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New Sufficient Conditions for the g -maximum Inequality

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Abstract

In this paper, we give new sufficient conditions for the existence of a solution of the g -maximum equality. As an application, we prove a new fixed point theorem.

Key words: Ky Fan inequality, g -maximum equality, fixed point.

PACS: C61, C62 and C72.

1 Introduction

Let X and Y be nonempty subsets of spaces E and F , respectively. Let $\Psi : X \times Y \longrightarrow \mathbb{R}$ and $g : X \longrightarrow Y$ be functions, and let $r \in \mathbb{R}$ be a constant. Consider the problem of finding \bar{x} such that

$$\Psi(\bar{x}, y) \leq r, \quad \forall y \in Y. \quad (1.1)$$

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Ky Fan [6] introduced and studied the minimax inequality problem of finding a solution $\bar{x} \in X$ of the inequality (1.1) in the case where $E = F$, $X = Y$, $g = id_X$ and $r = \sup_{x \in X} \Psi(x, x)$. The Ky Fan inequality has proven to be very useful in solving nonlinear problems in different areas. Due to its various applications, many researchers made efforts to generalize it. Indeed, many results have been obtained in this direction of research: we mention the results of Ding and Tan [5], Georgiev and Tanaka [10], Simons [14], Tian and Zhou [15], Yu and Yuan [16] and Yuan [17], and equilibrium problems studied by many authors as special cases, see [1], [4], [5], [7], [8], [9] and the references therein.

Note that in general all these works assume that $X = Y$ in (1.1). As far as we know there is only one result [13], where the author assumes $X \neq Y$, but considers the set X as an interval in the real line \mathbb{R} . In [12], the inequality (1.1) has been studied in the case where $E \neq F$ or $X \neq Y$. The same authors proved the following theorem.

Theorem 1.1 [12] (*g-Maximum Equality Theorem*) *Let X be a nonempty subset of a metric space E , Y be a nonempty convex, compact subset of a hausdorff locally convex vector space F and Ψ be a real-valued function defined on $X \times Y$. Suppose that there exists a nonempty compact subset X_0 of X and a continuous function g of X_0 into Y . Assume, in addition, that the following conditions are satisfied.*

- (1) $g(X_0)$ is convex in Y ,
- (2) the function Ψ is continuous on $X_0 \times Y$,
- (3) the function $y \mapsto \Psi(x, y)$ is quasi-concave on Y , for each $x \in X_0$,
- (4) for each $g(x) \in \partial g(X_0)$ and for each $y \in Y$, there exists $z \in Z_{g(X_0)}(g(x))$ such that $\Psi(x, y) \leq \Psi(x, z)$ where $Z_{g(X_0)}(g(x)) = \left[\bigcup_{h>0} \frac{g(X_0) - g(x)}{h} + g(x) \right] \cap Y$.

Then there exists $\bar{x} \in X_0$ such that

$$\sup_{y \in Y} \Psi(\bar{x}, y) = \Psi(\bar{x}, g(\bar{x})). \quad (1.2)$$

The main purpose of this paper is to establish the existence of a solution of the nonlinear *g-maximum equality* (1.2), under assumptions different from those of Theorem 1.1. As an application of this new result a new fixed point theorem is presented.

Let us first introduce some notations and definitions.

Consider a nonempty subset X of a metric space E and Y a nonempty subset of a locally convex space F . Let 2^Y be the set of all the parts of Y .

A set-valued $C : X \rightarrow 2^Y$ is said to be closed if the corresponding graph is closed in $X \times Y$, i.e. the set $\{(x, y) \in X \times Y \text{ such that } y \in C(x)\}$ is closed in $X \times Y$ [2]. A function $f : Y \rightarrow \mathbb{R}$ is said to be upper semicontinuous over Y if $\forall c \in \mathbb{R}$, the set $\{x \in Y, f(x) \geq c\}$ is closed; f is said to be lower semicontinuous over Y if $-f$ is upper semicontinuous and f is said to be continuous over Y if f and $-f$ are upper semicontinuous over Y . We say that f is quasi-concave on Y if for any y_1, y_2 in Y and for any $\theta \in [0, 1]$, we have $\min \{f(y_1), f(y_2)\} \leq f(\theta y_1 + (1 - \theta)y_2)$. And f is quasi-convex if $-f$ is quasi-concave.

Let f be a real-valued function defined on a metric space E . The *support* of f (denoted by $\text{supp}(f)$) is the smallest closed set S such that $f(x) = 0, \forall x \notin S$, i.e. $\text{supp}(f) = \{x \in E, \text{ such that } f(x) \neq 0\}$.

Let us consider an open finite covering $\{A_i\}_{i=1, \dots, n}$ of a set E . A continuous partition of unity associated to this finite covering, is a family of continuous functions $\{f_i\}_{i=1, \dots, n}$ defined from E into $[0, 1]$ such that:

$$\left\{ \begin{array}{l} 1) \forall x \in E, \sum_{i=1}^n f_i(x) = 1, \\ 2) \text{supp}(f_i) \subset A_i, i = 1, \dots, n. \end{array} \right.$$

We have the following Lemma.

Lemma 1.1 (*Theorem 4.1.31. page 187, [2]*) *For all open finite covering of a metric space E , there exists a continuous partition of unity associated to this finite covering.*

Zeidler [18] showed that this Lemma remains true if E is a locally convex Hausdorff space.

Let us consider a set-valued function C defined from X into X . A point $x \in X$ is called fixed point of F if $x \in C(x)$. If C is a single-valued function, then a fixed point x of C verifies $x = C(x)$.

We will use the following lemma.

Lemma 1.2 (*Kakutani-Fan-Glicksberg Fixed point Theorem*) *Let K be a nonempty compact convex subset of a locally convex Hausdorff space, and let $C : X \rightarrow 2^X$ be a closed set-valued function with nonempty convex values. Then the set of fixed points of C is nonempty and compact.*

2 The g -maximum Equality

Let us consider the following example.

Example 2.1 Let $X = [0, 1]$ and $Y =]-\infty, 0]$, $g(x) = -x$, $\forall x \in X$ and $\Psi(x, y) = -x^2 - y^2$.

It is clear that Theorem 1.1 cannot be applied because Y is not compact. Nevertheless, there exists $x = 0$ such that $\sup_{y \in Y} \Psi(x, y) = \Psi(x, -x)$. This example is an indication that the g -maximum equality (1.2) can have a solution under conditions different from those of Theorem 1.1.

In the following theorem we provide new sufficient conditions for which the g -maximum equality (1.2) has at least one solution; in particular, in the settings of Examples 2.1.

Theorem 2.1 Let X be a nonempty convex compact set of a locally convex Hausdorff space, and let Y be a nonempty set of a metric space. Consider two functions: $g : X \rightarrow Y$ continuous over X and $\Psi : X \times Y \rightarrow \mathbb{R}$ such that

- (1) $x \mapsto \Psi(x, y)$ is continuous over X , $\forall y \in Y$ and the function $z \mapsto \Psi(x, g(z))$ is lower semicontinuous over X , $\forall x \in X$
- (2) $x \mapsto \Psi(x, y)$ is quasi-concave over X , $\forall y \in Y$
- (3) $\forall (x, y) \in X \times Y$, $\exists z \in X$ such that $\Psi(x, y) \leq \Psi(z, g(x))$.

Then there exists $\bar{x} \in X$ such that

$$\sup_{y \in Y} \Psi(\bar{x}, y) = \Psi(\bar{x}, g(\bar{x})). \quad (2.1)$$

Proof. Suppose that (2.1) is not true, then

$$\forall x \in X, \exists y \in Y \text{ such that } \Psi(x, y) > \Psi(x, g(x)) \quad (2.2)$$

X can then be covered by the sets

$$\theta_y = \{ x \in X \text{ such that } \Psi(x, y) > \Psi(x, g(x)) \}, \quad y \in Y.$$

Let us prove that $\forall y \in Y$, θ_y is open. Indeed, let $x \in \overline{X/\theta_y}$, there exists a sequence $\{x_p\}_{p \geq 1}$ in X/θ_y converging to x , hence $\forall p \geq 1$, $\Psi(x_p, y) \leq \Psi(x_p, g(x_p))$. Taking into account condition (1) of Theorem 2.1, when $p \rightarrow +\infty$, we obtain $\Psi(x, y) \leq \Psi(x, g(x))$, i.e. $x \in X/\theta_y$, therefore X/θ_y is closed in X .

Since X is compact, it can be covered by a finite number n of subsets $\{\theta_{y_1}, \dots, \theta_{y_n}\}$ of type θ_y . Consider a continuous partition of unity $\{h_i\}_{i=1, \dots, n}$ associated to the finite covering $\{\theta_{y_1}, \dots, \theta_{y_n}\}$ (Lemma 1.1), *i.e.* $\{h_i\}_{i=1, \dots, n}$ verify

$$\left\{ \begin{array}{l} 1) \forall x \in X, \sum_{i=1}^n h_i(x) = 1, \\ 2) \text{supp}(h_i) \subset \theta_{y_i}, \quad i = 1, \dots, n. \end{array} \right.$$

Let us now consider the simplex S of \mathbb{R}^n

$$S = \{\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{R}^n \text{ such that } \sum_{i=1}^n \lambda_i = 1, \lambda_i \geq 0, i = 1, \dots, n\}.$$

Consider the following set-valued function

$$C : X \rightarrow X$$

$$\text{defined by } x \mapsto C(x) = \left\{ z \in X \text{ such that } \max_{\lambda \in S} \sum_{i=1}^n \lambda_i \Psi(x, y_i) \leq \Psi(z, g(x)) \right\}.$$

Now, we will prove step by step that the function C satisfies the conditions of Lemma 1.2 (Kakutani-Fan-Glicksberg fixed point Theorem):

- i) $\forall x \in X, C(x) \neq \emptyset$. Indeed, let be $x \in X$, the function $\lambda \mapsto \sum_{i=1}^n \lambda_i \Psi(x, y_i)$ is linear on \mathbb{R}^n , so it is continuous on the compact S and by the Weierstrass Theorem, there exists $\bar{\lambda} \in S$ such that

$$\max_{\lambda \in S} \sum_{i=1}^n \lambda_i \Psi(x, y_i) = \sum_{i=1}^n \bar{\lambda}_i \Psi(x, y_i) \leq \sum_{i=1}^n \bar{\lambda}_i \max_{i=1, \dots, n} \Psi(x, y_i) = \Psi(x, y_{i_0}) \quad (2.3)$$

where $y_{i_0} \in \{y_1, \dots, y_n\}$ hence

$$\max_{\lambda \in S} \sum_{i=1}^n \lambda_i \Psi(x, y_i) \leq \Psi(x, y_{i_0}).$$

Condition (3) of Theorem 2.1 implies, $\exists z \in X$ such that

$$\max_{\lambda \in S} \sum_{i=1}^n \lambda_i \Psi(x, y_i) \leq \Psi(x, y_{i_0}) \leq \Psi(z, g(x)).$$

Therefore, $z \in C(x)$, thus $C(x) \neq \emptyset$.

- ii) $\forall x \in X, C(x)$ is convex in X . Indeed, let $x \in X$ and $\bar{z}, \bar{\bar{z}}$ be two elements of $C(x)$ and $\theta \in [0, 1]$.

Let us prove that $\theta\bar{z} + (1 - \theta)\bar{z} \in C(x)$.
Since \bar{z} and \bar{z} are two elements in $C(x)$, we have

$$\max_{\lambda \in S} \sum_{i=1}^n \lambda_i \Psi(x, y_i) \leq \Psi(\bar{z}, g(x)) \text{ and } \max_{\lambda \in S} \sum_{i=1}^n \lambda_i \Psi(x, y_i) \leq \Psi(\bar{z}, g(x)),$$

hence

$$\max_{\lambda \in S} \sum_{i=1}^n \lambda_i \Psi(x, y_i) \leq \min \{ \Psi(\bar{z}, g(x)), \Psi(\bar{z}, g(x)) \}, \quad (2.4)$$

the condition (2) of Theorem 2.1 and the inequality (2.4) imply

$$\max_{\lambda \in S} \sum_{i=1}^n \lambda_i \Psi(x, y_i) \leq \Psi(\theta\bar{z} + (1 - \theta)\bar{z}, g(x)), \quad \forall \theta \in [0, 1];$$

thus $\theta\bar{z} + (1 - \theta)\bar{z} \in C(x)$.

iii) C has a closed graph in $X \times X$.

We have $\text{Graph}(C) \subset X \times X$. By assumption X is compact. Let $(x, z) \in \overline{\text{Graph}(C)}$, then there exists a sequence $\{(x_p, z_p)\}_{p \geq 1}$ in $\text{Graph}(C)$ which converges to (x, z) .

Hence we have $\forall p \geq 1, z_p \in C(x_p)$, *i.e.* $\forall p \geq 1, \max_{\lambda \in S} \sum_{i=1}^n \lambda_i \Psi(x_p, y_i) \leq \Psi(z_p, g(x_p))$.

Taking into account the condition (1) and the continuity of g of Theorem 2.1, when $p \rightarrow \infty$, we obtain

$$\max_{\lambda \in S} \sum_{i=1}^n \lambda_i \Psi(x, y_i) \leq \Psi(z, g(x)),$$

i.e. $z \in C(x)$, hence $(x, z) \in \text{Graph}(C)$, then $\text{Graph}(C)$ is closed in $X \times X$.

From (i)-(iii), we conclude that the function C satisfies all conditions of Lemma 1.2. Consequently, $\exists \tilde{x} \in X$ such that $\tilde{x} \in C(\tilde{x})$, *i.e.*

$$\max_{\lambda \in S} \sum_{i=1}^n \lambda_i \Psi(\tilde{x}, y_i) \leq \Psi(\tilde{x}, g(\tilde{x}))$$

hence $\forall \lambda \in S, \sum_{i=1}^n \lambda_i \Psi(\tilde{x}, y_i) \leq \Psi(\tilde{x}, g(\tilde{x}))$.

Using the continuous partition of unity $\{h_i\}_{i=1, \dots, n}$ given above, let $\tilde{\lambda} = (h_1(\tilde{x}), \dots, h_n(\tilde{x}))$, we have $\tilde{\lambda} \in S$ because $h_i(\tilde{x}) \geq 0$ and $\sum_{i=1}^n h_i(\tilde{x}) = 1$, therefore,

$$\sum_{i=1}^n h_i(\tilde{x}) \Psi(\tilde{x}, y_i) \leq \Psi(\tilde{x}, g(\tilde{x})).$$

Let $J = \{i \in \{1, \dots, n\} \text{ such that } h_i(\tilde{x}) > 0\}$, then $J \neq \emptyset$.

Note that $\sum_{i=1}^n h_i(\tilde{x}) \Psi(\tilde{x}, y_i) = \sum_{i \in J} h_i(\tilde{x}) \Psi(\tilde{x}, y_i)$.

We have $\forall i \in J, h_i(\tilde{x}) > 0$, therefore $\tilde{x} \in \text{supp}(h_i) \subset \theta_{y_i} \forall i \in J$, *i.e.*

$$\forall i \in J, \quad \Psi(\tilde{x}, y_i) > \Psi(\tilde{x}, g(\tilde{x})).$$

It follows that $\sum_{i \in J} h_i(\tilde{x}) \Psi(\tilde{x}, y_i) > \sum_{i \in J} h_i(\tilde{x}) \Psi(\tilde{x}, g(\tilde{x})) = \Psi(\tilde{x}, g(\tilde{x}))$ and then

$$\Psi(\tilde{x}, g(\tilde{x})) < \sum_{i \in J} h_i(\tilde{x}) \Psi(\tilde{x}, y_i) = \sum_{i=1}^n h_i(\tilde{x}) \Psi(\tilde{x}, y_i) \leq \Psi(\tilde{x}, g(\tilde{x})),$$

i.e. we obtain the following contradiction,

$$\Psi(\tilde{x}, g(\tilde{x})) > \Psi(\tilde{x}, g(\tilde{x})).$$

Therefore, (2.2) is not true. Hence

$$\exists \bar{x} \in X \text{ such that } \Psi(\bar{x}, y) \leq \Psi(\bar{x}, g(\bar{x})), \quad \forall y \in Y,$$

i.e. $\sup_{y \in Y} \Psi(\bar{x}, y) = \Psi(\bar{x}, g(\bar{x}))$. ■

Consider again Example 2.1. We have $X = [0, 1]$ and $Y =]-\infty, 0]$, $g(x) = -x$, $\forall x \in X$ and $\Psi(x, y) = -x^2 - y^2$.

It is clear that the assumptions and conditions (1)-(2) of Theorem 2.1 are satisfied. Let us verify condition (3) of Theorem 2.1. Indeed, let be $(x, y) \in X \times Y$, we have $\Psi(x, y) = -x^2 - y^2$ and $\Psi(z, g(x)) = -z^2 - x^2$. Since $-y^2 \leq 0$, $\forall y \in Y$, then there exists $z = 0 \in X$ such that $\Psi(x, y) = -x^2 - y^2 \leq -z^2 - x^2 = \Psi(z, g(x))$. Consequently $\exists x \in X$ such that $\sup_{y \in Y} \Psi(x, y) \leq \Psi(z, g(x))$. Indeed, $x = 0$ is such a point.

If the sets X and Y are identical and if we consider $g = id_X$, we obtain the following inequality similar to the Ky Fan inequality under other conditions.

Corollary 2.1 *Let X be a nonempty, convex and compact set in a locally convex Hausdorff space E and Ψ a real valued function defined on $X \times X$. Suppose that the following conditions are satisfied*

- (1) $x \mapsto \Psi(x, y)$ is continuous over X , $\forall y \in X$ and the function $y \mapsto \Psi(x, y)$ is lower semicontinuous over X , $\forall x \in X$
- (2) $x \mapsto \Psi(x, y)$ is quasi-concave over X , $\forall y \in X$
- (3) $\forall (x, y) \in X \times X$, $\exists z \in X$ such that $\Psi(x, y) \leq \Psi(z, x)$.

Then, there exists $\bar{x} \in X$ such that

$$\sup_{y \in X} \Psi(\bar{x}, y) = \Psi(\bar{x}, \bar{x}) \leq \sup_{y \in X} \Psi(y, y).$$

Remark 2.1 *If the function Ψ is semi-symmetrical, i.e. $\Psi(x, y) \leq \Psi(y, x)$, then the condition (3) of Corollary 2.1 is satisfied.*

3 Applications

In this section, we present a new fixed point theorem as an application of Theorem 2.1.

Let us consider the following example.

Example 3.1 *Consider the following function*

$$\begin{aligned} f : X = [\frac{6}{5}, 2] &\rightarrow \mathbb{R} \\ x &\mapsto f(x) = 1/(x - 1). \end{aligned}$$

We have $\max_{x \in [\frac{6}{5}, 2]} |f'(x)| = 25$, then f is a 25-lipschitz and also $f([\frac{6}{5}, 2]) \not\subseteq [\frac{6}{5}, 2]$ because $f(\frac{6}{5}) = 5 \notin [\frac{6}{5}, 2]$. Therefore the classical fixed point Theorems (Cauchy's, Banach-Cacciopoli-Picard's, Brouwer's, Browder's fixed point Theorem, ...) are not applicable.

The following theorem guarantees the existence of a fixed point for this type of functions.

Theorem 3.1 *Let X be a nonempty convex compact of a normed space $(E, \|\cdot\|_E)$. Let $f : X \rightarrow E$ be a continuous function such that*

- (1) $x \mapsto \|f(x) - y\|_E$ is quasi-convex over X , $\forall y \in E$,
- (2) $X \subset f(X)$.

Then f has a fixed point.

Proof. Let us consider the functions Ψ and g defined as follows:

$$\begin{aligned} \Psi : X \times E &\rightarrow \mathbb{R} \\ (x, y) &\mapsto \Psi(x, y) = -\|f(x) - y\|_E, \end{aligned}$$

$$g : X \rightarrow E$$

$$x \mapsto g(x) = x.$$

The function $\|\cdot\|_E$ is uniformly continuous over E , then the function Ψ is continuous over $X \times E$, and $x \mapsto \Psi(x, y)$ is quasi-concave over X (condition (1)), $\forall y \in E$.

Let us prove that $\forall (x, y) \in X \times E$, there exists $z \in X$ such that $\Psi(x, y) \leq \Psi(z, x)$. Indeed, according condition (2), we have $X \subset f(X)$, then $\forall x \in X$, $\exists z \in X$ such that $x = f(z)$, which implies $\|f(z) - x\|_E = 0$ and since $\forall x \in X$, $\forall y \in E$, we have $\|f(x) - y\|_E \geq 0$. Thus,

$$\forall x \in X, \forall y \in E, \exists z \in X \text{ such that } 0 = \|f(z) - x\|_E \leq \|f(x) - y\|_E,$$

i.e.

$$\forall x \in X, \forall y \in E, \exists z \in X \text{ such that } \Psi(x, y) \leq \Psi(z, x) = 0.$$

Since X is a nonempty, convex and compact subset of a normed space E , then according to Theorem 2.1, $\exists \bar{x} \in X$ such that

$$\|f(\bar{x}) - y\|_E \geq \|f(\bar{x}) - \bar{x}\|_E, \quad \forall y \in E.$$

Thus, if we let $y = f(\bar{x})$ in the last inequality, we obtain

$$\|\bar{x} - f(\bar{x})\|_E \leq 0.$$

Therefore $f(\bar{x}) = \bar{x}$, *i.e.* \bar{x} is a fixed point of function f . ■

Consider again the Example 3.1. The function $x \mapsto |1/(x - 1) - y|$ is quasi-convexe over $[\frac{6}{5}, 2]$, $\forall y \in \mathbb{R}$.

Since f is not increasing order, then $f([\frac{6}{5}, 2]) = [f(2), f(\frac{6}{5})] = [1, 5] \supset [\frac{6}{5}, 2]$. Thus according to Theorem 3.1, f has a fixed point in $[\frac{6}{5}, 2]$. Indeed, $\bar{x} = (1 + \sqrt{5})/2$ is such a point.

4 Conclusion

In this paper, through Theorem 2.1, we have established that the g -maximum equality has a least one solution under new conditions. This new Theorem (Theorem 2.1) is complimentary to Theorem 1.1. As an application of it, we have proved a new interesting fixed point theorem. We have exhibited examples where our results are applicable, but the well known fixed point

theorems are not applicable. This shows that our results enlarge the class of functions for which a fixed point exists. Finally, we hope that our results will be useful for solving theoretical and practical problems from various domains.

References

- [1] - A. N. Iusem and W. Soca, *New Existence Results for Equilibrium Problems*, *Nonlinear Analysis* 54, (2003) 621-635.
- [2] - J. P. Aubin and I. Ekeland, *Applied Nonlinear Analysis*. Wiley Interscience, New York, (1984).
- [3] - C. Berge, *Espaces Topologiques, Fonctions Multivoque*, Dunod, Paris, (1966).
- [4] - X. P. Ding, J.Y. Park and I.H. Jung, *Existence of Solutions for Nonlinear Inequalities in G -convex Spaces* *Appli. Math. Letters* 15, (2002), 735-741.
- [5] - X. P. Ding and K. K. Tan, *A Minimax Inequality with Application to Existence of Equilibrium Points and Fixed Point Theorems* . *Colloq. Math.* 63, (1992), 233-274.
- [6] - K. Fan, *Minimax Inequality and Application*, in "Inequalities, III" (O. Shisha Ed.) Academic Press, New York, (1972).
- [7] - L.J. Lin, *Applications of a Fixed Point Theorem in G -convex Space*, *Nonlinear Analysis* 46, (2001) 601-608.
- [8] - L.J. Lin and T.H. Chang, *S-KKM Theorems, Saddle Points and Minimax Inequalities*, *Nonlinear Analysis* 34, (1998) 73-86.
- [9] - L.J. Lin and S. Park, *On Some Generalized Quasi-Equilibrium Problems*, *Jour. Math. Anal. Appli.* 224, (1998) 167-181.
- [10] - Pando Gr. Georgiev and T. Tanaka, *Vector-valued Set-valued Variants of Ky Fan's Inequality*, *Journal of Nonlinear and Convex Analysis* 1, (2000) 245-254.
- [11] - R. Nessah and C. Chu, *Quasivariational Equation*, *Mathematical Inequalities and Applications* 7, (2004) 149-160.
- [12] - R. Nessah and M. Larbani *g -maximum Equality*, In proceedings of the NACA2003 conference, Takahashi W. and Tanaka T. (Eds), Yokohama Publishers (2004), 391-400.
- [13] - Ricceri B. *Existence Theorem for Nonlinear Problems*, *Memori di Mathamatica* 105, (1978) 77-99
- [14] - S. Simons, *Two Function Minimax Theorem and Variational Inequalities for Functions on Compact and Noncompact Sets, with some Comments on Fixed-point Theorems*, *Proc Symp. Pure Math.* 45, (1986) 377-392.

- [15] - G. Tian and J. Zhou, *Quasi-Inequalities without the Convexity Assumption*. J. Math. Annal 172, (1993) 289-299.
- [16] - J. Yu and X. Z. Yuan, *The Study of Pareto Equilibria for Multiobjective Games by Fixed Point Theorems and Ky Fan Minimax Inequality Methods*, Research Report, no 1/95, Department of institute of mathematics, guizhou Institute of technology, China, (1995).
- [17] - X. Z. Yuan, KKM principal, *Ky Fan Minimax Inequalities and Fixed Point Theorems*, Nonlinear World 2 (1995), 131-169.
- [18] - E. Zeidler, *Nonlinear Functional Analysis and its Applications*. Springer-Verlag, New York, 1986.