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version 19 December 2007

Abstract

The main purpose of this paper is to show that the generalized Schrödinger operator $\mathcal{A}^V f = \frac{1}{2}\Delta f + b\nabla f - Vf$, $f \in C_0^\infty(\mathbb{R}^d)$, is a pre-generator for which we can prove its $L^\infty\left(\mathbb{R}^d, dx\right)$-uniqueness. Moreover, we prove the $L^1(\mathbb{R}^d, dx)$-uniqueness of weak solutions for the Fokker-Planck equation associated with this pre-generator.

Key Words: C_0-semigroups; L^∞-uniqueness; generalized Schrödinger operators; Fokker-Planck equation.

1 Preliminaries

Let E be a Polish space equipped with a σ-finite measure μ on its Borel σ-field $\mathcal{B}$. It is well known that, for a C_0-semigroup $\{T(t)\}_{t\geq 0}$ on $L^1(E, d\mu)$, its adjoint semigroup $\{T^*(t)\}_{t\geq 0}$ is no longer strongly continuous on the dual topological space $L^\infty(E, d\mu)$ of $L^1(E, d\mu)$ with respect to the strong dual topology of $L^\infty(E, d\mu)$. In [10] Wu and Zhang introduce on $L^\infty(E, d\mu)$ *the topology of uniform convergence on compact subsets of* $(L^1(E, d\mu), \|\,.\,\|_1)$, denoted by $\mathcal{C}(L^\infty, L^1)$, for which the usual semigroups in the literature becomes C_0-semigroups. If $\{T(t)\}_{t\geq 0}$ is a C_0-semigroup on $L^1(E, \mu)$ with generator $\mathcal{L}$, then $\{T^*(t)\}_{t\geq 0}$ is a C_0-semigroup on $\left(L^\infty(E, d\mu), \mathcal{C}\left(L^\infty, L^1\right)\right)$ with generator $\mathcal{L}^*$. Moreover, on can proove that $\left(L^\infty(E, d\mu), \mathcal{C}\left(L^\infty, L^1\right)\right)$ is complete and that the topological dual of $\left(L^\infty(E, d\mu), \mathcal{C}\left(L^\infty, L^1\right)\right)$ is $\left(L^1(E, d\mu), \|\,.\,\|_1\right)$.

Let $\mathcal{A} : \mathcal{D} \longrightarrow L^\infty(E, d\mu)$ be a linear operator with its domain $\mathcal{D}$ dense in $\left(L^\infty(E, d\mu), \mathcal{C}\left(L^\infty, L^1\right)\right)$. $\mathcal{A}$ is said to be a *pre-generator* in $\left(L^\infty(E, d\mu), \mathcal{C}\left(L^\infty, L^1\right)\right)$, if there exists some C_0-semigroup on $\left(L^\infty(E, d\mu), \mathcal{C}\left(L^\infty, L^1\right)\right)$ such that its generator $\mathcal{L}$ extends $\mathcal{A}$. We say that $\mathcal{A}$ is an *essential generator* in $\left(L^\infty(E, d\mu), \mathcal{C}\left(L^\infty, L^1\right)\right)$

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(or $\left(L^\infty(E,d\mu),\mathcal{C}\left(L^\infty,L^1\right)\right)$-unique), if $\mathcal{A}$ is closable and its closure $\overline{\mathcal{A}}$ with respect to $\mathcal{C}\left(L^\infty,L^1\right)$ is the generator of some C_0-semigroup on $\left(L^\infty(E,d\mu),\mathcal{C}\left(L^\infty,L^1\right)\right)$. This uniqueness notion was studied in Arendt [1], Eberle [3], Djellout [2], Röckner [6], Wu [8] and [9] and others in the Banach spaces setting and Wu and Zhang [10] and Lemle [4] in the case of locally convex spaces.

2 L^∞-uniqueness of generalized Schrödinger operators

In this note we consider the generalized Schrödinger operator

$$\mathcal{A}^V f := \frac{1}{2}\Delta f + b\nabla f - Vf \quad , \quad \forall f \in C_0^\infty(\mathbb{R}^d) \tag{1}$$

where $b : \mathbb{R}^d \to \mathbb{R}^d$ is a measurable and locally bounded vector field and $V : \mathbb{R}^d \to \mathbb{R}$ is a locally bounded potential. In the case where $V = 0$, the essential self-adjointness of $\mathcal{A} := \frac{1}{2}\Delta + b\nabla$ has been completely charaterized in the works of Wielens [7] and Liskevitch [5]. L^1-uniqueness of this operator has been introduced and studied by Wu [9], its L^p-uniqueness has been studied by Eberle [3] and Djellout [2] for $p \in [1,\infty)$ and by Wu and Zhang [10] for $p = \infty$.
Our purpose is to find some sufficient condition to assure the $L^\infty(\mathbb{R}^d, dx)$-uniqueness of $(\mathcal{A}^V, C_0^\infty(\mathbb{R}^d))$ with respect to the topology $\mathcal{C}(L^\infty, L^1)$ in the case where $V \geq 0$.
At first, we must remark that the generalized Schrödinger operator $(\mathcal{A}^V, C_0^\infty(\mathbb{R}^d))$ is a pre-generator on $L^\infty(\mathbb{R}^d, dx)$. Indeed, if we consider the Feynman-Kac semigroup $\left\{P_t^V\right\}_{t\geq 0}$given by

$$P_t^V f(x) := \mathbb{E}^x 1_{[t<\tau_e]} f(X_t) e^{-\int_0^t V(X_s)\,ds} \tag{2}$$

where $(X_t)_{0\leq t<\tau_e}$ is the diffusion generated by $\mathcal{A}$ and τ_e is the explosion time, then by [10, Theorem 1.4] it follows that $\left\{P_t^V\right\}_{t\geq 0}$is a C_0-semigroup on $L^\infty(\mathbb{R}^d, dx)$ with respect to the topology $\mathcal{C}(L^\infty, L^1)$. By Ito's formula on can prove that f belongs to the domain of the generator $\mathcal{L}_{(\infty)}^V$ of C_0-semigroup $\left\{P_t^V\right\}_{t\geq 0}$on $(L^\infty(\mathbb{R}^d, dx), \mathcal{C}(L^\infty, L^1))$.
The main result of this note is

Theorem 2.1. *Suppose that there is some mesurable locally bounded function $\beta : \mathbb{R}^+ \to \mathbb{R}$ such that*

$$\frac{b(x)x}{|x|} \geq \beta(|x|) \quad , \quad \forall x \in \mathbb{R}^d,\ x \neq 0. \tag{3}$$

Let $\tilde{\beta}(r) = \beta(r) + \frac{d-1}{2r}$. If the one-dimensional diffusion operator

$$\mathcal{A}_1^V = \frac{1}{2}\frac{d^2}{dr^2} + \tilde{\beta}(r)\frac{d}{dr} - V(r) \tag{4}$$

is $L^\infty(0,\infty;dx)$*-unique, then* $\left(\mathcal{A}^V, C_0^\infty(\mathbb{R}^d)\right)$ *is* $L^\infty(\mathbb{R}^d,dx)$*-unique. Moreover, for any* $f\in L^1(\mathbb{R}^d,dx)$ *the Fokker-Planck equation*

$$\begin{cases} \partial_t u(t,x) = \frac{1}{2}\Delta u(t,x) - (div\, b + V)\, u(t,x) \\ u(0,x) = f(x) \end{cases} \tag{5}$$

has one $L^1(\mathbb{R}^d,dx)$*-unique weak solution given by* $u(t,x) = P_t^V f(x)$.

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