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Local interaction between vorticity and shear in a perfect incompressible fluid

P. Vieillefosse

Laboratoire de Physique Théorique des Liquides (*), Université P. et M. Curie, 4, place Jussieu, 75230 Paris Cedex 05, France

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Résumé. — Nous montrons par un modèle simple tiré des équations d'Euler que l'écoulement d'un fluide parfait incompressible devient divergent au bout d'un temps fini. Pour cela nous considérons l'interaction locale entre la vorticité et le cisaillement en négligeant les gradients de ces deux grandeurs dans leurs équations du mouvement. Il en résulte un système de 8 équations différentielles non linéaires du premier ordre dont le comportement asymptotique peut être facilement extrait. Les 3 valeurs propres du tenseur de cisaillement divergent vers $+\infty$ pour les deux plus grandes et $-\infty$ pour la plus petite. Le vecteur vorticité diverge aussi en se couchant sur le vecteur propre du tenseur de cisaillement qui correspond à la valeur propre intermédiaire (positive), donnant ainsi un signe positif à la fonction de transfert de l'énergie de l'équation de von Kármán-Howarth. En même temps la rotation du référentiel des axes propres du cisaillement s'arrête.

Abstract. — We show from a simple model related to the Euler equations that the flow of an incompressible and inviscid fluid diverges in a finite time. For this we look at the local interaction between vorticity and shear by neglecting the gradients of these two quantities in their equations of motion. A non linear system of 8 first order differential equations is obtained whose asymptotic behaviour can be easily obtained. The two largest eigenvalues of the shear tensor diverge to $+\infty$ and the smallest one to $-\infty$. The vorticity vector also diverges and lies along the eigenvector of the shear tensor which corresponds to the (positive) intermediate eigenvalue, thus giving a positive sign to the energy spreading function of the von Kármán-Howarth equation. At the same time the rotation of the shear principal axis stops.

1. **Introduction.** — In turbulent flows, the interaction between vorticity and shear plays an important part in the energy spreading from large to small scales. As a matter of fact, the shearing flow around a point leads to a variation of the inertia moments of the fluid. If the viscosity is negligible the angular momentum is constant in time. The fluid rotation around an axis, about which the inertia moment decreases, leads to an increasing angular velocity. On the other hand if shear increases the inertia moment about the rotation axis, the angular velocity will decrease. We then have an increasing or decreasing angular velocity according to the coupling between the vorticity ω and the shear u. We show from a simple model related to the Euler equations that the first possibility does occur, leading to a divergence in a finite time.

In this study we shall be concerned with incompressible and invicid fluids. The neglect of viscosity is valid in the initial phase of high-Reynolds-number flow where the energy is confined to large scales.

Several authors [1, 2] have conjectured that the Euler equations lead to a divergence in a finite time. Morf *et al.* [3] have shown by a numerical calculation

(*) Equipe associée au C.N.R.S.

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that spontaneous singularities occur in the Taylor-Green vortex model [4] (1).

In part 2 we derive the model of local interaction between vorticity and shear from the Euler equations by neglecting the gradients of these two quantities in the equations of motion. A non-linear system of 8 first order differential equations (3 + 5) is obtained. In parts 3 and 4 these equations are transformed by introducing the rotating principal axis of the shear tensor u and the traces of matrices which are functions of ω and u. In part 5 we solve this system and show that a divergence occurs in a finite time. The vorticity length indefinitely increases and the vector lies along on the shear principal axis which corresponds to the intermediate eigenvalue. The 3 eigenvalues of u are also indefinitely increasing with two positive and one negative values. At the same time the principal axis rotation stops.

2. **The model.** — The flow of inviscid incompressible fluids is a solution of the Euler and continuity equations

$$\frac{\partial}{\partial t} v_i = - v_j \, \partial_j v_i - \frac{1}{\rho} \partial_i P \qquad \partial_i v_i = 0 \tag{1}$$

where v_i is the i-component of the velocity field, ρ is the mass density and P is the pressure field. The nine components of the tensor formed by the velocity gradient $\partial_i v_j$ can be divided into three sets; the scalar $\partial_i v_i$, the vector $\omega_i = (\text{curl } v)_i$ which is the vorticity and the symmetrical and traceless tensor $u_{ij} = (\partial_i v_j + \partial_j v_i - \frac{2}{3} \delta_{ij} \, \partial_l v_l)$ which is the shear. Taking the gradient of the Euler equation we obtain three sets of equations which correspond to each set of components

$$\frac{1}{\rho} \partial_k^2 P = \frac{1}{4} (2 \, \omega_l^2 - u_{ij} \, u_{ij}) \tag{2}$$

$$\frac{\partial}{\partial t} \omega_i = - v_j \, \partial_j \omega_i + \frac{1}{2} \omega_j \, u_{ji} \tag{3}$$

$$\frac{\partial}{\partial t} u_{ij} = - v_k \, \partial_k u_{ij} - \frac{1}{2} (u_{ki} \, u_{kj} - \frac{1}{3} \delta_{ij} u_{kl} \, u_{kl}) - - \frac{1}{2} (\omega_i \, \omega_j - \frac{1}{3} \delta_{ij} \omega_k^2) - \frac{2}{\rho} (\partial_i \, \partial_j - \frac{1}{3} \delta_{ij} \, \partial_k^2) \, P \tag{4}$$

where δ_{ij} is the unity tensor.

We neglect the terms linked to the gradients of vorticity and shear and the term of pressure gradient in (4) (2). Thus we obtain a closed set of eight equations for vorticity and shear (3)

$$\frac{\partial}{\partial t} \omega_i = \frac{1}{2} \omega_j \, u_{ji} \tag{5}$$

$$\frac{\partial}{\partial t} u_{ij} = - \frac{1}{2} \left(u_{ki} \, u_{kj} - \frac{1}{3} \delta_{ij} u_{kl} \, u_{kl} \right) - - \frac{1}{2} \left(\omega_i \, \omega_j - \frac{1}{3} \delta_{ij} \omega_k^2 \right). \tag{6}$$

It must be noticed that in some way the neglected terms are not correlated with vorticity and shear in the homogeneous and isotopic turbulence hypothesis. Taking into account the fluid incompressibility, the following mean quantities are zero

$$\begin{aligned} \overline{\omega_i (v_j \, \partial_j) \, \omega_i} &= - \overline{\frac{1}{2} \partial_j v_j \, \omega_i^2} = 0 \\ \overline{u_{ij} (v_k \, \partial_k) \, u_{ij}} &= - \overline{\frac{1}{2} \partial_k v_k \, u_{ij}^2} = 0 \\ \overline{u_{ij} \left(\partial_i \, \partial_j - \frac{1}{3} \delta_{ij} \, \partial_k^2 \right) P} &= \overline{P \left(\partial_i \, \partial_j - \frac{1}{3} \delta_{ij} \, \partial_k^2 \right) u_{ij}} = 0 . \end{aligned} \tag{7}$$

Whereas the kept terms play an important part in the conversion of the energy of large scale components into the energy of small-scale components. In particular the following equation can be written :

$$\overline{\omega_i \, \omega_j \, u_{ij}} = 2 \int_0^{+\infty} k^2 \, \mathrm{d}k T(k) \tag{8}$$

where $T(k)$ is the energy conversion function of the von Kármán-Howarth equation [1].

Our model which supposes that the gradients of vorticity and shear are negligible exhibits the local interaction between vorticity and shear.

Introducing the matrix Ω

$$\Omega_{ij} = \omega_i \, \omega_j \tag{9}$$

the following equations for the symmetrical matrices Ω and u are obtained

$$\begin{aligned} \dot{\Omega} &= \frac{1}{2} (\Omega u + u \Omega) \\ \dot{u} &= - \frac{1}{2} \left(u^2 - \frac{1}{3} \mathbb{1} \, \text{trace} \, u^2 \right) - \frac{1}{2} \left(\Omega - \frac{1}{3} \mathbb{1} \, \text{trace} \, \Omega \right) \end{aligned} \tag{10}$$

where the derivative with respect to the time is noted by a dot and $\mathbb{1}$ is the unity matrix.

(1) This result which was obtained by analytic continuation of the series for the enstrophy using the Padé approximant is at present questioned by an other numerical calculation based on the analyticity strip method [5]. However a singularity cannot be ruled out. Here we do not assert that a spontaneous singularity does or does not occur in the Taylor-Green vortex model (or in other models) but we only show that the above mentioned mechanism of variation of the inertia moments leads to a divergence in a finite time whatever the initial conditions (except for a set of zero measure). Moreover this singularity can disappear if the effects of the gradients are taken into account.

(2) Both hypotheses are related because of (2).

(3) Léorat [6] has previously studied this system of equations by means of numerical calculations and has mentioned, without giving details that a divergence of the enstrophy occurs in a finite time.

3. **Principal axis transformation.** — The vorticity and shear matrices, Ω and u, can both be diagonalized and the eigenvectors are orthogonal in each case. But the simultaneous diagonalization can only be performed if the commutator

$$C = [\Omega, u] \tag{11}$$

is zero. From (10) it can be easily seen that this commutator is a constant

$$\dot{C} = 0 . \tag{12}$$

From writing the characteristic polynomial of C it is possible to derive a first integral (constant function of u_{ij} and ω_i). This first integral E will later be obtained. In the general case where C is not zero, it turns out to be more interesting to take the eigenvectors of shear as base vectors. With this coordinate system the number of coupled variables goes from 8 to 5. All the quantities relative to this base will be primed.

The orthogonal and normalized eigenvectors $\hat{e}'_i$ of u are expanded on the old base $\hat{e}_i$ with the help of the orthogonal matrix R :

$$\hat{e}'_i = R_{ji}\, \hat{e}_j . \tag{13}$$

Then the equation of motion of R is

$$\dot{R} = R\Delta' \tag{14}$$

where Δ' is an antisymmetric matrix whose dual vector $\Delta'_k = -\frac{1}{2}\varepsilon_{ijk}\, \Delta'_{ij}$ is the instantaneous vector of rotation of the new base (ε_{ijk} is the Levi-Civita density).

From (10) we deduce the equations of motion for the vorticity components ω'_i and for the shear components u'_i which satisfy the trace condition

$$u'_1 + u'_2 + u'_3 = 0 \tag{15}$$

$$\begin{aligned} \dot{u}'_1 + \frac{1}{2}(u'^2_1 + \Omega'_1) &= \dot{u}'_2 + \frac{1}{2}(u'^2_2 + \Omega'_2) \\ &= \dot{u}'_3 + \frac{1}{2}(u'^2_3 + \Omega'_3) \\ &= \frac{1}{6}(u'^2_1 + u'^2_2 + u'^2_3 + \Omega'_1 + \Omega'_2 + \Omega'_3) \\ \frac{\dot{\Omega}'_1}{\Omega'_1} &= u'_1 - \left\{ \frac{\Omega'_2}{u'_1 - u'_2} + \frac{\Omega'_3}{u'_1 - u'_3} \right\} \\ \frac{\dot{\Omega}'_2}{\Omega'_2} &= u'_2 - \left\{ \frac{\Omega'_1}{u'_2 - u'_1} + \frac{\Omega'_3}{u'_2 - u'_3} \right\} \\ \frac{\dot{\Omega}'_3}{\Omega'_3} &= u'_3 - \left\{ \frac{\Omega'_1}{u'_3 - u'_1} + \frac{\Omega'_2}{u'_3 - u'_2} \right\} \end{aligned} \tag{16}$$

where Ω'_i stands for ω'^2_i. The antisymmetric matrix Δ' which governs the rotation of the principal axis of u is given by

$$\Delta'_{ij} = \frac{1}{2}\, \frac{\omega'_i\, \omega'_j}{u'_i - u'_j} . \tag{17}$$

4. **Equations for the traces.** — Another set of equations can be obtained by considering the traces of the matrices u^n and $u^n\,\Omega$; these equations, of course, do not depend on the base choice :

$$\begin{aligned} t_n &= \text{trace}\,(u^n\, \Omega) \\ s_n &= \text{trace}\, u^{n+2} . \end{aligned} \tag{18}$$

From (10) or (16) and taking into account the particular form (9) of the matrix Ω we obtain

$$\begin{aligned} \dot{t}_n &= -\frac{n-2}{2}\, t_{n+1} + \frac{n}{6}\, t_{n-1}(s_0 + t_0) - \\ &\qquad - \frac{1}{2} \sum_{p=1}^{n} t_{n-p}\, t_{p-1} \\ \dot{s}_n &= -\frac{n+2}{2}(s_{n+1} + t_{n+1}) + \frac{n+2}{6}(s_0 + t_0)\, s_{n-1} . \end{aligned} \tag{19}$$

In fact the number of independent variables is 5, for example s_0, s_1, t_0, t_1 and t_2. This number is obvious if we consider the expressions of these 5 quantities as functions of u'_i and Ω'_i with the supplementary condition (15); we get a system of 6 equations for 6 unknowns. By solving this system we can express u'_i and Ω'_i as functions of s_0, s_1, t_0, t_1 and t_2. Thus the other quantities s_n and t_n are functions of these 5 initial variables

$$\begin{aligned} 0 &= \Sigma_i\, u'_i & t_0 &= \Sigma_i\, \Omega'_i \\ s_0 &= \Sigma_i\, u'^2_i & t_1 &= \Sigma_i\, u'_i\, \Omega'_i \\ s_1 &= \Sigma_i\, u'^3_i & t_2 &= \Sigma_i\, u'^2_i\, \Omega'_i . \end{aligned} \tag{20}$$

The eigenvalues u'_i are the roots of the polynomial

$$P(u') = u'^3 - \frac{1}{2}\, s_0\, u' - \frac{1}{3}\, s_1 \tag{21}$$

and the quantities Ω'_i are given by :

$$\Omega'_i = 2\, \frac{t_0\, u'^2_i + t_1\, u'_i + t_2 - \frac{1}{2}\, t_0\, s_0}{6\, u'^2_i - s_0} . \tag{22}$$

By taking $P(u'_i) = 0$ into account we deduce

$$\begin{aligned} s_2 &= \frac{1}{2}\, s_0^2 \quad s_3 = \frac{5}{6}\, s_0\, s_1 \\ s_p &= \frac{1}{2}\, s_0\, s_{p-2} + \frac{1}{3}\, s_1\, s_{p-3} \; (p \geqslant 3) . \end{aligned} \tag{23}$$

For t_p it is easier to take it from (19). In particular we have

$$t_3 = \frac{1}{3}\, s_1\, t_0 + \frac{1}{2}\, s_0\, t_1 . \tag{24}$$

As the closure problem is solved for the system (19), we obtain, for the 5 independent variables, the following equations

$$\dot{s}_0 = -(s_1 + t_1) \qquad \dot{t}_0 = t_1$$
$$\dot{s}_1 = -\frac{3}{2}t_2 + \frac{1}{2}s_0 t_0 - \frac{1}{4}s_0^2 \quad \dot{t}_1 = \frac{1}{2}t_2 + \frac{1}{6}s_0 t_0 - \frac{1}{3}t_0^2$$
$$\dot{t}_2 = \frac{1}{3}t_1 s_0 - \frac{2}{3}t_0 t_1 . \qquad (25)$$

This system is equivalent to (16). Compared to the initial equations (5), (6), the rotation of the principal axis of u is missing. Finally we mention that t_0 is equal to twice the enstrophy.

5. **Solution of the equations of motion.** — In fact the system (25) can be solved by introducing the two quantities

$$a = t_0 - \frac{1}{2}s_0$$
$$b = -\left(t_1 + \frac{1}{3}s_1\right). \qquad (26)$$

It is easily seen that a and b satisfy the equations

$$2\dot{a} = -3b$$
$$3\dot{b} = a^2 \qquad (27)$$

or

$$\ddot{a} = -\frac{1}{2}a^2 = -\frac{\partial}{\partial a}\frac{a^3}{6}. \qquad (28)$$

The motion of a is the same as that of a particle with unit mass in a potential $\frac{1}{6}a^3$. Let us call $\frac{1}{6}a_0^3$ the constant « energy » of such a particle

$$\frac{1}{2}\dot{a}^2 + \frac{1}{6}a^3 = \frac{1}{6}a_0^3 = \frac{1}{24}(4a^3 + 27b^2). \qquad (29)$$

We shall see later that the sign of a_0 is related to the form of streamlines a_0 defines the time scale of the problem ($\sim |a_0|^{-1/2}$). Then the motion of a is very simple; if $\dot{a}$ is negative at the initial time, a will monotonically decrease and tend to $-\infty$ in a finite time. If $\dot{a}$ is initially positive, a will first increase, then rebound against the potential and then will monotonically decrease as in the first case. The asymptotic behaviour can be obtained easily from (29) by neglecting a_0

$$a \sim \frac{-12}{(\tau_0 - t)^2}. \qquad (30)$$

We remark that only the divergence time τ_0 depends on the initial condition ($\tau_0 \sim |a_0|^{-1/2}$). The equation (29) can be integrated in the following form :

$$a = a_0\left(1 - \varepsilon\sqrt{3}\,\mathrm{tg}^2\frac{\varphi}{2}\right) \qquad \varepsilon = a_0/|a_0|$$
$$\mathrm{d}t = \pm\frac{3^{1/4}}{|a_0|^{1/2}}\frac{\mathrm{d}\varphi}{\sqrt{1 - k^2\sin^2\varphi}} \qquad k = \frac{1}{2}\sqrt{2 + \varepsilon\sqrt{3}}. \qquad (31)$$

In the special case where $\dot{a}$ is zero at the initial time, τ_0 is given by

$$\tau_0 = \frac{3^{1/4}}{|a_0|^{1/2}}F(k) \qquad (32)$$

where F is the elliptic function of first kind.

Another first integral (different from a_0) can be directly obtained from (25)

$$E = t_0 t_2 - t_1^2. \qquad (33)$$

A simple calculation leads to

$$E = \Omega_1'\Omega_2'(u_1' - u_2')^2 + \Omega_1'\Omega_3'(u_1' - u_3')^2 + \Omega_2'\Omega_3'(u_2' - u_3')^2 > 0. \qquad (34)$$

We also have the following first integral :

$$F = t_4 t_2 t_0 - t_4 t_1^2 - t_3^2 t_0 - t_2^3 + 2t_3 t_2 t_1 = \Omega_1'\Omega_2'\Omega_3'(u_1' - u_2')^2(u_1' - u_3')^2(u_2' - u_3')^2 \qquad (35)$$

which shows that two eigenvalues of u cannot be equal without any diverging quantity. Therefore the order of the three eigenvalues does not change in time. We shall suppose $u_3' < u_2' < u_1'$.

From (25) we deduce

$$t_1 = \dot{t}_0 \quad \frac{1}{2}t_2 = \dot{t}_1 + \frac{1}{3}t_0^2 - \frac{1}{6}s_0 t_0 = \ddot{t}_0 + \frac{1}{3}t_0 a \qquad (36)$$

taking (33) into account we obtain the second order differential equation for t_0

$$\ddot{e} + \frac{1}{6}ae = \frac{E}{4e^3} \qquad e = \sqrt{t_0}. \qquad (37)$$

Solving this equation is sufficient to solve the problem completely because all the quantities can be expressed as functions of a, e and their first derivatives

$$t_0 = e^2 \quad t_1 = 2e\dot{e} \quad t_2 = 4\dot{e}^2 + \frac{E}{e^2}$$
$$s_0 = 2(e^2 - a) \quad s_1 = 2\dot{a} - 6e\dot{e}. \qquad (38)$$

However the full solution of (37) is quite difficult to obtain. Nevertheless the asymptotic behaviour is simple — when a is negative, $\ddot{e}$ is positive : then e diverges towards $+\infty$. Neglecting the right hand side of (37) and taking (30) into account we are led to

$$e \sim \sqrt{\alpha}\,\tau^{-1} \qquad \tau = \tau_0 - t. \tag{39}$$

The positive constant α depends on the initial conditions. All the quantities t_n and s_n diverge with the following power laws :

$$t_0 \sim \alpha\tau^{-2} \quad t_1 \sim 2\,\alpha\tau^{-3} \quad t_2 \sim 4\,\alpha\tau^{-4}$$
$$s_0 \sim (2\,\alpha + 24)\,\tau^{-2} \quad s_1 \sim -(6\,\alpha + 48)\,\tau^{-3}. \tag{40}$$

The eigenvalues of the shear tensor, which are the roots of the polynomial $P(u')$ (21) have the following asymptotic behaviours :

$$u'_3 \sim -(1 + \sqrt{\alpha + 9})\,\tau^{-1} \quad u'_2 \sim 2\,\tau^{-1}$$
$$u'_1 \sim (\sqrt{\alpha + 9} - 1)\,\tau^{-1}. \tag{41}$$

Two of these values are positive and u'_2 only depends on the initial conditions *via* τ_0. As for vorticity, the equation (22) leads to

$$\Omega'_2 \sim \alpha\tau^{-2}. \tag{42}$$

As the numerators of Ω'_1 and Ω'_3 are zero according to this order of expansion, the following terms in t_n and s_n are needed to get the asymptotic behaviour of Ω'_1 and Ω'_3. From the first integral E (34) we already see that Ω'_1 and Ω'_3 go to zero at least as fast as τ^4

$$\Omega'_1 < \frac{E}{\Omega'_2(u'_1 - u'_2)^2} \sim \frac{E}{\alpha(\sqrt{\alpha + 9} - 3)^2}\,\tau^4.$$

In the equation of motion of Ω'_1 and Ω'_3 (16) we neglected the ratios Ω'_1/Ω'_2 and Ω'_3/Ω'_2

$$\mathrm{d}\,\mathrm{Log}\,\Omega'_1 = \mathrm{d}\,\mathrm{Log}\,\Omega'_3 = 4\,\mathrm{d}\,\mathrm{Log}\,\tau \tag{43}$$

Ω'_1 and Ω'_3 go to zero as τ^4.

Thus we have shown that the length of the vorticity diverges as τ^{-1} (enstrophy diverges as τ^{-2}) and, that it lies along the shear principal axis which corresponds to the intermediate eigenvalue. The three eigenvalues of u diverge as τ^{-1}. It is important to remark that this intermediate value u'_2 is positive. The function $\omega_i\,\omega_j\,u_{ij} = t_1 \sim \Omega'_2\,u'_2$ which is proportional to the energy conversion function (8), so is really positive.

Now we look at the rotation of the principal axis of u. From (17) we deduce that $\Delta'_{12} = -\Delta'_3$ and $\Delta'_{23} = -\Delta'_1$ tend to zero as τ^3 and that $\Delta'_{13} = \Delta'_2$ tends to zero as τ^9. The instantaneous vector of rotation becomes perpendicular to vorticity and its length tends to zero as τ^3. Thus rotation of the principal axis of u stops when the divergence time approaches.

Finally we consider the form of streamlines as τ tends to zero. The velocity field is given in the rotating base by

$$\begin{pmatrix} v'_x \\ v'_y \\ v'_z \end{pmatrix} = \frac{1}{2}\begin{pmatrix} u'_1 & -\omega'_3 & \omega'_2 \\ \omega'_3 & u'_2 & -\omega'_1 \\ -\omega'_2 & \omega'_1 & u'_3 \end{pmatrix}\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix} = \frac{1}{2}\Lambda\begin{pmatrix} x' \\ y' \\ z' \end{pmatrix}. \tag{44}$$

The characteristic polynomial of the matrix Λ is

$$P(\lambda) = -(\lambda^3 + a\lambda + b) \tag{45}$$

where a and b are defined in (26). Its discriminant $4\,a^3 + 27\,b^2 = 4\,a_0^3$ is a constant (29) which determines the time scale of the problem ($\sim |\,a_0\,|^{-1/2}$). As t approaches τ_0, ω'_1 and ω'_3 become very small compared to the other quantities and

$$\Lambda \sim \begin{pmatrix} u'_1 & 0 & \omega'_2 \\ 0 & u'_2 & 0 \\ -\omega'_2 & 0 & u'_3 \end{pmatrix}. \tag{46}$$

The eigenvalues of Λ are approximatively equal to $2\,\tau^{-1}$, $2\,\tau^{-1}$, $-4\,\tau^{-1}$ and the corresponding eigenvectors are given by

$$\begin{aligned} \hat{e}_X &= \cosh\chi\;\hat{e}'_1 - \sinh\chi\;\hat{e}'_3 \\ \hat{e}_Y &= \hat{e}'_2 \\ \hat{e}_Z &= -\sinh\chi\;\hat{e}'_1 + \cosh\chi\;\hat{e}'_3 \end{aligned} \tag{47}$$

where χ is defined by

$$\tau\omega'_2 \sim \pm\sqrt{\alpha} = 3\sinh 2\,\chi. \tag{48}$$

In this coordinate system the streamlines equations are :

$$\frac{X}{X_0} = \frac{Y}{Y_0} = \sqrt{\frac{Z_0}{|Z|}} \tag{49}$$

X_0, Y_0 and Z_0 being constants.

The fluid comes along the Z axis from large positive and negative values of Z and goes back to infinity in the X-Y plane. A rotating movement around the Y axis arises from the non zero value of $\hat{e}_X\cdot\hat{e}_Z = -\sinh 2\,\chi$. Notice that the form of streamlines only depends on the initial conditions from χ i.e. α.

6. **Conclusion.** — Neglecting the gradients of vorticity and shear, we have deduced from the Euler equations a model for local interaction between these two quantities. This model leads to a singularity (a divergence) in a finite time with critical exponents which are integers. Moreover the geometry of the divergence is explained.

The divergence time depends on a_0 as defined in equation (29). If a_0 is not a constant in space the divergence time will vary from one point to another and the gradients of vorticity and shear will surely no longer be negligible. The way of approaching the divergence will certainly be modified. This is all the more true since the evolution of the system is quite sensitive to the details of the equations. Unfortunately we first looked at the system (16) with missing terms. Then the evolution was much more complicated and fractional exponents arose.

If we try to compare our results with the numerical results of Morf, Orszag and Frisch [3] we see that in their situation $b = 0$ and $a = 4 \cos^2 x_3 \cos (x_1 + x_2) \cos (x_1 - x_2)$ at the initial time. The smallest divergence time, which corresponds to $a = -4$, is $\tau_0 = 0.26$. Their divergence time for the whole flow is 5.2. For the enstrophy we get a critical exponent equal to 2 compared to their value of 0.8. It is reasonable to think that the gradients of vorticity and shear lead to a divergence which is both slower and weaker than in our calculation.

Finally we note that our problem exhibits an analogy with the motion of an asymmetric top. The antisymmetrical matrix C is related to a pseudo vector which corresponds to the angular momentum of the top. If we connect the angular velocity Δ of the principal axis of the shear with the angular velocity of the top we get as inertia tensor, $I = 4(\text{trace } u^2)\, \mathbb{1} - 6\, u^2$, whose principal axes coincide with those of u. The eigenvalues of I are related to the squares of the differences of the eigenvalues of u, $I_1' = 2(u_2' - u_3')^2$. Contrary to the top case the inertia tensor I here is not a constant.

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