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Abstract – We define, for any group G, finite approximations ; with this tool, we give a new presentation of the profinite completion $\widehat{\pi} : G \to \widehat{G}$ of an abtract group G. We then prove the following theorem : if k is a finite prime field and if V is a k-vector space, then, there is a natural isomorphism between $\widehat{V}$ (for the underlying additive group structure) and the additive group of the double-dual V^{**}. This theorem gives counter-examples concerning the iterated profinite completions of a group. These phenomena don't occur in the topological case.

(1) Introduction.

In this paper, we study the profinite completion of a certain class of groups, namely, the additive groups of vector spaces over $\mathbf{F}_p$. The principal result is that, in this case, the profinite completion equals the double-dual. This study is based on a "dual" definition of the profinite completion of a group.

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(2) Brief survey of the classical point of view for profinite completion.

As explained in [Ser02] or [RZ00], one usually defines the profinite completion of a group[1] G as follows. The profinite completion $\widehat{G}$ of G is the projective limit (*ie* the inverse limit) of the finite quotients of G :

$$\widehat{G} = \varprojlim_{\substack{N \triangleleft G \\ [G:N]<\infty}} G/N.$$

There is a more explicit form for this definition. Indeed, if N, M are two normal subgroups of G with $N \subset M$, we have a natural factorisation $\varphi_{N\subset M}$ of the canonical projection π_M :

$$\begin{tikzcd} & G/N \arrow[d, "\varphi_{N\subset M}"] \\ G \arrow[ur, "\pi_N"] \arrow[r, "\pi_M"'] & G/M \end{tikzcd}$$

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[1] If we set in thecategory of *topological* groups, we should precise : "... of a *discrete* group G".

One can then write :

$$\widehat{G} = \left\{ (x_N) \in \prod_{\substack{N \lhd G \\ [G:N]<\infty}} G/N \mid \forall N \subset M, \quad \varphi_{N \subset M}(x_N) = x_M \right\}.$$

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(3) Finite approximations and profinite completion.

In this paper, we will use a "dual" (but equivalent) point of view for the profinite completion of a group. To begin with, we introduce the notion of "finite approximation", which will lead naturally to the concept of profinite completion.

(3.1) Definition. *If G is a group, we call finite approximation of G every couple $\mathbf{v} = (F, \varphi)$ where F is a finite group and $\varphi : G \to F$ a morphism. We denote $F = F_{\mathbf{v}}$ and $\varphi = \varphi_{\mathbf{v}}$. We say that $f : \mathbf{v} \to \mathbf{v}'$ is a morphism between $\mathbf{v}$ and $\mathbf{v}'$ if it is an arrow that makes the following diagram commute :*

$$\begin{tikzcd} & F_{\mathbf{v}} \arrow[d, "f"] \\ G \arrow[ur, "\varphi_{\mathbf{v}}"] \arrow[r, "\varphi_{\mathbf{v}'}"'] & F_{\mathbf{v}'} \end{tikzcd}.$$

*We denote **App**$_f(G)$ the category of finite approximations of G.*

Intuitively, a finite approximation of G allows the mathematician to get some information about G by only dealing with finite objects. Here are some examples, from various aeras of mathematics, of finite approximations :

a) $\begin{array}{rcl} \mathbf{R}^* & \longrightarrow & \mathbf{Z}/2\mathbf{Z} \\ x & \longmapsto & \mathrm{sgn}(x) \end{array}$ the sign of a real number.

b) The reduction modulo n, $\mathbf{Z} \to \mathbf{Z}/n\mathbf{Z}$ and all the derived morphisms and generalizations, such that $\mathbf{Z}_{(p)} \to \mathbf{Z}/p\mathbf{Z}$, such that $GL_m(\mathbf{Z}) \to GL_m(\mathbf{Z}/n\mathbf{Z})$ or such that $\mathcal{O}_K \to \mathcal{O}_K/\mathfrak{P}$ if K is a number field ;

c) If X a topological space with a finite number of connected components, we can consider the "trace" on $\pi_0(X)$ of an automorphism : $\begin{array}{rcl} \mathrm{Aut}(X) & \longrightarrow & \mathfrak{S}_{\pi_0(X)} \\ \phi & \longmapsto & \pi_0(\phi) \end{array}$.

d) If we denote $\mathfrak{S}_{(\mathbf{N})} = \varinjlim_n \mathfrak{S}_n$ the group of permutation of $\mathbf{N}$ with finite support, we can still define a signature $\mathfrak{S}_{(\mathbf{N})} \to \mathbf{Z}/2\mathbf{Z}$.

e) Finally, if $K/\mathbf{Q}$ is a Galois extension, then $\begin{array}{rcl} \mathrm{Gal}(\overline{\mathbf{Q}}/\mathbf{Q}) & \longrightarrow & \mathrm{Gal}(K/\mathbf{Q}) \\ \sigma & \longmapsto & \sigma|_K \end{array}$ is a finite approximation.

(3.2) Profinite completion. Then, one can define very naturally the profinite completion of G as the projective limit of all the finite approximations of G. More precisely, (and without dealing with any problem of set theory)

$$\widehat{G} = \left\{ (g_{\mathbf{v}})_{\mathbf{v}\in\mathbf{App}_f(G)} \in \prod_{\mathbf{v}} F_{\mathbf{v}} \mid \forall\psi : \mathbf{v} \to \mathbf{w},\ \psi(g_{\mathbf{v}}) = g_{\mathbf{w}} \right\}$$

which comes with the *profinite projection*

$$\widehat{\pi} : \begin{array}{l} G \longrightarrow \widehat{G} \\ g \longmapsto (\varphi_{\mathbf{v}}(g))_{\mathbf{v}\in\mathbf{App}_f(G)} \end{array} .$$

Intuitively, this object is what remains from G when one can only deal with information of finite type ; some elements will be identified but, at the same time, some new elements will appear. Formally, in general, $\widehat{\pi}$ is not surjective or injective.

(3.3) Surjective finite approximations. Among the finite approximations, some are surjective ; they form a full subcategory $\mathbf{App}_f^s(G)$ of $\mathbf{App}_f(G)$. In the same way that we have defined the profinite completion, we can then define the "surjective" profinite completion

$$\widehat{G}^s = \varprojlim_{\mathbf{v}\in\mathbf{App}_f^s(G)} F_{\mathbf{v}}.$$

The important fact about this object is that we have the following fact, whose proof is not difficult.

(3.4) Proposition. *The natural morphism $\widehat{G} \to \widehat{G}^s$ is an isomorphism.*

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(4) Profinite completion of the additive group of a vector space over $\mathbf{F}_p$.

(4.1) Profinite completion and double-dual. Before looking at what happens in the situation where the base field is $\mathbf{F}_p$, let us remark that, in the general case, there is a morphism of comparison between the profinite completion of an "additive group" and its double-dual. Let k be a finite field and V a vector space over k. We still denote by V the underlying additive group.

Let f and g be two linear forms of V and let $\lambda \in k$. The forms f, g and $f+\lambda\cdot g$ are, in particular, finite approximations of V (in the additive group of k) and we denote by $\mathbf{v}_f$, $\mathbf{v}_g$ and $\mathbf{v}_{f+\lambda\cdot g}$ the corresponding approximations. Now, let $\mathbf{x} = (x_{\mathbf{v}})_{\mathbf{v}} \in \widehat{V}$ be a "profinite" element.

(4.2) Fact. $x_{\mathbf{v}_{f+\lambda\cdot g}} = x_{\mathbf{v}_f} + \lambda \cdot x_{\mathbf{v}_g}$.

Proof : Indeed, we have the following diagram of morphisms of finite approximations

$$\begin{tikzcd}
 & k^2 \arrow[d, bend left, "p_1"] \arrow[dd, bend left=50, "p_2"] \arrow[ddd, bend left=70, "p_1+\lambda\cdot p_2"] \\
V \arrow[ur, "{(f,g)}"] \arrow[r, "f"] \arrow[dr, "g"] \arrow[ddr, "f+\lambda\cdot g"'] & k \\
 & k \\
 & k
\end{tikzcd} .$$

Then, if we denote by $\mathbf{w}$ the approximation $V \xrightarrow{(f,g)} k^2$, the definition of the profinite completion imposes that $x_{\mathbf{v}_f} = p_1(x_{\mathbf{w}})$ and $x_{\mathbf{v}_g} = p_2(x_{\mathbf{w}})$ and $x_{\mathbf{v}_{f+\lambda\cdot g}} = (p_1 + \lambda \cdot p_2)(x_{\mathbf{w}})$, that is

$$x_{\mathbf{v}_{f+\lambda\cdot g}} = x_{\mathbf{v}_f} + \lambda \cdot x_{\mathbf{v}_g}.$$

■

Using this fact, one can define the morphism of comparison :

$$\begin{array}{rcl} \Psi : \quad \widehat{V} & \longrightarrow & V^{**} \\ (x_{\mathbf{v}})_{\mathbf{v}} & \longmapsto & \left(\begin{array}{rcl} V^* & \longrightarrow & k \\ f & \longmapsto & x_{\mathbf{v}_f} \end{array} \right) \end{array}$$

(4.3) The case where $k = \mathbf{F}_p$. From now on, p is a prime number and $k = \mathbf{F}_p$. The interesting case is when V is of infinite dimension. A good way to understand what happens is to consider $V = (\mathbf{Z}/2\mathbf{Z})^{\mathbf{N}}$.

The first thing to do is to see that if $\varphi : V \to F$ is a finite *surjective* approximation, then F is isomorphic to (the additive group of) $(\mathbf{F}_p)^n$ for some n. Indeed, first of all, since F is the homomorphic image of V, F is abelian. Moreover, all the elements of F satisfy $x^p = e$. Thus, the classification of the abelian finite groups gives the conclusion.

We can now prove :

(4.4) Theorem *Let V be a vector space over $\mathbf{F}_p$. Then, $\Psi : \widehat{V} \to V^{**}$ is an isomorphism.*

Proof : We first prove that Ψ is injective : let $\mathbf{x} = (x_{\mathbf{v}})_{\mathbf{v}} \in \widehat{V}$ such that for all linear form $f : V \to k$, $x_{\mathbf{v}_f} = 0$. Let $\mathbf{v}$ be a finite surjective approximation of V ; we can suppose that $\mathbf{v} = (k^n, \varphi)$, where $\varphi : V \to k^n$ is any morphism. By composing φ with the n projections p_i to the factors k, one obtain n morphisms. If we prove that the n corresponding elements are equal to 0, then, it will follow that $x_{\mathbf{v}}$ is equal to 0 and, thus, that Ψ is injective. But, and it is the (easy) key point, a morphism $V \to k$ of groups is actually a linear form, since we can rewrite the condition $\varphi(\lambda \cdot \vec{v}) = \lambda \cdot \varphi(\vec{v})$ as $\varphi(\vec{v} + \cdots + \vec{v}) = \varphi(\vec{v}) + \cdots + \varphi(\vec{v})$, for our base field is $\mathbf{F}_p$. And, by assumption, all the $x_{\mathbf{v}_f} = 0$.

For the surjectivity, let $\Theta \in V^{**}$ be a double-dual element. We would like to find a profinite element $\mathbf{x} = (x_{\mathbf{v}})_{\mathbf{v}} \in \widehat{V}$ such that, for all linear form f of V, one have $x_{\mathbf{v}_f} = \Theta(f)$. So, let $\mathbf{v} = (k^n, \varphi)$ (as we can suppose it) be a finite approximation of V. Let denote $p_1, \cdots, p_n$ the n projections of k^n to the factors k. Naturally, we define $x_{\mathbf{v}}$ by reconstructing it from the linear forms $p_i \circ \varphi$:

$$x_{\mathbf{v}} := (\Theta(p_1 \circ \varphi), \Theta(p_2 \circ \varphi), \ldots, \Theta(p_n \circ \varphi)) \in k^n.$$

Now, we just have to check that the family $(x_{\mathbf{v}})$ is “compatible“. So let $\mathbf{v} = (k^n, \varphi)$ and $\mathbf{w} = (k^n, \psi)$ be two finite approximations and g a morphism between them :

$$\begin{tikzcd} & k^n \arrow[d, "g"] \\ V \arrow[ur, "\varphi"] \arrow[r, "\psi"'] & k^m \end{tikzcd}.$$

We want to prove that $g(x_{\mathbf{v}}) = x_{\mathbf{w}}$. By composing with the m projections q_i of k^m, it suffices to prove it in the case where $m = 1$:

$$\begin{tikzcd} & k^n \arrow[d, "g"] & \\ V \arrow[ur, "\varphi"] \arrow[r, "\psi"'] & k^m \arrow[r, "q_1"] \arrow[dr, "q_m"'] & k \\ & & \vdots \\ & & k \end{tikzcd}.$$

So, we are brought to this situation

$$\begin{tikzcd}
 & k^n \arrow[d, "g"] \\
V \arrow[ur, "\varphi"] \arrow[r, "\psi"'] & k
\end{tikzcd} ,$$

where we know that g can be written as $g = \lambda_1 \cdot p_1 + \cdots + \lambda_n \cdot p_n$, with $\lambda_i \in k$. The fact that the previous diagram commutes tells us that $\psi = \sum_i \lambda_i \cdot (p_i \circ \varphi)$; and, now :

$$\begin{aligned}
g(x_{\mathbf{v}}) &= g\left(\left(\Theta\left(p_1 \circ \varphi\right), \Theta\left(p_2 \circ \varphi\right), \ldots, \Theta\left(p_n \circ \varphi\right)\right)\right) \\
&= \sum_i \lambda_i \cdot \Theta\left(p_i \circ \varphi\right) \\
&= \Theta\left(\sum_i \lambda_i \cdot \left(p_i \circ \varphi\right)\right) = \Theta(\psi) \\
&= x_{\mathbf{w}},
\end{aligned}$$

which concludes the proof. ■

(4.5) $\widehat{\pi}$ and the canonical injection $i : V \to V^{**}$. We denote $i : V \to V^{**}$ the canonical injection defined by $i(\vec{v})(f) = f(\vec{v})$. One can improve a bit the theorem **4.4** : the isomorphism Ψ between $\widehat{V}$ and V^{**} through Ψ identifies $\widehat{\pi}$ with i. The proof is easy.

(4.6) Theorem. *Let V be a vector space over $\mathbf{F}_p$. Then, $\Psi : \widehat{V} \to V^{**}$ is an isomorphism and the diagram*

$$\begin{tikzcd}
 & \widehat{V} \arrow[d, "\Psi", "\wr"'] \\
V \arrow[ur, "\widehat{\pi}"] \arrow[r, "i"'] & V^{**}
\end{tikzcd}$$

commutes.

(4.7) Remark. One can prove the theorem **4.4** with more abstracted arguments. To begin with, we know (*cf.* for example [Par70, §2.7, theorem 2]) that, in a general category $\mathcal{C}$, if the limits exist, we always have the natural isomorphism

$$\operatorname{Hom}\left(\varinjlim_i X_i, X\right) \cong \varprojlim_i \operatorname{Hom}\left(X_i, X\right).$$

Moreover, in the case of k-vector spaces, this isomorphism is linear ; thus, for a system of k-vector space V_i, we have :

$$\left(\varinjlim_i V_i\right)^* \cong \varprojlim_i \left({V_i}^*\right).$$

Let k, from now on, be a field and V a k-vector space. If we denote by $\left(Y_i\right)_i$ the system of finite-dimensional subvector spaces of V^*, we have $V^* = \varinjlim_i Y_i$ and thus, thanks the previous isomorphism :

$$V^{**} \cong \varprojlim_i \left({Y_i}^*\right).$$

Moreover, there is a natural bijection between the finite-dimensional subspaces of V^* and the finite-codimensional subspaces of V, *via* the application

$$Y \mapsto Y^{\perp} := \left\{v \in V \mid \forall \varphi \in Y,\, \varphi(v) = 0\right\}.$$

Besides, if Y is a finite-dimensional subspace of V^* then the dual Y^* is naturally isomorphic to $V/Y^\perp$. Consequently, if we denote by $(Z_j)_j$ the system of finite-codimensional subspaces of V, we have :

$$V^{**} \cong \varprojlim_j (V/Z_j)\,.$$

But, if $k = \mathbf{F}_p$ for a prime number p, one can identify the k-vector space V with its underlying additive group[2] $\omega(V)$, its dual V^* with $\mathrm{Hom}_{\mathbf{Gr}}(\omega(V), \omega(\mathbf{F}_p))$, and its finite dimensional quotients with the finite quotient of $\omega(V)$. We thus finally get the expected alternative proof of the theorem **4.4**.

(5) A family of counter-examples.

One would like to know if, given a group G, one have $\widehat{\widehat{G}} \simeq \widehat{G}$. This fact is known to be false (*cf.* example 4.2.13 of [RZ00]), but as we will see, it is still false, in general, after taking i times the profinite completion.

(5.1) The sequence of i-th profinite completions. We introduce the following notation. If G is a group, we denote $\widehat{G}^{[1]} = \widehat{G}$ and $\widehat{G}^{[i+1]} = \widehat{\widehat{G}^{[i]}}$. These groups come with projections, as follows :

$$G \xrightarrow{\widehat{\pi}^{[1]}} \widehat{G}^{[1]} \xrightarrow{\widehat{\pi}^{[2]}} \widehat{G}^{[2]} \longrightarrow \cdots \longrightarrow \widehat{G}^{[i]} \xrightarrow{\widehat{\pi}^{[i+1]}} \widehat{G}^{[i+1]} \longrightarrow \cdots\,.$$

We will prove that, in general, none of the $\widehat{\pi}^{[i]}$ is an isomorphism.

(5.2) Proposition. *Let p be a prime number and $k = \mathbf{F}_p$. Let V be (the additive group of) a k-vector space of infinite dimension. Then, in the following sequence*

$$V \xrightarrow{\widehat{\pi}^{[1]}} \widehat{V}^{[1]} \xrightarrow{\widehat{\pi}^{[2]}} \widehat{V}^{[2]} \longrightarrow \cdots \longrightarrow \widehat{V}^{[i]} \xrightarrow{\widehat{\pi}^{[i+1]}} \widehat{V}^{[i+1]} \longrightarrow \cdots$$

all the $\widehat{\pi}^{[i]}$ are injective but non-surjective morphisms.

Proof : This follows from the identification of the arrows $\widehat{\pi}^{[i]}$ with the canonical injections of a vector space in its double-dual, and from the fact that these injections are injective but non-surjective when the vector spaces are of infinite dimension, cf. Théorème 6, §7, n°5 of [Bou62]. ■

[2] We denote $\omega : k-\mathbf{Vs} \to \mathbf{Gr}$ the forgetful functor from the category of k-vector spaces to the category of groups.

(6) Conclusion : abstract setting vs. topological setting.

This study has been given for groups but a similar point of view can be applied to *topological* groups. In this case, we start with a topological group $\mathcal{G}$ and we consider the category $\mathbf{App}_{discr}(G)$ of finite and *discrete* approximations : they are couples $\mathbf{v} = (F, \varphi)$, where F is a discrete and finite topological group and $\varphi : \mathcal{G} \to F$ a continuous morphism of groups.

One obtain the (topological) profinite completion of $\mathcal{G}$, wich is, as well-known, a topological group, compact and totally disconnected (cf. [Ser02]), and one obtain a profinite projection, which is a continuous morphism :

$$\widehat{\pi}^{top} : \mathcal{G} \to \widehat{\mathcal{G}}^{top}.$$

More generally, as previously done, one can define the sequence of iterated (topological) profinite completions :

$$\mathcal{G} \xrightarrow{\widehat{\pi}^{[1],top}} \widehat{\mathcal{G}}^{[1],top} \longrightarrow \cdots \longrightarrow \widehat{\mathcal{G}}^{[i],top} \xrightarrow{\widehat{\pi}^{[i+1],top}} \widehat{\mathcal{G}}^{[i+1],top} \longrightarrow \cdots.$$

The situation is then totally different than before. Indeed, we have :

(6.1) Proposition. *Let $\mathcal{G}$ be a topological group. Then, for all $i \geq 2$, the arrows $\widehat{\pi}^{[i],top}$ are isomorphisms of topological groups.*

(6.2) Profinite groups : abstract setting and topological setting. There is a synthetical way to see the fundamental difference between the propositions **4.2** and **6.1**. For this sake, we introduce two notions of profinite groups. We will say that a group G is *profinite* if it is the projective limit of a system of finite groups ; we will say that a topological group $\mathcal{G}$ is *topologically profinite* if it is the projective limit of a system of finite and discrete groups. We then have :

(6.3) Theorem *Let $\mathcal{G}$ be a topological group. Then :*

$$\mathcal{G} \textit{ is topologically profinite } \iff \widehat{\pi}^{top} : \mathcal{G} \to \widehat{\mathcal{G}}^{top} \textit{ is an isomorphism.}$$

(6.4) Proposition *Let G be a group. Then :*

$$G \textit{ is profinite } \Leftarrow \widehat{\pi} : G \to \widehat{G} \textit{ is an isomorphism}$$

$$G \textit{ is profinite } \not\Rightarrow \widehat{\pi} : G \to \widehat{G} \textit{ is an isomorphism.}$$

(6.5) A positive answer. One could legitimately be disapointed by the non-equivalence of G being profinite and of $\widehat{\pi}$ being an isomorphism. Indeed, on the one hand, there is the very classical definition of a profinite group and, on the other hand, there is the deep property for a group to have its profinite projection $\widehat{\pi}$ to be an isomorphism (such a group, in a way, is separated — for $\widehat{\pi}$ is injective — and complete — for $\widehat{\pi}$ is surjective). One would have expected these two to coincide...

Fortunately, there is a positive result in this direction. It is a difficult result, which has been published in 2007 by Nikolay Nikolov and Dan Segal, *cf.* [NS07a] and [NS07b], and whose proof uses the classification of finite simple groups. In order to state their result, let us remark that if G is an (abstract) profinite group, if we write $G = \varprojlim_i F_i$, where the F_i's are finite, and if we endow each of the F_i's with the discrete topology, then we can view G as a topological group.

(6.6) Theorem *Let G be an (abstract) profinite group, which is topologically of finite type for the associated topology. Then, $\widehat{\pi} : G \to \widehat{G}$ is an isomorphism.*

(6.7) Acknowledgments. My first acknowledgments go to Xavier Caruso for many helpful discussions. I would like also to thank the referee for many valuable comments and for making me known the alternative proof **4.7**.

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