# Modeling of Electrical Field Modified by Injected Space Charge

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**This paper presents the numerical solution of the coupled Poisson equation and charge conservation equation. We present an algorithm to obtain the distributions of electric field and charge density resulting from a corona discharge in the two-dimensional hyperbolic blade-ground plate configuration. We use finite elements method (FEM) to determine the potential distribution, finite volume method (FVM) and method of characteristics (MOC) to determine the distribution of charge density. The structured mesh is redefined at each iteration step to decrease artificial numerical diffusion. For solving the conservation equation, MOC with redefinition of structured mesh appears to be the best technique.**

***Index Terms*—Finite element, finite volume, method of characteristics, Poisson equation.**

## I. Introduction

In SOME electrostatic applications, like electrostatic precipitation, one or several electrodes are injecting charge carriers in an insulating medium (for example through corona effect in gases). The injected space charge $\rho$ modifies the electrical field distribution which, in turn, controls the distribution of space charge. Generally there is no analytical solution for this problem. Previous attempts to obtain numerical solutions are using classical methods to solve Poisson equation and the equation of charges (which is of first order in $\rho$). These attempts using structured or unstructured mesh were limited to two-dimensional (2-D) electrodes configurations where the injecting electrodes are wires characterized by an approximately constant electric field at their surface. The problem is much more difficult when the injecting electrode is a blade or a needle because the electric field is very inhomogeneous and the charge generation (through corona effect in a gas) is restricted to a small region close to the blade edge or to the needle tip.

Anagnostopulos and Bergeles used the finite difference method (FDM) to solve Poisson equation and MOC to solve Poisson equation with structured mesh in wire-duct electrostatic precipitator [1]. Levin and Hoburg used FEM to solve Poisson equation and FVM (donor-cell) for the charge equation with unstructured mesh in wire-duct electrostatic precipitator [2]. Butler *et al.* employed FEM and MOC with unstructured mesh in wire-duct configuration and wire-cylinder precipitator. Adamiak and Atten used FEM and a particular MOC with structured mesh in point-plane configuration [4]. Abdel-Salam and Al-Hamouz solved the problem with a mesh which depends on tubes of field lines in the case of conductor-to-plane configuration [5]. Cristina *et al.* used a structured mesh to solve Poisson equation in pipe-type configuration [6]. Davidson *et al.* solved the problem of barbed plate-plate precipitator and point-plane (3-D configuration) by using a structured mesh with FEM and MOC [7]. Elmoursi *et al.* used the method of charge simulation and MOC to solve the problem of rod-plane geometry with structured mesh to evaluate the electrical field and the charge distribution [8].

Here, we treat the hyperbolic Blade-Plane problem of two electrodes. We present the two methods of calculations of charges, FVM and MOC, with some details. Our work focuses on the redefinition of the structured mesh and on the best technique to solve the problem.

## II. Formulation of the Problem

Here, the model for corona modified field is simplified by neglecting the thickness of the ionization layer on the corona electrode and considering only one ionic species, moving with a constant mobility. The two coupled equations (Poisson equation (1) and charge conservation equation (2), governing the electrical potential $V$ and the charge density $\rho$ in the drift zone, are

$$\nabla^2 V = -\rho/\varepsilon \tag{1}$$

$$\nabla \boldsymbol{J} = 0 \tag{2}$$

where $\boldsymbol{J} = \sigma \boldsymbol{E} + \rho(\boldsymbol{u} + K\boldsymbol{E}) - D\nabla\rho$ with $\boldsymbol{E} = -\nabla V$

- $V$ electrical potential (volt);
- $\rho$ volume charge density (C/m$^3$);
- $\boldsymbol{J}$ density of current (A/m$^2$);
- $\boldsymbol{E}$ electrical field (V/m);
- $\sigma$ electrical conductivity (S/m);
- $K$ mobility of charge carriers (m$^2$/V.s);
- $D$ diffusion coefficient of charge carriers (m$^2$/s);
- $\boldsymbol{u}$ velocity field of the medium (m/s).

Assuming that the medium conductivity is zero and that the diffusion and convection currents are negligible compared to the drift current $\boldsymbol{J} = K\rho\boldsymbol{E}$, (2) leads to

$$\boldsymbol{E} \cdot \nabla\rho = -\rho^2/\varepsilon. \tag{3}$$

Digital Object Identifier 10.1109/TMAG.2005.844546

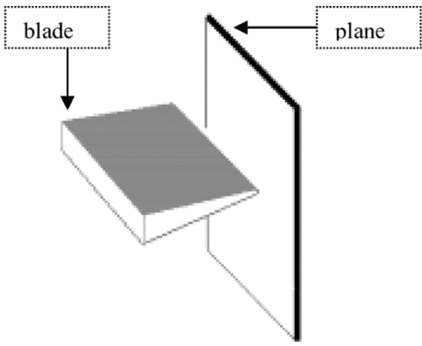


Fig. 1. Schematic view of the two electrode configuration: hyperbolic blade-plane.

The boundary conditions associated with (1) are of Dirichlet type ($V = \text{Const.}$ on the electrodes). From the mathematical viewpoint, only one boundary condition concerning $\rho$ is associated with (3): the charge density must be prescribed at the injector [4]. Physically, with the blade-plane (2-D) configuration considered here (see Fig. 1), the corona discharge takes place in a narrow zone along the blade edge where the small radius of curvature $r$ induces a very strong electric field. The small ionizing zone is disregarded and the boundary condition simplified into prescribing the charge density $\rho_{\text{inj}}$ of injected ions at the blade surface (this distribution can have steep gradients). Finally, by taking appropriate references for the different variables, the problem is put in non dimensional form.

## III. Numerical Technique

As it is usually done in this space charge problem, the governing equations are solved by a procedure of successive approximations. For a given $\rho$ distribution, the Poisson equation (1) is solved, which gives a new potential distribution. With the new field distribution, we determine a new distribution of space charge, by solving the charge conservation equation (in one of the forms (2) or (3) depending on the retained numerical method). For strong injection levels, an under-relaxation procedure is used to avoid instability of the iteration scheme.

A structured mesh is used; its nodes are the intersections of two sets of orthogonal field and equipotential lines. The initial mesh is defined from the solution of Laplace equation for the electrodes configuration. Here, by taking a blade of hyperbolic cross section, the coordinates of the nodes are easily obtained using the corresponding conformal mapping (see Fig. 2). Some attempts to obtain the solution were developed with this initial mesh but we show later that an appropriate way of solving the space charge problem requires to redefine the mesh at each iteration step.

### A. Poisson Equation

The boundary conditions associated with the Poisson equation are (in nondimensional form) of Dirichlet type on the electrodes: $V = 1, V_p = 0$ and Neumann condition $\partial V/\partial n = 0$ on the axis of the symmetry and on the outer boundary defined as a field line of the harmonic solution (Fig. 2). The problem

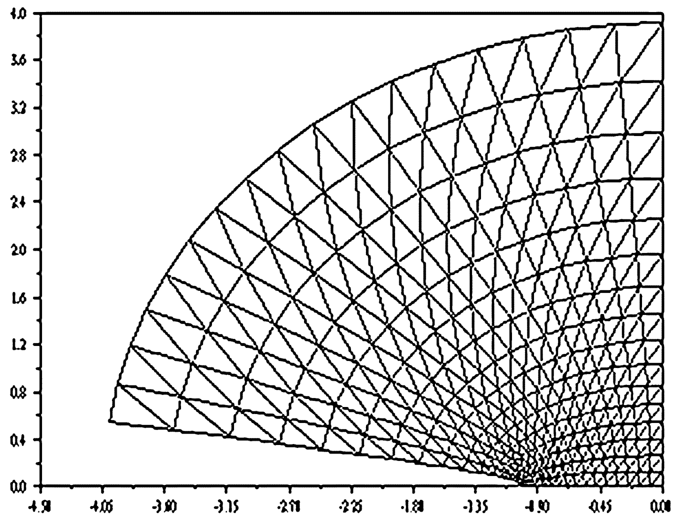

Fig. 2. Structured mesh for the FEM (hyperbolic blade with a normalized radius of curvature $r = 0.02$).

is solved using the FEM with first order triangular finite elements (see Fig. 2). The FEM procedure was tested on a configuration of two coaxial cylinders, for which an analytical solution is available.

### B. Charge Conservation Equation

In order to solve the first order partial differential equation for the charge density, we examined the two numerical techniques: finite volume method (FVM) working on (2) and the method of characteristics (MOC) applied to the form (3) of the charge conservation. As it is observed experimentally that the current density on the collecting plate drops suddenly to zero at some distance from the axis [4], particular attention has been devoted to the ability of the numerical procedure to give solutions with quasi-discontinuity of $\rho$. Before considering injection laws relating the injected charge density with the local electric field on the injecting electrode [4], we examined two distributions of $\rho_{\text{inj}}$: a continuous one with Gauss distribution as a function of the curvilinear coordinate $s$ along the hyperbola and a "rectangular" discontinuous one with $\rho_{\text{inj}} = \rho_0$ for $0 \le s \le s_0$ and $\rho_{\text{inj}} = 0$ for $s > s_0$ (origin $s = 0$ being on the symmetry axis).

*1) FVM Technique:* For the FVM, the dual mesh is defined by the centers of the quadrangles of the finite elements mesh [9].

The FVM is based on the integral form of the charge conservation law

$$\int_{abcd} \vec{\boldsymbol{J}} \cdot d\vec{S} = 0. \tag{4}$$

The upwind method used to determine $\rho$ gives results that verify the current conservation, but at the price of a numerical diffusion when keeping the initial mesh (not readapted): in the case of a rectangular distribution of $\rho_{\text{inj}}$, the obtained $\rho$ distribution exhibit smooth gradients in the direction normal to the field lines.

*2) MOC Technique:* In this method, we use the mesh defined for solving the Poisson equation. We solve the second form of the charge conservation equation which is, in nondimensional form

$$\boldsymbol{E}\nabla(1/\rho) = 1. \tag{5}$$

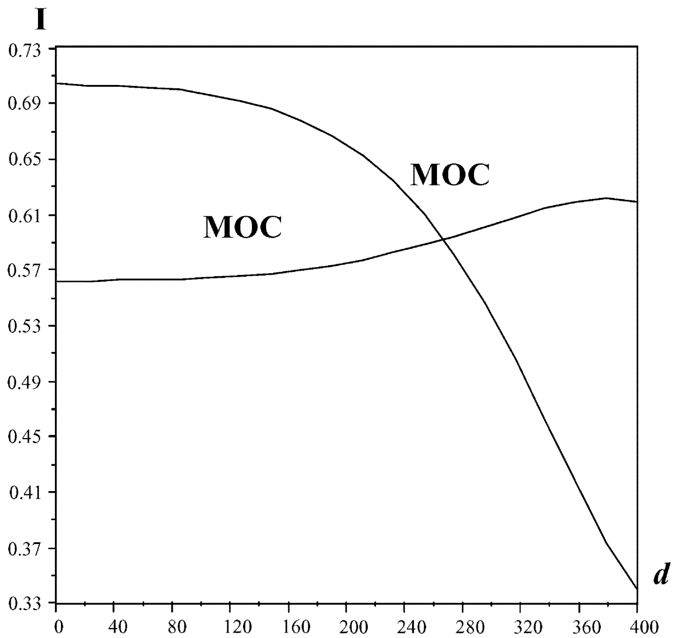


Fig. 3. Total current crossing the equipotential lines as a function of the distance from blade edge on the axis (results obtained with the fixed initial mesh).

In this approach, we used two different interpolations for the charge density. The first one (named MOC1) interpolates in $\rho$, the second one (MOC2) interpolates in $1/\rho$. We note that, without redefining the mesh, the method MOC1 give results which approximately verify the current conservation (Fig. 3) but the numerical diffusion is strong. The numerical diffusion with MOC2 is much less marked than with MOC1, but the current conservation is not at all verified in this case (note the dramatic decrease of current in Fig. 3).

The difficulties in solving (2) or (3) arise from their first order character. A way to overcome the problems is to integrate along the characteristic lines. This is possible if sets of successive nodes lie on the same characteristic lines. It is therefore crucial to adapt the mesh during the iteration.

## IV. Mesh Redefinition

The redefinition of the structured mesh will help us to practically eliminate the numerical diffusion. The trajectories of the charge carriers are the characteristic lines for (3). These trajectories and the orthogonal equipotential lines determine the structured mesh. We determine first the field lines and, then, the new nodes uniformly distributed along them.

To determine the field lines, we build a local linear approximation of the potential by least squares method on the nodal values of two neighboring quadrangles. For instance, to determine the field line (dotted line) in quadrangle #1 of Fig. 4, we use the potential values at the six vertices of the two quadrangles 1 and 2. For the field line in quadrangle 2, we use quadrangles 2 and 3, etc. This method is efficient but sensitive to the edge curvature and to the charge value on the injector. When always starting from the same nodes on the injecting electrode, after some iterations the field lines exhibit a strong divergence in the charged zone (Fig. 5) and the computation fails (numerical instability).

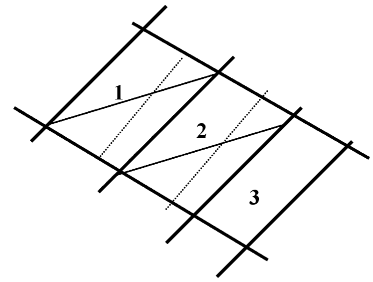


Fig. 4. Representation of quadrangles of the FE mesh.

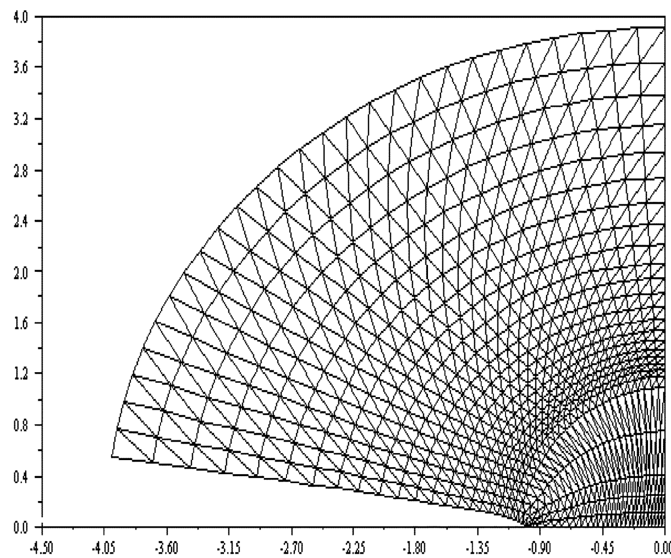

Fig. 5. Mesh deformation after several iterations (fixed nodes on the injecting blade).

To obtain a more regular mesh, it is necessary to redefine the nodes position on the injecting electrode. This was performed through a relation involving the previous nodes separations on the plane (where the electrical lines arrive) and by using an under-relaxation process with a parameter $\Omega$. The value of $\Omega$ depends on the curvature and on the injected charge density; for example, $\Omega < 0.4$ for a radius of curvature $r = 0.02$ and $\rho_0 > 10$. With this technique we could build structured mesh after each new determination of the potential until convergence. Several kinds of subdivisions along the field and equipotential lines were tested to obtain a regular enough mesh with fine distribution of nodes in the zones of high field and charge density values.

## V. Results

By adequately choosing the various parameters influencing the rate of convergence, we obtained good convergence (the mean relative difference between two successive approximations of the charge density distribution becomes lower than $10^{-4}$ after about 40 iterations for a $40 \times 40$ mesh).

The results obtained with the FVM and upwind technique to calculate $\rho$ give a very good conservation (to a few $10^{-4}$) of the current flowing from blade to plate. For a rectangular

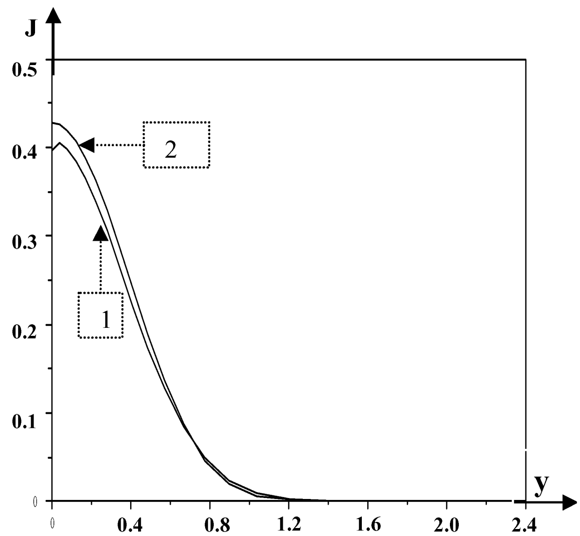


Fig. 6. Current density on the plane versus distance $y$ from the symmetry axis for Gauss distribution for $\rho_{\rm inj}$. Method of charge density determination: (1) FVM. (2) MOC.

$\rho_{\rm inj}$ distribution there is no noticeable diffusion of the charge density. However there is a local minimum of charge density at the axis of the symmetry (see in Fig. 6 the corresponding minimum of current density on the plate).

The MOC technique with redefinition of the mesh gives the most satisfactory results: the current flowing from the blade to the plate is conserved (fluctuations $\sim 10^{-3}$). In the case of rectangular distribution for $\rho_{\rm inj}$, there is no diffusion of charge across the field line separating the charged and charge free zones and Fig. 7 clearly shows the sudden drop of current density on the collecting plate.

## VI. Conclusion

The use of FEM to solve the Poisson equation and of MOC to solve the charge conservation equation combined with the redefinition of the structured mesh at each iteration step of the successive approximations, define an efficient algorithm which eliminates the spurious numerical diffusion associated to the use of MOC or FVM on a fixed structured or unstructured mesh.

This algorithm has been applied successfully to two distributions of charge on the injector, a Gauss distribution and a rectangular (discontinuous) one. It should also work efficiently with an injection law for $\rho_{\rm inj}$ reflecting the main physical characteristics of the corona discharges on blades. It appears well suited to treat the case of blades with a very small radius of curvature. Last but not least, it should be generalized without major difficulty to the case of 3-D electrodes configurations.

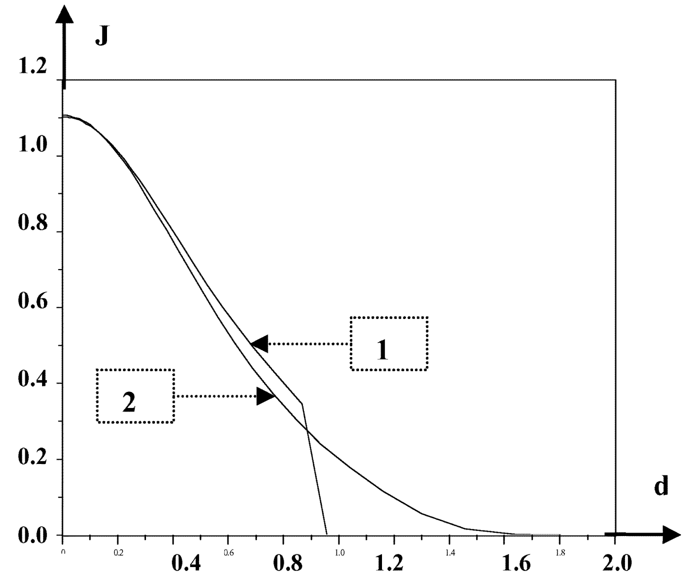


Fig. 7. Current density on the plane with charge density determined using the MOC. Distribution $\rho_{\rm inj}$ on the blade: (1) Rectangular. (2) Gauss ($\rho_{\max} = 15, 21 \times 21$ mesh).

## References

[1] J. Anagnostopulos and G. Bergeles, "Corona discharge simulation in wire-duct electrostatic precipitator," *J. Electrostat.*, vol. 54, pp. 129–147, 2002.

[2] P. L. Levin and J. F. Hoburg, "Donor cell-finite element description of wire-duct precipitator fields, charges, and efficiencies," *IEEE Trans. Ind. Appl.*, vol. 26, no. 4, pp. 662–670, Jul./Aug. 1990.

[3] J. Butler, Z. J. Cendes, and J. F. Hoburg, "Interfacing the finite-element with the method of characteristics in self-consistent electrostatics field models," *IEEE Trans. Ind. Appl.*, vol. 25, no. 3, pp. 533–538, May/Jun. 1989.

[4] K. Adamiak and P. Atten, "Simulation of discharge corona in point-plane configuration," *J. Electrostat.*, vol. 61, pp. 85–98, 2004.

[5] M. Abdel-Salam and Z. Al-Hamouz, "Finite-element analysis of monopolar ionized fields including ion diffusion," *J. Phys. D: Appl. Phys.*, vol. 26, pp. 2202–2211, 1993.

[6] S. Cristina, G. Dinelli, and M. Feliziani, "Numerical computation of corona space charge and V-I characteristic in DC electrostatic precipitators," *IEEE Trans. Ind. Appl.*, vol. 27, no. 1, pp. 147–153, Jan. 1991.

[7] J. H. Davidson, P. J. McKinney, and P. Linnebur, "Three-dimensional (3-D) model of electric field and space charge in the barbed plate-to-plate precipitator," *IEEE Trans. Ind. Appl.*, vol. 32, no. 4, pp. 858–866, Jul./Aug. 1996.

[8] A. A. Elmoursi and C. E. Speck, "Simulation of space charge in unbounded geometries," *IEEE Trans. Ind. Appl.*, vol. 26, no. 2, pp. 384–392, Mar./Apr. 1990.

[9] A. M. Meroth, T. Gerber, C. D. Munz, P. L. Levin, and A. J. Schwab, "Numerical solution of non stationary charge coupled problems," *J. Electrostat.*, vol. 45, pp. 177–198, 1999.

Manuscript received June 8, 2004.