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Valderio Reisen, Eric Moulines, Philippe Soulier, Glaura Franco

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Valderio A. Reisen Eric Moulines Philippe Soulier Glaura Franco

Departamento de Estatística - CCE - UFES - Av. Fernando Ferrari , s/n - Vitória - ES - Brazil - CEP: 29065-900. e-mail: valderio@cce.ufes.br

GET/Télécom Paris, 46 rue Barrault, 75634 Paris, Cédex 13. e-mail: moulines@tsi.enst.fr

Department of Statistics, Universidade Federal de Minas Gerais, Belo Horizonte, MG, Brazil. e-mail: glaura@est.ufmg.br

Abstract: This paper introduces a semiparametric regression estimator of the memory parameter for long-memory time series process. It is based on the regression in a neighborhood of the zero-frequency of the periodogram averaged over epochs. The proposed estimator is theoretically justified and empirical Monte Carlo investigation gives evidence that the method is very promising to estimate the long-memory parameter.

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1. Introduction

Let $\{X_t\}$ be a covariance stationary process with spectral density

$$f(\omega) \triangleq |\omega|^{-2d} f_\star(\omega)\ , \omega \in [-\pi, \pi] \quad \text{and} \quad d \in (-1/2, 1/2)\ , \tag{1}$$

where $f_\star(\omega)$ is continuous at zero, $0 < f_\star(0) < \infty$ and $\int f_\star(\omega)\mathrm{d}\omega < \infty$. To maintain generality of the short-run dynamics, we do not impose a specific functional form on $f_\star(\omega)$. Equation (1) is referred to as a *semiparametric* model for $f(\omega)$, specifying its form only near zero frequency. Time series with spectral density satisfying (1) can be observed in many areas of applications; see for example Beran (1994), Doukhan et al. (2003) and the references therein. The process is said to have *short-memory* when $d = 0$, *long-memory* when $d \in (0, 1/2)$ and *negative* (or *intermediate*) memory if $d \in (-1/2, 0)$. Equation 1 is satisfied leading models for long and negative memory such as Fractional Autoregressive Integrated Moving Averages (FARIMA) and fractional noise. These, however, are parametric models, specifying $f(\omega)$ up to finitely many unknown parameters over all frequencies $(-\pi, \pi]$. The memory parameter d (like the scale parameter $f_\star(0)$) is typically unknown and should be estimated. Many works have been devoted to the estimation of the memory parameter in the semiparametric context. One of the most popular is the ordinary least squares (OLS) estimator, due to Geweke & Porter-Hudak (1983) (GPH). This method employs the periodogram to obtain, through a regression equation, the OLS estimator of the memory parameter. Since the GPH estimator has been introduced, other variants of this method have been suggested with the aim of improving the quality of estimates and to achieve better asymptotic sample properties; see Doukhan et al. (2003) for an in depth survey and Nielsen & Frederiksen (2005) for a detailed experimental study.

In some areas of application, extremely long time series (hundreds of thousands to millions of samples are not uncommon, for example, in the analysis of teletrafic time-series or in high-frequency finance) has to be dealt with. Often, the analysis of these data has to be done on-line. Instead of computing the periodogram on the whole data set, a simple solution to the problem consists in partitioning the sample into subsets, referred to as *epochs*, computing the periodogram over each epoch and averaging these to obtain an averaged version of the periodogram. This line of research has been pursued in the parametric context by Beran & Terrin (1994) but has not, up to our best knowledge, been explored in the semiparametric context.

In this contribution, we study the averaged periodogram spectral estimator, based on the division of series into epochs, to obtain the memory parameter estimate of a long-memory process. The estimation method follows the GPH procedure, where the periodogram is replaced by the averaged periodogram in the regression equation. Some desirable asymptotic properties of the proposed estimator are derived and empirical investigation gives evidence to support the use of the procedure as an alternative method to GPH to reduce the variance of the fractional memory parameter.

All these topics are presented in the paper as follows: Section 2 outlines some properties of the averaged periodogram. Section 3 presents the proposed estimator of the memory parameter d and discuss the statistical performance of the estimator. Section 4 presents the set-up of the Monte-Carlo experiments and assesses the finite sample properties of our estimator.

Notations: In the paper, $a \triangleq b$ denotes that a is defined as b, $\lfloor a \rfloor$ is the integer part of a and $a \wedge b \triangleq \min(a, b)$.

2. Some properties of the averaged periodogram of a fractional difference process

Let $\{X_t\}_{t=1}^N$ be a realization of the process X_t and $N = gn$, where g and n are integer values that correspond , respectively, to the number of epochs and the sample size of each epoch. The most natural tool for inference in the spectral domain is the periodogram, defined as the square modulus of the discrete Fourier transform (DFT). Instead of computing the periodogram over the whole dataset, the DFT and the periodogram of each individual epoch are obtained at the Fourier frequencies $\omega_k = 2\pi k/n$, $1 \leq k \leq [n/2]$, *i.e.* for $\ell = 0, \ldots, g-1$,

$$d_{\ell,n}(\omega_k) \triangleq (2\pi n)^{-1/2} \sum_{t=1}^{n} X_{t+\ell n} \mathrm{e}^{\mathrm{i}t\omega_k} \;, \tag{2}$$

$$I_{\ell,n}(\omega_k) \triangleq |d_{\ell,n}(\omega_k)|^2 = (2\pi n)^{-1} \left| \sum_{t=1}^{n} X_{t+\ell n} \mathrm{e}^{\mathrm{i}t\omega_k} \right|^2 \;. \tag{3}$$

The averaged periodogram is then defined as the sample mean of the periodogram obtained on the successive epochs,

$$\bar{I}_{g,n}(\omega_k) = g^{-1} \sum_{\ell=0}^{g-1} I_{\ell,n}(\omega_k) \;. \tag{4}$$

The averaged periodogram serves as the basis of the Welch estimator of the spectral density. Since in the sequel the spectral density f can be either zero ($d < 0$) or infinite ($d > 0$) at zero frequency, we have found more appropriate to state results using the normalized periodogram, *i.e.* the raw periodogram normalized by the inverse of the spectral density. For short-memory processes, *i.e.* $d = 0$ in (1), it is well known that

(i) the periodogram in each epoch is an asymptotically unbiased estimate of the spectral density, *i.e.* $f^{-1}(\omega_k)\mathbb{E}[I_{0,n}(\omega_k)] = 1 + O(n^{-1})$, $1 \leq k \leq [n/2]$, where the $O(n^{-1})$ term is uniform with respect to (*w.r.t.*) the frequency coordinates k,

(ii) the DFT ordinates in a single epoch at distinct Fourier frequencies are asymptotically uncorrelated,

$$\mathrm{Cov}\left(f^{-1/2}(\omega_j) d_{0,n}(\omega_j), f^{-1/2}(\omega_k) d_{0,n}(\omega_k)\right) = \delta_{j,k} + O(n^{-1}) \;, \quad 1 \leq j \leq k < [n/2],$$

where the $O(n^{-1})$ term is uniform *w.r.t.* the frequency indices j, k.

(iii) the DFT ordinates over different epochs are asymptotically uncorrelated, i.e. for any $\ell > 0$,

$$\mathrm{Cov}\left(f^{-1/2}(\omega_j)d_{0,n}(\omega_j), f^{-1/2}(\omega_k)d_{\ell,n}(\omega_k)\right) = O(n^{-1}) \ , \qquad 1 \leq j \leq k < [n/2],$$

where the $O(n^{-1})$ term is uniform *w.r.t.* the frequency indices j, k and the epoch index ℓ.

Assuming that the number of epochs g is a fixed integer, the above results imply that

(iv) the averaged periodogram is an asymptotically unbiased estimator of the spectral density, *i.e.* $f^{-1}(\omega_k)\mathbb{E}[\bar{I}_{g,n}(\omega_k)] = 1 + O(n^{-1})$, where the $O(n^{-1})$ term is uniform in k;

(v) the averaged periodogram ordinates are asymptotically uncorrelated and its variance is equal to the square of the spectral density divided by the number of epochs,

$$\mathrm{Cov}\left(f^{-1}(\omega_j)\bar{I}_{g,n}(\omega_j), f^{-1}(\omega_k)\bar{I}_{g,n}(\omega_k)\right) = g^{-1}\delta_{j,k} + O(n^{-1}).$$

We now discuss some periodogram properties for the process (1) with $d \neq 0$.

It was first shown by Künsch (1986) and exhaustively investigated by Hurvich & Beltrão (1993) in the single epoch case that the asymptotic behavior of low frequency DFT ordinates departs strongly from the weak dependence situation: in particular, the DFT ordinates computed at Fourier frequencies $\omega_j = 2\pi j/n$ and $\omega_k = 2\pi k/n$ for fixed positive integers j and k are correlated as $n \to \infty$. Not surprisingly, the same departure from the weak dependence behavior is also observed in a multiple epochs scenario. The dependence in particular implies that the correlation among DFT coefficients over different epochs does not asymptotically vanish. More precisely, the following theorem shows that the correlation of the DFT coefficients evaluated at Fourier frequencies ω_j and ω_k, for *fixed* j and k, computed over different epochs does not vanish as $n \to \infty$.

Theorem 1. *Assume that the spectral density f is given by* (1). *Let $\ell \geq 0$ and $1 \leq j \leq k$ be fixed positive integers. Then,*

$$\lim_{n\to\infty} \omega_j^d \omega_k^d \mathbb{E}\left[d_{0,n}(\omega_j)\bar{d}_{\ell,n}(\omega_k)\right] = \frac{(2\pi j)^d(2\pi k)^d}{2\pi} D_1(d; j, k, \ell) \tag{5}$$

$$\lim_{n\to\infty} \omega_j^d \omega_k^d \mathbb{E}\left[d_{0,n}(\omega_j)d_{\ell,n}(\omega_k)\right] = \frac{(2\pi j)^d(2\pi k)^d}{2\pi} D_2(d; j, k, \ell) \ . \tag{6}$$

where

$$D_1(d; j, k, \ell) \triangleq \int_{-\infty}^{\infty} |\omega|^{-2d} \Delta(\omega - 2\pi j)\Delta(2\pi k - \omega)\mathrm{e}^{-\mathrm{i}\ell\omega}\mathrm{d}\omega \ , \tag{7}$$

$$D_2(d; j, k, \ell) \triangleq \int_{-\infty}^{\infty} |\omega|^{-2d} \Delta(\omega - 2\pi j)\Delta(-2\pi k - \omega)\mathrm{e}^{-\mathrm{i}\ell\omega}\mathrm{d}\omega \tag{8}$$

with $\Delta(\omega) \triangleq (\mathrm{e}^{\mathrm{i}\omega} - 1)/(\mathrm{i}\omega)$.

Proof. See Section 5. □

Theorem 1 combined with the results of Theorem 5 given in Hurvich & Beltrão (1993) implies that the averaged periodogram is an asymptotically biased estimate of the spectral density, *i.e.* $\lim_{n\to\infty} f^{-1}(\omega_j)\bar{I}_{g,n}(\omega_j) \neq 1$ for any given j, $1 \leq j \leq k$, and the correlation of the averaged periodogram ordinates does not asymptotically vanish, *i.e.*

$$\lim_{n\to\infty} f^{-1}(\omega_j) f^{-1}(\omega_k) \mathrm{Cov}(\bar{I}_{g,n}(\omega_j), \bar{I}_{g,n}(\omega_k)) \neq \delta_{j,k} ,$$

for any given j and k. Nevertheless, under appropriate regularity condition for the spectral density of the short-memory process $f_\star$, it can established that there exist two sequences $\{r(f;k)\}$ and $\{r(f;j,k)\}$ satisfying, for all n and all $1 \leq j < k \leq [n/2]$,

$$\begin{aligned}
&|\mathbb{E}[\bar{I}_{g,n}(\omega_k)/f(\omega_k)] - 1| \leq r(f;k) , \\
&|\mathrm{Cov}(f^{-1}(\omega_j)\bar{I}_{g,n}(\omega_j), f^{-1}(\omega_k)\bar{I}_{g,n}(\omega_k)| \leq r(f;g,k,j) ,
\end{aligned}$$

such that

- $\lim_{k\to\infty} r(f;k) = 0$, which means that the bias is small for frequencies sufficiently far away from zero, and
- for any sequence m_n such that $\lim_{n\to\infty} m_n = \infty$ and $\lim_{n\to\infty} m_n/n < 1/2$, $\sum_{1\leq j<k\leq m_n} r(f;g,j,k) = O\left(\log^r(m_n)\right)$, for some $r > 0$. This shows that whereas the dependence does not vanish it is small when the Fourier frequencies are sufficiently far apart.It is also possible (see the results below) to get bounds of this quantity as a function of g.

To derive the above results, some regularity assumptions need to be imposed on the spectral density of the short-memory process. Consider the set of functions, which is adapted from (Robinson, 1995, Theorem 2) (see also (Soulier, 2001) and (Moulines & Soulier, 2003)).

Definition 1. *For $\mu \geq 1$ and $\nu \in (0,1]$. Let $\mathcal{L}_\star(\mu,\nu)$ be the set of functions $\phi : [-\pi,\pi] \to \mathbb{R}_+$ satisfying for all $\omega, \omega' \in [-\pi,\pi] \setminus \{0\}$*

$$\max_{\omega\in[-\pi,\pi]} \phi(\omega) \leq \mu\phi(0) , \tag{9}$$

$$|\phi(\omega) - \phi(\omega')| \leq \mu\phi(0) \frac{|\omega - \omega'|}{|\omega| \wedge |\omega'|} , \tag{10}$$

$$|\phi(\omega) - \phi(\omega')| \leq \mu\phi(0) \left||\omega| - |\omega'|\right|^\nu , \tag{11}$$

The set of functions $\mathcal{L}_\star(\mu,\nu)$ contains all the functions which are strictly positive and continuously differentiable on $[-\pi,\pi]$. In particular, if $f_\star$ is the spectral density of an stationary and invertible ARMA process, then $f_\star \in \mathcal{L}_\star(\mu,1)$. More interestingly, it also contains functions of the form $\tilde{f}_\star(\omega) = f_\star(\omega) + \sigma^2|\omega|^\nu$, where $f_\star$ is a strictly positive continuously differentiable function. Note that such spectral density appears in the so-called signal-plus-noise model, where a fractional process with smooth spectral density $f_\star$ is observed in presence of white noise, uncorrelated from the process.

Theorem 2. *Let $\delta_-, \delta_+ \in [0, 1/2)$, $\mu \geq 1$,and $\nu \in (0, 1]$ be constants. Assume that f is given by* (1) *with $f_\star \in \mathcal{L}_\star(\mu)$ and $d \in [-\delta_-, \delta_+]$. Then, there exists a constant C (depending on the constants δ_-, δ_+ and μ) such that, for any $1 \leq j < k \leq n/4$ and $\ell \geq 0$,*

$$\left|\mathbb{E}[f^{-1}(\omega_k) I_{0,n}(\omega_k)] - 1\right| \leq C \log(1+k)/k \ , \tag{12}$$

$$f^{-1/2}(\omega_j) f^{-1/2}(\omega_k) \left|\mathbb{E}[d_{0,n}(\omega_j)\bar{d}_{\ell,n}(\omega_k)] + \mathbb{E}[d_{0,n}(\omega_j) d_{\ell,n}(\omega_k)]\right| \\ \leq C \log(k) j^{-|d|} k^{|d|-1} \left(\ell^{-\{(1-2d)\wedge 1\}} + (\ell n)^{-\nu}\right) \ . \tag{13}$$

Proof. See section 6. □

It is worthwhile to note that the dependence among successive epochs does not asymptotically vanish as $n \to \infty$, in strong contrast with the short-memory case. Also, the strength of the dependence among the epochs depends on the memory coefficient d.

Using Corollary 2.1 in Soulier (2001) and under the additional assumption that the process is Gaussian, it is possible to translate the results above to non-linear transforms of the DFT ordinates, for instance to the "log" function of the average periodogram. These are presented in the following corollary which lead to results that provide a theoretical justification for the estimator proposed in this paper. Some additional notations are required to state the results. Let U be a central chi-square with $2g$ degrees of freedom. Then, $\mathbb{E}[\log(U/2)] = \psi(g)$ and $\mathrm{Var}[\log(U/2)] = \psi'(g)$ where ψ is the digamma function (see (Johnson & Kotz, 1970, p. 198)). For instance, $\psi(1) = -\gamma$, where γ is the Euler constant and $\psi'(1) = \pi^2/6$. It is well-known that $\lim_{g\to\infty} g\psi'(g) = 1$. Hence, for large g, $\mathrm{Var}[\log(U/2)] = O(g^{-1})$. Let

$$\xi_{n,k} \triangleq \log\left[\bar{I}_{g,n}(\omega_k)\right] - \log\left[f_\star(\omega_k)\right] - \psi(g) + \log(g) \ . \tag{14}$$

The following corollary establishes the statistical properties of $\xi_{n,k}$.

Corollary 3. *Assume that X_t is a Gaussian process. Then, there exists an integer K and a constant C (depending only on the constants δ_-, δ_+, μ, and K) such that, for any $K \leq j < k \leq [(n-1)/2]$*

$$\begin{aligned} &|\mathbb{E}[\xi_{n,j}]| \leq C \log(1+j)/j \ , \\ &|\mathrm{Var}[\xi_{n,j}] - \psi'(g)| \leq C \log^2(1+j)/j^2 \ , \\ &|\mathrm{Cov}(\xi_{n,j}, \xi_{n,k})| \leq C \log^2(k) j^{-2|d|} k^{2|d|-2} \ . \end{aligned}$$

Proof. See Section 6. □

3. Estimation of the memory parameter based on log-periodogram regression

As an application of the results obtained above, it is argued in this section that the averaged periodogram is a simple mean to reduce the variance of semiparametric estimator of the memory parameter based on log-periodogram regression. For simplicity, in this contribution the focus will be on the GPH estimator proposed by Geweke & Porter-Hudak (1983) (GPH) and further analyzed by Robinson (1995) and Hurvich et al. (1998). The same reduction of variance holds for the bias reduced log-periodogram estimator introduced by Andrews & Guggenberger (2003), which is based on regression of $\log f_\star(\omega)$ by an even polynomial of degree $2r$, and for the estimator introduced by Guggenberger & Sun (2006), which is obtained by taking a weighted average of GPH estimators over different bandwidths.

The GPH estimator is the ordinary least square (OLS) regression estimator obtained from an approximated regression equation of the logarithm of the spectral density (1), having the logarithm of the spectral density as the dependent variable and $\log(\omega)$ as the independent variable. Taking the logarithm of (1), the log-spectral density can be expressed as

$$\log f(\omega) = \log f_\star(0) - 2d\log(\omega) + \log\left[f_\star(\omega)/f_\star(0)\right] \ . \tag{15}$$

For the proposed estimator, the spectral density $f(\omega)$ is replaced by the averaged periodogram $\bar{I}_{g,n}(\omega)$, and using the decomposition (14), an estimate of d is obtained from the regression equation

$$\log\left[\bar{I}_{g,n}(\omega_k)\right] = a_0 - 2d\log(\omega_k) + \log\left[f_\star(\omega_k)/f_\star(0)\right] + \xi_{n,k} \ , \tag{16}$$

where the intercept is $a_0 = \log f_\star(0) + \psi(g) - \log(g)$ and the random variables $\{\xi_{n,k}\}$ are defined in Corollary 3. The GPH estimate of the memory parameter d is thus given by

$$\hat{d}_{m_n,g} = \sum_{k=1}^{m_n} a_{k,n}(m_n)\log\left[\bar{I}_{g,n}(\omega_k)\right] \tag{17}$$

where $\{m_n\}$, the bandwidth in the regression equation (16), is a sequence of integers and the weights $a_k(m_n)$ are given by

$$a_{k,n}(m_n) \triangleq \frac{[-2\log(\omega_k)] - m_n^{-1}\sum_{j=1}^{m_n}[-2\log(\omega_j)]}{\sum_{k=1}^{m_n}\left\{[-2\log(\omega_k)] - m_n^{-1}\sum_{j=1}^{m_n}[-2\log(\omega_j)]\right\}^2} \ . \tag{18}$$

We will now derive a central limit theorem for the above estimator. To do this, it is required to state some additional regularity conditions of the spectral density of the short-memory process; see Giraitis et al. (2000).

Theorem 4. *Assume that $\{X_t\}$ is a Gaussian process with spectral density f satisfying* (1) *with $f_\star \in \mathcal{L}_\star(\mu, \beta)$ for some $\mu < \infty$ and $\beta \in (0, 1]$ and*

$$\text{for all} \quad \omega \in [-\Omega_0, \Omega_0], \quad |f_\star(\omega) - f_\star(0)| \le \mu f_\star(0)|\omega|^\beta \ .$$

Let $\{m_n\}$ be a non-decreasing sequence of integers such that

$$\lim_{n\to\infty}(m_n^{-1} + m_n^{2\beta+1}n^{-2\beta}) = 0 \ . \tag{19}$$

Then $\sqrt{m_n}(\hat{d}_{m_n,g} - d)$ is asymptotically distributed as Gaussian with zero-mean and variance $\psi'(g)/4$.

Proof. see section 7. □

Using similar arguments as those given in Hurvich et al. (1998), the bias and variance of $\hat{d}_{m_n,g}$ are computed by assuming that $f_\star$ is three times differentiable in a neighborhood of the zero frequency. Using once again Corollary 3,

$$\mathbb{E}\left[\hat{d}_{m_n,g}\right] - d = \sum_{j=1}^{m_n} a_{j,n}(m_n) \log f_\star(\omega_j) + \sum_{j=1}^{m_n} a_{j,n}(m_n)\mathbb{E}[\xi_{n,j}] \tag{20}$$

where $\xi_{n,j}$ is defined in (14) and

$$\mathrm{Var}(\hat{d}_{m_n,g}) = \sum_{j=1}^{m_n} a_{j,n}^2(m_n)\mathrm{Var}(\xi_{n,j}) + \sum_{k=1}^{m_n}\sum_{j=k+1}^{m_n} a_{j,n}(m_n)a_{k,n}(m_n)\mathrm{Cov}(\xi_{n,j}, \xi_{n,k}) \ . \tag{21}$$

Along the same lines as Hurvich et al. (1998) (Lemma 1 to Lemma 8), we establish an explicit expression for the mean-square error (MSE) of the proposed estimator.

Theorem 5. *Assume that $f_\star \in \mathcal{L}_\star(\mu)$ and satisfies the conditions $f_\star'(0) = 0$, $|f_\star''(\omega)| < \infty$ and $|f_\star'''(\omega)| < \infty$ for any $\omega \in [-\Omega_0, \Omega_0]$ where $\Omega_0 \in (0, \pi]$. Then,*

$$\mathbb{E}[\hat{d}_{m_n,g} - d] = -\frac{2\pi^2 f_\star''(0)m_n^2}{9f_\star(0)n^2} + o\left(\frac{m_n^2}{n^2}\right) + O\left(\frac{\log^3(m_n)}{m_n}\right) \tag{22}$$

and

$$\mathrm{Var}(\hat{d}_{m_n,g}) = \frac{\psi'(g)}{4m_n} + o\left(\frac{1}{m_n}\right) \ . \tag{23}$$

Neglecting the remainder terms in the bias and variance, and assuming that $f_\star''(0) \neq 0$ minimizing the approximate expression for the MSE, i.e.

$$\mathrm{MSE}(n, g) = \left[\frac{2\pi^2 f_\star''(0)m_n^2}{9f_\star(0)n^2}\right]^2 + \frac{\psi'(g)}{4m_n} \tag{24}$$

with respect to the bandwidth parameter m_n for a given number of epochs g yields the asymptotically optimal choice for the bandwidth $m_n(g)$

$$m_n(g) \triangleq \left(\frac{\psi'(g)}{16B_\star}\right)^{1/5} n^{4/5} \ , \tag{25}$$

where $B_\star \triangleq (4/81)\pi^4 \{f_\star''(0)/f_\star(0)\}^2$. With this choice for $m_n(g)$, the optimal value for the mean-square error is

$$\mathrm{MSE}(n, g) = C_\star\{\psi'(g)\}^{4/5} n^{-4/5} \ , \tag{26}$$

where $C_\star \triangleq \left\{(16)^{-4/5}B_\star^{-2/5} + (16B_\star)^{1/5}/4\right\}$. We will now discuss the potential advantages in performance obtained by dividing the series into epochs. The optimal MSE of the classical GPH (using a single epoch) is given by $\mathrm{MSE}(N, 1) = C_\star\{\psi'(1)\}^{4/5}N^{-4/5}$.

Dividing the series into g epochs each of size $n = N/g$, the optimal MSE is given by

$$\mathrm{MSE}(N/g, g) = C_\star\{\psi'(g)\}^{4/5}(N/g)^{-4/5} = C_\star\{g\psi'(g)\}^{4/5}N^{-4/5}\ . \tag{27}$$

Since $g \mapsto g\psi'(g)$ is a decreasing function, the optimal MSE is also a decreasing function of the number of epochs, which means that *dividing the series into epochs is a very simple way to improve the MSE*. The quantity $g\psi'(g)$ decreases from $\pi^2/6$ to 1 and as g goes to infinity (more precisely, $g\psi'(g) = 1+1/(2g)+O(g^{-2})$). For $g = 3$, $g\psi'(g)$ is 1.1848 and its value changes slowly thereafter, as it is shown in the following tabulation.

m	1	2	3	4	5	6	7	8
$m\psi'(m)$	1.646	1.290	1.185	1.138	1.108	1.090	1.080	1.070

4. Monte-Carlo results

This section provides a limited Monte Carlo experiment to support our claims. For this purpose, realizations of a Gaussian white noise sequence ε_t, $t = 1, \cdots, n$, with unit variance, were generated by IMSL-FORTRAN subroutine DRNNOR and trajectories of Gaussian processes $\{X_t\}$ with spectral density satisfying (1) were simulated according to the procedure outlined by Hosking (1981). To assess the performance of $\hat{d}_{m_n,g}$, we compute the bias, the mean-square error (mse) and the coverage rates (cr) of the asymptotic confidence interval based on the normal distribution (see Theorem (4)). The quantities were calculated based on 2,000 replications for different sample sizes N and number of epochs g. The results are displayed in Tables 1 to 3. In each experiment, the sampling distribution for the standardized $\hat{d}_{m_n,g}$ estimator was calculated to obtain the coverage rate, which refers to the percentage of cases where the true value of d ($d = 0.3$) lies inside the 95% asymptotic confidence interval ($\hat{d}_{m_n,g} \pm 1.96\sigma_{e,n}$), where $\sigma^2_{e,n}$ is the asymptotic variance. We use two different approximations of the asymptotic variances, which are asymptotically equal but different for finite sample size. The variances are $\sigma^2_{a,n} = \psi'(g)/4m_n$ (which is the limiting variance in Theorem 4) and

$$\sigma^2_{r,n} = \frac{\psi'(g)}{\sum_{k=1}^{m_n}\left\{[-2\log(\omega_k)] - m_n^{-1}\sum_{j=1}^{m_n}[-2\log(\omega_j)]\right\}^2}$$

which is the variance of the regression obtained as in the case where the averaged periodogram ordinates are independent with equal variance $\psi'(g)$. For the ARFIMA process with short-memory parameters, the bandwidth m_n is the one given in the previous section ($m_n(g)$) that minimizes the asymptotic mean-square error of the estimator; we assume here that the parameters specifying the short-memory component $f_\star$ of the ARFIMA model are known, to avoid discussing the separate issue of the optimal choice of the bandwidth (this may be seen as an oracle estimator in this semiparametric context). In the case where

$f_\star(\omega) \equiv 1$, short-memory dynamics are not present, hence two large and fixed bandwidths were used, $m_1 = n^{0.7}$ and $m_2 = \frac{n-1}{2}$. These bandwidth choices were considered with the aim to verify the finite property of the bandwidth on the estimates and the convergence of the standardized estimator to the normal distribution.

Results from Table 1 support the asymptotic properties discussed in the previous sections for the ARFIMA$(0, d, 0)$ model. As it can be observed, the mean-square error decreases as the number of epochs g increases. Breaking the series in a fixed number of epochs can produce a significant reduction in the mean-square error (this effect is similar to the pooling in frequency domain advocated in Robinson (1995)). Although large g does not bring too much gain in terms of the mean-square error, it does not cause penalty in the estimates unless it reaches very large value, as it has been already discussed in the previous section.

Even if the sample sizes used in this limited Monte-Carlo experiments are not large, the coverage rates of the asymptotic confidence intervals are reasonably accurate. The bandwidth m_1 produces estimates which are (as expected) less accurate than m_2. For both bandwidths, the coverage of the asymptotic confidence intervals based on $\sigma_{r,n}$ are precise, even for relatively small sample sizes. The coverage rates of the standardized estimators using the asymptotic standard deviation (cr_a) is reduced as g increases. This indicates that the convergence of $\hat{d}_{m_n,g}$, standardized by σ_a, to the N(0,1) is affected by the sample size reduction.

Table 1
ARFIMA$(0, d, 0)$

			$d = 0.3$								
N	g	m_1	*mean*	*mse*	cr_r	cr_a	m_2	*mean*	*mse*	cr_r	cr_a
512	1	78	0.3032	0.00681	94.7	92.1	255	0.3035	0.00222	96.0	97.9
	2	48	0.3068	0.00443	95.7	91.5	127	0.3051	0.00181	95.2	95.6
	4	29	0.3073	0.00371	95.2	87.4	63	0.3034	0.00193	94.0	91.1
2048	1	207	0.3004	0.00215	95.2	94.6	1023	0.3006	0.00052	94.6	98.6
	2	128	0.3027	0.00148	95.8	92.7	511	0.3007	0.00041	94.8	98.1
	4	78	0.3032	0.00109	96.2	93.1	255	0.3027	0.00036	96.0	97.4
	8	48	0.3034	0.00096	94.5	89.6	127	0.3022	0.00039	94.8	95.9
	16	29	0.3043	0.00084	95.7	88.8	63	0.3047	0.00046	94.1	90.7
8192	1	548	0.3007	0.00078	94.5	94.3	4095	0.3007	0.00013	95.2	99.6
	2	337	0.3021	0.00054	95.6	94.4	2047	0.3006	0.00010	94.9	99.4
	4	207	0.3016	0.00039	94.3	93.0	1023	0.3008	0.00009	94.9	99.0
	8	128	0.3036	0.00032	94.1	92.3	511	0.3016	0.00009	94.3	98.3
	16	78	0.3034	0.00027	94.2	90.3	255	0.3021	0.00009	94.2	97.3

Table 2 gives the results for the ARFIMA process when non-trivial short-memory components are present in the model. As it has often been reported in the long-memory literature, the short-memory component causes significant bias in the estimator of the long-memory parameter, especially if the bandwidth is not properly set; see, for example, Hurvich & Beltrão (1994), Hurvich & Ray (1995) and Reisen et al. (2001) among others. As shown by the asymptotic analysis, the bias of the estimator is not significantly affected by the division of the sample into epochs, *i.e.* at least when the number of epochs g is not large compared to N. This experiment shows that the improvement of the estimates in terms of mean-square error depends on the sample size N and the number g of epochs. For all cases, a decrease of the mean-square error is observed when taking $g = 2, 3$, but the improvement becomes marginal when $g \geq 4$. This empirical property of the estimator is not surprising and it was justified theoretically in the previous section. For example, for $g = 2$ the decrease of the mse predicted from the asymptotic expression (27) when using the optimal bandwidth is 0.783, which is consistent with the values found in the Monte-Carlo experiments. The coverage rate is stable when using the regression variance in the standardized estimator, by the other hand the values of cr_a are reduced as g increases, which is presumable due to the sample size reduction.

Table 2
Estimation in ARFIMA$(1, d, 0)$ model using the optimal bandwidth

	$d = 0.3$, $\phi = -0.3$						$d = 0.3$, $\phi = 0.3$				
N	g	m_n	$mean$	mse	cr_r	cr_a	m_n	$mean$	mse	cr_r	cr_a
512	1	103	0.26682	0.00574	92.8	90.2	62	0.3440	0.01083	91.7	87.3
	2	49	0.26973	0.00511	92.7	88.6	29	0.3445	0.01049	91.8	83.3
	4	23	0.27282	0.00572	92.7	84.9	14	0.3517	0.01252	91.6	76.4
2048	1	312	0.2824	0.00175	93.2	91.6	190	0.3247	0.00299	92.5	91.1
	2	148	0.2856	0.00151	92.6	90.5	90	0.3256	0.00275	91.6	88.6
	4	72	0.2847	0.00148	93.1	89.1	44	0.3272	0.00290	91.1	85.0
	8	35	0.2852	0.00153	93.2	87.1	21	0.3283	0.00346	89.9	80.2
	16	29	0.2864	0.00186	93.9	83.9	10	0.3349	0.00458	91.2	73.2
8192	1	947	0.2905	0.00053	93.1	92.7	577	0.3135	0.00091	93.1	91.6
	2	451	0.2919	0.00044	93.4	92.4	275	0.3146	0.00084	90.7	89.3
	4	219	0.2925	0.00041	92.9	91.5	134	0.3158	0.00082	90.7	88.3
	8	108	0.2932	0.00041	93.6	91.2	66	0.3155	0.00089	90.7	86.0
	16	53	0.2939	0.00043	93.0	88.9	32	0.3184	0.00106	89.9	82.2

The optimal bandwidth depends on the parameters of the model, thus it is not possible to obtain this quantity in practical situations. Indeed, it is well known that the semiparametric estimators are bandwidth driven estimation procedures. Due to these peculiarities, an empirical investigation was considered for the bandwidth $m_n = (\frac{N}{g})^{0.5}$, and the results are in 3. It is not surprising that the use of this bandwidth produces estimates with larger mean-square errors. However, there is an empirical evidence that the reduction of the mean-square error can be obtained even for $g > 4$. In addition, the cr_r is more accurate than the previous case, which can be explained by the fact that the reduction in the number of the periodogram ordinates in the regression mitigates the effect of the AR coefficient. The reduction of the impact of the AR coefficient for this choice of the bandwidth is also manifested by the similarity between the estimates of the two models.

TABLE 3
Estimation in ARFIMA$(1, d, 0)$ model using $m_n = (\frac{N}{g})^{0.5}$

		$d = 0.3$, $\phi = -0.3$					$d = 0.3$, $\phi = 0.3$				
N	g	m_n	*mean*	*mse*	cr_r	cr_a	m_n	*mean*	*mse*	cr_r	cr_a
512	1	22	0.3016	0.02873	95.0	88.7	22	0.3116	0.02782	96.0	90.0
	2	16	0.3039	0.01656	95.8	87.1	16	0.3205	0.01781	94.6	86.2
	4	11	0.2916	0.01388	94.5	81.8	11	0.3336	0.01441	93.6	80.3
2048	1	45	0.3008	0.01197	95.8	91.4	45	0.3053	0.01185	94.8	92.2
	2	32	0.3075	0.00714	95.0	90.6	32	0.3064	0.00748	94.4	89.2
	4	22	0.3064	0.00470	95.3	88.4	22	0.3117	0.00495	95.6	86.9
	8	16	0.2999	0.00337	96.0	87.2	16	0.3216	0.00414	93.2	82.8
	16	11	0.2926	0.00305	94.7	83.9	11	0.3390	0.00455	88.9	72.6
8192	1	90	0.3066	0.00573	94.7	91.9	90	0.3076	0.00536	95.0	93.5
	2	64	0.3057	0.00316	94.8	91.7	64	0.3014	0.00312	95.4	92.5
	4	45	0.3051	0.00217	94.4	90.6	45	0.3057	0.00212	95.4	91.2
	8	32	0.3046	0.00158	94.3	90.1	32	0.3061	0.00158	94.0	89.5
	16	22	0.3045	0.00122	94.3	85.0	22	0.3141	0.00140	92.4	83.5

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5. Proof of Theorem 1

Denote by
$$D_n(\omega) = \mathrm{e}^{\mathrm{i}\omega}(1 - \mathrm{e}^{\mathrm{i}n\omega})/(1 - \mathrm{e}^{\mathrm{i}\omega}) \tag{28}$$
the Dirichlet kernel. Straightforward calculations show that
$$\mathbb{E}[d_{n,0}(\omega_j)\bar{d}_{n,\ell}(\omega_k)] = \int_{-\pi}^{\pi} f(\omega)E_{n,j,k}(\omega)\mathrm{e}^{-\mathrm{i}\ell n\omega}\mathrm{d}\omega \ , \tag{29}$$
where
$$E_{n,j,k}(\omega) \triangleq (2\pi n)^{-1}D_n(\omega - \omega_j)\bar{D}_n(\omega - \omega_k) \ . \tag{30}$$
The proof then follows from the change of variable $\omega \to n\omega$, using the dominated convergence theorem.

6. Proof of Theorem 2

We preface the proof by two technical lemmas, which are used in the sequel. Throughout this section, C is a constant, depending only on μ, ν, δ_- and δ_+, but which may take different values upon each appearance.

Lemma 6. *There exists a constant C (depending only on δ_-, δ_+, μ) such that, for any $d \in [-\delta_-, \delta_+]$, $f_\star \in \mathcal{L}_\star(\mu, \nu)$, and $\omega, \omega' \in [-\pi, \pi] \setminus \{0\}$,*
$$\left| |\omega|^{-2d} f_\star(\omega) - |\omega'|^{-2d} f_\star(\omega') \right| \leq C f_\star(0)(|\omega| \wedge |\omega'|)^{-1-2d}|\omega - \omega'| \ . \tag{31}$$

In addition, there exists a constant C, depending only on μ and ν such that, for all integers $\ell \geq 1$ and $j \in \{1, \dots, \tilde{n}/2\}$,

$$\int_{-\omega_j}^{\omega_j} |f(\omega) - f(\omega + \pi/\ell n)| \mathrm{d}\omega \\ \leq C f_\star(0) \left[(\ell n)^{-1} \left\{ (\ell n)^{2d} + (j/n)^{-2d} \right\} + (\ell n)^{-\nu} (j/n)^{1-2d} \right] . \quad (32)$$

Proof. The proof of (31) is obvious and is omitted for brevity. Note first that, for any $\omega, \omega' \in [-\pi, \pi]$,

$$|f(\omega) - f(\omega')| \leq C f_\star(0) \left\{ \left| |\omega|^{-2d} - |\omega'|^{-2d} \right| + |\omega|^{-2d} \left| |\omega| - |\omega'| \right|^{\nu} \right\} . \quad (33)$$

Applying this inequality with $\omega' = \omega + \pi/\ell n$, yields to

$$\int_{-\omega_j}^{\omega_j} |f(\omega) - f(\omega + \pi/\ell n)| \mathrm{d}\omega \\ \leq C f_\star(0) \left\{ \int_{-\omega_j}^{\omega_j} \left| |\omega|^{-2d} - |\omega + \pi/\ell n|^{-2d} \right| \mathrm{d}\omega + (\ell n)^{-\nu} \int_{-\omega_j}^{\omega_j} |\omega|^{-2d} \mathrm{d}\omega \right\} \\ = C f_\star(0) \left\{ \int_{-\omega_j}^{\omega_j} \left| |\omega|^{-2d} - |\omega + \pi/\ell n|^{-2d} \right| \mathrm{d}\omega + (1-2d)^{-1} (\ell n)^{-\nu} |\omega_j|^{1-2d} \right\} .$$

On the intervals $[-\omega_j, -2\pi/\ell n]$ and $[\pi/\ell n, \omega_j]$, we use the bound

$$\left| |\omega|^{-2d} - |\omega + \pi/\ell n|^{-2d} \right| \leq C|d| \left\{ |\omega|^{-1-2d} + |\omega + \pi/\ell n|^{-1-2d} \right\} (\ell n)^{-1} ,$$

which yields

$$\int_{-\omega_j}^{-2\pi/\ell n} + \int_{\pi/\ell n}^{\omega_j} \left| |\omega|^{-2d} - |\omega + \pi/\ell n|^{-2d} \right| \mathrm{d}\omega \leq C \left\{ (\ell n)^{2d} + n^{2d} j^{-2d} \right\} (\ell n)^{-1} .$$

On the interval $[-2\pi/\ell n, \pi/\ell n]$, we use the bound

$$\left| |\omega|^{-2d} - |\omega + \pi/\ell n)|^{-2d} \right| \leq C \left\{ |\omega|^{-2d} + |\omega + \pi/\ell n|^{-2d} \right\} ,$$

which yields

$$\int_{-\pi/2\ell n}^{\pi/\ell n} \left| |\omega|^{-2d} - |\omega + \pi/\ell n)|^{-2d} \right| \mathrm{d}\omega \leq C(1-2d)^{-1} (\ell n)^{-1+2d} ,$$

which concludes the proof. □

Define

$$\Delta_{\ell,n}(\omega) \triangleq (2\pi n)^{-1} \left\{ D_n(\omega + \pi/\ell n) - D_n(\omega) \right\} , \quad (34)$$

where D_n is the Dirichlet kernel defined in (28).

Lemma 7. *There exists a constant C such that, for all $n, \ell \geq 1$ and ω such that $0 < \omega \leq \omega + \pi/(\ell n) \leq \pi$,*

$$|\Delta_{\ell,n}(\omega)| \leq C \ell^{-1} (1 + n|\omega|)^{-1} . \quad (35)$$

Proof. For any $\omega, \omega' \in (0, \pi]$,

$$
\begin{aligned}
|D_n(\omega) - D_n(\omega')| &= \left| \sum_{k=1}^{n} \left\{ \mathrm{e}^{\mathrm{i}\omega k} - \mathrm{e}^{\mathrm{i}\omega' k} \right\} \right| = \left| \int_{\omega}^{\omega'} \sum_{k=1}^{n} k\, \mathrm{e}^{\mathrm{i}k\lambda} \mathrm{d}\lambda \right| \\
&\le \int_{\omega}^{\omega'} \left| \sum_{k=1}^{n} k\, \mathrm{e}^{\mathrm{i}k\lambda} \right| \mathrm{d}\lambda \le C \int_{\omega}^{\omega'} \frac{n^2}{1+n\lambda} \mathrm{d}\lambda \\
&= Cn \left| \log(1+n\omega') - \log(1+n\omega) \right| \; .
\end{aligned}
$$

Thus, if $0 < \omega \le \omega + \pi/(\ell n) \le \pi$,

$$
|\Delta_{\ell,n}(\omega)| \le C \left| \log(1 + n\omega + \pi/\ell) - \log(1+n\omega) \right| \le C'(1+n\omega)^{-1}\ell^{-1} \; .
$$

□

Proof of Theorem 2. The proof for $\ell = 0$ follows from Moulines & Soulier (1999) (which is a refinement of (Robinson, 1995, Theorem 2)). We consider the case $\ell \neq 0$. Since the function $\omega \mapsto f(\omega) E_{n,j,k}(\omega) \mathrm{e}^{-\mathrm{i}\ell n\omega}$ is 2π-periodic and $\mathrm{e}^{-\mathrm{i}\ell n(\omega+\pi/\ell n)} = -\mathrm{e}^{-\mathrm{i}\ell n\omega}$, we may rewrite (29) as

$$
\begin{aligned}
&2\mathbb{E}[d_{n,0}(\omega_j)\bar{d}_{n,\ell}(\omega_k)] \\
&\quad = \int_{-\pi}^{\pi} \left\{ f(\omega) E_{n,j,k}(\omega) - f(\omega + \pi/\ell n) E_{n,j,k}(\omega + \pi/\ell n) \right\} \mathrm{e}^{-\mathrm{i}\ell n\omega} \mathrm{d}\omega \\
&\quad = A(n,j,k) + B(n,j,k) \; ,
\end{aligned}
$$

where $E_{n,j,k}(\omega)$ is defined in (30) and the two terms $A(n,j,k)$ and $B(n,j,k)$ are respectively defined by

$$
A(n,j,k) \triangleq \int_{-\pi}^{\pi} \left\{ f(\omega) - f(\omega + \pi/\ell n) \right\} E_{n,j,k}(\omega) \mathrm{e}^{-\mathrm{i}\ell n\omega} \mathrm{d}\omega \; , \tag{36}
$$

$$
B(n,j,k) \triangleq \int_{-\pi}^{\pi} f(\omega + \pi/\ell n) \left\{ E_{n,j,k}(\omega) - E_{n,j,k}(\omega + \pi/\ell n) \right\} \mathrm{e}^{-\mathrm{i}\ell n\omega} \mathrm{d}\omega \; . \tag{37}
$$

First consider $A_{n,jk}$. Denote $g_\star = f_\star(w) - f_\star(\omega + \pi(\ell n))$.

We proceed like in the proof of (Robinson, 1995, Theorem 2). We decompose A as the following sum

$$
\begin{aligned}
A(n,j,k) &\triangleq \sum_{i=1}^{6} \int_{W_i(n,j,k)} \left\{ f(\omega) - f(\omega + \pi/\ell n) \right\} E_{n,j,k}(\omega) \mathrm{e}^{-\mathrm{i}\ell n\omega} \mathrm{d}\omega \\
&\triangleq \sum_{i=1}^{6} q_i(n,j,k) \; ,
\end{aligned}
$$

where $W_1(n,j,k) \triangleq \{-\omega_j/2 \le \omega \le \omega_j/2\}$, $W_2(n,j,k) \triangleq \{\omega_j/2 \le \omega \le (\omega_j + \omega_k)/2\}$, $W_3(n,j,k) \triangleq \{(\omega_j + \omega_k)/2 \le \omega \le 3\omega_k/2\}$, $W_4(n,j,k) \triangleq \{3\omega_k/2 \le \omega \le \pi\}$, $W_5(n,j,k) \triangleq \{-\pi \le \omega \le -\omega_k\}$, and $W_6(n,j,k) \triangleq \{-\omega_k \le \omega \le -\omega_j/2\}$. Note that, for $\omega_0 < \pi$, there exists a constant $C < \infty$ (depending only on ω_0) such that, for all $\omega \in [-\omega_0, \omega_0]$,

$$
|D_n(\omega)| \le Cn(1 + n|\omega|)^{-1} \; , \tag{38}
$$

$$
|E_{n,j,k}(\omega)| \le Cn(1 + n|\omega - \omega_j|)^{-1}(1 + n|\omega - \omega_k|)^{-1}. \tag{39}
$$

For $\omega \in W_1(n,j,k)$, (39) implies that $|E_{n,j,k}(\omega)| \le Cnk^{-1}j^{-1}$. Using the bound (32), we therefore obtain

$$\omega_j^d \omega_k^d |q_1(n,j,k)| 1 \le j^{d-1}k^{d-1}n^{1-2d} \int_{-\omega_j/2}^{\omega_j/2} |f(\omega) - f(\omega+\pi/\ell n)|\, \mathrm{d}\omega \\ \le C f_\star(0) \left(\ell^{-(1-2d)} j^{d-1}k^{d-1} + \ell^{-1} j^{-d-1}k^{d-1} + (\ell n)^{-\nu} j^{-d}k^{d-1} \right) .$$

For $d \in [0,1/2)$, we have $j^{d-1} \le j^{-d}$; for $d \in [0,1/2)$, $j^{-d}k^{d-1} \le j^{-|d|}k^{|d|-1}$. Therefore, for $d \in (-1/2,1/2)$,

$$\omega_j^d \omega_k^d |q_1(n,j,k)| \le C f_\star(0) \left\{ \ell^{-\{(1-2d)\wedge 1\}} + (\ell n)^{-\nu} \right\} j^{-|d|}k^{|d|-1} . \tag{40}$$

By (33), for $\omega \in [\omega_j/2, (\omega_j+\omega_k)/2]$,

$$|f(\omega) - f(\omega+\pi/\ell n)| \le C f_\star(0) n^{2d} j^{-1-2d} \ell^{-1} .$$

For $\omega \in W_2(n,j,k)$, we use the bounds $|D_n(\omega-\omega_k)| \le Cn\{1+(k-j)\}^{-1}$, and

$$\int_{\omega_j/2}^{(\omega_k+\omega_j)/2} |D_n(\omega-\omega_j)| \mathrm{d}\omega \le C \log(k)$$

(see (Robinson, 1994, Lemma 5)). Thus

$$\omega_j^d \omega_k^d |q_2(n,j,k)| \le C f_\star(0)\, \ell^{-1} j^{-1-d} k^d \{1+(k-j)\}^{-1} \log(k) .$$

We consider separately the cases $j \le k \le 2j$ (j and k are close) and $j > 2k$ (j and k are far apart). If $j \le k \le 2j$, then $j^{-1} \le 2k^{-1}$, so that $j^{-1-d}k^d\{1+(k-j)\}^{-1} \le 2j^{-d}k^{d-1}$. On the other hand, if $k \ge 2j$, $j \le k/2$, $(k-j)^{-1} \le 2k^{-1}$. Therefore, $j^{-1-d}k^d\{1+(k-j)\}^{-1} \le 2j^{-d}k^{d-1}$. Combining these two inequalities and using, for $d<0$ that $j^{-d}k^{d-1} \le j^d k^{-d-1}$ yields, for $d \in (-1/2,1/2)$,

$$\omega_j^d \omega_k^d |q_2(n,j,k)| \le C f_\star(0)\, \ell^{-1} j^{-|d|} k^{|d|-1} \log(k) . \tag{41}$$

The bound for q_3 can be obtained exactly along the same lines. For $\omega \in W_4(n,j,k)$, we use (31) which shows that

$$|f(\omega) - f(\omega+\pi/\ell n)| \le C f_\star(0) \ell^{-1} n^{2d} k^{-1-2d} ,$$

and, by (39), $|E_{n,j,k}(\omega)| \le Cn^{-1}(\omega-\omega_k)^{-2}$, which imply

$$\omega_j^d \omega_k^d |q_4(n,j,k)| \le C f_\star(0) \ell^{-1} j^d k^{-1-d} n^{-1} \int_{3\omega_k/2}^{\pi} (\omega-\omega_k)^{-2} \mathrm{d}\omega \\ \le C f_\star(0) \ell^{-1} j^d k^{-2-d} .$$

For $d \ge 0$, $j^d k^{-2-d} \le j^{-d}k^{d-1}$ and for $d<0$, $j^d k^{-2-d} \le j^d k^{-d-1}$. Therefore, for any $d \in (-1/2,1/2)$,

$$\omega_j^d \omega_k^d |q_4(n,j,k)| \le C f_\star(0)\, \ell^{-1} j^{-|d|} k^{|d|-1} \log(k) . \tag{42}$$

The bound for q_5 can be obtained exactly along the same lines. For $\omega \in W_6(n,j,k)$, we use the bounds $|f(\omega) - f(\omega + \pi/\ell n)| \leq C f_\star(0) \ell^{-1} n^{2d} j^{-1-2d}$, $|D_n(\omega - \omega_k)| \leq C n k^{-1}$, and $\int_{-\omega_k}^{-\omega_j/2} |D_n(\omega - \omega_j)| \mathrm{d}\omega \leq C \log(k)$, which imply

$$\omega_j^d \omega_k^d |q_6(n,j,k)| \leq C f_\star(0)\, \ell^{-1} j^{-1-d} k^{d-1} \log(k) \; .$$

For $d \geq 0$, $j^{-1-d} \leq j^{-d}$ and for $d \in (-1/2, 0)$, $j^{-1-d} k^{d-1} \leq j^d k^{-d-1}$, which implies, for $d \in (-1/2, 1/2)$ that

$$\omega_j^d \omega_k^d |q_6(n,j,k)| \leq C f_\star(0)\, \ell^{-1} j^{-|d|} k^{|d|-1} \; . \tag{43}$$

By combining the bounds obtained in (40), (41), (42) and (43), we therefore obtain the following bound

$$\omega_j^d \omega_k^d |A(n,j,k)| \leq C f_\star(0) \left\{ \ell^{-(1-2d)\wedge 1} + (\ell n)^{-\nu} + \ell^{-1} \log(k) \right\} j^{-|d|} k^{|d|-1} \; . \tag{44}$$

We now consider the second term $B(n,j,k)$ defined in (37). Note that

$$\begin{aligned} & E_{n,j,k}(\omega) - E_{n,j,k}(\omega + \pi/\ell n) \\ & \qquad = \Delta_{\ell,n}(\omega - \omega_j) \bar{D}_n(\omega - \omega_k) + D_n(\omega + \pi/\ell n - \omega_j) \bar{\Delta}_{\ell,n}(\omega - \omega_k) \; , \end{aligned} \tag{45}$$

where D_n is the Dirichlet kernel and $\Delta_{\ell,n}$ is defined in (34). In addition, since $D_n(\omega)$ and $\Delta_{\ell,n}(\omega)$ are polynomial in $\mathrm{e}^{\mathrm{i}\omega}$ of degree at most n, for any $\ell > 0$, and any $1 \leq j, k \leq [n/2]$,

$$\int_{-\pi}^{\pi} \Delta_{\ell,n}(\omega - \omega_j) \bar{D}_n(\omega - \omega_k) \mathrm{e}^{-\mathrm{i}\ell n \omega} \mathrm{d}\omega = 0 \; . \tag{46}$$

Using this identity together with (45), we may split $B(n,j,k)$ into two terms $B_1(n,j,k)$ and $B_2(n,j,k)$ which are defined as follows:

$$\begin{aligned} B_1 &\triangleq \int_{-\pi}^{\pi} \left\{ f\left(\omega + \frac{\pi}{\ell n}\right) - f\left(\omega_j + \frac{\pi}{\ell n}\right) \right\} \Delta_{\ell,n}(\omega - \omega_j) \bar{D}_n(\omega - \omega_k) \mathrm{e}^{-\mathrm{i}\ell n \omega} \mathrm{d}\omega \; , \\ B_2 &\triangleq \int_{-\pi}^{\pi} \left\{ f\left(\omega + \frac{\pi}{\ell n}\right) - f\left(\omega_k + \frac{\pi}{\ell n}\right) \right\} D_n(\omega - \omega_j) \bar{\Delta}_{\ell,n}(\omega - \omega_k) \mathrm{e}^{-\mathrm{i}\ell n \omega} \mathrm{d}\omega \; . \end{aligned}$$

These two terms can be handled similarly. We consider $B_1(n,j,k)$. We decompose this term as the sum $B_1(n,j,k) \triangleq \sum_{i=1}^{6} \tilde{q}_i(n,j,k)$, with

$$\begin{aligned} & \tilde{q}_i(n,j,k) = \\ & \int_{W_i(n,j,k)} \left\{ f\left(\omega + \frac{\pi}{\ell n}\right) - f\left(\omega_j + \frac{\pi}{\ell n}\right) \right\} \Delta_{\ell,n}(\omega - \omega_j) \bar{D}_n(\omega - \omega_k) \mathrm{e}^{-\mathrm{i}\ell n \omega} \mathrm{d}\omega \; , \end{aligned}$$

where the intervals $W_i(n,j,k)$, $i = 1, 6$ are defined as above. For $\omega \in W_1(n,j,k)$, we have

$$|f(\omega + \pi/\ell n) - f(\omega_j + \pi/\ell n)| \leq C f_\star(0) \left(|\omega|^{-2d} + \omega_j^{-2d} \right) .$$

Moreover, Lemma 7 shows that $|\Delta_{\ell,n}(\omega - \omega_j)| \leq C \ell^{-1} j^{-1}$ and (38) implies $|D_n(\omega - \omega_k)| \leq C n k^{-1}$. Therefore,

$$|\tilde{q}_1(n,j,k)| \leq C f_\star(0) n (jk\ell)^{-1} \int_{-\omega_j/2}^{\omega_j/2} (|\omega|^{-2d} + \omega_j^{-2d}) \mathrm{d}\omega \leq C f_\star(0) n^{2d} j^{-2d} k^{-1} \ell^{-1} \; .$$

Since for $d < 0$, $j^{-d}k^{d-1} \le j^d k^{-d-1}$, this bound implies, for any $d \in (-1/2, 1/2)$,

$$\omega_j^d \omega_k^d |\tilde{q}_1(n,j,k)| \le C f_\star(0)\, \ell^{-1} j^{-|d|} k^{|d|-1} \; . \tag{47}$$

Consider now $\omega \in W_2(n,j,k)$. We consider separately the case $j \le k \le 2j$ (j and k are close) and $k > 2j$ (j and k are far apart). In the first case (j and k close), we use the bounds

$$|f(\omega + \pi/\ell n) - f(\omega_j + \pi/\ell n)| \le C f_\star(0) \omega_j^{-1-2d} |\omega - \omega_j|$$

(see (31)), $|\Delta_{\ell,n}(\omega - \omega_k)| \le C\ell^{-1} n^{-1} |\omega - \omega_j|^{-1}$ (Lemma 7), and $|D_n(\omega - \omega_k)| \le Cn\{1 + (k-j)\}^{-1}$, which imply that

$$\begin{aligned} \omega_j^d \omega_k^d |\tilde{q}_2(n,j,k)| &\le C f_\star(0)\, \omega_j^d \omega_k^d \{1 + (k-j)\}^{-1} \ell^{-1} \int_{\omega_j/2}^{(\omega_k+\omega_j)/2} \omega_j^{-1-2d} \mathrm{d}\omega \\ &\le C f_\star(0)\, \ell^{-1} j^{-1-d} k^d \; . \end{aligned}$$

Since $j^{-1} \le 2k^{-1}$, $j^{-1-d}k^d \le 2 j^{-d} k^{d-1} \le 2 j^{-|d|} k^{|d|-1}$. Therefore, for $j \le k \le 2j$ and all $d \in (-1/2, 1/2)$,

$$\omega_j^d \omega_k^d |\tilde{q}_2(n,j,k)| \le C f_\star(0) \ell^{-1} j^{-|d|} k^{|d|-1} \; . \tag{48}$$

In the second case (j and k far apart), we use the bounds $|f(\omega + \pi/\ell n) - f(\omega_j + \pi/\ell n)| \le C f_\star(0)(\omega_k^{-2d} + \omega_j^{-2d})$, $|D_n(\omega - \omega_k)| \le Cnk^{-1}$, and

$$\int_{\omega_j/2}^{(\omega_k+\omega_j)/2} |\Delta_{\ell,n}(\omega - \omega_j)| \mathrm{d}\omega \le C\ell^{-1} \int_{\omega_j/2}^{(\omega_k+\omega_j)/2} (1 + n\omega)^{-1} \mathrm{d}\omega \le C\ell^{-1} n^{-1} \log(k) \; .$$

This implies that

$$\begin{aligned} \omega_j^d \omega_k^d |\tilde{q}_2(n,j,k)| &\le C f_\star(0) \ell^{-1} \left(j^{-d} k^{d-1} + j^d k^{-d-1} \right) \log(k) \\ &\le C f_\star(0)\, \ell^{-1} j^{-|d|} k^{|d|-1} \log(k) \; , \end{aligned} \tag{49}$$

For $\omega \in W_3(n,j,k)$, and $j \le k \le 2j$, we use the bounds $|f(\omega + \pi/\ell n) - f(\omega_j + \pi/\ell n)| \le C \omega_j^{-1-2d} |\omega - \omega_j|$, $|\Delta_{\ell,n}(\omega - \omega_j)| \le C\ell^{-1} n^{-1} |\omega - \omega_j|^{-1}$, and

$$\int_{(\omega_j+\omega_k)/2}^{3\omega_k/2} |D_n(\omega - \omega_k)| \mathrm{d}\omega \le C \log(k) \; . \tag{50}$$

Therefore, since $k \le 2j$,

$$\omega_j^d \omega_k^d |\tilde{q}_3(n,j,k)| \le C f_\star(0) \log(k) \omega_j^{-1-d} \omega_k^d \ell^{-1} n^{-1} \le C f_\star(0) \ell^{-1} j^{-|d|} k^{|d|-1} \; .$$

For $\omega \in W_3(n,j,k)$, and $j < k/2$, we use the bounds $|f(\omega + \pi/\ell n) - f(\omega_j + \pi/\ell n)| \le C f_\star(0) \left(\omega_k^{-2d} + \omega_j^{-2d} \right)$, $|\Delta_{\ell,n}(\omega - \omega_j)| \le C\ell^{-1} n^{-1} \omega_k^{-1}$, and (50). Therefore, for $j < k/2$,

$$\begin{aligned} \omega_j^d \omega_k^d |\tilde{q}_3(n,j,k)| &\le C f_\star(0) \log(k) \ell^{-1} k^{-1} \left((k/j)^d + (j/k)^d \right) \\ &\le C f_\star(0) \ell^{-1} j^{-|d|} k^{|d|-1} \; . \end{aligned}$$

For $\omega \in W_4(n,j,k)$, we use the bounds $|f(\omega+\pi/\ell n) - f(\omega_j+\pi/\ell n)| \leq Cf_\star(0)(\omega^{-2d}+\omega_j^{-2d})$, $|\Delta_{\ell,n}(\omega-\omega_j)| \leq C\ell^{-1}n^{-1}\omega^{-1}$, and $|D_n(\omega-\omega_k)| \leq C\omega^{-1}$, which implies that

$$
\begin{aligned}
|\tilde{q}_4(n,j,k)| \leq Cf_\star(0)\ell^{-1}n^{-1}\int_{3\omega_k/2}^{\pi}(\omega^{-2d}+\omega_j^{-2d})\omega^{-2}\mathrm{d}\omega \\
\leq Cf_\star(0)\ell^{-1}k^{-1}(\omega_k^{-2d}+\omega_j^{-2d})\ .
\end{aligned}
$$

Hence,

$$
\omega_j^d\omega_k^d|\tilde{q}_4(n,j,k)| \leq Cf_\star(0)\ell^{-1}k^{-1}\left((k/j)^d+(j/k)^d\right) \leq Cf_\star(0)\ell^{-1}j^{-|d|}k^{|d|-1}\ . \tag{51}
$$

The bound for $\tilde{q}_5$ can be obtained exactly along the same lines. For $\omega \in W_6(n,j,k)$, we use the bounds $|f(\omega+\pi/\ell n)-f(\omega_j+\pi/\ell n)| \leq Cf_\star(0)(\omega_k^{-2d}+\omega_j^{-2d})$, $\int_{W_6(n,j,k)}|\Delta_{\ell,n}(\omega-\omega_j)|\mathrm{d}\omega \leq C\ell^{-1}n^{-1}\log(k)$ and $|D_n(\omega-\omega_k)| \leq Cnk^{-1}$, which implies that

$$
\begin{aligned}
\omega_j^d\omega_k^d|\tilde{q}_6(n,j,k)| \leq Cf_\star(0)\ell^{-1}\left(j^{-d}k^{d-1}+j^dk^{-d-1}\right)\log(k) \\
\leq Cf_\star(0)\ell^{-1}j^{-|d|}k^{|d|-1}\log(k)\ .
\end{aligned}
$$

This concludes the proof. □

7. Proof of Theorem 4

The proof is similar to the one used for the pooled periodogram given in Moulines & Soulier (2003) (which simplifies the arguments given Robinson (1995) being based on the central limit theorem for non-linear functions of Gaussian vectors given in Soulier (2001)). The error $\hat{d}_{g,n}-d$ is naturally decomposed into a stochastic and a bias terms as follows

$$
\begin{aligned}
\hat{d}_{g,n}-d = \sum_{k=1}^{m_n} a_{k,n}(m_n)\log\left[\bar{I}_{g,n}(\omega_k)/f(\omega_k)\right] \\
+\sum_{k=1}^{m_n} a_{k,n}(m_n)\log\left[f_\star(\omega_k)/f_\star(0)\right] = S_n(m_n,g)+B_n(m_n,f_\star)\ ,
\end{aligned}
$$

where the coefficients $\{a_{k,n}(m)\}$ is defined in (18). In the previous expression, $S_n(m_n,g)$ is a stochastic fluctuation term (which depends in particular on the number of epochs) and $B_n(m_n,f_\star)$ is the bias caused by the approximation in the neighborhood of the zero frequency of $f_\star$ by a constant. The result will follow from the weak convergence of the stochastic term $S_n(m_n,g)$ and from a bound for the bias term $B_n(m_n,f_\star)$. By (Moulines & Soulier, 2003, Lemma 6.1), there exists a constant $C(\Omega_0,\beta,\mu)$ such that, for any $f_\star \in \mathcal{F}_\star(\Omega_0,\beta,\mu)$ and any non-negative integer m satisfying $2\pi m/n \leq \Omega_0$, $|B_n(m,f_\star)| \leq C(\Omega_0,\beta,\mu)(m/n)^\beta$. Therefore, under the condition (19), $\lim_{n\to\infty}\sqrt{m_n}B_n(m_n,f_\star)=0$.

To establish the weak convergence result, we use (Moulines & Soulier, 2003, Theorem 9.7) (which is based on (Soulier, 2001, Theorem 4.1)). To simplify the notations, put $a_{k,n} = a_{k,n}(m_n)$. Since $\lim_{n\to\infty} m_n^{-1} + n^{-1}m_n = 0$, it follows from Robinson (1995) that

$$\lim_{n\to\infty} m_n \sum_{k=1}^{m_n} a_{k,n}^2 = 1/4 \quad \text{and} \quad \max_{1\le k\le m_n} |a_{k,n}| = O\left(m_n^{-1/2}\log(m_n)\right) .$$

Set $v_n = \lfloor m_n^\gamma \rfloor$ for some $\gamma \in (1/2, 1)$ and $\bar{g}_n \triangleq m_n^{-1}\sum_{k=1}^{m_n}[-2\log(\omega_k)]$, where the function g is defined in (18). Note that

$$\max_{1\le k\le m_n} |a_{k,n}| \sum_{k=1}^{v_n} |a_{k,n}| \log^2(v_n) \le m_n^\gamma \log^2(m_n) \left(\max_{1\le k\le m_n} |a_{k,n}|\right)^2 \to 0 ,$$

$$\sum_{k=1}^{m_n} |a_{k,n}| \log(v_n)/v_n \le m_n^{1/2-\gamma} \log(m_n) \frac{m_n^{-1}\sum_{k=1}^{m_n} |-2\log(\omega_k) - \bar{g}_n|}{m_n^{-1}\sum_{k=1}^{m_n} \{-2\log(\omega_k) - \bar{g}_n\}^2} \to 0 ,$$

To apply (Moulines & Soulier, 2003, Theorem 9.7), we finally need to prove that there exists a constant C such that, for all $k \in \{1, \ldots, m_n\}$, $\mathbb{E}\left[\log^2\left(\bar{I}_{g,n}(\omega_k)/f(\omega_k)\right)\right] \le C$. Corollary 3 shows that this bounds holds for any $k \in \{K, \ldots, m_n\}$, where K is a fixed integer. For the first K frequencies, we use Theorem 1 to show that, for any given k and uniformly in n, the minimal eigenvalue of the covariance matrix of the random vector

$$[\mathrm{Re}\{d_{n,0}(\omega_k)\}, \mathrm{Im}\{d_{n,0}(\omega_k)\}, \ldots, \mathrm{Re}\{d_{n,g-1}(\omega_k)\}, \mathrm{Im}\{d_{n,g-1}(\omega_k)\}]$$

is bounded away from zero.

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