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January 3, 2008

Abstract

We prove that the types in Separably Closed Hasse Fields which have transcendance degree 1 are separably thin

1 Introduction

It is well-known that the theory of Separably Closed Hasse Fields, introduced in [Zie03], is not superstable, as any theory of non perfect fields. However, some interesting types, as the generic type of $p^\infty A(K)$ for some semi-abelian variety A defined over a model K, have finite rank in a strong sense, namely the differential field generated by a realisation of such a type has finite transcendance degree over a set of parameters (see [Hru96]), such types are called "thin". In [PZ03], a stronger version of this condition was introduced, involving separability. This condition was called "very thin" by the authors, we call it here "separably thin" in order to be more explicit. It is a particulary relevant condition to obtain applications in algebraic geometry, illustrated by the differential algebraic proof of Mordell-Lang conjecture given in [PZ03] under this assumption.

The first natural example given in [PZ03] of a thin type which is not very thin is of U-rank and transcendance degree 2. Two questions then arose. First, is it possible to have such an example of smaller rank? And second, can we find an example of such a type in a semi-abelian variety?

Let us look for example at the generic type t of $p^\infty E(K)$ for an elliptic curve, it is a type of U-rank and transcendance degree 1. Using the well-known classification of elliptic curves , which are either ordinary or defined over a finite field, up to isomorphism, it is easy to prove that t is separably thin. What is the broader reason for this fact?

It is neither because t is of U-rank 1, nor because t lives in an abelian variety. In [BK04], an example was given of a type of U-rank 1 and transcendance degree 2, hence thin, which is not separably thin. In [BD07], an abelian variety A is exhibited with the property that $p^\infty A(K)$ is not separably thin. In this paper, we generalize the previous fact concerning elliptic curves as follows: any type of transcendance degree 1 is separably thin.

2 Preliminaries

Definition 1 *We denote by $HF_{p,e}$, for p a prime number and $e \geq 1$ an integer, the theory of fields of characteristic p with e pairwise commuting Hasse derivation (or D-fields) in the language $\mathcal{L} := (0, 1, +, -, ., (D_{i,j})_{\substack{1\leq i\leq e\\ j\in\mathbb{N}}})$. It means that each $D_i = (D_{i,j})_{j\in\mathbb{N}}$ is a Hasse derivation:*

- *for each $j \in \mathbb{N}$, $D_{i,j}$ is a map from K to K with $D_{i,0} = id_K$*
- *for each $j \in \mathbb{N}$, $D_{i,j}$ is additive*
- *for each $j \in \mathbb{N}$, and $x, y \in K$, $D_{i,j}(xy) = \sum_{l+m=j} D_{i,l}(x)D_{i,m}(y)$ (Leibniz rule)*
- *for each $j, k \in \mathbb{N}$, $D_{i,j} \circ D_{i,k} = \binom{k+j}{j} D_{i,k+j}$ (iteration rule)*

Furthermore, we require that $D_{i,j} \circ D_{k,l} = D_{k,l} \circ D_{i,j}$ for every i, j, k, l.

Fact 1 ([Oku63])

- $D_{i,j}(x_1 \dots x_m) = \sum_{j_1+\dots+j_m=j} D_{i,j_1}(x_1) \dots D_{i,j_m}(x_m)$
- *If $j = \sum_{l=0}^{s} j_l p^l$ is the p-adic expansion of j, then*

$$D_{i,1}^{j_0} \circ \dots \circ D_{i,s}^{j_s} = \underbrace{\frac{j!}{(p!)^{j_0} \dots (p^s!)^{j_s}}}_{\neq 0} D_{i,j}$$

The theories that we will work with are the model completions of $HF_{p,e}$, called $SCH_{p,e}$ in [Zie03].

Definition 2 *We denote by $SCH_{p,e}$ the theory of Separably Closed Hasse Fields in the language $\mathcal{L}$. It is axiomatized by:*

- *the axioms of $HF_{p,e}$*
- *the field K has degree of imperfection e, that is the dimension of K as a K^p-vector space is e*
- *the field K is strict, that is $(\bigwedge_{1\leq i\leq e} D_{i,1}(x) = 0) \Leftrightarrow (\exists y, x = y^p)$*
- *the field K is separably closed*

Theorem 1 ([Zie03], Theorem 1.1) *For p and e fixed, the theory $SCH_{p,e}$ is complete and admits Quantifier Elimination. It is the model completion of $HF_{p,e}$.*

Because of Quantifier Elimination, the types in $SCH_{p,e}$ are described using "D-polynomials". The following will be useful for our proof.

Definition 3 *For $k \models HF_{p,e}$, we denote by*

$$k\{X\} := k[X^{(j_1,\dots,j_e)}]_{j_1,\dots,j_e\in\mathbb{N}}$$

the k-algebra of D-polynomials. The Hasse derivation $D_1, \dots, D_e$ are naturally extended from k to $k\{X\}$ by

$$D_{i,j}(X^{(j_1,\dots,j_e)}) = \binom{j + j_i}{j} X^{(j_1,\dots,j_i+j,\dots,j_e)}.$$

If P is a D-polynomial different from 0, we define the order of P as the maximum integer j such that $X^{(j_1,\dots,j_e)}$ appears in P with non zero coefficient where one of the j_i is j. By convention, the order of P is -1 is P is a non zero element of k.

We will denote by $k(\{X\})$ the quotient field of $k\{X\}$. Similarly, the Hasse field generated by an element a over k will be denoted by $k(\{a\})$.

Lemma 1 *Let f be in $k(X) \subseteq k(\{X\})$. Then*

$$D_{i,j}(f) = \frac{df}{dX} D_{i,j}(X) + g,$$

for some $g \in k(\{X\})$ of order less than j (that is g can be written as a quotient of two D-polynomials of order less than j).

Proof If f is a monomial, the result is an easy consequence of Fact 1. We obtain the result for any polynomial f by linearity.
If $f = P/Q$ for two polynomials P and Q, the Leibniz rule gives that

$$D_{i,j}\Big(\frac{P}{Q}\Big) = \frac{D_{i,j}(P) - \sum_{h<j} D_{i,h}(P/Q) D_{i,j-h}(Q)}{Q},$$

and it comes, using the previous case and an induction over j, that

$$D_{i,j}\Big(\frac{P}{Q}\Big) = \frac{\frac{dP}{dX} D_{i,j}(X) - \frac{P}{Q}\frac{dQ}{dX} D_{i,j}(X) + g}{Q} = \frac{df}{dX} D_{i,j}(X) + \frac{g}{Q},$$

for some rational fraction g of order less than j. □

Definition 4 *For t a type over $k \models HF_{p,e}$, and a a realisation of t, we define the transcendance degree of t as the transcendance degree of $k(\{a\})$ over k.*
If it is finite, t is said to be thin.
We say that t is separably thin (or very thin in [PZ03]) if $k(\{a\})$ is a separable algebraic extension of a field finitely generated over k.

Lemma 2 *Let $k \models HF_{p,e}$ and a be an element in a Hasse field extending k which is algebraic separable over k. Then $k(\{a\}) \subseteq k(a)$.*

Proof Let P be the minimal polynomial of a over k. For each i, we prove by induction over j that $D_{i,j}(a) \in k(a)$: using Lemma 1, we know that

$$D_{i,j}(P) = \frac{dP}{dX} D_{i,j}(X) + Q$$

for some D-polynomial Q of order less than j. As P is separable, $\frac{dP}{dX}(a) \neq 0$, and $Q(a) \in k(a)$ by the induction hypothesis, hence $D_{i,j}(a) \in k(a)$. □

3 The result

Proposition 1 *Let K be a strict model of $HF_{p,e}$, and $t \in S(K)$ be a type of transcendance degree 1 over K. Then t is separably thin.*

Proof We begin with some reductions:
Let $\overline{a} \in L^m$ be a realisation of t in some extension L of K. Since K is strict, $K(\{\overline{a}\})$ is a separable extension of K (see Corollary 2.2 of [Zie03]). Hence, if $\overline{a}$ is algebraic over K, it is separable algebraic over K, and then $K(\{\overline{a}\}) = K(\overline{a})$ (by Lemma 2) is algebraic over K, which is impossible since the transcendance degree of t is 1.
Then $K(\overline{a})$ is a separable extension of K of transcendance degree 1, over which $K(\{\overline{a}\})$ is algebraic. We can extract from $\overline{a}$ an element a which is

a separating transcendance basis over K. By Lemma 2, for every other element b from $\overline{a}$, $K(\{b\}) \subseteq K(\{a\}, b)$, hence it suffices to show that the 1-type $tp(a/K)$ is separably thin.
More precisely, we will prove that $K(\{a\}) \subseteq K(a)^{sep}$. Note that this is equivalent to prove that $K(a)^{sep}$ is stable under every $D_{i,j}$, $1 \leq i \leq e$, $j \in \mathbb{N}$. The latter condition is obviously sufficient; and it is necessary using Lemma 2.
Let us fix $1 \leq i \leq e$. We want to prove by induction over j that $K(a)^{sep}$ is stable under $D_{i,j}$. It is sufficient to consider j of the form p^r: by Fact 1, if $K(a)^{sep}$ is stable under $D_{i,1}, D_{i,p}, \ldots, D_{i,p^{r-1}}$, it is stable under every $D_{i,j}$ for every $j < p^r$.

We have then to show that $D_{i,p^r}(a) \in K(a)^{sep}$, assuming that $K(a)^{sep}$ is stable under $D_{i,j}$, $1 \leq i \leq e, j < p^r$.
Let us assume the opposite. As $D_{i,p^r}(a)$ is algebraic over $K(a)$, we can choose n minimal such that $D_{i,p^r}(a)^{p^n} \in K(a)^{sep}$, with $n \geq 1$. Let us denote by

$$f(X) := X^d + f_{d-1}(a)X^{d-1} + \ldots + f_0(a)$$

the unitary minimal separable polynomial of $D_{i,p^r}(a)^{p^n}$ over $K(a)$, where the f_i's are rational functions over K (and $f_d := 1$).
Let us apply D_{i,p^r} to the equality $f(D_{i,p^r}(a)^{p^n}) = 0$:

$$\sum_{h=0}^{d} D_{i,p^r}\left(f_h(a) D_{i,p^r}(a)^{hp^n}\right) = 0,$$

and use that $D_{i,j}(x^{hp^n}) = D_{i,j/p^n}(x^h)^{p^n}$ if p^n divides j, 0 otherwise, to obtain:

$$D_{i,p^r}\left(f_h(a) D_{i,p^r}(a)^{hp^n}\right) = \sum_{j=0}^{p^{r-n}} D_{i,p^r - p^n j}\Big(f_h(a)\Big) D_{i,j}\left(D_{i,p^r}(a)^h\right)^{p^n}. \quad (*)_h$$

If $r < n$, the only term which appears above is for $j = 0$.
We have to consider the terms of the form $D_{i,j}(D_{i,p^r}(a)^h)$, with $j \leq p^{r-n} < p^r$ because we have assumed $n \geq 1$. Using our induction hypothesis that $K(a)^{sep}$ is stable under $D_{i,l}$, $l < p^r$, we show that $D_{i,j}(D_{i,p^r}(a)^h)$ can be written as $q(D_{i,p^r}(a))$, for some polynomial q with coefficients in $K(a)^{sep}$:
First note that it suffices to prove this claim for $h = 1$, because of the equality

$$D_{i,j}(x^h) = \sum_{l=0}^{j} D_l(x) D_{j-l}(x^{h-1}).$$

As $j < p^r$, we know that $D_{i,j}(a) \in K(a)^{sep}$; let us denote by u the minimal (separable) polynomial of $D_{i,j}(a)$ over $K(a)$. We view u as a polynomial in $L[X] \subseteq L\{X\}$ for $L := K(\{a\})$. Note that, for a monomial bX^h,

$$D_{i,p^r}(bX^h) = D_{i,p^r}(b)X^h + bD_{i,p^r}(X^h) + \sum_{l=1}^{p^r-1} D_{i,l}(b) D_{i,p^r-l}(X^h),$$

where $D_{i,p^r}(X^h)$ is the sum of $hD_{i,p^r}(X)X^{h-1}$ and a D-polynomial over $\mathbb{Z}$ of order less than p^r. Hence, by linearity,

$$D_{i,p^r}(u(X)) = u^{D_{i,p^r}}(X) + \frac{du}{dX} D_{i,p^r}(X) + v(X),$$

where $u^{D_{i,p^r}}$ is the polynomial obtained by applying D_{i,p^r} to the coefficients of u, and v is a D-polynomial of order at most p^r-1, with coefficients in $K(a)^{sep}$ (because this field is stable under $D_{i,l}$, $l < p^r$); in particular, $v(D_{i,j}(a)) \in K(a)^{sep}$.
We have to look at what we obtain when we apply D_{i,p^r} to the coefficients of u. For a rational function t over K, we know by Lemma 1 that $D_{i,p^r}(t(a)) \in K(a)D_{i,p^r}(a) + K(a)^{sep}$.
Hence we find $\alpha, \beta \in K(a)^{sep}$ such that

$$0 = D_{i,p^r}\Big(D_{i,j}(a)\Big) = \frac{du}{dX}\Big(D_{i,j}(a)\Big)(D_{i,p^r} \circ D_{i,j})(a) + \alpha D_{i,p^r}(a) + \beta.$$

Since $\frac{du}{dX}(D_{i,j}(a))$ is a non zero element of $K(a)^{sep}$, we obtain that $D_{i,j}(D_{i,p^r}(a)) \in K(a)^{sep}D_{i,p^r}(a) + K(a)^{sep}$.

We look at the terms of the sum $(*)_h$. For $0 < j \leq p^{r-n}$, we have just proved that

$$D_{i,p^r-p^nj}(a)D_j\Big(D_{i,p^r}(a)^h\Big)^{p^n} \in K(a)^{sep}[D_{i,p^r}(a)^{p^n}].$$

For $j = 0$, we have by Lemma 1 that

$$D_{i,p^r}(f_h(a)) \in \frac{df_h}{dX}(a)D_{i,p^r}(a) + K(a)^{sep}.$$

Hence, by summing over all h, we find a polynomial g with coefficients in $K(a)^{sep}$ such that

$$0 = D_{i,p^r}\Big(f(D_{i,p^r}(a))\Big) = g\Big(D_{i,p^r}(a)^{p^n}\Big) + \sum_{h=0}^{d-1}\frac{df_h}{dX}(a)D_{p^r}(a)^{hp^n+1}.$$

Now write

$$P = g(X^{p^n}) + \sum_{h=0}^{d-1}\frac{df_h}{dX}(a)X^{hp^n+1} \in K(a)^{sep}.$$

We have $P(D_{i,p^r}(a)) = 0$ and

$$\frac{dP}{dX}\Big(D_{i,p^r}(a)\Big) = \sum_{h=0}^{d-1}\frac{df_h}{dX}(a)D_{i,p^r}(a)^{hp^n} = 0$$

since $D_{i,p^r}(a)$ is not algebraic separable over $K(a)^{sep}$.
It means that we have obtained a polynomial $\sum_{h=0}^{d-1}\frac{df_h}{dX}(a)X^h$ over $K(a)$ which has $D_{i,p^r}(a)^{p^n}$ as a root. By minimality of d, the degree of f, it implies that this polynomial is zero, that is $\frac{df_h}{dX}(a) = 0$ for each h. Because a is transcendental over K, $\frac{df_h}{dX} = 0$, hence $f_h = g_h(X^p)$ for some rational function g_h over K.

We want to prove that f_h is the p-th power of some element in $K(X)$. Let us apply $D_{i',1}$ to the equality $f(D_{i,p^r}(a)^{p^n}) = 0$ for every $1 \leq i' \leq e$. We get

$$\sum_{h=0}^{d-1} D_{i',1}\Big(g_h(a^p)\Big)D_{i,p^r}(a)^{hp^n} = 0.$$

As $D_{i',1}(g_h(a^p)) \in K(a)$, and f is the minimal polynomial of $D_{i,p^r}(a)^{p^n}$ over $K(a)$, we must have $D_{i',1}(g_h(a^p)) = 0$ for each h. Write $g_h = \frac{P_h}{Q_h}$ for

P_h and Q_h two relatively prime polynomials in $K[X]$, P_h being unitary. We have

$$0 = D_{i',1}\Big(\frac{P_h(a^p)}{Q_h(a^p)}\Big) = \frac{P_h^{D_{i',1}}(a^p)Q_h(a^p) - P_h(a^p)Q_h^{D_{i',1}}(a^p)}{Q_h(a^p)^2}.$$

Since a is transcedental over K, it means that $P_h^{D_{i',1}}Q_h - P_hQ_h^{D_{i',1}} = 0$. But the degree of $P_h^{D_{i',1}}$ is smaller than the degree of P_h (because P_h is unitary), then, by the irreducibility of $\frac{P_h}{Q_h}$, we must have $Q_h^{D_{i',1}} = P_h^{D_{i',1}} = 0$. Since it is true for each i' and since K is strict, it means that P_h and Q_h have coefficients in K^p. Hence there is l_h in $K(X)$ such that $f_h = g_h(X^p) = l_h(X)^p$.

But then, extracting the p-th root of the equality $f(D_{i,p^r}(a)^{p^n}) = 0$, we find

$$\sum_{h=0}^{d} l_h(a)D_{i,p^r}(a)^{hp^{n-1}} = 0.$$

But the polynomial $\tilde{f} := \sum_{h=0}^{d} l_h(a)X^h$ is over $K(a)$, and $D_{i,p^r}(a)^{p^{n-1}}$ is a simple root of $\tilde{f}$ since

$$\frac{d\tilde{f}}{dX}\Big(D_{i,p^r}(a)^{p^{n-1}}\Big) = \Big(\frac{df}{dX}\Big(D_{i,p^r}(a)^{p^n}\Big)\Big)^{1/p} \neq 0.$$

It means that $D_{i,p^r}(a)^{p^{n-1}}$ is algebraic separable over $K(a)$, which contradicts the minimality of n. $\square$

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