A BOTANICAL QUASICRYSTAL

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Resume - L'arrangement des florets de fleurs composées (phyllotaxie) est une cristallographie en symétrie cylindrique, engendrée par un algorithme simple. Leur structure expose le nombre d'or, les nombres de Fibonacci, elle est auto-similaire et a des joints de grain quasicristallins, pour des raisons cristallographiques (homogeneité) et biologiques élémentaires. Les joints de grain sont aussi des disinclinaisons de 2π.

Abstract - Daisies, and composite flowers, exhibit an ordered arrangement of florets (phyllotaxis) generated by a simple algorithm. Their structure has self-similarity (inflation), golden mean, Fibonacci numbers, and quasicrystalline grain boundaries, for elementary crystallographic (homogeneity) and biological reasons. Grain boundaries are also 2π-disclinations.

I - INTRODUCTION

The cellular structure of daisies, sunflowers, etc., called phyllotaxis, is aperiodic, and its grain boundaries are one-dimensional quasicrystals. Even though this last observation is, as far as I know, original, it could have been made on photographs of composite flowers [1,2,3], and is obvious in computer representations of the structure obtained 4 years ago [4]. Phyllotaxis consists of a sufficiently simple class of structures, so that the arrangement of florets in the natural, botanical specimen, corresponds exactly to the model, drawn by computer from a three-line algorithm [4].

We shall see that self-similarity (or the inflation-deflation operation familiar in quasicrystals), has an obvious botanical reason, the need for the same algorithm or genetic coding to describe both a small (young) and large (old) flower. Since these structures are crystallographic (filling space with florets or cells, in cylindrical symmetry imposed by the stem of the flower), self-similarity has also an obvious crystallographic reason, the requirement of as much homogeneity as is compatible with cylindrical symmetry [4]. This, in turn, imposes the value τ, the golden mean (or the similar, "noble" numbers [5]) for the only parameter of the structure.

The structure consists of regular crystalline grains which form concentric annuli, separated by grain boundaries which are quasicrystalline. Inflation imposes itself both between grains, and

within the grain boundary. First of all, by the Fibonacci numbers of spirals (parastichies), and of dislocations on the grain boundary. But also, by the structure of the grain boundary (quasi-crystalline sequence of dislocations and isolated hexagonal cells of the type studied by de Bruijn [6]). Clearly, the fact that quasicrystals occur at grain boundaries is an important cyrstallographic observation, which is probably of much wider occurrence than in phyllotaxis, and corresponds, as we shall see, to a self-similar transformation between neighbouring grains.

The grain boundary surrounds a 2π-disclination. It is also a glide circle for the dislocations, and therefore a closed geodesic of the structure. We shall study, in the second part of this paper, the nature and role of 2π-disclinations in Euclidean space. It is well established that 2π-disclinations (or odd lines) occur in the structure of glasses and play a part in their physical properties [7,21], but their precise geometrical (rather than botanical) representation has yet to be exhibited. We shall show that 2π-disclinations occur in conventional quasicrystals [8]. Complete space decurving requires 4π-disclinations, which do indeed occur in quasicrystals as in many regular (periodic) patterns. Their geometrical form in infinite, 2D patterns, will be shown.

In one single structure (the Daisy of Fig.1), we can see self-similarity (inflation), golden mean, Fibonacci numbers and quasicrystals. The aim of this paper is to explain the relation between these numerical, geometrical and crystallographic concepts, and why they necessarily occur in botany or in crystallography in cylindrical symmetry.

II - PHYLLOTAXY (Arrangement of Leaves or Florets)

The inner florets of an aster, a daisy or a sunflower, the scales of a pinecone or a pineapple, form an area-filling structure, in which neighbouring "cells" are arranged on visible spirals (parastichies). Hexagonal florets (as in daisies, asters or pineapples) belong to three parastichies, rhombus cells (as in pinecones or sunflowers), to two, alternatively left- and right-handed. In most cases the number of parastichies in each family are consecutive Fibonacci numbers, even though "light" errors do occur (e.g. 6% of Norwegian spruce cones have the Lucas, rather than the Fibonacci series [1]). Important errors (false phyllotaxis [4,9]) are eliminated, presumably by natural selection. In the largest sunflowers [1-3] or daisies, the structure stretches from one pair or triplet of consecutive Fibonacci numbers to the next higher.

The florets are generated at regular intervals from the stem, and at a given angle ($2\pi\alpha$) from each other. Younger florets push older ones away from the centre so that the shape of each floret, and the structure itself, must remain fluid. They are only the result of filling space. The only structural information which is encoded is the angle $2\pi\alpha$ between successive florets, and the fact that they are generated at regular time intervals. In addition, the role of the stem imposes cylindrical symmetry.

A family of structures is therefore generated by a simple code or algorithm, able to cope with affine transformations (growth, size and breeding), and the Fibonacci series is the visible manifestation of this code. The code is probably a practical translation of some

botanical variational principle, for example sharing an area in order to maximize the flux of sun, rain, air [1]. But, as far as we are concerned, we have a genuine crystallography (in cylindrical symmetry) with a few area-filling structures containing a large number of cells and characterized by a simple algorithm or code. The structure is automatically axial [10] and dominated by self-similarity. Its crystallographic ingredients (crystallographic grains, and quasi-crystallographic grain boundaries) are obviously archetypical.

III - THE DAISY

1. The Algorithm. The florets are labelled by the positive integer $l=1,2,\ldots$ (older florets have larger l). The "seed" of floret l is given in cylindrical coordinates by

$$r(l) = a\sqrt{l}$$
$$\varphi(l) = 2\pi\alpha l \qquad (1)$$

The precise radial law is not very important structurally (except that non-hexagonal cells can be avoided if the seeds lie on a logarithmic spiral [11]). It has been chosen here so that the cell size remains uniform (a is its typical linear dimension). Curvature of the cellular substrate (as in pinecones), or growth of the florets as they age, can be accommodated by any monotonic function $r(l)$, with the main structural features remaining unaffected. α ($0<\alpha<1$) is the only structural parameter. The floret seeds lie on a spiral (1), called generative spiral, which, unlike the parastichies, is not usually visible in the structure (Fig.1).

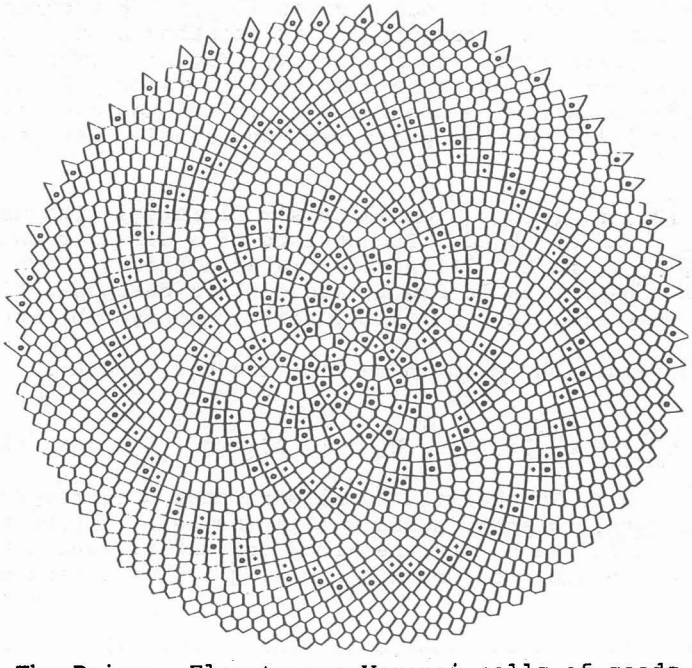


Fig. 1 - The Daisy. Florets are Voronoi cells of seeds given by eq.(1). All are hexagonal, except o = pentagon, + = heptagon. +o = dislocation. From R. Occelli, ref. [12].

The cellular structure is then constructed around the seeds by Voronoi construction (Dirichlet domain). The shape of each cell is therefore fluid, and depends on the position of its neighbours' seeds. The seed is not necessarily at the centre of mass of the cell (this constitutes an efficient algorithm for relaxing the structure, if necessary [4]).

A typical Daisy structure can be seen in Fig. 1. It is identical to the real thing. The Voronoi algorithm (together with a topological signature, o for pentagom, + for heptagon, and thus o+ denoting a dislocation [4]) is due to R. Occelli, as is Fig. 1 [12].

2. The Code. One can define concentric, circular shells containing an integer number of florets, q, say. Then, floret 1+q is neighbour to floret 1, $\varphi(1+q) \approx \varphi(1)$ are approximatively, but not exactly equal. Consequently, α cannot be rational. If it were ($\alpha = p/q$, say), $\varphi(1+q) = \varphi(1)$, the cell seeds would lie on the same radial axis, and the cellular structure ressembles a spider web [4], rather than a daisy.

Any irrational $\alpha<1$ can be uniquely written as an infinite continued fraction, $\alpha = 1/[q_1+(q_2+(q_3+(\ldots)))]$, where q_i are positive integers. Thus $\alpha = \{q_i\}$, is represented uniquely by the set of integers or code $\{q_i\}$. The continued fraction can be truncated at level m, say. This yields a succession of rational approximants or convergents to $\alpha \approx A_m/B_m = 1/[q_1+(\ldots+1/q_m)]$. In the daisy, B_m is the number of florets in a circular shell, or of parastichies, and A_m, the number of turns in the generative spiral necessary to fill the shell. Floret $1 + B_m$ is neighbour to floret 1.

Large daisies will use large B_m (and A_m) on the outside, and small B_m (and A_m) close to the stem. Since a given floret 1 has three older (outside) neighbours, the structure in its neighbourhood will be described by three approximants simultaneously (Fig.2). (This flexibility is a property of the infinite continued fraction. A rational number, described by a finite continued fraction, has clearly one best approximant (itself), and the spider web, one single, best set of parastichies).

A Grain $(m) = \{B_m, B_{m-1}, B_{m-2}\}$ contains all florets 1 with neighbours $1 + B_m$, $1 + B_{m-1}$ and $1 + B_{m-2}$. It consists of a circular annulus with B_m (steepest), B_{m-1} and B_{m-2} (flattest) parastichies in each family. The grains are conspicuous in Fig.1, and contain hexagonal florets exclusively. They are, topologically, perfect honeycomb lattices.

There exists a recursion relation between the B's (and A's):

$$\begin{aligned} A_m &= q_m A_{m-1} + A_{m-2} \quad , A_o = 0, A_1 = 1 \\ B_m &= q_m B_{m-1} + B_{m-2} \quad , B_o = 1, B_1 = q_1 \end{aligned} \qquad (2)$$

Thus $\alpha = \{q_i\} = \{B_i\}$: To the structural parameter α corresponds a code $\{q_i\}$, or the crystallographic numbers $\{B_i\}$ (numbers of parastichies). If $q_i = 1$ for all i, the B_i are the Fibonacci series (and $\alpha = 1/\tau$). They constitute the Lucas series if $q_1 = 3$, $q(i>1) = 1$ (and $\alpha = 1/(3+1/\tau)$).

There is a crystallographic requirement for $q_i = 1$, for $i > s$ (the first few q_i can be arbitrary), namely, homogeneity of the structure [4]. The three numbers B_i constitute a local frame of reference inside the grain (they label the neighbours to a given floret), and

the grain is homogeneous, if neighbours can be labelled independently of the path. Consider floret 1, and go to its neighbour $1+B_m$. This can be done either directly (via a B_m parastichy), or by using the other two sides of a triangle, through floret $1+B_{m-1}$ (via B_{m-1} and B_{m-2} parastichies successively) (Fig.2). The corresponding labels must be independent of the path, $1+B_m = (1+B_{m-1}) + B_{m-2}$, independently of 1, thus

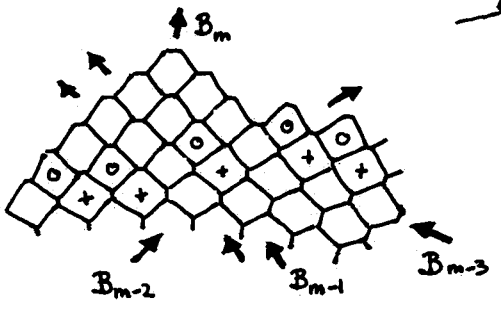
$$B_m = B_{m-1} + B_{m-2} \quad , \; q_m = 1 \qquad (3)$$

the Fibonacci recursion relation!

The irrational numbers satisfying (3), $\alpha = \{q_1,q_2,\ldots,q_s,1,1,1,1\ldots\}$ have been called noble by Percival [5]. The simplest $\alpha = \{1\ldots\} = 1/\tau$ is the golden mean. Noble numbers are similar to $1/\tau$, as noble metals (Cu, Ag, Au) are similar to gold. Pinecones can be found with Lucas instead of Fibonacci numbers of parastichies [1].

The arbitrariness in the choice of noble number, and in the first few q_i can be understood because the centre of the structure appears very disordered, the grains are very thin, and an additional 6 pentagons are required topologically (Euler relation) [4]. At the very least, recursion relation (3) can have arbitrary initial condition $B_1 = q_1$.

Fig. 2 - Local reference frame. (Cells labelled arcwise along parastichies).



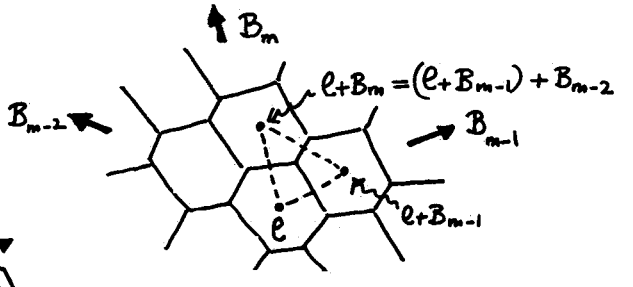


Fig. 3 - Grain boundary (detail) between grains (m-1) and (m). It contains $d_m = B_{m-2}$ dislocations (+o) and $s_m = B_{m-3}$ isolated hexagons.

The homogeneity requirement is so overriding, that should one select an α which is not noble (eg. $\alpha = 4/(11 + \sqrt{5}) = \{3,3,4,4\ldots\}$), the computed structure takes up intermediate convergents, additional $B_m^{(i)}$ satisfying the homogeneity requirement (3) [4,9]. The structure is then very strongly spiralling (dominated by one family of very long spirals) and the grain boundaries are no longer self-similar (inflatable) [4]. But these serious coding errors never occur in nature.

IV - GRAIN BOUNDARIES ARE QUASICRYSTALS

Additional material (i.e. parastichies) must be added as one goes out from the centre of the Daisy. Numerically, this is done by switching from one approximant A_{m-1}/B_{m-1} of α, to the next higher A_m/B_m. Visually, (Fig.1), the flattest parastichies (B_{m-3}) of grain (m -1) = $\{B_{m-1},B_{m-2},B_{m-3}\}$ are replaced by the steepest (B_m) ones in the next

grain (m) = $\{B_m, B_{m-1}, B_{m-2}\}$ with families B_{m-1} and B_{m-2} going through. In crystallography, grain boundaries are arrays of dislocations, and edge dislocations are the means to insert additional material. The boundary between annular grains (m-1) and (m) is a circle of d_m dislocations, or pentagon-heptagon pairs, with s_m solitary hexagons in between: The flattest parastichies, B_{m-3}, of grain (m-1), end on the $s_m = B_{m-3}$ solitary hexagons of the boundary, to be replaced by B_m steepest spirals in grain (m), sprouting from all the $2d_m + s_m$ florets in the grain boundary, while parastichies B_{m-1} and B_{m-2} go through. Thus, (Fig.3),

$$B_{m-3} = s_m,\ B_{m-2} = d_m,\ B_{m-1} = d_m + s_m,\ B_m = 2d_m + s_m \qquad (4)$$

This corresponds once again to the Fibonacci recursion relation (3) (or eq.(2) with $q_m = 1$). B_{m-3} and B_m have opposite chirality. Equation (4) is a rule of articulation between grains valid throughout the structure. A given family of parastichies extends through 3 grains, (m), (m+1) and (m+2) and 4 grain boundaries, starting from the 2d+s cells of the inner boundary and ending on the s isolated hexagons of the outer boundary:

$$2d_m + s_m = B_m = d_{m+1} + s_{m+1} = B_m = d_{m+2} = B_m = s_{m+3} \qquad (5)$$

[4]. Self-similarity between successive grains and their boundaries is manifest and faultless. It is governed, numerically, by the Fibonacci recursion relation (3), which is equivalent to the crystallographic requirement of homogeneity within each grain, to a matching rule at grain boundaries, and to a noble structural parameter α.

With the numerical problem solved, let us see how a quasicrystal solves the geometrical problem of interspacing additional material in the outer grain, ie. dislocations, between existing parastichies of the inner grain, given that successive B's are co-prime ($B_m/B_{m-1} = q_m + B_{m-2}/B_{m-1}$, from eq.(2)), so that dislocations cannot be interspaced regularly between isolated hexagons. (Fig.4).

The one-dimensional version of the strip, or cut-and-projection method [13-16] intespaces horizontal intervals among vertical intervals (in general, the vectors defining the unit cell of the projected 2D lattice). The irregular staircase obtained (the 1D equivalent of Wieringa ceiling in 2D Penrose tilings [14]) distributes these intervals in a unique, most economical fashion, because it is a geodesic (in the "Manhattan" metric of the 2D lattice) between any two lattice points contained in the strip.

Consider the (finite) boundary (4) between grains (m-1) and (m). The strip method interspaces $d_m = B_{m-2}$ dislocations among $s_m = B_{m-3}$ isolated hexagonal cells. The irregular staircase describes the boundary. It contains $d_m + s_m = B_{m-2} + B_{m-3}$ elements, thus $2d_m + s_m = 2B_{m-2} + B_{m-3}$ cells (each dislocation is a pentagon-heptagon dipole), distributed in the most uniform fashion, because it is a geodesic between points (0,0) and (B_{m-2}, B_{m-3}) in the 2D lattice. If α is a noble number, the grain boundary has B_{m-1} elements and B_m cells, as required (4). [If α is not noble (false phyllotaxis [4,9]), the number of elements and cells in the grain boundary are intermediate convergents [9]. One family of parastichies - that given by the principal convergents (2) - persists through many, $(q_m + 2)$, grains, giving the

structure its strongly spiralling appearance. Grains, and grain boundaries, are no longer self-similar [4,12].]

The strip method is therefore an algorithm for constructing grain boundary structures: Let the adjacent grains contain a and b (>a) lattice planes, respectively, on either side of a given segment of boundary. b-a is the number of additional cells, and (b-a)/2, that of dislocations in the boundary, whose structure is given by the strip method, where the azimuth of the strip is given by $\tan \mu = a/[(b-a)/2]$ (here = means equality, or rational convergent - principal or intermediate). The algorithm above was designed for hexagonal lattice, but applies, mutatis mutandis, to any Bravais lattice. If α is noble, $b = B_m$, $a = B_{m-3}$, and the azimuth $1/\tau = \tan \mu = B_{m-3}/B_{m-2}$ is the golden mean (regardless of the actual value of the noble α, or of the actual series of B's (Fibonacci, Lucas...), as long as B satisfy recursion relation (3), since $1-B_{m-3}/B_{m-2} = (B_{m-4}/B_{m-3})(B_{m-3}/B_{m-2})$ satisfies, in the limit $m=\infty$, the same equation as $1/\tau$).

[When (b-a)/2<a, the strip method still yields the structure of the grain boundary, with azimuth $\tan \mu = (b-a)/2a$. It has few dislocations, and successive grain boundaries are no longer self-similar. This structure arises in false phyllotaxis [4,12]].

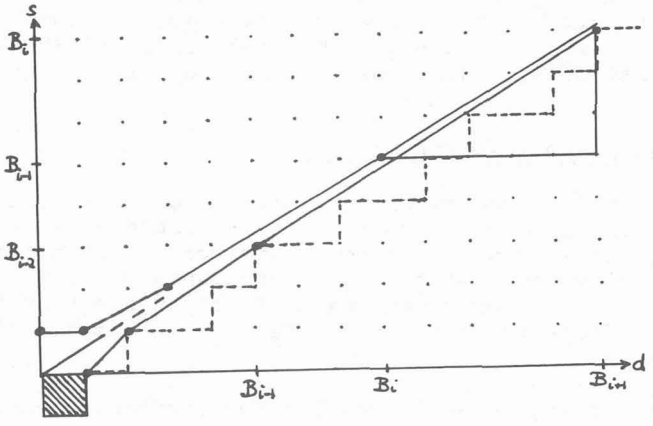


Fig. 4 - Quasicrystal grain boundary (broken line), obtained by the strip method. The strip has golden azimuth, collimated by 2 polygonal lines constructed on its rational convergents. There are no lattice point in between: Any azimuth within the collimation has identical, finite grain boundary.

Each grain boundary (the irregular staircase obtained by the strip method) is a (finite, circular) quasicrystal, which can be generated by inflation (s->d, d->ds). This can be demonstrated by suitable modification of the primitive cell of the 2D lattice in the strip method [13,6], or by the geometric method given below. But, when the structure is self-similar (α is noble and the Fibonacci recursion relation (3) holds), the sequence of all grain boundaries is itself generated by inflation, recursively outwards from the centre. Their structure has the familiar, aperiodic form, eg. [dsddsdsddsdds], but on circles. This is manifest in Fig.1.

Inflation generates a sequence of finite structures w_i (words dsd...), such that $w_i = w_{i-1}w_{i-2}$ (as is easily shown by induction). Let us demonstrate this property geometrically by the strip method (Fig.4). Finite grain boundary structures are generated by rational convergents of the azimuth of the strip. Successive convergents are alternatively larger and smaller than the irrational which they approximate.

Between successive convergents to an irrational azimuth, there is a region without any 2D lattice point, which collimate the irrational azimuth (this construction is due to F. Klein [17]). Any finite strip within this region has therefore the same irregular staircase and grain boundary. Compare now the azimuths B_i/B_{i+1} and B_{i-1}/B_i (which yields staircase structures w_i and w_{i-1} respectively). The beginning of w_i is identical to the whole structure w_{i-1}. Its end is identical to that of azimuth $(B_i-B_{i-1})/(B_{i+1}-B_i)$, which, if α is noble, equals B_{i-2}/B_{i-1}, and is itself another convergent of the azimuth of the strip, with structure w_{i-2}. Thus, $w_i = w_{i-1}w_{i-2}$.

Other quasicrystalline grain boundaries, generated by different radial law, or α, in algorithm (1), are presented in refs [4] and [12]. In particular, their topological stability (stability under relaxation of the structure and thermal fluctuations) is qualitatively discussed.

We have seen that large angle grain boundaries (d>s) have, in 1D, a quasicrystalline structure. Since the strip method, used to establish this result, can also generate quasicrystals in 2 and 3D [13-16,18], it is likely that quasicrystals will turn up on grain boundaries of real 3D materials.

V - QUASICRYSTALS AND 2π-DISCLINATIONS

Consider now a grain boundary of the Daisy (Fig.1) as structural constituent of the 2D packing of florets. Regarded as a Burgers contour, it surrounds a 2π-disclination (a vector in the outer grain, transported parallel to itself around the boundary, is rotated by 2π). It is also a geodesic line within each of its adjacent grains. Thus, a geometrical indication of the presence of a 2π-disclination is a closed geodesic.

The best local packing in 3D (a regular tetrahedron) cannot fill Euclidean space, but only an ideal space of positive curvature [18]. Real closed packed structures are resolution of this geometrical frustration. The ideal space, and the structure filling it, are decurved by disclinations [19].

Among all disclinations, 2π-disclinations have three major advantages: They are the only non-trivial disclinations which are independent of the rotation axis. Their contribution to decurving is therefore isotropic. They restore a local frame to its original orientation after circumnavigation (even though its connections with the rest of the structure are entangled). In glasses, they are the only linear structures ("defects") compatible with a trivial space group [20,21]. Their counterparts in discrete, random structures are odd lines [7]. These advantages offset (in glasses, at any rate) the inconvenience of large distortion. It is therefore of interest to see whether, and how, 2π-disclinations occur in other packings or tilings.

2D quasicrystals can be regarded as aperiodic superstructures of 2π-

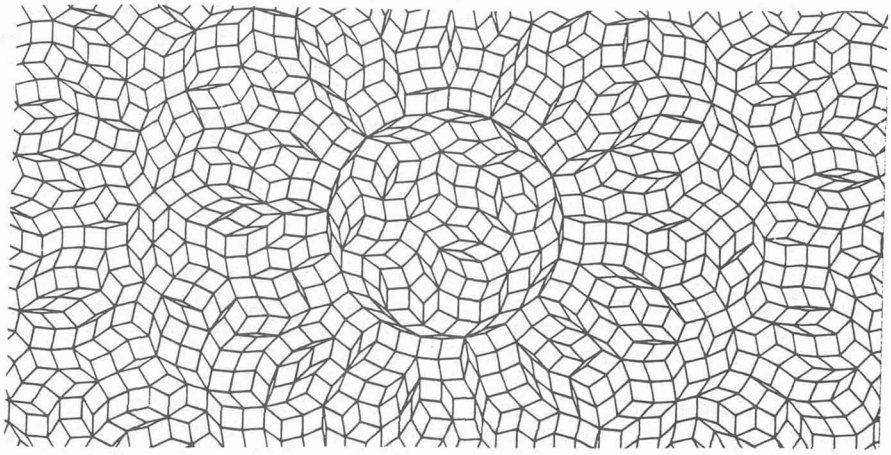


Fig. 5 - 2π-disclination in a 17-quasicrystal (Gähler and Rhyner).

disclinations. Fig.5 (Gähler and Rhyner, private communication) is a quasicrystal with orientational order along n = 17 directions (Penrose 's has n = 5), which can be constructed by duality [14] from a 17-grid (with all phases approximately equal). It is a tiling of (n-1)/2 = 8 types of rhombi. One notices a nearly perfect 2n = 34-gon, which is the projection of a hemispherical half bubble tiled with rhombi. The 34-gon has n = 17 rhombi of each type, 2 along each direction with the thinnest ones on its boundary. Parallel transport around the bubble demonstrates that its equator surrounds a 2π-disclination.

Similar 2π-disclinations occur in Penrose tilings (n = 5), but, being more common in the structure, they are less obvious. The hemispherical bubbles project as regular decagons, containing 5 each of the two types of rhombi, with the thin ones on the rim. The decagons are projections of (nearly) half rhombic triacontahedra, or half icosidodecahedra in the (dual) pentagrid (Fig.6). The 5 central grid lines form 5 geodesics on the hemisphere, and the sixth geodesic (absent in the pentagrid) is its equator, surrounding a 2π-disclination.

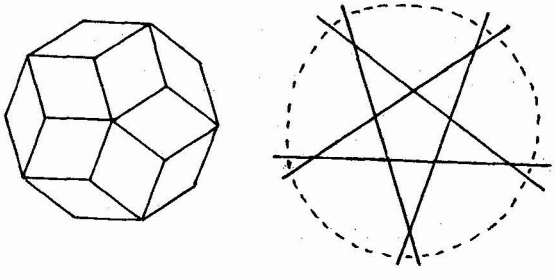


Fig. 6 - 2π-disclination in a Penrose tiling and in its dual.

Another example of 2π-disclinations may be found in the experiment of Dozier and Chaikin [22]: Latex in a cylindrical tube is put under oscillatory pressure. The structure is crystalline (colloidal crystal) away from the axis of the tube. For the crystalline structure to

be compatible with cylindrical boundary condition, it must either polygonize in wedge-shaped grains, with radial grain boundaries, as suggested recently [23] - in which case there is no disclination - or, as in the Daisy, grains are concentric annuli and the axis of the cylinder is a 2π-disclination. Both solutions are possible geometrically, but only the 2π-disclination is continuous and defect-free on the cylindrical boundary.

VI - 4π-DISCLINATIONS

4π-disclinations are trivial defects and can therefore be accommodated in tiling of Euclidean space. How do 2π-disclinations combine to form a 4π-disclination, and are these Euclidean combinations visible in tilings, whether crystalline or quasicrystalline?

The answer is elementary: n closed geodesics (each surrounding a 2π-disclination) intersecting at one point, constitute a 4π-disclination (Parallel transport rotates a vector by $2\pi n - 2\Sigma\beta_i = 4\pi$, where β_i are the inner angles of the polygon joining the centres of the geodesics, $\Sigma\beta_i = (n-2)\pi$).

This pattern occurs in many regular patterns and in Penrose tilings (Fig.7). A region of flat space is obtained by three decagons intersecting at one point, and sharing an edge or a thin rhombus. The 3D analogue is the model of Guyot and Audier [24] for Al_4Mn, of rhombic tricontahedra, sharing a face or an obtuse rhombohedron. Quasicrystals may thus be regarded as superstructures of 2π-disclinations, which combine locally as 4π-disclinations in order to flatten space.

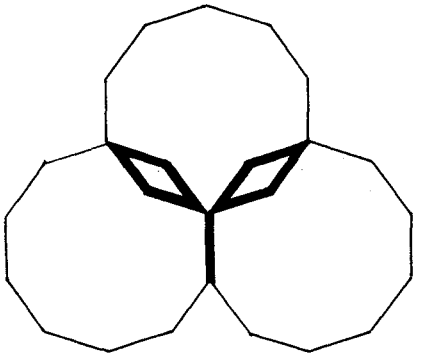


Fig. 7 - 4π-disclination in a Penrose tiling, made of 3 2π-disclinations intersecting at a point and sharing elements indicated by a heavy line.

VII - CONCLUSIONS AND ACKNOWLEDGEMENTS

The Daisy has circular grain boundaries which are quasicrystalline. There is self-similarity (inflation) within each boundary, between boundaries and also in the interplay of grains and boundaries. The Daisy exhibits in a single structure (Fig.1) golden (or noble) number, Fibonacci (or Lucas) series, quasicrystallinity and self-similarity, all resulting from the crystallographic requirement of homogeneity (Fig.2). Grain boundaries are closed geodesics, surrounding a 2π-disclination. The explanation for quasicrystalline grain boundaries, using the strip method, is sufficiently general and simple to suggest that large angle grain boundaries should be quasi- crystals, even in 3D structures and without cylindrical symmetry.

The numerical work on the Daisy has been done in collaboration with A. Lissowski and R. Occelli [4], who obtained Fig.1 [12]. Fig.5 is due to Gähler and Rhyner. Useful discussions and correspondence with these colleagues, and with A.L. Mackay, A.J.A. Lawrence, F. Rothen, J.F. Sadoc and R. Schilling are gratefully acknowledged.

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