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ON THE EXISTENCE OF APPROXIMATED EQUILIBRIA IN DISCONTINUOUS ECONOMIES

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Abstract In this paper, we prove an existence theorem for approximated equilibria in a class of discontinuous economies. The existence result is a direct consequence of a discontinuous extension of Brouwer's fixed point Theorem (1912), and is a refinement of several classical results in the standard General Equilibrium with Incomplete markets (GEI) model (e.g., Bottazzi (1995), Duffie and Shafer (1985), Husseini et al. (1990), Geanakoplos and Shafer (1990), Magill and Shafer (1991)). As a by-product, we get the first existence proof of an approximated equilibrium in the GEI model, without perturbing the asset structure nor the endowments. Our main theorem rests on a new topological structure result for the asset equilibrium space and may be of interest by itself.

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1 Introduction

Since Hart's seminal paper (1975), it is well known that equilibria may fail to exist in the standard GEI model[2] because of the discontinuity of the market excess demand at prices (called Hart points) at which the rank of the asset payoff matrix drops. On a technical level, this discontinuity prevents from using standard fixed point arguments, as it is done in the Arrow-Debreu General Equilibrium (GE) model (1954). The principal approach to this problem has been to prove the generic existence of an equilibrium, i.e. to prove the existence for almost all characteristics of the economy.

This paper proposes another approach : it proves, for broad classes of asset structures and for every endowment, the existence of an approximated equilibrium.

[1]I wish to thank the anonymous referees for valuable comments. Errors are mine.

[2]For an exposition of this model, see, for example, Brown et al. (1996), Duffie and Shafer (1985), Husseini et al. (1990), Geanakoplos and Shafer (1990), Magill and Shafer (1991), Magill and Quinzi (1996) or Hens' survey (in Kirman, 1998).

More precisely, let J, S and L be integers such that $J \leq S$ and $1 \leq L$. The asset structure V is a mapping assigning a $S \times J$ matrix to each price vector $p \in S_+^{L-1}$ (the positive part of the sphere of $\mathbb{R}^L$). In addition, this mapping is supposed to be smooth and transverse to the manifolds of low rank matrices[3]. For example, this transversality condition is true for generic real asset structures, or if the asset structure has constant rank (see Bottazzi (1995)).

Moreover, the aggregate excess demand of our economy is a mapping $z : S_{++}^{L-1} \to \mathbb{R}^L$ that satisfies the classical assumptions of the Arrow-Debreu model : it is bounded below, blows up at the boundary of the price set and satisfies Walras' Law. Besides, it satisfies a new partial continuity assumption, called V-continuity, and which is true in the GEI model : the mapping z is called V-continuous if for every convergent sequence of full rank prices[4] $(p_\ell)_{\ell \in \mathbb{N}}$ with $(\mathrm{span} V(p_\ell))_{\ell \in \mathbb{N}}$ convergent[5] the sequence $(z(p_\ell))_{\ell \in \mathbb{N}}$ converges. This last definition is economically relevant and can be related to the following continuity principle : when the full rank price p and the market subspace $\mathrm{span} V(p)$ vary infinitesimally, the variation of the excess demand is infinitesimal.

Finally, a price vector p is called an approximated equilibrium if there exists a sequence of full rank prices $(p_\ell)_{\ell \in \mathbb{N}}$ converging to p with $(z(p_\ell))_{\ell \in \mathbb{N}}$ convergent to 0. Since an approximated equilibrium is (generically) an equilibrium, our main existence result entails the standard generic existence results in the GEI model.

The main result rests on a discontinuous generalization of Brouwer's fixed point Theorem (see Theorem 1). The variational form of our fixed point theorem says that every inward V-continuous vector field on S_+^{L-1} that is continuous on a neighborhood of the boundary of S_+^{L-1} has an approximated equilibrium (see Theorem 2). This theorem, applied to the aggregate excess demand of our axiomatized GEI model, entails the existence of an approximated equilibrium (see Theorem 3), exactly as Brouwer's Theorem[6] entails the existence of an equilibrium in the GE model.

Lastly, our fixed point theorem rests on the following topological structure result : the set $\{(p, E) \in S_{++}^{L-1} \times G^J(\mathbb{R}^S),\ \mathrm{span} V(p) \subset E,\ \mathrm{rank}\ V(p) = J\}$ is dense in the asset equilibrium space $\{(p, E) \in S_{++}^{L-1} \times G^J(\mathbb{R}^S),\ \mathrm{span} V(p) \subset E\}$, and the latter is a $(L-1)$-manifold (see Proposition 4).

The remainder of this paper is organized as follows. In Section 2, we give the definition of a V-continuous economy and the definition of an approximated equilibrium. Then, we prove that the standard GEI economy is a particular case of a V-continuous economy, and that approximated equilibria are generically equilibria. In Section 3 is stated the discontinuous generalization of Brouwer's

[3]This assumption has been introduced by Bottazzi (1995).

[4]A price p is said to be a full rank price if rank $V(p) = J$.

[5]for the Hausdorff distance d_H defined on $G^J(\mathbb{R}^S)$, the set of all J linear subspaces of $\mathbb{R}^S$. See the appendix for a precise definition.

[6]More exactly, Kronecker's Theorem, which says that a continuous inward vector field on a closed ball of a Euclidean space has an equilibrium. This is almost immediately equivalent to Brouwer's Theorem.

fixed point Theorem, and, as a consequence, its variational form. In Section 4 is proved, as a corollary of Section 3, the existence of approximated equilibria in V-continuous economies. The classical existence results in the GEI model are then derived from this result. The last section provides the proof of our discontinuous extension of Brouwer's fixed point Theorem, and in particular presents the new topological structure result for the asset equilibrium space.

2 The model

2.1 Definition of a V-continuous economy

We consider[7] a pure exchange economy with a positive number L of commodities. Let $S^{L-1}_{++} = \{(p_1, ..., p_L) \in \mathbb{R}^L, \sum_{i=1}^{L} p_i^2 = 1, \forall i = 1, ..., L, p_i > 0\}$ be the set of normalized prices of the L commodities and let $S^{L-1}_{+} = \{(p_1, ..., p_L) \in \mathbb{R}^L, \sum_{i=1}^{L} p_i^2 = 1, \forall i = 1, ..., L, p_i \geq 0\}$.

Let S and J be two positive integers such that $J \leq S$ and let $V : S^{L-1}_{+} \to \mathcal{M}(S \times J)$, where $\mathcal{M}(S \times J)$ is the set of all $S \times J$ matrices. For example, the mapping V may be seen as the asset structure of the standard GEI model, which specifies the financial returns of J assets at S states of nature. In the following, a price $p \in S^{L-1}_{++}$ is called a full rank price if rank $V(p) = J$.

For every $p \in S^{L-1}_{++}$, we suppose that only the aggregate excess demand of the economy $z(p) = (z_1(p), ..., z_L(p)) \in \mathbb{R}^L$ is observable, and do not suppose that it derives from the utility maximization of the agents under their budget constraints. Besides, z is assumed to satisfy the following assumptions :

(i) (Walras Law) For every full rank price $p \in S^{L-1}_{++}$, $p \cdot z(p) = 0$.
(ii) (Boundary condition) For every sequence of full rank prices $(p_\ell)_{\ell \in \mathbb{N}}$ *of* S^{L-1}_{++} *converging to* $\bar{p} \notin S^{L-1}_{++}$, *one has* $lim_{\ell \to +\infty} \|z(p_\ell)\| = +\infty$.
(iii) (Bounded below) There exists $M \in \mathbb{R}$ *such that for every* $i = 1, ..., L$ *and for every full rank price* $p \in S^{L-1}_{++}$, $z_i(p) \geq M$.

[7] In this paper, if $x = (x_1, ..., x_n)$ and $y = (y_1, ..., y_n)$ belong to $\mathbb{R}^n$, we denote by $x \cdot y = \sum_{i=1}^{n} x_i y_i$, the scalar product of $\mathbb{R}^n$, $\|x\| = \sqrt{x \cdot x}$, the Euclidean norm. If E is a vector subspace of $\mathbb{R}^n$, we denote by $E^\perp = \{u \in \mathbb{R}^n, \forall x \in E, x \cdot u = 0\}$ the orthogonal space to E. If $u_1, ..., u_k$ belong to E, a vector space, we denote by span$\{u_1, ..., u_k\}$ the vector subspace of E spanned by $u_1, ..., u_k$. If M is a matrix, we denote by span$M$ the vector subspace spanned by M. If f is a mapping from a set X to a set Y, then for every $X' \subset X$, $f\mid_{X'}$ denotes the restriction of f to X'. If X is a topological space, we denote by int(X) the interior of X. If J and S are two positive integers such that $J \leq S$, we denote by $G^J(\mathbb{R}^S)$ the set consisting of all the linear subspaces of $\mathbb{R}^S$ of dimension J, called the (J-)Grassmannian manifold of $\mathbb{R}^S$. We denote by d_H the Hausdorff distance defined on $G^J(\mathbb{R}^S)$ (see the appendix for a precise definition). If f is a mapping differentiable at x, then we denote by $Df(x)$ the derivative of f at x. Finally, in all the paper, if M is a Banach manifold (resp. a finite dimensional Euclidean space), we say that a property P_λ, depending upon a parameter $\lambda \in M$, holds generically (or for generic $\lambda \in M$) if there exists an open and dense subset M' of M (resp. an open and full measure subset M' of M, for the Lesbegue measure on M) such that for every $\lambda \in M'$, P_λ is true.

In the Arrow-Debreu GE model, it is well known that the excess demand mapping z satisfies Assumptions (i), (ii) and (iii). In the standard GEI model, classically, one can suppose that it is the case by considering the notion of normalized no-arbitrage equilibrium.[8] The main difference between the two models rests on the continuity property of the aggregate excess demand : it is continuous on S^{L-1}_{++} in the GE model, although the aggregate excess demand of the GEI model does not need to be continuous at Hart points. The aim of the following definition is to provide a simple and new notion of weak continuity that encompasses the two previous cases.

Definition 1 *A mapping $f : S^{L-1}_{++} \to I\!R^L$ is V-continuous if for every convergent (in S^{L-1}_{++}) sequence $(p_\ell)_{\ell \in I\!N}$ of full rank prices with $(spanV(p_\ell))_{\ell \in I\!N}$ convergent the sequence $(f(p_\ell))_{\ell \in I\!N}$ converges.*

Remark 1 More generally, if X and Y are metric spaces and $V : X \to \mathcal{M}(S \times J)$ is a mapping, then one can similarly define V-continuity of any mapping $f : X \to Y$.

Remark 2 Clearly, any continuous mapping $f : S^{L-1}_{++} \to \mathrm{I\!R}^L$ is V-continuous. On the other hand, a V-continuous mapping does not need to be continuous : for example, in the next section, we will prove that the aggregate excess demand of the standard GEI model is V-continuous. At best, one can say that if $V : S^{L-1}_{+} \to \mathcal{M}(S \times J)$ is a continuous mapping and $f : S^{L-1}_{++} \to \mathrm{I\!R}^L$ is V-continuous, then f is continuous on the (open) subset of full-rank prices (see the proof in the appendix).

We now give the axiomatized definition of our economy :

Definition 2 *A V-continuous economy $\mathcal{E}(z, V)$ is characterized by a smooth mapping $V : S^{L-1}_{+} \to \mathcal{M}(S \times J)$, and a V-continuous mapping $z : S^{L-1}_{++} \to I\!R^L$ satisfying assumptions (i), (ii) and (iii).*

Remark 3 As in the standard GEI model, the discontinuity of the aggregate excess demand rests on the possible discontinuity of the correspondence span$V(.)$. However, notice that Definition 2 does not require an explicit dependence between z and span$V(.)$.

2.2 Definition of an approximated equilibrium

Since the excess demand mapping of our economy does not need to be continuous, one needs to weaken the classical definition of an equilibrium in order to obtain some existence results :

[8] which is defined with respect to a single agent's present value price system, where the agent acts as if he were facing complete contingent markets (e.g., Magill and Shafer, 1991). Notice that the primitive aggregate excess demand mapping may not satisfy the boundary condition.

Definition 3 *Let $\mathcal{E}(z, V)$ be a V-continuous economy. An approximated equilibrium $\bar{p} \in S_{++}^{L-1}$ of $\mathcal{E}(z, V)$ is the limit of a sequence of full rank prices $(p_\ell)_{\ell \in \mathbb{N}}$ of S_{++}^{L-1} such that the sequence $(z(p_\ell))_{\ell \in \mathbb{N}}$ converges to 0. We let $AE(\mathcal{E}(z, V))$ be the set of approximated equilibria of the economy.*

Remark 4 Similarly, if X and Y are metric spaces and $V : X \to \mathcal{M}(S \times J)$ is a mapping, then one can define the notion of approximated equilibrium of any mapping $f : X \to Y$.

The notion of approximated equilibrium clearly generalizes the notion of equilibrium in the standard Arrow-Debreu model, since any approximated equilibrium of a continuous mapping z is an equilibrium. The next section will give the link between the notion of approximated equilibrium and the classical notions of equilibrium and pseudo equilibrium in the standard GEI model.

2.3 GEI economies are V-continuous

As in previous works (e.g., Chichilnisky and Heal (1996), Duffie and Shafer (1985)), we formalize the GEI model in an abstract fashion, only specifying the main properties of the no-arbitrage aggregate excess demand mapping[9] :

Definition 4 *The GEI economy $\mathcal{E}(z^{GEI}, V)$ is defined by :*

1)A continuous mapping $Z : S_{++}^{L-1} \times G^J(\mathbb{R}^S) \to \mathbb{R}^L$, which is bounded below, satisfies the Walras Law and the following boundary condition : for every sequence $(p_\ell, E_\ell)_{\ell \in \mathbb{N}}$ of $S_{++}^{L-1} \times G(\mathbb{R}^S)$ converging to $(\bar{p}, \bar{E})$, $\bar{p} \notin S_{++}^{L-1}$, then $lim_{\ell \to +\infty} \|Z(p_\ell, E_\ell)\| = +\infty$.

2) A smooth mapping $V : S_{+}^{L-1} \to \mathcal{M}(S \times J)$.

3) A mapping $z^{GEI} : S_{++}^{L-1} \to \mathbb{R}^L$ which satisfies :

$$\forall p \in S_{++}^{L-1} \mid \text{rank } V(p) = J, z^{GEI}(p) = Z(p, \text{span}V(p)).^{10} \tag{1}$$

It is well known that the excess demand z^{GEI} may be discontinuous at every Hart point, i.e. on an uncountable subset of S_{++}^{L-1}. We now prove that z^{GEI} defines a V-continuous economy.

Proposition 1 *The economy $\mathcal{E}(z^{GEI}, V)$ is a V-continuous economy.*

Proof. By definition, the mapping z^{GEI} clearly satisfies Assumption (i) (Walras Law) and Assumption (iii) (Bounded below) of the definition of V-continuity.

[9] which is obtained by considering a single agent's present value price system, where this agent acts as if he were facing complete contingent markets. The no-arbitrage excess demand mapping satisfies more properties than the primitive aggregate excess demand, and its equilibria are equilibria of the initial economy.

[10] The value of $z^{GEI}(p)$ for rank $V(p) < J$ does not matter in the perspective of equilibrium existence result.

Assumption (ii) (Boundary condition) of the definition of V-continuity is true by Assumption (1) of Definition 4 and because $G^J(\mathbb{R}^S)$ is a compact set. Thus, to end the proof, one only has to check that z^{GEI} is V-continuous. Let $(p_\ell)_{\ell\in\mathbb{N}}$ be a convergent (in S^{L-1}_{++}) sequence of full-rank prices of S^{L-1}_{++} such that the sequence $(\text{span}V(p_\ell))_{\ell\in\mathbb{N}}$ converges to $E \in G^J(\mathbb{R}^S)$. We want to prove that $z(p_\ell) = Z(p_\ell,\text{span}V(p_\ell))$ converges, which is a consequence of the continuity of Z and the convergence of $(p_\ell,\text{span}V(p_\ell))_{\ell\in\mathbb{N}}$ in $S^{L-1}_{++} \times G^J(\mathbb{R}^S)$.

Now, we relate the notions of approximated equilibrium, equilibrium and pseudo-equilibrium in the GEI model. Let first recall that an equilibrium of $\mathcal{E}(z^{GEI}, V)$ is a price $p \in S^{L-1}_{++}$ such that $z^{GEI}(p) = 0$. An equilibrium price p is said to be a full-rank equilibrium if rank $V(p) = J$. We let $E(\mathcal{E}(z^{GEI}, V))$ be the set of full-rank equilibria of $\mathcal{E}(z^{GEI}, V)$. Besides, let recall that a pseudo-equilibrium of $\mathcal{E}(z^{GEI}, V)$ is $(p, E) \in S^{L-1}_{++} \times G^J(\mathbb{R}^S)$ such that $Z(p, E) = 0$ and $\text{span}V(p) \subset E$. If (p, E) is a pseudo-equilibrium, then p is called a pseudo-equilibrium price, and we let $PE(\mathcal{E}(z^{GEI}, V))$ be the set of pseudo-equilibrium prices of $\mathcal{E}(z^{GEI}, V)$.

Proposition 2

$$E(\mathcal{E}(z^{GEI}, V)) \subset AE(\mathcal{E}(z^{GEI}, V)) \subset PE(\mathcal{E}(z^{GEI}, V)).$$

Besides, if for every $p \in S^{L-1}_{++}$, rank $V(p) = J$, then these inclusions are equalities.

Proof. See the Appendix.

Remark 5 If V is a continuous asset structure, the two first sets in the inclusions above may be empty[11], although the pseudo-equilibrium set is known to be nonempty. Thus, the notion of approximated equilibrium is a strict refinement of the notion of pseudo-equilibrium.

We now prove that the inclusions in Proposition 2 are generically, in some sense, equalities. For this, we will need to introduce two regularity assumptions, one on the asset structure, another on the excess demand mapping.

In the following, for every $\rho = 0, ..., J$, let $\mathcal{M}^\rho(S \times J)) := \{M \in \mathcal{M}(S \times J),$ $\text{rank}M = J - \rho\}$. If X is a smooth manifold (possibly with boundary) then a smooth mapping $V : X \to \mathcal{M}(S \times J)$ is called a transverse mapping [12] if :

[11] For example, it suffices to consider a continuous financial structure such that rank $V(p) < J$ for every price $p \in S^{L-1}_{++}$.

[12] This assumption has been introduced by Bottazzi (1995) who proved the existence of an equilibrium for generic endowments and for transverse financial asset structures. See section 6.2 for the topological definition of transversality.

(T) Transversality Assumption : for every $\rho = 1, ..., J$, $V \mid_{int(X)}$ is transverse to the manifold $\mathcal{M}^\rho(S \times J)$.

It is known that Assumption (T) is true for generic real asset structures, commodity forward contracts or more generally for generic smooth asset structures (Bottazzi (1995), p.66). Now, we show that Assumption (T) is locally equivalent to a regularity assumption : let $\bar{x} \in \mathrm{int}(X)$ such that rank $V(\bar{x}) = J - \rho$ for some $\rho \in \{1, ..., J\}$. Without any loss of generality, up to a permutation of the rows and of the columns of $V(\bar{x})$, one can suppose that for every x in a neighborhood $U \subset \mathrm{int}(X)$ of $\bar{x}$,

$$V(x) = \begin{pmatrix} a(x) & b(x) \\ c(x) & d(x) \end{pmatrix} \tag{2}$$

where $a(x)$ is a $(J - \rho) \times (J - \rho)$ invertible matrix.

Lemma 1 *The transversality condition (T) is equivalent to the regularity*[13] *of the mapping $f : U \to \mathcal{M}((S - J + \rho) \times \rho)$ defined by $f(x) = d(x) - c(x)a^{-1}(x)b(x)$ for every $x \in U$. Besides, $f^{-1}(0) \cap U = \{x \in U,\ rank\ V(x) = J - \rho\}$.*

Proof. The proof is given in the appendix[14].

We now recall a regularity property on the aggregate excess demand, which is true in the standard GEI model (e.g., Duffie and Shafer, 1985, p.293) :

(R) Regularity Assumption : the mapping z^{GEI} depends smoothly on a parameter $e \in I\!R^L_{++}$, and for every $(p, E, e) \in S^{L-1}_{++} \times G^J(I\!R^S) \times I\!R^L_{++}$, rank $D_e z^{GEI}(p, E, e) = L$.[15]

The existence result of Bottazzi (1995) says that, under Assumption (R), for generic $e \in \mathrm{I\!R}^L_{++}$ and for every transverse financial structure, each pseudo-equilibrium is a full rank equilibrium. Thus, from Proposition 2, one obtains :

Proposition 3 *Under Assumptions (R) and (T), for generic $e \in I\!R^L_{++}$, one has :*

$$E(\mathcal{E}(z^{GEI}, V)) = AE(\mathcal{E}(z^{GEI}, V)) = PE(\mathcal{E}(z^{GEI}, V)).$$

Remark 6 Classically, the set $PE(\mathcal{E}(z^{GEI}, V))$ is nonempty for every continuous asset structure. Thus, one obtains, for generic $e \in \mathrm{I\!R}^L_{++}$ and under Assumptions (R) and (T), the existence of an approximated equilibrium. The aim of this paper is to obtain the (non-generic) existence of an approximated equilibrium, without Assumption (R).

[13]The mapping f is said to be regular if for every $x \in f^{-1}(0)$, $Df(x)$ in onto.

[14]A similar statement can be found, without proof, in Bottazzi (1995).

[15]In the standard GEI model, the parameter e is the endowment vector of the unconstrained agent of the model.

3 A discontinuous generalization of Brouwer's fixed point Theorem and its variational form

3.1 A discontinuous generalization of Brouwer's fixed point Theorem

Let $B \subset \mathbb{R}^n$ be the closed unit ball centered at 0. Brouwer's fixed point Theorem says that every continuous mapping from B to B admits a fixed point. The aim of this section is to give a discontinuous generalization of this theorem, in order to obtain the existence of an approximated equilibrium for V-continuous economies. Let n, S and J be positive integers such that $J \leq S$ and let $L = n+1$.

First, we weaken the classical notion of fixed point. In the following definition, let $V : B \to \mathcal{M}(S \times J)$ be a smooth mapping.

Definition 5 *Let $f : B \to B$ be a mapping. The point $\bar{x} \in B$ is said to be an approximated fixed point of f if there is a sequence $(x_\ell)_{\ell \in \mathbb{N}}$ in $\{x \in B$, rank $V(x) = J\}$ converging to $\bar{x}$ and such that $\lim_{\ell \to +\infty} f(x_\ell) = \bar{x}$.*

Remark 7 If f is continuous, then an approximated fixed point of f is clearly a fixed point of f.

We now state our discontinuous generalization of Brouwer's Theorem :

Theorem 1 *If $V : B \to \mathcal{M}(S \times J)$ is a transverse mapping, then every V-continuous mapping $f : B \to B$ admits an approximated fixed point.*

Proof. see section 5.

An immediate extension is the case where f is defined on a set C diffeomorphic to B.

Corollary 1 *If $V : C \to \mathcal{M}(S \times J)$ is a transverse mapping, then every V-continuous mapping $f : C \to C$ admits an approximated fixed point.*

Proof. Let $h : B \to C$ be a diffeomorphism, and let $\bar{f} : B \to B$ be defined by $\bar{f} = h^{-1} \circ f \circ h$. Let $V_1 : B \to \mathcal{M}(S \times J)$ be defined by $V_1 = V \circ h$. We first check that $\bar{f}$ is V_1-continuous. Let $(x_\ell)_{\ell \in \mathbb{N}}$ be a sequence of elements of $\{x \in B$, rank $V_1(x) = J\}$ such that the sequence $(\text{span}V_1(x_\ell))_{\ell \in \mathbb{N}}$ is convergent. From the definition of V_1, for every $\ell \in \mathbb{N}$, rank $V(y_\ell) = J$ and $(\text{span}V(y_\ell))_{\ell \in \mathbb{N}}$ converges, where $y_\ell = h(x_\ell)$ for every $\ell \in \mathbb{N}$. Thus, since $(y_\ell, \text{span}V(y_\ell))_{\ell \in \mathbb{N}}$ converges, and from the V-continuity of f, this implies that the sequence $(f(y_\ell))_{\ell \in \mathbb{N}}$ converges, i.e. $(f(h(x_\ell))_{\ell \in \mathbb{N}}$ converges. By continuity of h^{-1}, the sequence $(\bar{f}(x_\ell))_{\ell \in \mathbb{N}}$ converges, which finally proves that $\bar{f}$ is V_1-continuous.

Now, the reader can easily check that V_1, as V, is a transverse mapping. Thus, from Theorem 1 applied to the V_1-continuous mapping $\bar{f}$, there exists an approximated fixed-point $\bar{x} \in B$ of $\bar{f}$. This means that there exists a sequence $(x_\ell)_{\ell \in \mathbb{N}}$ of $\{x \in B, \text{ rank } V_1(x) = J\}$ which converges to $\bar{x}$ and such that the sequence $(\bar{f}(x_\ell))_{\ell \in \mathbb{N}}$ converges to $\bar{x}$. From the definition of $\bar{f}$, it clearly implies that $h(\bar{x})$ is an approximated equilibrium of f.

3.2 Variational form of Theorem 1

Given a convex subset K of $\mathbb{R}^L$, recall that the Bouligand contingent cone $T_K(x)$ to K at $x \in K$ can be written

$$T_K(x) = \overline{\{\nu(e - x), e \in K, \nu > 0\}}.$$

A vector field on K is a mapping $z : K \to \mathbb{R}^L$ such that for every $x \in K, z(x) \in T_K(x)$. It is well known that Brouwer's fixed point Theorem is equivalent to saying that every continuous vector field on S_+^{L-1} admits an equilibrium (e.g., Magill and Quinzi, 1996, p.114-115). This variational form of Brouwer's fixed point Theorem can be used to prove the existence of an equilibrium in the standard Arrow-Debreu model (e.g., Magill and Quinzi, 1996, p.56-58 and p.119). The aim of this subsection is to state a variational form of Theorem 1.

Theorem 2 *If $V : S_+^{L-1} \to \mathcal{M}(S \times J)$ is a transverse mapping, then every inward V-continuous vector field $z : S_+^{L-1} \to \mathbb{R}^L$ that is continuous on a neighborhood of the boundary of S_+^{L-1} admits an approximated equilibrium $\bar{x} \in S_+^{L-1}$.*

Proof. Let $\epsilon > 0$ and $\bar{y} \in S_+^{L-1}$. We claim that for $\nu > 0$ large enough, one can define a V-continuous mapping $f : S_+^{L-1} \to S_+^{L-1}$ by

$$\forall x \in S_+^{L-1} \mid \text{rank } V(x) = J, f(x) = \frac{x + \dfrac{z(x) + \epsilon(1, ..., 1)}{\nu}}{\|x + \dfrac{z(x) + \epsilon(1, ..., 1)}{\nu}\|}$$

and

$$\forall x \in S_+^{L-1} \mid \text{rank } V(x) < J, f(x) = \bar{y}.$$

First, since z is a vector field on S_+^{L-1}, one cannot have $x + \dfrac{z(x) + \epsilon(1, ..., 1)}{\nu} = 0$.

Now, let prove that for ν large enough, for every $x \in S_+^{L-1}$, one has $f(x) \in S_+^{L-1}$. Let suppose on the contrary that there exists a sequence $(x_\ell)_{\ell \in \mathbb{N}}$ in $\{x \in S_+^{L-1}, \text{ rank } V(x) = J\}$ such that the first component (to fix ideas) of $x_\ell + \dfrac{z(x_\ell) + \epsilon(1, ..., 1)}{\ell}$ is negative. It implies

$$z^1(x_\ell) \leq -\ell x_\ell^1 \tag{3}$$

and

$$z^1(x_\ell) \leq -\epsilon \tag{4}$$

where $z^1(x_\ell)$ and x_ℓ^1 denote the first components of $z(x_\ell)$ and x_ℓ.

Now, since $S_+^{L-1} \times G^J(\mathbb{R}^S)$ is a compact manifold, without any loss of generality (up to an extraction), one can assume that the sequence $(x_\ell,\mathrm{span}V(x_\ell))_{\ell\in\mathbb{N}}$ converges. So, from the V-continuity of z, the sequence $(z(x_\ell))_{\ell\in\mathbb{N}}$ converges, and from Equation 3, the sequence $(x_\ell^1)_{\ell\in\mathbb{N}}$ must converge to 0. Thus, the sequence $(x_\ell)_{\ell\in\mathbb{N}}$ converges to $\bar{x} \in \partial S_+^{L-1}$, and from the continuity of z on a neighborhood of the boundary of S_+^{L-1}, the sequence $(z(x_\ell))_{\ell\in\mathbb{N}}$ converges to $z(\bar{x}) \in \mathbb{R}^L$. Besides, since z is a vector field on S_+^{L-1} and since $\bar{x}^1 = 0$, one has $z^1(\bar{x}) \geq 0$, which is a contradiction with Equation 4.

Therefore, the mapping f is well defined, and from the V-continuity of z, it is clearly a V-continuous mapping. Consequently, for ν large enough, one can apply Corollary 1 to the mapping f : there exists a sequence $(x_\ell)_{\ell\in\mathbb{N}}$ in $\{x \in S_+^{L-1}$, rank $V(x) = J\}$ converging to $\bar{x} \in S_+^{L-1}$, and such that $(f(x_\ell))_{\ell\in\mathbb{N}}$ converges to $\bar{x}$. Besides, from the compactness of $S_+^{L-1} \times G^J(\mathbb{R}^S)$ and from the V-continuity of z, without any loss of generality, one can suppose that the sequence $(z(x_\ell))_{\ell\in\mathbb{N}}$ converges to $\bar{z} \in \mathbb{R}^L$ (see the argument above). Consequently, from the definition of f, one obtains

$$\bar{x} = \frac{\bar{x} + \dfrac{\bar{z} + \epsilon(1,...,1)}{\nu}}{\|\bar{x} + \dfrac{\bar{z} + \epsilon(1,...,1)}{\nu}\|}. \tag{5}$$

So, from $\bar{z} \cdot \bar{x} = 0$ one easily obtains $\bar{z} = (\epsilon \sum_{i=1}^L \bar{x}^i)\bar{x} - \epsilon(1,...,1)$, which implies $\|\bar{z}\| \leq 2\sqrt{L}\epsilon$. This proves that for ℓ large enough, one has $\|z(x_\ell)\| \leq \epsilon(1 + 2\sqrt{L})$. For every $\ell \in \mathbb{N}^*$, it allows to define $y_\ell \in \{x \in S_+^{L-1}$, rank $V(x) = J\}$ such that $\|z(y_\ell)\| \leq \dfrac{1}{\ell}$, and without any loss of generality (up to an extraction), one can suppose that the sequence $(y_\ell)_{\ell\in\mathbb{N}^*}$ converges to an approximated equilibrium of z. This ends the proof of Theorem 2.

Now, we are ready to prove the existence of an approximated equilibrium for V-continuous economies.

4 Existence of an approximated equilibrium

Theorem 3 *If V is a transverse financial structure, then every V-continuous economy $\mathcal{E}(z,V)$ admits an approximated equilibrium, i.e. $AE(\mathcal{E}(z,V)) \neq \emptyset$.*

Proof. Let $\mathcal{E}(z,V)$ be a V-continuous economy and define $\tilde{z} : S_+^{L-1} \to \mathbb{R}^L$ by

$$\tilde{z}(p) = \alpha(p)z(p) + (1 - \alpha(p))z^*(p),$$

where z^* is any continuous vector field on S_+^{L-1}, and $\alpha : S_+^{L-1} \to \mathbb{R}$ is any continuous mapping equal to 0 on a neighborhood $\mathcal{V}$ of the boundary of S_+^{L-1}, and equal to 1 on a compact subset K of S^{L-1}_{++}.[16] The mapping $\tilde{z}$ is continuous on a neighborhood of the boundary of S^{L-1}_{+}, and is a V-continuous vector field on S^{L-1}_{+}. Thus, from Theorem 2, there exists $\bar{p}_K \in S^{L-1}_{+}$, an approximated equilibrium of $\tilde{z}$. We now claim that there exists K such that $\alpha(\bar{p}_K) = 1$. Otherwise, there would exist a sequence $(p_\ell)_{\ell \in \mathbb{N}}$ converging to the boundary of S^{L-1}_{++} and such that $z(p_\ell)$ converges to 0, which is a contradiction with the boundary condition (ii).

To finish, just notice that if $\bar{p}_K$ is an approximated equilibrium of $\tilde{z}$ and if $\alpha(\bar{p}_K) = 1$, then $\bar{p}_K$ is clearly an approximated equilibrium of z.

A first corollary of Theorem 3 is the existence of a pseudo-equilibrium for every continuous asset structure :

Corollary 2 *For every GEI excess demand z^{GEI} and for every continuous asset structure V, there exists a pseudo-equilibrium $(p, E) \in S^{L-1}_{++} \times G^J(\mathbb{R}^S)$.*

Proof. Let $M = C^0(S^{L-1}_{+}, \mathcal{M}(S \times J))$, equipped with $\|.\|_\infty$, and let $V \in M$. From a transversality theorem, the set of transverse mappings in M is dense in M. Thus, their exists a sequence $(V_\ell)_{\ell \in \mathbb{N}}$ of transverse mappings that converges to V. From Theorem 3, for every $l \in \mathbb{N}$, the economy $\mathcal{E}(z^{GEI}, V_\ell)$ admits an approximated equilibrium, and consequently, from Proposition 2, a pseudo-equilibrium $(p_\ell, E_\ell) \in S^{L-1}_{++} \times G^J(\mathbb{R}^S)$. Thus, one has $Z(p_\ell, E_\ell) = 0$ and $\text{span}V_\ell(p_\ell) \subset E_\ell$. Without any loss of generality, from the compactness of $S^{L-1}_{+} \times G^J(\mathbb{R}^S)$, and since Z satisfies the boundary condition, one can suppose that the sequence (p_ℓ, E_ℓ) converges to $(\bar{p}, \bar{E}) \in S^{L-1}_{++} \times G^J(\mathbb{R}^S)$. Consequently, one obtains $Z(\bar{p}, \bar{E}) = 0$ and $\text{span}V(\bar{p}) \subset \bar{E}$, i.e. $(\bar{p}, \bar{E})$ is a pseudo-equilibrium.

A second corollary of Theorem 3 is the following existence result of Bottazzi (1995) :

Corollary 3 *If z^{GEI} satisfies Assumption (R) and if the asset structure V is transverse, then for generic $e \in \mathbb{R}^L_{++}$, there exists an equilibrium of z^{GEI}.*

Proof. From Proposition 3, under Assumption (R) and Assumption (T), one has $E(\mathcal{E}(z^{GEI}, V)) = AE(\mathcal{E}(z^{GEI}, V))$ for generic $\mathbb{R}^L_{++}$. From Theorem 3 and Assumption (T), the set $AE(\mathcal{E}(z^{GEI}, V))$ is nonempty. This proves that under Assumption (R), for generic $e \in \mathbb{R}^L_{++}$ and for transverse financial structures, there exists an equilibrium.

5 Proof of Theorem 1

The proof is given in three steps, corresponding to the three following subsections. The general idea is to associate to any V-continuous mapping $f : B \to B$

[16]This construction is standard in equilibrium existence proofs. See, for example, Magill and Quinzi, 1996, p.119-120.

a continuous mapping $\tilde{f} : \tilde{B} \to B$, where $\tilde{B}$ is, in some sense, the blowing up of B at the discontinuity points of f, and $\tilde{f}$ is some extension of f to $\tilde{B}$. The first step defines the blowing up $\tilde{B}$ of B, and provides some important properties of $\tilde{B}$. The second step extends the discontinuous mapping f to a continuous mapping $\tilde{f} : \tilde{B} \to B$. The last step proves the existence of an approximated fixed point of f by classical topological degree techniques applied to $\tilde{f}$.

5.1 Construction of $\tilde{B}$ and its main properties

In the following, if $V : B \to \mathcal{M}(S \times J)$ is a smooth mapping, then one defines the set $\tilde{B} = \{(x, E) \in int(B) \times G^J(\mathrm{I\!R}^S),\ spanV(x) \subset E\}$ and for $\rho \in [0, J]$, the set $\tilde{B}_\rho = \{(x, E) \in \tilde{B}, rank\ V(x) = J - \rho\}$.

Proposition 4 *If $V : B \to \mathcal{M}(S \times J)$ is a transverse mapping, then :*
i) The set $\tilde{B}$ is a smooth n-submanifold of $int(B) \times G^J(\mathrm{I\!R}^S)$.
ii) For every $\rho \in [0, J]$, the set $\tilde{B}_\rho$ is a smooth $(n - \rho^2)$-submanifold of $int(B) \times G^J(\mathrm{I\!R}^S)$.
iii) If $\overline{\tilde{B}_0}$ denotes the closure of $\tilde{B}_0$ in $int(B) \times G^J(\mathrm{I\!R}^S)$ then one has $\overline{\tilde{B}_0} = \tilde{B}$.

Remark 8 A statement similar to Part ii) of Proposition 4 can be found in Bottazzi (1995, p.70-71). Besides, if $V : S_+^{L-1} \to \mathcal{M}(S \times J)$ is a transverse mapping, then one can clearly obtain the following result from Proposition 4 : the set $\{(p, E) \in S_{++}^{L-1} \times G^J(\mathrm{I\!R}^S),\ \mathrm{span}V(p) \subset E, \text{ rank } V(p) = J\}$ is dense in the asset equilibrium space $\{(p, E) \in S_{++}^{L-1} \times G^J(\mathrm{I\!R}^S),\ \mathrm{span}V(p) \subset E\}$, and the latter is a $(L - 1)$-manifold.

Proof of (i). We will exhibit a smooth parametrization of $\tilde{B}$ on a neighborhood of $(\bar{x}, \bar{E}) \in \tilde{B}$. One has span$V(\bar{x}) \subset \bar{E}$ and rank $V(\bar{x}) = J - \rho$ for some $\rho \in \{0, ..., J\}$. If $\rho = 0$ then from the continuity of V, for every $(x, E) \in \tilde{B}$ in some neighborhood of $(\bar{x}, \bar{E})$, one has E =span$V(x)$. It clearly allows to parametrize (smoothly) $(x, E) \in \tilde{B}$, in a neighborhood of $(\bar{x}, \bar{E})$, by x.

Now, let suppose that $\rho \geq 1$. Without any loss of generality, up to a permutation of the rows and of the columns of $V(\bar{x})$, one can suppose that for every x in a neighborhood $U \subset \mathrm{int}(B)$ of $\bar{x}$,

$$V(x) = \begin{pmatrix} a(x) & b(x) \\ c(x) & d(x) \\ e(x) & f(x) \end{pmatrix} \tag{6}$$

where $a(x)$ is an invertible $(J - \rho) \times (J - \rho)$ matrix and $c(x)$ a $\rho \times (J - \rho)$ matrix. Thus, from Lemma 1, the condition rank $V(x) = J - \rho$ is equivalent, on U, to the following regular equation[17] :

[17] We say that the equation $F(x) = 0$ is regular if the mapping F is regular.

$$\begin{pmatrix} d(x) \\ f(x) \end{pmatrix} - \begin{pmatrix} c(x)a^{-1}(x)b(x) \\ e(x)a^{-1}(x)b(x) \end{pmatrix} = 0 \tag{7}$$

Now, the condition $\mathrm{span}V(\bar{x}) \subset \bar{E}$ means exactly that there exists a $S \times \rho$ matrix $\begin{pmatrix} b_1 \\ d_1 \\ f_1 \end{pmatrix}$, where b_1 is a $(J-\rho) \times \rho$ matrix and d_1 is a $\rho \times \rho$ matrix[18], such that

$$\bar{E} = \ \mathrm{span} \begin{pmatrix} a(\bar{x}) & b_1 \\ c(\bar{x}) & d_1 \\ e(\bar{x}) & f_1 \end{pmatrix} \tag{8}$$

Without any loss of generality, one can suppose that the first $J \times J$ submatrix of this last matrix is invertible, which implies that $Y(\bar{x}) := c(\bar{x})a^{-1}(\bar{x})b_1 - d_1$ is invertible. Then, multiplying the matrix of Equation 8 by the two $J \times J$ invertible matrices $\begin{pmatrix} a^{-1}(\bar{x}) & a^{-1}(\bar{x})b_1 \\ 0 & -I_\rho \end{pmatrix}$ and $\begin{pmatrix} I_{J-\rho} & 0 \\ -Y(\bar{x})^{-1}c(\bar{x})a^{-1}(\bar{x}) & Y(\bar{x})^{-1} \end{pmatrix}$, one obtains

$$\bar{E} = \ \mathrm{span} \begin{pmatrix} I_{J-\rho} & 0 \\ 0 & I_\rho \\ -\bar{X}c(\bar{x})a^{-1}(\bar{x}) + e(\bar{x})a^{-1}(\bar{x}) & \bar{X} \end{pmatrix} \tag{9}$$

for some $(S-J) \times \rho$ matrix $\bar{X}$. Now, recall that if $\bar{E} = \ \mathrm{span} \begin{pmatrix} I_J \\ \bar{A} \end{pmatrix}$, where $\bar{A}$ is a $(S-J) \times J$ matrix, then every $J-$subspace E of IR^S in a neighborhood of $\bar{E}$ can be written $E = \ \mathrm{span} \begin{pmatrix} I_J \\ A \end{pmatrix}$ for a unique $(S-J) \times J$ matrix A. Besides, it classically allows to define a local chart from a neighborhood of $\mathcal{M}((S-J) \times J)$ to a neighborhood of $\bar{E}$.

Now, the construction that leads to Equation 9 can be done for every (E, x) in a neighborhood of $(\bar{E}, \bar{x})$ and such that $\mathrm{span}V(x) \subset E$. Thus, E can be locally and smoothly parametrized by a $(S-J) \times \rho$ matrix X and by $x \in \mathrm{int}(B)$, with

$$E = \ \mathrm{span} \begin{pmatrix} I_{J-\rho} & 0 \\ 0 & I_\rho \\ -Xc(x)a^{-1}(x) + e(x)a^{-1}(x) & X \end{pmatrix}. \tag{10}$$

Consequently, on a neighborhood of $(\bar{E}, \bar{x})$, the equation $\mathrm{span}V(x) \subset E$ is equivalent to

$$\mathrm{rank} \begin{pmatrix} I_{J-\rho} & 0 & b(x) \\ 0 & I_\rho & d(x) \\ -Xc(x)a^{-1}(x) + e(x)a^{-1}(x) & X & f(x) \end{pmatrix} = J \tag{11}$$

[18] Remark that b_1, d_1 and f_1 are not univocally defined and cannot be used in the local parametrization of $\tilde{B}$.

which is finally equivalent to $f(x)+Xc(x)a^{-1}(x)b(x)-e(x)a^{-1}(x)b(x)-Xd(x)=0$, i.e. to

$$(f(x)-e(x)a^{-1}(x)b(x))-X(d(x)-c(x)a^{-1}(x)b(x))=0. \tag{12}$$

Equation 12 is clearly a regular equation in (X,x) because Equation (7) is regular in x. Thus, $\tilde{B}$ can be locally described by a regular and smooth system of $\rho(S-J)$ equations with $n+\rho(S-J)$ variables, i.e. $\tilde{B}$ is a smooth n-manifold.

Proof of (ii). From the implicit function theorem and since V is a transverse mapping, one obtains from Lemma 1 that for every $\rho=1,...,J$, the set B_ρ is a smooth $(n-\rho(S-J+\rho))$-submanifold of $\text{int}(B)$. From the continuity of V, it is also true for $\rho=0$.

Now, let define, for every $\rho=0,...,J$,

$$\tilde{B}'_\rho=\{(x,E)\in B_\rho\times G^{S-J}(\mathbb{R}^S),\ E\subset\ \text{span}V(x)^\perp\}.$$

Observe that the mapping $\Phi:\text{int}(B)\times G^{S-J}(\mathbb{R}^S)\to\text{int}(B)\times G^J(\mathbb{R}^S)$ defined by $\Phi(x,E)=(x,E^\perp)$ is a smooth diffeomorphism. Besides, $\tilde{B}'_\rho$ is a fiber bundle[19] : its basis is B_ρ, which is a smooth $(n-\rho(S-J+\rho))$-dimensional manifold, and its fiber at x is the smooth $\rho(S-J)$-manifold $G^{S-J}((\text{span}V(x))^\perp)$. Hence, $\tilde{B}'_\rho$ is a smooth $(n-\rho^2)$-submanifold of $B\times G^{S-J}(\mathbb{R}^S)$. Finally, $\tilde{B}_\rho=\Phi(\tilde{B}'_\rho)$ is a smooth $(n-\rho^2)$-submanifold of $B\times G^J(\mathbb{R}^S)$, which ends the proof of Statement ii).

The proof of iii) is a clear consequence of the two first statements and of $\tilde{B}=\tilde{B}_0\cup(\cup_{\rho=1}^J\tilde{B}_\rho)$.

5.2 Extension of f to $\tilde{f}$

We now define a mapping $\tilde{f}:\tilde{B}\to B$ as follows : let $(x,E)\in\tilde{B}$; from $\overline{\tilde{B}_0}=\tilde{B}$, there exists a sequence $(x_\ell,\text{span}V(x_\ell))_{\ell\in\mathbb{N}}$ in $\tilde{B}_0$ converging to (x,E). We then define

$$\tilde{f}(x,E)=\lim_{\ell\to+\infty}f(x_\ell).$$

Remark that the limit above is well defined because the mapping f is V-continuous and because the sequence $(x_\ell,\text{span}V(x_\ell))_{\ell\in\mathbb{N}}$ converges.

Now, we prove that this construction does not depend on the choice of the sequence $(x_\ell,\text{span}V(x_\ell))_{\ell\in\mathbb{N}}$. Let $(x_\ell,\text{span}V(x_\ell))_{\ell\in\mathbb{N}}$ and $(y_\ell,\text{span}V(y_\ell))_{\ell\in\mathbb{N}}$ be two sequences in $\tilde{B}_0$ converging to (x,E). Then define the sequence $(z_\ell)_{\ell\in\mathbb{N}}$ in B by $z_{2\ell}=x_\ell$ and $z_{2\ell+1}=y_\ell$ for every $\ell\in\mathbb{N}$. Since $(z_\ell,\text{span}V(z_\ell))_{\ell\in\mathbb{N}}$ converges to (x,E), and since f is V-continuous, the sequence $(f(z_\ell))_{\ell\in\mathbb{N}}$ converges. This clearly proves that $(f(x_\ell))_{\ell\in\mathbb{N}}$ and $(f(y_\ell))_{\ell\in\mathbb{N}}$ converge to the same limit. So, the previous definition of $\tilde{f}$ is correct.

[19] see Bottazzi (1995, p.70-71) to have an explicit parametrization of this manifold.

We now prove that $\tilde{f}$ is continuous. Recall that d_H denotes the Hausdorff distance defined on $G^J(\mathbb{R}^S)$, and consider δ' the metric defined on $\tilde{B}$ by

$$\forall((x,E),(x',E')) \in \tilde{B} \times \tilde{B}, \delta'((x,E),(x',E')) = \|x - x'\| + d_H(E,E').$$

Let $(x_\ell, E_\ell)_{\ell\in\mathbb{N}}$ be a sequence in $\tilde{B}$ converging to $(x,E) \in \tilde{B}$, and prove that $(\tilde{f}(x_\ell, E_\ell))_{\ell\in\mathbb{N}}$ converges to $\tilde{f}(x,E)$. From the definition of $\tilde{f}$ and from $\overline{\tilde{B}_0} = \tilde{B}$, for every integer $\ell \in \mathbb{N}$, there exists $(y_\ell,\text{span}V(y_\ell))$ in $\tilde{B}_0$ such that

$$\delta'((x_\ell, E_\ell),(y_\ell, \text{span}V(y_\ell))) \leq \frac{1}{\ell} \tag{13}$$

and

$$\|\tilde{f}(x_\ell, E_\ell) - f(y_\ell)\| \leq \frac{1}{\ell} \tag{14}$$

From Equation 13, the sequence $(y_\ell, \text{span}V(y_\ell))_{\ell\in\mathbb{N}}$ converges to (x,E). Thus, from the definition of $\tilde{f}$, one has $\tilde{f}(x,E) = \lim_{\ell\to+\infty} f(y_\ell)$. Consequently, from Equation 14, one obtains $\lim_{\ell\to+\infty}\tilde{f}(x_\ell, E_\ell) = \tilde{f}(x,E)$, which proves that $\tilde{f}$ is continuous.

5.3 Proof of the existence of an approximated equilibrium

We now prove that there exists an approximate solution $(x_\ell, E_\ell) \in \tilde{B}$ of the equation[20] $\tilde{f}(x,E) = x$ (in the sense that $\|\tilde{f}(x_\ell,E_\ell) - x_\ell\| \leq \frac{1}{\ell}$) by a topological degree argument[21]. Then, we will prove that any limit of a convergent subsequence of $(x_\ell)_{\ell\in\mathbb{N}}$ is an approximated fixed point of f.

Let $\ell \in \mathbb{N}$ with $\ell \geq 2$. In order to overcome the fact that, in general, the mapping $\tilde{f}$ may not be defined on the adherence of $\tilde{B}$, we will work on $\tilde{B}^\ell := \{(x,E) \in \tilde{B}, \|x\| < 1 - \frac{1}{\ell}\}$. To prove the existence of $(x_\ell, E_\ell) \in \tilde{B}$ such that $\|\tilde{f}(x_\ell, E_\ell) - x_\ell\| \leq \frac{1}{\ell}$, suppose that the set $\{(x,E) \in \tilde{B}, \|\tilde{f}(x,E) - x\| \leq \frac{1}{\ell}, \|x\| = 1 - \frac{1}{\ell}\}$ is empty and prove that there exists $(x_\ell, E_\ell) \in \tilde{B}^\ell$ such that $\|\tilde{f}(x_\ell, E_\ell) - x_\ell\| \leq \frac{1}{\ell}$.

From $\overline{\tilde{B}_0} = \tilde{B}$, there exists $\bar{x} \in B$ such that rank $V(\bar{x}) = J$ and $\|\bar{x}\| < 1 - \frac{1}{\ell}$. Let then define the continuous mapping $H : [0,1] \times \tilde{B}^\ell \to \mathbb{R}^n$ by

$$\forall(t,x,E) \in [0,1] \times \tilde{B}^\ell, H(t,x,E) = (1-t)((1-\frac{1}{\ell})\tilde{f}(x,E) - x) - t(x - \bar{x}).$$

[20] Actually, at this stage, another (more complicated) approach to prove the existence of an approximated equilibrium would be to extend $\tilde{f}$ to a continuous mapping $\bar{f} : B \times G^J(\mathbb{R}^S) \to B$, and to find $(x,E) \in B \times G^J(\mathbb{R}^S)$ such that $\bar{f}(x,E) = x$ and span$V(p) \subset E$ by classical methods (e.g., Husseini et al. (1990) or Hirsch et al. (1990)). We thank an anonymous referee for this remark. Anyway, being able to define $\tilde{f}$ and to prove that a solution x of $\bar{f}(x,E) = x$ and span$V(p) \subset E$ is an approximated equilibrium rests on a fine topological description of the set $\tilde{B}$, which is the difficult part of our proof.

[21] We only are able to give an approximate solution of this equation because the mapping $\tilde{f}$ is defined on a non-compact manifold.

Let prove that H is compactly rooted, which means that $H^{-1}(0)$ is compact in $[0,1] \times \tilde{B}^\ell$. If not, then, from the continuity of $\tilde{f}$ on $\tilde{B}$, there clearly exists $(t,x,E) \in [0,1] \times \tilde{B}$ with $\|x\| = 1 - \frac{1}{\ell}$, such that $0 = (1-t)((1-\frac{1}{\ell})\tilde{f}(x,E) - x) - t(x - \bar{x})$. Notice that $t \neq 1$, because $\|x\| = 1 - \frac{1}{\ell}$ and $\|\bar{x}\| < 1 - \frac{1}{\ell}$. Thus, one can write :

$$\tilde{f}(x,E) = \frac{\ell}{\ell - 1} \frac{x - t\bar{x}}{(1-t)}.$$

Besides, since by assumption there is no solution of $\|\tilde{f}(x,E) - x\| \leq \frac{1}{\ell}$ with $\|x\| = 1 - \frac{1}{\ell}$, one has $t \neq 0$. Hence, from $\|\bar{x}\| < 1 - \frac{1}{\ell}$ and from a triangle inequality, one has :

$$\|\tilde{f}(x,E)\| \geq \frac{\ell}{\ell - 1} \frac{\|x\| - t\|\bar{x}\|}{(1-t)} > \frac{\ell}{\ell - 1} \frac{1 - \frac{1}{\ell} - t(1 - \frac{1}{\ell})}{(1-t)} > \frac{(1-t)}{(1-t)} = 1,$$

which is a contradiction, and proves that H is compactly rooted.

Since $H : [0,1] \times \tilde{B}^\ell \to \mathbb{R}^n$ is a continuous and compactly rooted homotopy, and since $\tilde{B}^\ell$ is a boundaryless smooth n-manifold, one has $\deg H(0,.) = \deg H(1,.)$, where deg is the modulo 2 degree[22] of continuous and compactly rooted mappings from $\tilde{B}^\ell$ to $\mathbb{R}^n$.[23] Besides, $H(1,.)$ has only one zero $(\bar{x}, \mathrm{span} V(\bar{x}))$ in $\tilde{B}^\ell$, and $DH(1,.)(\bar{x}, \mathrm{span} V(\bar{x}))$ is clearly bijective. Thus, $\deg H(1,.) = 1$, which proves that $\deg H(0,.) \neq 0$. Consequently, there exists $(x,E) \in \tilde{B}_\ell$ such that $H(0,x,E) = 0$, i.e. $\tilde{f}(x,E) = \frac{\ell}{\ell-1}x$. It clearly implies $\|\tilde{f}(x,E) - x\| \leq \frac{1}{\ell}$.

Finally, one has proved that there is $(x_\ell, E_\ell) \in \tilde{B}$ such that

$$\|\tilde{f}(x_\ell, E_\ell) - x_\ell\| \leq \frac{1}{\ell} \tag{15}$$

Lastly, from the definition of $\tilde{f}$, there exists a sequence $(x'_\ell, \mathrm{span} V(x'_\ell))_{\ell \in \mathbb{N}}$ in $\tilde{B}_0$ such that

$$\|\tilde{f}(x_\ell, E_\ell) - f(x'_\ell)\| \leq \frac{1}{\ell} \tag{16}$$

and

$$\|x_\ell - x'_\ell\| \leq \frac{1}{\ell} \tag{17}$$

From the compactness of the set $\{(x,E) \in B \times G^J(\mathbb{R}^S),\ \mathrm{span} V(x) \subset E\}$, one may suppose, without any loss of generality, that the sequence $(x_\ell, E_\ell)_{\ell \in \mathbb{N}}$ converges to $(x,E) \in B \times G^J(\mathbb{R}^S)$ with $\mathrm{span} V(x) \subset E$. From Equations 16, 15 and 17, one obtains

$$\|f(x'_\ell) - x'_\ell\| \leq \frac{3}{\ell} \tag{18}$$

[22]Since $\tilde{B}^\ell$ may not be orientable, one has to use modulo 2 degree.

[23]See Villanacci et al. (2002, p.196) for an axiomatic definition of the topological degree which is here used.

Besides, from Equation 17, the sequence $(x'_\ell)_{\ell\in\mathbb{N}}$ converges to x. Thus, x is an approximated fixed point of f.

6 Appendix

6.1 The Grassmannian manifold $G^J(\mathbb{R}^S)$

First, we briefly recall how $G^J(\mathbb{R}^S)$ can be topologized, where J and S are two integers such that $0 < J \leq S$. A J-frame in $\mathbb{R}^S$ is a J-uple of linearly independent vectors of $\mathbb{R}^S$. The collection of all J-frames in $\mathbb{R}^S$ forms an open subset of the J-fold Cartesian product $\mathbb{R}^S \times ... \times \mathbb{R}^S$, called the Stiefel manifold $V_J(\mathbb{R}^S)$. By definition, we give $G^J(\mathbb{R}^S)$ the finest topology that makes the mapping π continuous, where

$$\pi : V_J(\mathbb{R}^S) \to G^J(\mathbb{R}^S)$$

is defined by

$$\pi(x_1, ..., x_J) = \mathrm{span}\{x_1, ..., x_J\}.$$

One can then prove that the set $G^J(\mathbb{R}^S)$, equipped with this topology, is a compact smooth $J(S-J)$-manifold, and that π is smooth (e.g., Milnor et al. (1974), p.55-71). Besides, there exists a metric d_H on $G^J(\mathbb{R}^S)$, called the Hausdorff metric, which is compatible with the topology defined above and that can be directly defined, for every (L_1, L_2) in $G^J(\mathbb{R}^S) \times G^J(\mathbb{R}^S)$, by :

$$d_H(L_1, L_2) = \max\{\max\{d(x, L_1 \cap B) \mid x \in L_2 \cap B\}, \max\{d(x, L_2 \cap B) \mid x \in L_1 \cap B\}\},$$

where $d(x, K)$ denotes the distance between $x \in \mathbb{R}^S$ and K, a compact subset of $\mathbb{R}^S$, and B denotes the closed unit ball of $\mathbb{R}^S$.

6.2 Proof of Lemma 1

Let $\bar{x} \in \mathrm{int}(X)$ such that rank $V(\bar{x}) = J - \rho$ for some $\rho \in \{1, ..., J\}$ and suppose that for every x in a neighborhood $U \subset \mathrm{int}(X)$ of $\bar{x}$,

$$V(x) = \begin{pmatrix} a(x) & b(x) \\ c(x) & d(x) \end{pmatrix}, \tag{19}$$

where $a(x)$ is an invertible $(J-\rho) \times (J-\rho)$ matrix.

Multiplying this matrix by the invertible $J \times J$ matrix $\begin{pmatrix} a^{-1}(x) & a^{-1}(x)b(x) \\ 0 & -I_\rho \end{pmatrix}$ one obtains

$$\begin{pmatrix} I_{J-\rho} & 0 \\ c(x)a^{-1}(x) & c(x)a^{-1}(x)b(x) - d(x) \end{pmatrix}$$

Thus, the condition rank $V(x) = J - \rho$ is equivalent, on U, to the equation $f(x) := d(x) - c(x)a^{-1}(x)b(x) = 0$. Now, the transversality condition at $\bar{x}$ can be written :

$$T_{V(\bar{x})}\mathcal{M}^{\rho}(S \times J) + DV(\bar{x})(T_{\bar{x}}X) = T_{V(\bar{x})}\mathcal{M}(S \times J) \tag{20}$$

Let then consider

$$F : \mathcal{M}((J-\rho)\times(J-\rho))\times\mathcal{M}((J-\rho)\times\rho)\times\mathcal{M}((S-J+\rho)\times(J-\rho)) \to \mathcal{M}((S-J+\rho)\times\rho)$$

by $F(A, B, C) = CA^{-1}B$, well defined if A is invertible. From above, one obtains

$$T_{V(\bar{x})}\mathcal{M}^{\rho} = \{\begin{pmatrix} a' & b' \\ c' & DF(a(\bar{x}), b(\bar{x}), c(\bar{x}))(a', b', c') \end{pmatrix}, a' \in \mathcal{M}((J-\rho) \times (J-\rho)),$$

$$b' \in \mathcal{M}((J-\rho) \times \rho), c' \in \mathcal{M}((S-J+\rho) \times (J-\rho))\}$$

and clearly

$$T_{V(\bar{x})}\mathcal{M} = \{\begin{pmatrix} a'' & b'' \\ c'' & d'' \end{pmatrix}, a'' \in \mathcal{M}((J-\rho) \times (J-\rho)), b'' \in \mathcal{M}((J-\rho) \times \rho),$$

$$c'' \in \mathcal{M}((S-J+\rho) \times (J-\rho)), d'' \in \mathcal{M}((S-J+\rho) \times \rho)\},$$

and finally

$$DV(\bar{x})(T_{\bar{x}}X) = \{\begin{pmatrix} Da(\bar{x})(h) & Db(\bar{x})(h) \\ Dc(\bar{x})(h) & Dd(\bar{x})(h) \end{pmatrix}, h \in T_{\bar{x}}X\}$$

Thus, a simple computation proves that the transversality equation 20 is equivalent to : for every a'', b'', c'' and d'' given as above, there exists $h \in T_{\bar{x}}X$ such that

$$Df(\bar{x})(h) = d'' - DF(a(\bar{x}), b(\bar{x}), c(\bar{x}))(a'', b'', c'').$$

But d'' can be taken independently of a'', b'' and c'', so the previous condition is equivalent to the ontoness of $Df(\bar{x})$, which ends the proof.

6.3 Proof of Remark 2

Suppose that $V : S_{+}^{L-1} \to \mathcal{M}(S \times J)$ is continuous and that $f : S_{++}^{L-1} \to \mathbb{R}^L$ is a V-continuous mapping. Let $\bar{p} \in S_{++}^{L-1}$ such that rank $V(\bar{p}) = J$, and let $(p_\ell)_{\ell\in\mathbb{N}}$ be a sequence in S_{++}^{L-1} converging to $\bar{p} \in S_{++}^{L-1}$. To prove that the mapping f is continuous on $\{p \in S_{++}^{L-1}$, rank $V(p) = J\}$, we have to prove that the sequence $(f(p_\ell))_{\ell\in\mathbb{N}}$ converges to $f(\bar{p})$. From the continuity of V, the set $\{p \in S_{++}^{L-1}$, rank $V(p) = J\}$ is open in S_{++}^{L-1}. Thus, there exists $L \in \mathbb{N}$ such that for every $\ell \geq L$, rang$V(p_\ell) = J$. Then, notice that from the definition of the topology of $G^J(\mathbb{R}^S)$

(see Section 6.1.), the mapping $\Phi : \{M \in \mathcal{M}(S \times J), \mathrm{rang} M = J\} \to G^J(\mathbb{R}^S)$ defined by

$$\Phi(M) = \mathrm{span} M$$

is continuous. Therefore, one has $\lim_{\ell \to +\infty} \mathrm{span} V(p_\ell) = \mathrm{span} V(\bar{p})$. Now, let define the sequence $(q_\ell)_{\ell \in \mathbb{N}}$, in $\{p \in S^{L-1}_{++}$, rank $V(p) = J\}$ for ℓ large enough, by

$$\forall \ell \in \mathbb{N}, q_{2\ell} = p_\ell \text{ and } q_{2\ell+1} = \bar{p}.$$

Since $(q_\ell, \mathrm{span} V(q_\ell))_{\ell \in \mathbb{N}}$ converges, and from the V-continuity of f, the sequence $(f(q_\ell))_{\ell \in \mathbb{N}}$ converges, which clearly proves that $(f(p_\ell))_{\ell \in \mathbb{N}}$ converges to $f(\bar{p})$. This ends the proof.

6.4 Proof of Proposition 2

The first inclusion $E(\mathcal{E}(z^{GEI}, V)) \subset AE(\mathcal{E}(z^{GEI}, V))$ is clear. Now, let $p \in S^{L-1}_{++}$ be an approximated equilibrium of z^{GEI}, and let us prove that p is a pseudo-equilibrium price. From the definition of an approximated equilibrium, there exists $(p_\ell)_{\ell \in \mathbb{N}}$, a sequence of S^{L-1}_{++} converging to p, such that for every $\ell \in \mathbb{N}$, rank $V(p_\ell) = J$ and such that $(z^{GEI}(p_\ell))_{\ell \in \mathbb{N}}$ converges to 0. Then recall that from Equation 1, one has

$$\forall \ell \in \mathbb{N}, z^{GEI}(p_\ell) = Z(p_\ell, \mathrm{span} V(p_\ell)). \tag{21}$$

From the compactness of $G^J(\mathbb{R}^S)$, there exists a subsequence of $(\mathrm{span} V(p_\ell))_{\ell \in \mathbb{N}}$ that converges to $E \in G^J(\mathbb{R}^S)$. Thus, without any loss of generality, one can suppose that the sequence $(p_\ell, \mathrm{span} V(p_\ell))_{\ell \in \mathbb{N}}$ converges to $(p, E) \in S^{L-1}_{++} \times G^J(\mathbb{R}^S)$. Hence, from Equation 21 and from the continuity of Z, one obtains at the limit

$$Z(p, E) = 0. \tag{22}$$

Now, if we define the continuous mapping $H : S^{L-1}_{++} \times G^J(\mathbb{R}^S) \to (\mathbb{R}^S)^J$ by

$$\forall (p, E) \in S^{L-1}_{++} \times G^J(\mathbb{R}^S), H(p, E) = (\mathrm{proj}_{E^\perp} V_1(p), ..., \mathrm{proj}_{E^\perp} V_J(p)),$$

then for every $\ell \in \mathbb{N}$, one has $H(p_\ell, \mathrm{span} V(p_\ell)) = 0$. Consequently, one obtains at the limit $H(p, E) = 0$, which means

$$\mathrm{span} V(p) \subset E. \tag{23}$$

Thus, from Equations 22 and 23, one has $p \in PE(\mathcal{E}(z^{GEI}, V))$.

References

[1] Arrow, J.K., Debreu, G, 1954. Existence of an equilibrium for a competitive economy. Econometrica 22, 265-290.

[2] Balasko, Y., 1988. Foundations of the theory of General equilibrium. Academic Press, Boston.

[3] Bottazzi, J.M., 1995. Existence of equilibria with incomplete markets: the case of smooth returns. Journal of Mathematical Economics 24, 59-72.

[4] Bich, P., 2004. An extension of Brouwer's fixed point theorem allowing discontinuities. C.R.Acad. Paris, Ser. I 338, 673-678.

[5] Brown, D., Demarzo, P., Eaves, C., 1996. Computing Equilibria when asset market are incomplete. Econometrica 64, 1-27.

[6] Brouwer, L.E.J., 1912. Über Abbildung von Mannigfaltigkeiten. Mathematische Annalen 71, 97-115.

[7] Chichilnisky, G., Heal, G., 1996. On the existence and the structure of the pseudo-equilibrium manifold. Journal of Mathematical Economics 26, 171-186.

[8] Duffie, D., Shafer, W., 1985. Equilibrium in incomplete markets I. A basic model of generic existence. Journal of Mathematical Economics 14 (3), 285-300.

[9] Hart, O.D., 1975. On the optimality of Equilibrium when the market structure is incomplete. Journal of Economic Theory 11, 418-443.

[10] Husseini, S.Y., Lasry, J.M., Magill, M., 1990. Existence of Equilibrium with incomplete markets. Journal of Mathematical Economics 19, 39-67.

[11] Geanakoplos, J., Shafer, W., 1990. Solving systems of simultaneous equations in economics. Journal of Mathematical Economics 19, 69-93.

[12] Hirsch, M.W., Magill, M., Mas-Colell, A., 1990. A geometric approach to a class of equilibrium existence theorems. Journal of Mathematical Economics. 19 (1-2), 95-106.

[13] Hens, T. 1998. Incomplete markets. In: Kirman, A. (ed.) Elements of general equilibrium analysis. New York, Blackwell.

[14] Hirsch M.W., 1976. Differentiel topology. Springer Verlag, New-York.

[15] Magill, M., Shafer, W., 1991. Incomplete Markets. Handbook of Mathematical Economics, Volume 4, edited by W. Hildenbrand and H. Sonnenschein, 167-194.

[16] Magill, M., Quinzi, M., 1996. Theory of Incomplete Markets, Volume 1.The MIT Press, Cambridge, Massachusetts.

[17] Milnor, J.W., Stasheff, J.D., 1974. Characteristic classes. Annals of Mathematics Studies 76, Princeton University Press, New Jersey.

[18] Villanacci, A., Carosi, L., Benevieri, P., Batinelli, A., 2002. Differential topology and general equilibrium with complete and incomplete markets. Kluwer Academic Publishers, Boston.