (maximal number of triangle adjacent to an edge) is bounded, then φ_A is bounded.
3. *If X has local rank 2 (i.e. if X is a triangle building) then φ_A vanishes identically for any $A \in X$.*

Proof. (1) Assume that there is some vertex $A \in X$ such the support of φ_A is unbounded. Let $S \subset \mathbf{R}$ be the support φ_A and $(r_n)_n$ be a sequence of points in S converging to ∞. For each n let D_{r_n} be a flat disk in X with center A and radius r_n. Note that by the local compactness assumption, for every $r \in \mathbf{R}$ the number of flat disks of radius r in X with center A is finite. In particular there exists an infinite subsequence S_1 of $(r_n)_n$ such that every disk D_r for $r \in S_1$ coincide on the ball of radius 1 and center A. Iterating, there exists for every $k \in \mathbf{N}^*$ a infinite subset S_{k+1} of S_k such that every disk D_r for $r \in S_{k+1}$ coincide on the ball of radius $k+1$ with some fixed flat disk F_k of center A and radius k. Now the increasing union $F = \cup_k F_k$ is a flat in X. Thus X contains a flat and hence is not hyperbolic by the no flat criterion.

(2) Assume now that the local rank of X is $\leq \frac{3}{2}$. Let A be a vertex of X. We claim that there is a finite number of flat hexagon in X (simplicially isometric to hexagons in $\mathbf{R}^2$ endowed with the tessellation by equilateral triangles) centered at a given vertex A, which are not included in a flat of X.

Indeed let us prove that any two hexagons of X of same simplicial radius which coincide on the ball of simplicial radius 2 are equal. We argue by recurrence. So let H_1 and H_2 be two flat hexagons of X and let B_n be a simplicial hexagon of radius n in X on which H_1 and H_2 coincide. We assume that $n \geq 2$ and prove that H_1 and H_2 coincide on an hexagon of radius $n+1$ (provided $n+1$ is no greater than the common radius of H_1 and H_2). As $n \geq 2$ there is a vertex A' in the boundary of B_n whose internal angle in B_n is π. In the link $L_{A'}$ of X at A' the two hexagons H_1 and H_2 generate two path of length π creating with the path corresponding to B_n two cycles of length 2π in $L_{A'}$. By our local rank assumption (see the definition in the introduction), this cycles must coincide. But this is easily seen to imply that H_1 coincide with H_2 up to radius $n+1$. Thus $H_1 = H_2$.

This shows that the number of hexagons of X of radius n is bounded by the number of hexagons of X of radius 2. Consider the family $\mathcal{F}_A$ of maximally flat hexagons (i.e. not included in larger flat hexagons) centered at A of radius at most 2. The preceding paragraph shows that the map on $\mathcal{F}_A$ which associated to a maximally flat hexagon its simplicial sphere of radius 2 is injective. This proves the claim.

Let $r_0 = r_0' + \frac{\sqrt{3}}{2}$, where r_0' is the maximal radius of the elements in $\mathcal{F}_A$.

Let D be a flat disk of radius $r > r_0$ and center A in X which is not included in a flat, and let $H' \supset D$ be the reunion of all triangle of X whose interiors have non empty intersection with D, so H' is flat as D is. Let H be a maximal hexagon of center A in H'. Then H is included in a flat Π by definition of r_0. As D is not included in flat there is a triangle t of $H'\backslash H$ which is not included in Π. Let U and V be the two vertices of t which belong to H. Our assumption that both links L_U and L_V have rank $\leq \frac{3}{2}$ shows that the disk D must actually be included in $H \cup t$. On the other hand the interior of t has non empty intersection with D by definition. A 2-dimensional computation provides the strong constraint on the radius of D stated in the lemma, i.e. $r \in I_p$ for some $p \in \mathbf{N}$ where

$$I_p = \left](2p+1)\frac{\sqrt{3}}{2}, \sqrt{3p(p+1)+1}\right].$$

But this also shows that the subdisk of D of radius $(2p+1)\frac{\sqrt{3}}{2}$ is contained in Π. By [7] there is only a finite number of distinct flat in X containing A. As the number of triangle on every edge is uniformly bounded, we conclude that φ_A is indeed bounded on $[r_0, \infty[$. More precisely one has

$$\varphi_A \leq Nq^6$$

where N is the number of flats and $q+1$ is the maximal valency of edges (i.e. q is the order of X).

The above proof can be easily adapted to the case of φ_A for any point A of X (i.e. not necessarily a vertex). Then the intervals I_p vary accordingly depending on the position of A on its face.

(3) Assume finally first that X has local rank 2. We have to show that any flat disk in X is included in a flat of X. Let $H_0 \supset D$ be the reunion of all triangle of X whose interiors have non empty intersection with D. Then H_0 is flat as D is, and the internal angle at every point in the boundary of H_0 is at most $4\pi/3$. By applying local rank 2 a finite number of times (at most the number of vertices in the boundary of H_0), we deduce that there is a flat simplicial set H_1 whose interior contains H_0 and which has interior angle at most $4\pi/3$ at every point of its boundary. Iterating this we get a sequence of flat simplicial sets $H_0 \subset H_1 \subset H_2 \ldots$ converging to a flat containing D. □

Remarks 55. (a) The estimates in the proof of Assertion (2) can be made precised. Explicit computations in the case of the polyhedron of [6] (which is of rank $\frac{3}{2}$) are summarized on Figure 2.

(b) The jumps in φ_A may be considered as side effects as they are inherent of the fact we chose Euclidean disks in the definition of mesoscopic rank—what we did so as to have a definition applying to any CAT(0) polyhedron of dimension 2 (and higher). In the triangle case they can be removed be choosing hexagons instead, as we saw along the proof. Observe however the mesoscopic rank behavior of Theorem 11 are *not* side effects and cannot be removed by choosing hexagons.

(c) The above proofs can be generalized to any CAT(0) space of dimension 2 under suitable uniform boundedness geometry assumptions.

(d) The converse of Assertion (3) is true. In fact it is enough to assume that $\varphi_A(\frac{1}{2}) = 0$ for any middle point A of any edge of X, as is easily seen.

In the remaining part of this Section we prove Theorem 11. We start with the rank $\frac{7}{4}$ case for we already are familiar with it from Section 4.

6.1. **Proof of Theorem 11, Item (b).** Recall that the complex V_0^1 admits the following presentation:

$$V_0^1 = [[1,2,3],[1,4,5],[1,6,4],[2,6,8],[2,8,5],[3,6,7],[3,7,5],[4,8,7]].$$

We write $a_1, \ldots, a_8$ for the corresponding generators of the fundamental group Γ of V_0^1. The 1-skeleton of the universal cover X of V_0^1 coincide with the Cayley graph of

Γ with respect to $\{a_1, \ldots a_8\}$ and each (oriented) edge in X is labelled by the index in $\{1, \ldots, 8\}$ of its generator. A singular geodesic of X is said to be of the form i^∞, $i \in \{1, \ldots, 8\}$, if all its edges are labelled by i.

Recall (cf. [13, p. 182]) that a simplicial subset S of X is called a (flat) *strip* if it is isometric to a product $I \times \mathbf{R} \subset \mathbf{R}^2$ where I is a compact interval of $\mathbf{R}$. The boundary of S is a reunion of two (parallel) geodesics, say g and h, and is denoted (g, h). The *height* of S is the simplicial distance between g and h. We say that a strip is *periodic* if there is a non trivial $\gamma \in \Gamma$ such that $\gamma(S) = S$. Then there is $\gamma \neq e$ of smallest length satisfying this condition, and we call *period* of S the simplicial length of γ.

Lemma 56. *There are 3 distinct strips of height 1 and period 1 in X whose boundaries are of the form $(5^\infty, 6^\infty)$.*

Proof. A immediate computation shows that the identities

$$a_1a_5 = a_6a_1,\ a_2a_5 = a_6a_2,\ a_3a_5 = a_6a_3,$$

and

$$a_5a_4 = a_4a_6,\ a_5a_8 = a_8a_6,\ a_5a_7 = a_7a_6,$$

holds in Γ. This gives exactly 3 distinct strips of height 1 and period 1 whose boundaries are of the form $(5^\infty, 6^\infty)$ in X. □

In particular there are uncountably many flats in X. Extending the notation i^∞ we encode simplicial paths in X (more precisely their classes modulo Γ) by their corresponding sequence of labels and write g^∞, given such a path g, for the bi-infinite path obtained by juxtaposing of g to itself on both sides infinitely many times.

Lemma 57. *Consider a simplicial path of X of the form*

$$g = 271834.$$

Then g^∞ is geodesic and there are 2 distinct strips of height 1 and period 6 in X whose boundaries are of the form (g^∞, g^∞).

Proof. Let $u = a_1a_8$, $v = a_2a_7$ and $w = a_3a_4$. Then a computation shows that u, v, and w commute to a_6 in Γ. The lemma readily follows. □

In particular the geodesic g^∞ is included in a flat Π of X on which the group

$$\Lambda = \langle wvu, a_6 \rangle \simeq \mathbf{Z}^2$$

acts. The quotient space Π/Λ has 12 triangles. Part of the flat Π and the geodesic segment g are represented on Figure 3.

Lemma 58. *Keep the notations of Lemma 57 and let h be a simplicial path of X of the form*

$$h = 65.$$

Then h^∞ is a CAT(0) geodesics of X and there are 3 distinct strips of height 1 and period 6 in X whose boundaries have the form (g^∞, h^∞).

Proof. Let $w_g = vuw$ and $w_h = a_5a_8$ be the words corresponding to g and h. The Lemma follows from the identity

$$w_h^3 w_g = w_g w_h^3$$

in Γ, which is easily checked. □

In particular there are uncountably many semi-flats on the geodesic g^∞. The following Lemma establishes Item (b) of Theorem 11 (as well as exponential mesoscopic rank).

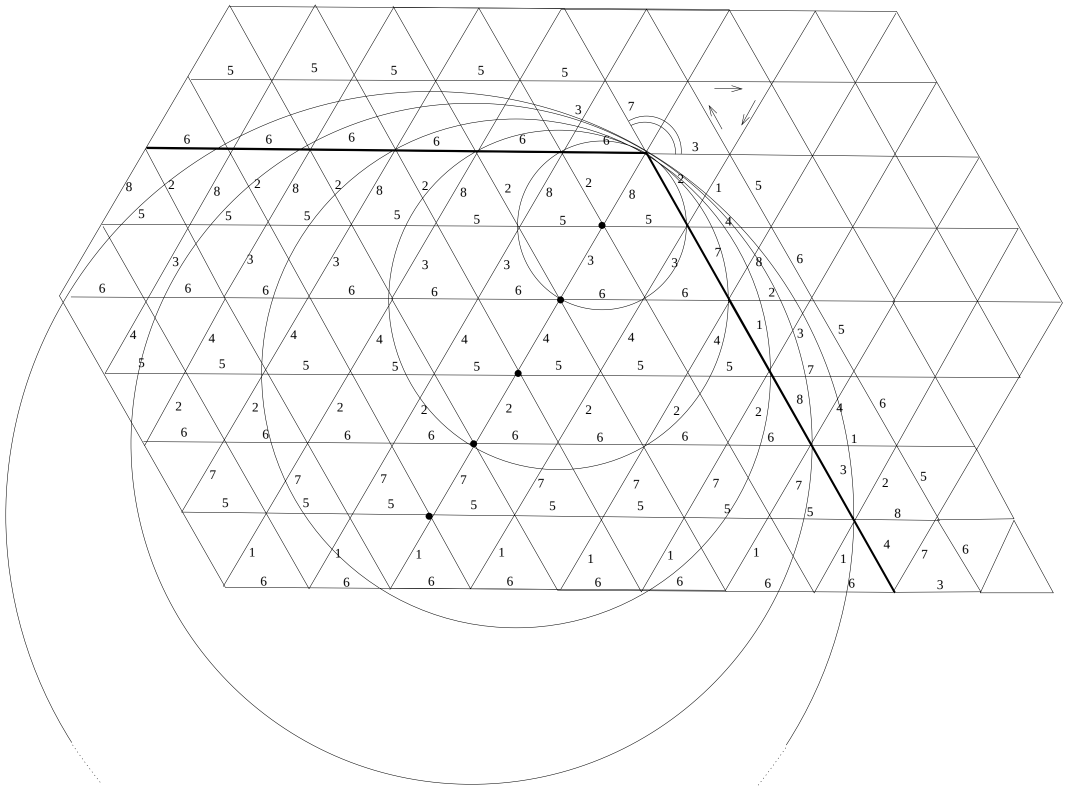

FIGURE 3. Exponential mesoscopic rank for V_0^1

Lemma 59. *Let A be a vertex of X and k be an integer greater than $\frac{\sqrt{3}}{2-\sqrt{3}} = 7.46....$ Then on the interval $]k-\sqrt{3}, k]$ of $\mathbf{R}_+$ one has*

$$\varphi_A \geq 2^{2\mu_k - 4}$$

where

$$\mu_k = \left\lceil k(\frac{2}{\sqrt{3}} - 1) \right\rceil .$$

In particular φ_A has continuous support starting from 7, and exponential growth.

Proof. Let Π be the flat of X containing A defined after Lemma 57. The geodesics of the form 6^∞ and g^∞ intersect with internal angle $2\pi/3$ in Π, and there is a unique vertex B of Π whose distance to A is k, and such that the line segment $(AB]$ of Π is the bisector of this angle (see Figure 3). Denote by d_1 and d_2 the lines of Π issued from B corresponding to 6^∞ and g^∞ respectively, and by Π_0 the sector of angle $2\pi/3$ at B whose boundary is included in $d_1 \cup d_2$.

Let μ_k be the integer defined in the statement and let $\nu_k = 2^{\mu_k - 1}$. By Lemma 56 one can find ν_k distinct strips $\{S_1, \ldots S_{\nu_k}\}$ in X of height μ_k whose boundaries all contain d_1, and which contain the strip of height 1 on d_1 in Π which is opposite to Π_0. On the other hand by Lemma 58 one can find ν_k distinct strips $\{T_1, \ldots T_{\nu_k}\}$ in X of height μ_k whose boundary contains d_2, which contains d_1 as well as the strip of height 1 on d_1 defined in Lemma 58.

Let $i \in \{1, \ldots, \nu_k\}^2$ and consider the subset Π_i of X defined by $\Pi_i = \Pi_0 \cup S_{i_1} \cup T_{i_2}$. Then our choice of μ_k shows, by an elementary exercise in Euclidean geometry (in $\mathbf{R}^2$), that

- the set D_i of points of Π_i at distance $\leq k$ from A in Π_i is a flat disk in X whose boundary contains B,

- the disks D_i are pairwise distinct when i varies in $\{1,\dots,\nu_k\}^2$.

For $r \in [0,k]$ write D_i^r for the concentric disk of radius r in D_i. Assume that $\mu_k \geq 2$. Then it is not hard to show that for any fixed $r \in]k-\sqrt{3},k]$ the family of disks $\{D_1^r, \dots D_{\nu_k}^r\}$ contains at least $2^{2\mu_k-4}$ distinct elements. Observe that $\mu_k \geq 2$ if $k > \frac{\sqrt{3}}{2-\sqrt{3}}$.

Let $j \in \{1,\dots,\nu_k\}^2$ and $r \in]k-\sqrt{3},k]$. Let us now show that the disk D_j^r is not included in a flat.

For a disk D_i, $i \in \{1,\dots,\nu_k\}^2$, define $\tilde{D}_i$ to be the sector of center A and angle $2\pi/3$ inside D_i whose bisector is the segment $[A,B]$.

Claim 60. *Every flat that contains D_j^r must contain one of the sectors $\tilde{D}_i$ for some $i \in \{1,\dots,\nu_k\}^2$.*

Proof of Claim 60. Note that as $r > k-\sqrt{3}$ the disk D_j^r intersects some strips T_{i_1} and S_{i_2}, $i \in \{1,\dots,\nu_k\}^2$, up to height 2 at least. Let Π' be a flat that contains D_j^r and $\tilde{\Pi}'$ be the sector of center A and angle $2\pi/3$ inside Π' whose bisector is the segment $[A,B]$. A local argument (along d_1 and d_2) shows that $\tilde{\Pi}'$ contains A as well as the triangles in T_{i_1} and S_{i_2} adjacent to d_1 and d_2. Then as the transverse valency of the two sets $\cup_{i\in\{1,\dots,\nu_k\}} T_i$ and $\cup_{i\in\{1,\dots,\nu_k\}} S_i$ is maximal (i.e. equal to 3), the flatness of Π' shows that the intersection of $\tilde{\Pi}'$ with the disk of center A and radius k in Π' is of the form $\tilde{D}_i$ for some $i \in \{1,\dots,\nu_k\}^2$. □

Hence it is enough to prove that $\tilde{D}_i$ is not included in a flat. We prove that $\tilde{D}_i$ is actually maximally flat in the sense that it is not included in any open flat disk of X centered at A. Indeed assume that there is such a disk D_i'. This gives a path of length 2π in the link L_A at A in X. The construction of S_i and S_0 shows that L_A contains the path 3251 (see Figure 3, where we identified L_A with the simplicial sphere of radius 1 at A so L_A is included in the 1-skeleton of X and inherits of its labelling). It is easily seen that there is no such a cycle in L_A.

This shows that φ_A is at least equal to $2^{2\mu_k-4}$ on $[k-\sqrt{3},k]$ and proves the lemma. □

6.2. **Proof of Theorem 11, Item (a).** Let us first give some more details on the complex of friezes $\tilde{V}_{\bowtie}$.

Let P be the complex constructed in Section 3 of [5]. Recall that P is compact with 2 vertices A_1, A_2 whose universal cover is an exotic triangle building Δ of order 2. By a theorem of Tits ([58], see also Théorème 1 in [5]) there are exactly two isomorphism classes of spheres of radius 2 of this family of triangle buildings. They correspond to the 2-sphere of the building of $\mathrm{PSL}_3(K)$ in the two cases $K = \mathbf{Q}_2$ or $K = \mathbf{F}_2((t))$. It is shown in [5, Théorème 6] that the sphere of radius 2 at a vertex A of Δ corresponds to $\mathbf{Q}^2$ if and only if A projects to (say) A_1.

Let S be the *median section* which is described on Figure 25 of page 599 in [5]. This is a (metric) graph with 6 vertices and 9 edges, and $P \backslash S$ is a disjoint union of two complex with one vertex and boundary isometric to S. Let P_1 be the closure of the complex which contains A_1 (i.e. which associated to $\mathbf{Q}^2$).

Definition 61. The complex $V_{\bowtie}$ is defined to be the complex obtained by gluing together two copies of P_1 along S via the identity map.

We denote by O and O' its two vertices corresponding to A_1 and number from 1 to 6 the vertices corresponding to $S \subset V_{\bowtie}$.

Lemma 62. *The universal cover $X = \tilde{V}_{\bowtie}$ of complex $V_{\bowtie}$ is a CAT(0) space.*

Proof. It is easily checked that the links at the vertices of S all are isometric to a trivalent graph with two vertices, and 3 edges between these vertices of respective length $2\pi/3$, $4\pi/3$, and $4\pi/3$. Thus all cycles have length $\geq 2\pi$. $\square$

The above surgery creates 3 shapes in $V_{\bowtie}$ (represented on Figure 4): a triangle, a lozenge, and a bow tie (the latter we translate a 'queue d'aronde' in french). They are unions of 4, 2, and 6 equilateral triangles respectively.

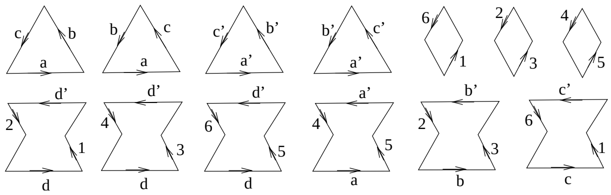

FIGURE 4. The group of friezes $\Gamma_{\bowtie}$

The complex $V_{\bowtie}$ contains 4 triangles, 3 lozenges and 6 bow ties. One checks that two of the triangles $t_1 = abc$ and $t_2 = acb$ are glued on O while the two others $t_1' = a'b'c'$ and $t_2' = a'c'b'$ are glued on O'. Three bow ties are adjacent to a loop d on O and a loop d' on O'. The three others are glued alongside to triangles as indicated on Figure 4.

The following shows exponential mesoscopic rank of the group $\Gamma_{\bowtie}$.

Lemma 63. *Let A be a vertex of X which projects down to O in $V_{\bowtie}$ and $k \geq 3$ be an integer. Then on the interval $](k-1)\sqrt{3}, k\sqrt{3}]$ of $\mathbf{R}_+$ one has*

$$\varphi_A \geq 2^{2k-4}.$$

The function φ_A has continuous support starting from $\frac{\sqrt{3}}{2} = 0.86...$, and exponential growth.

Proof. Let T be a flat sector of angle $\pi/3$ in X built out of t_1 and t_2, whose boundary is a reunion of two half-lines d_1 and d_2 intersecting at a point B. We can assume that d_1 and d_2 are labelled by a and b respectively. Let $[B, \infty[$ be the bisector of T.

Consider the following three strips of height 1 in X:

- St_1, corresponding to the fourth bow tie, with boundaries of the form $(a^{\infty}, a'^{\infty})$,
- St_2, corresponding to the triangles t_1, t_2, with boundaries of the form (a^{∞}, a^{∞}),
- St_3, corresponding to the triangles t_1', t_2', with boundaries of the form $(a'^{\infty}, a'^{\infty})$,

with the labelling of Figure 4 and the notations of Lemma 59. Then, given a integer $k \geq 1$, a computation shows that there are precisely $\nu_k = 2^{k-1}$ flat strips $S_1, \ldots, S_{\nu_k}$ of simplicial height k on the line d_1 which start with St_1.

The configuration is symmetric relatively $[B, \infty[$, i.e. denoting $\mathrm{St}_1^*, \mathrm{St}_2^*, \mathrm{St}_3^*$ the three strips defined analogously to $\mathrm{St}_1^*, \mathrm{St}_2^*, \mathrm{St}_3^*$, there precisely ν_k flat strips $S_1^*, \ldots, S_{\nu_k}^*$ of simplicial height k on the line d_2 which start with St_1^*.

Let A be a vertex of $]B, \infty[$, so that $|A - B| = k\sqrt{3}$ for some integer $k \geq 1$ (note that by construction all vertices of $[B, \infty[$ project down to O in $V_{\bowtie}$). We assume that $k > 2$, so in particular

$$(k-1)\sqrt{3} > k\frac{\sqrt{3}}{2}.$$

As in the proof of Lemma 59 for $i \in \{1, \ldots, \nu_k\}^2$ we let Π_i be the subset of X defined by $\Pi_i = T \cup S_{i_1} \cup S_{i_2}^*$. Then

- the set D_i of points of Π_i at distance $\leq k\sqrt{3}$ from A in Π_i is a flat disk in X whose boundary contains B,
- the disks D_i are pairwise distinct when i varies in $\{1, \ldots, \nu_k\}^2$.

Then for $r \in [0, k]$ the family of concentric disks D_i^r of radius r in D_i, for any fixed $r \in](k-1)\sqrt{3}, k\sqrt{3}]$ contains at least ν_{k-2}^2 distinct elements.

But for $j \in \{1, \ldots, \nu_k\}^2$ and $r \in]k - \sqrt{3}, k]$ the disk D_j^r is not included in a flat, because the disks D_i, $i \in \{1, \ldots, \nu_k\}^2$ are not (for a similar, albeit more geometrical, reason to that of Lemma 59: a singularity arises at the apex B of T).

This shows that φ_A is at least equal to $\nu_{k-2}^2 = 2^{2k-4}$ on $[(k-1)\sqrt{3}, k\sqrt{3}]$. The fact that φ_A has continuous support as soon as $r > \sqrt{3}/2$ is not hard to show. This proves the lemma. □

Remark 64. If in the above proof one had enumerated the strips S_i, S_i^* up to simplicial isomorphic (i.e. up to 'type of frieze') rather than up to equality in X, then we would have found precisely F_k such strips of simplicial height k, where $(F_k)_{k\geq 0}$ is the Fibonacci sequence

$$0, 1, 1, 2, 3, 5, 8, 13, \ldots$$

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