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ON ONE PROPERTY OF DISTANCES IN THE INFINITE RANDOM QUADRANGULATION

MAXIM KRIKUN

ABSTRACT. We show that the Schaeffer's tree for an infinite quadrangulation only changes locally when changing the root of the quadrangulation. This follows from one property of distances in the infinite uniform random quadrangulation.

1. PREREQUISITES

1.1. Quadrangulations and Schaeffer's bijection. Let Q be a rooted quadrangulation. Denote by $V(Q)$ the set of vertices of Q and by x the root vertex. For every $z \in V(Q)$ let $d_x(z)$ be the (graph) distance from z to x .

Quadrangulations are necessarily bipartite, so the increments of d_x along any path in Q are ± 1 . Two vertices on the same distance from the root can not be joined by an edge, but only by a diagonal in some face. Denote by Q_\times the graph obtained by adding to Q the two diagonals of each face. Note that Q_\times is not a planar map anymore but only a graph.

The tree $Sch_x(Q)$ is defined as follows [9]: for each face of Q consider the values of d_x on the vertices of this face in counterclockwise order. If for some R these distances match the pattern $(R, R+1, R, R-1)$, add the edge $(R, R+1)$ to Sch_x ; if the distances match the pattern $(R, R+1, R, R+1)$, add the diagonal $(R+1, R+1)$. The graph thus obtained is a tree, spanning all the vertices of Q except x . In general Sch_x lives on the graph Q_\times , but not on the map Q itself, however, since Sch_x (as an embedded graph) has no self-intersections, it inherits naturally a planar map structure from Q .

The main theorem concerning the tree Sch_x asserts that if for a finite sphere quadrangulation Q one takes Sch_x (viewed as a planar tree) and the values of d_x on the vertices of Sch_x , this information is sufficient to reconstruct Q . We omit the description of the actual reconstruction procedure, although it's very simple.

This bijection proved to be an important tool in the study of random quadrangulations, and was used by many authors [2, 6, 7, 4, 5, 8].

On the other hand, the construction of Sch_x depends in an essential way on the choice of the root in the quadrangulation. The main goal of this note is to gain a deeper understanding of this dependence. In short, we show that this dependence is essentially continuous — small displacement of the root in a typical random quadrangulation only leads to small perturbation in the structure of the tree. To make this statement precise, we need to place ourselves in the framework of infinite random maps.

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1.2. Infinite quadrangulation. Consider a uniform measure μ_N on the set of rooted quadrangulations of a sphere with N faces. The local weak limit of this sequence is a probability measure μ_∞ supported on infinite planar quadrangulations. A sample from this measure is the *uniform infinite planar quadrangulation*.

More precisely, let \mathcal{Q} be the space of all finite rooted quadrangulations, endowed with the ultrametric distance

$$D(q, q') = \inf\{2^{-R} | R \in \mathbb{Z}, B_R(q) = B_R(q')\},$$

$B_R(q)$ denoting the ball of radius R around the root in the quadrangulation q . In this metric the sequence of quadrangulations $\{q_n\}_{n \in \mathbb{Z}}$ converges, if for every R all of it's elements, starting from some index $n(R)$, coincide on a ball of radius R around the root. Elements of the completion $\bar{\mathcal{Q}}$, different from finite quadrangulations, are called *infinite quadrangulations*, and the measure μ_∞ is supported on $\bar{\mathcal{Q}} \setminus \mathcal{Q}$. This construction was first considered in [1] in the case of triangulations; the treatment for quadrangulations is completely analogous [3].

1.3. Invariance with respect to the random walk. Let Q be a finite quadrangulation, and let $\{v_t\}_{t \in \mathbb{Z}}$ be the simple random walk on Q (viewed as a graph). Let e_t be the edge traversed by a random walk prior to arriving at v_t , then $\{e_t\}_{t \in \mathbb{Z}}$ is a random walk on the set of directed edges of Q (the *edge random walk*), and it's stationary measure is uniform on this set.

Therefore, if one acts upon the space of probability measures on finite quadrangulations by displacing the root randomly according to one-step transitions of the edge random walk, the measures μ_N are invariant under this action, and so is the limiting measure μ_∞ .

From this observation we conclude, in particular, that if some property holds with probability one for a uniform infinite quadrangulation, this property will hold with probability one for the same quadrangulation with the root displaced for a uniformly bounded distance according to any rule, deterministic or random.

1.4. Topological ends. An infinite connected graph G has *one end* if for any finite subgraph $H \subset G$ at most one (and therefore exactly one) component of the complement $G \setminus H$ is infinite. In particular, if G is a tree then from each vertex of G there is exactly one infinite simple path. If G is planar and has one end, then it's planar dual also has one end.

Both the uniform infinite triangulation and the uniform infinite quadrangulation have one end [1, 3].

1.5. Skeleton of the infinite quadrangulation. Given an infinite rooted quadrangulation Q , consider the set of vertices situated at distance R from the root and the edges of Q between these vertices. Denote the graph thus obtained by Γ_R . Γ_R cuts Q into a number of connected components, only one of which is infinite a.s. (this follows from the a.s. one-endedness of Q). Let γ_R be a cycle in Γ_R that separates the root from this infinite component; if some vertices of Γ_R are connected by more than one edge, and such a cycle happens to be non-unique, let us choose among all alternatives the edge which is closer to the root.

Each edge of γ_R is an (R, R) -diagonal in some face of type $(R, R-1, R, R+1)$, and the vertex corresponding to $(R-1)$ belongs necessarily to the cycle γ_{R-1} . Let e_1, e_2, \dots be the edges of γ_R , and w_1, w_2, \dots the associated vertices on γ_{R-1} .

We introduce a directed tree structure on the union of edges of γ_R , by declaring the edges between w_n and w_{n+1} to be the offspring of the edge e_n . The tree thus obtained is called the *skeleton* of the quadrangulation Q and is written $\text{Sk}(Q)$ or $\text{Sk}_x(Q)$ to emphasise the dependence on the choice of the root. Note that $\text{Sk}(Q)$ is directed from infinity to the root.

When Q is the uniform infinite quadrangulation, $\text{Sk}(Q)$ is distributed as the tree of a certain time-reversed critical Galton-Watson process, whose offspring distribution can be calculated explicitly [3]. In particular it follows that for every R with probability one all of the individuals on γ_R have a single common ancestor, and therefore tree $\text{Sk}(Q)$ has one end (see 5.1 for details).

2. MAIN THEOREM

Let x and y be two distinct vertices of the quadrangulation Q ; we may assume that x is the root vertex. Let $f(z) = d_x(z) - d_y(z)$ be a function on the vertices of $V(Q)$.

From the definition of Sch_x it follows that for each face of Q the structure of Sch_x inside it only depends on the increments of d_x around this face. Therefore if the quadrangulation Q is infinite and f is constant in Q except for a finite number of vertices, then the trees Sch_x and Sch_y coincide almost everywhere in Q .

For simplicity let's take y to be a neighbour of x , i.e. $d_x(y) = 1$. Then we can formulate the main result as follows:

Theorem 1. *Let (x, y) be the root in a uniform infinite random quadrangulation Q . With probability one the function $f(z) = d_x(z) - d_y(z)$ is constant almost everywhere in Q .*

3. PROOF

3.1. Trees and geodesics within quadrangulations. A *geodesic emanating from x* is a finite or infinite path starting at x and such that d_x is strictly increasing along this path.

Let $z \in V(Q)$. An *x -slit from z* is an infinite simple path in Q_\times starting at z , such that d_x is non-decreasing along this path. Denote by $\text{Sl}_x(Q)$ the set of vertices of Q admitting an x -slit.

Note that since the number of vertices at any finite distance from x is finite a.s., d_x tends to infinity along any x -slit. Also note that the x -slit does not necessarily exist for every $z \in V(Q)$.

Given a point $z \in \text{Sl}_x(Q)$, we define the *rightmost geodesic* from z to x , denoted Rg_x^z , recursively as following: let a be the first point on the x -slit from z , and let b be the first point after A in the clockwise order around z , such that $d_x(b) = d_x(z) - 1$. Then Rg_x^z consists of the edge (z, b) followed by the rightmost geodesic from b to x .

Clearly, the choice the vertex b does not depend on the choice of the x -slit; and given the x -slit from z exists, an x -slit from b exists as well, so the rightmost geodesic from b is well-defined.

We call an infinite geodesic γ , emanating from x , *rightmost*, if for every $z \in \gamma$ it is the rightmost geodesic from z to x .

All rightmost geodesics on Q form a tree; denote this tree by \mathcal{R} . This tree can be seen as a dual tree to $\text{Sk}(Q)$, and therefore it has one end. Thus we obtain

Lemma 1. *The rightmost infinite geodesic Rg_X^∞ is unique a.s.*

Now let y be some vertex in Q different from x , and let $f(z) = d_x(z) - d_y(z)$. For simplicity we'll take $d_x(y) = 1$, then f only takes two values ± 1 . There are two alternatives – either f takes the same value for all $z \in V(Q)$ except a finite subset, either each of the two sets $V^\pm := \{z \in V(Q) | f(z) = \pm 1\}$ is infinite. Our goal is to prove that the second case doesn't happen, therefore from now on we assume that both V^+ and V^- are infinite.

Lemma 2. *f is non-decreasing along any geodesic emanating from x ; f is non-increasing along any geodesic emanating from y .*

Proof. Let γ be a geodesic emanating from x . The increments of d_x along γ are all equal to $+1$, while the increments of d_y are ± 1 , so the statement follows. The statement for y follows by symmetry. \square

Lemma 3. *f is constant almost everywhere in $V(Q)$ if and only if f is constant on γ_R for some $R > d(x, y)$.*

Proof. If f is constant on $V(Q)$ except the finite set W , then f is constant on γ_R for all $R > \text{diam}(W)$.

Suppose now that f is constant on some cycle γ_R , $R > d(x, y)$. As mentioned above, γ_R separates Q in two parts, and the part W containing the root x is finite a.s. Since $R > d(x, y)$, W also contains y . Pick a point $z \notin W$ and some geodesic from z to x ; this geodesic intersects γ_R at some point x' . Similarly, take some geodesic from z to y and let y' be the point where this geodesic intersects γ_R .

By the previous lemma, f is non-decreasing on the geodesic from x' to z , and non-increasing on the geodesic from y' to z , and since f is constant on γ_R we conclude that $f(x') = f(y') = f(z)$, thus f is constant on the complement of W . \square

Lemma 4. *Assume that both V^- and V^+ are infinite. Then $f = -1$ on Rg_x^∞ .*

Proof. Assume the opposite, i.e. that $f = +1$ on Rg_x^∞ starting from some point $v \in \text{Rg}_x^\infty$. Since f is nondecreasing along each geodesic emanating from x , $f = +1$ on the whole subtree of \mathcal{R} above v . But since \mathcal{R} is locally finite and has one end a.s., it follows that $f = -1$ for at most a finite number of vertices of \mathcal{R} . Therefore $f = +1$ on γ_R for some large R and by the previous lemma f is constant a.e. in $V(Q)$ — a contradiction. \square

Because of the invariance with respect to the random walk, every property that holds a.s. for Q rooted at x holds also for Q rooted at y . Using this, and the fact that f changes its sign if we swap x with y , we obtain the following

Lemma 5. *There exists an infinite rightmost geodesic Rg_y^∞ , and $f = +1$ on this geodesic unless V^+ is finite.*

Note that the rightmost geodesic to y should be considered with respect to the y -slits, and as an x -slit is not necessarily a y -slit, $\text{Sl}_x(Q)$ does not necessarily coincide with $\text{Sl}_y(Q)$.

3.2. Now assuming both V^+ and V^- are infinite, we have the following picture: there exists a rightmost geodesic for x , and a rightmost geodesic for y , and these two are disjoint.

Observe that Rg_x^∞ is a geodesic for y : since f is constant, and d_x is strictly increasing along Rg_x^∞ , d_y is strictly increasing as well, so adding the edge (x, y) to Rg_x^∞ we obtain an honest infinite geodesic emanating from y .

By Lemma 5 the rightmost infinite geodesic for y is unique, so Rg_x^∞ should not be rightmost for y . More precisely, for every point $z \in Rg_x^\infty$ define the *divergence point* $\delta(z)$ as the lowest common point (lowest in the sense of d_x and d_y) of the geodesics Rg_y^z and Rg_x^z . Then the set of divergence points

$$\Delta := \{v \mid v = \delta(z) \text{ for some } z \in Rg_x^\infty\}$$

has to be infinite.

Let $v \in \Delta$ be a divergence point, and let $R = d_x(v)$. Let b be the first point on the geodesic Rg_y^v . Then we have

- $d_y(b) = d_y(v) - 1 = R$,
- $d_x(b) > d_x(v) - 1$ therefore $d_x(b) = R + 1$,
- $f(b) = +1$.

Let's now compare the rightmost geodesic Rg_y^b with the rightmost geodesic Rg_x^b . $Rg_y^b \cup (y, x)$ is a geodesic from b to x , therefore Rg_x^b either coincides with Rg_y^b or diverges from it and goes to the right¹

Note that Rg_x^b and Rg_y^b may coincide only for a finite number of divergence points z — otherwise it would follow that Rg_y^∞ is a second infinite rightmost geodesic for x , which as we know does not happen.

On the other hand, it's clear that Rg_x^b and Rg_y^b may not diverge until Rg_y^b merges with Rg_x^∞ (otherwise Rg_x^b would cross Rg_y^∞ in some point, and Rg_y^b would not be the rightmost geodesic from b to y).

Therefore (for a.e. divergence point in Δ) the geodesic Rg_x^b crosses Rg_y^∞ , and eventually merges with Rg_x^∞ but on the other side, so that parts of the geodesics Rg_x^v , Rg_x^b together with the edge (v, b) form a loop that separates the root x from infinity. It follows that the trunk of the skeleton $Sk_x(Q)$ passes between v and b .

Since $v \in \gamma_R$, we obtain that the trunk of $Sk_x(Q)$ intersects γ_R at an edge immediately next to v , and v is the point where Rg_x^∞ intersects γ_R .

To conclude the proof of the main theorem we will show that this can only happen finitely many times.

Lemma 6. *For a given R let γ_R be the cycle in Q_\times that is situated at distance R from the root and that separates the root from the infinite part of Q . Let v_R be the vertex where Rg_x^∞ intersects γ_R , and let e_R be edge where the trunk of $Sk_x(Q)$ intersects γ_R .*

Then for any d_0 with probability one $d(v_R, e_R) < d_0$ only for finitely many R .

Proof. The cycle γ_R separates the quadrangulation into two parts, and it's known [3] that given the length of γ_R these parts are conditionally independent.

Observe that the position of e_R on γ_R only depends on the finite part (the one containing x), while the position of v_R only depends on the infinite part. The measure μ_∞ is invariant with respect to cutting the quadrangulation along γ_R and gluing the parts back with a uniformly distributed twist. Due to such rotational symmetry, the probability that e_R and v_R are close is of order $1/|\gamma_R|$.

¹i.e. to the left, from the point of view of an observer moving along the geodesic towards y .

Since $|\gamma_R|$ grows as R^2 (and it can be checked, see 5.2, that $\mathbb{E}|\gamma_R|^{-1} = O(R^{-2})$) it follows by Borel-Cantelli that only a finite number of events $\{d(v_R, e_R) < d_0\}_{R=0}^\infty$ can occur simultaneously. \square

This finishes the proof of the main theorem.

4. CONCLUDING REMARKS

For every directed edge (x, y) in Q define

$$\Delta h(x, y) = \lim_{z \rightarrow \infty} d(x, z) - d(y, z),$$

then there exists a function h on $V(Q)$, defined up to an additive constant by

$$\Delta h(x, y) = h(x) - h(y).$$

Applying the construction of the tree Sch with respect to the increments of h , one obtains a spanning forest of Q_\times . Denote this graph by Sch_∞ . It seems natural to interpret Sch_∞ as a limit of trees $\text{Sch}_z(Q)$, when z is sent to infinity, and to conjecture that

Conjecture 1. $\text{Sch}_\infty(Q)$ is connected a.s.

From a slightly different point of view, one can consider Sch_∞ as the (limit of the) Schaeffer's tree in a large random quadrangulation, seen from a uniformly chosen point.

Conjecture 2. Let Q_n be a uniform rooted quadrangulation with n faces, and z_n a uniformly chosen vertex in Q_n . Then the pair $(Q_n, \text{Sch}_{z_n}(Q_n))$ converges as $n \rightarrow \infty$ to (Q, Sch_∞) (in the sense of local weak convergence).

Provided that the first conjecture above holds, it would be useful to know the distribution of the tree $\text{Sch}_\infty(Q)$, seen as a planar tree with a $-1, 0, 1$ -valued anti-symmetric function Δh on directed edges. The representation of the tree $\text{Sch}_x(Q)$ in [2] strongly suggests the following

Conjecture 3. The pair $(\text{Sch}_\infty, \Delta h)$ is distributed as a Galton-Watson tree with geometric-1/2 offspring distribution, conditioned to be infinite, and with increments distributed independently and uniformly among $\{-1, 0, 1\}$.

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5. SOME ADDITIONAL CALCULATIONS

5.1. Let Q be a uniform infinite quadrangulation, and γ_R be a sequence of cycles in Q_\times , located at distance R from the root and separating the root from infinity, as defined in Section 1.5. In [3] the following theorem was proved:

Theorem 2. $|\gamma_R|$ is a Markov chain with transition probabilities given by

$$\mathbb{P}\left\{|\gamma_{r+n}| = k \mid |\gamma_r| = l\right\} = \frac{[t^k]F(t)}{[t^l]F(t)} \cdot \mathbb{P}\{\xi_n = l \mid \xi_0 = k\},$$

where ξ is a critical branching process with offspring generating function

$$\varphi(t) = \frac{1}{2t} \left(\sqrt{(t-9)(t-1)^3} - 3 + 6t - t^2 \right),$$

and

$$F(t) = \frac{3}{4} \left(\sqrt{\frac{9-t}{1-t}} - 3 \right).$$

Moreover, the tree structure $\text{Sk}(Q)$ restricted to $\gamma_1, \dots, \gamma_R$, and conditioned on $|\gamma_R| = m$ has the distribution of a forest of m Galton-Watson trees, conditioned to have exactly one individual at height R and no individuals higher than R .

Let A_R^h be the number individuals at $\gamma(R+h)$, having nonempty offspring at $\gamma(R)$. Then the generating function for A_R^{R+h} is given by

$$\mathbb{E} y^{A_R^{R+h}} = [t]F(\varphi_h(0) + y(\varphi_{R+h}(t) - \varphi_h(0))),$$

where φ_R is the R -fold iteration of φ , which can be written explicitly as

$$(1) \quad \varphi_R(t) = 1 - \frac{8}{\left(\sqrt{\frac{9-t}{1-t}} + 2R \right)^2 - 1}.$$

In particular we obtain

$$\begin{aligned} \mathbb{P}\{A_R^h = 1\} &= [y][t]F(\varphi_h(0) + y(\varphi_{R+h}(t) - \varphi_h(0))) \\ &= \frac{(h+2)^2(h+1)^2(2R+2h+3)}{(2h+3)(R+h+2)^2(R+h+1)^2}. \end{aligned}$$

It follows that

$$\lim_{h \rightarrow \infty} \mathbb{P}\{A_R^h = 1\} = 1,$$

so all individuals at γ_R have a single common ancestor a.s., thus proving that the graph $\text{Sk}(Q)$ is connected a.s., and

$$\sum_{h \geq 0} \mathbb{P}\{A_R^h = 1\} = \infty,$$

so the distance to this common ancestor has infinite mean.

5.2. Let

$$\Phi(t) = \int_0^t \frac{F(y)}{y} dy,$$

then

$$\mathbb{E} |\gamma_R|^{-1} = [t]\Phi(\varphi_R(t)) = \frac{2(2R+3)}{(R+1)(R+2)(R+3)} = O(R^{-2}).$$

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