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Revision of the basic equations of wave distribution function analysis

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Abstract. The basic equations of wave distribution function analysis are rewritten in forms that treat the electric and magnetic fields of the waves in a more symmetrical way than the original equations do, and are slightly better for computing.

Key words. Radio science (electromagnetic metrology) · Electromagnetics (plasmas; signal processing and adaptive antennas)

This note revises a set of equations first published in two papers in the now defunct *Geophysical Journal of the Royal Astronomical Society.* The corrections, however, are made to the form of the equations rather than to their content, i.e., they concern errors of presentation, not of material fact.

For a linear random field of electromagnetic (EM) waves in a magnetoplasma, and for each of the EM wave modes that this medium supports, the *wave distribution function (WDF)* describes how the energy density at a given time t and position $\mathbf{r}$ is distributed with respect to the wave number vector $\mathbf{k}$, or, more conveniently, with respect to the frequency and to the direction of propagation. In this latter form, the WDF is the function $F_m(t, \mathbf{r}, \omega, \boldsymbol{\kappa})$ such that

$$F_m(t, \mathbf{r}, \omega, \boldsymbol{\kappa})\, \mathrm{d}^3r\, \mathrm{d}\omega\, \mathrm{d}\sigma$$

is the average energy in the mode m at the time t, in the volume element d^3r at the position $\mathbf{r}$, due to waves with their angular frequencies in the range from ω to $\omega + \mathrm{d}\omega$ and their normal directions in the element of solid angle $\mathrm{d}\sigma$ centred on the unit vector $\boldsymbol{\kappa}$ (which is $\mathbf{k}/|\mathbf{k}|$). This definition was first given in a paper by Storey and Lefeuvre (1979), hereinafter referred to as Paper I. Their main findings were embodied in two integral equations, numbered (24) and (25); expressions for the kernels in these equations were given in a later paper (Storey and Lefeuvre, 1980), which will be called Paper II.

Wave distribution function analysis is a set of procedures for determining the WDFs from measurements of the electromagnetic field. So far, it has been developed only for measurements of different field components, all made over the same brief interval of time at the same (possibly moving) point, as for instance on an artificial satellite; only this case will be considered here. At least three EM field components are needed. These may be all electric, all magnetic, or one of one kind and two of the other. At most, measurements of all three electric and three magnetic field components can be made and analysed. Thus the basic data are simultaneous measurements of N field components versus time, where N can be any number from 3 to 6.

In a preliminary step, these N time series are processed to yield the best possible estimates of their N auto-spectra and $N(N-1)$ cross-spectra, a total of N^2 power spectra; precisely how these spectra are defined will be discussed in a moment. They are the input to the actual analysis, for which purpose it is convenient to arrange them in an $N \times N$ matrix known as the *spectral matrix.* Then, in this main part of the procedure, the spectral matrix is analysed to determine the WDFs. The analysis involves the inversion of equation (24) or (25) of Paper I, which is a badly posed inverse problem.

From here on, in discussing WDF analysis, the particular case where $N = 6$ will be taken. The generalisation to other values of N should be obvious.

The basic equations of WDF analysis comprise, firstly, the expression for the spectral matrix in terms of the field components, and secondly, the equations that relate the WDFs to the spectral matrix. All of them were stated correctly in Paper I, but in forms that are now seen to be awkward in some respects. The purpose of the present note is to restate them in more serviceable forms, which will be done by changing the definitions of some of the variables.

In deciding on the exact form of the spectral matrix, the problem is that the electric and magnetic field components

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have different physical dimensions, whereas, to simplify the analysis, it is desirable that the matrix elements should all have the same dimensions. To this end, the electric or the magnetic components or both must be transformed in some way so as to make their dimensions the same, before using them to estimate the spectral matrix. In Paper I, the authors chose to transform the magnetic field **H** of the wave into an equivalent electric field, by multiplying it by the wave impedance Z_0 of free space; this factor is $Z_0 = (\mu_0/\varepsilon_0)^{1/2}$, where μ_0 and ε_0 are the permeability and permittivity of free space, respectively. Using a right-handed Cartesian coordinate system Oxyz with its z-axis parallel to the static magnetic field, as shown in Fig. 1 of that paper, a 6-component generalized electric field vector $\mathscr{E}$ was defined as follows:

$$\left.\begin{array}{lll} \mathscr{E}_1 = E_x & \mathscr{E}_2 = E_y & \mathscr{E}_3 = E_z \\ \mathscr{E}_4 = Z_0 H_x & \mathscr{E}_5 = Z_0 H_y & \mathscr{E}_6 = Z_0 H_z \end{array}\right\}. \qquad (1)$$

This was equation (6) in Paper I. From measurements of these 6 field components, the 6 auto-spectra and 30 cross-spectra that form the elements of the 6×6 spectral matrix **S**, all of which had the dimensions of (electric field)^2, were to be estimated by the usual means; see, for instance, Priestley (1981).

This way of giving the electric and magnetic field components the same dimensions before defining the spectral matrix is inelegant, because it treats the two kinds of field differently. A better way to make their dimensions the same was devised by Suchy and Altman (1975). In the present context, it involves multiplying the right-hand sides of the six equations in the set (1) by the factor $\varepsilon_0^{1/2}$, thus replacing those definitions of the components of $\mathscr{E}$ by the following ones:

$$\left.\begin{array}{llll} \mathscr{E}_1 = \varepsilon_0^{1/2} E_x & \mathscr{E}_2 = \varepsilon_0^{1/2} E_y & \mathscr{E}_3 = \varepsilon_0^{1/2} E_z \\ \mathscr{E}_4 = \mu_0^{1/2} H_x & \mathscr{E}_5 = \mu_0^{1/2} H_y & \mathscr{E}_6 = \mu_0^{1/2} H_z \end{array}\right\}. \qquad (2)$$

Unlike the previous set, these definitions are symmetrical with respect to the electric and magnetic fields: each field component is multiplied by the square root of its related free-space parameter.

When redefined in this way, the vector $\mathscr{E}$ is no longer a generalized electric field: its components $\mathscr{E}_i$ all have the dimensions of (energy density)$^{1/2}$. None the less, since this 6-component vector still represents both the electric and the magnetic wave fields, here it will be called the *electromagnetic field vector*, or *EM field vector* for short.

With this new definition of $\mathscr{E}$, the elements S_{ij} of the spectral matrix **S** are ε_0 times the ones given by the previous definition. They now have the dimensions of energy density.

The abandonment of the definition (1) for $\mathscr{E}$ leads to the redefinition of other quantities besides **S** in Papers I and II, and it also affects some, though not all, of the other equations in these papers. In Paper I, the only other quantities that need redefining are, firstly, the generalized electric field vector $\mathscr{E}$ of an elementary monochromatic plane wave, and secondly, its complex amplitude vector **e**, both of which are introduced in equation (10) of that paper. Both should now be redefined as EM field vectors in the same way as $\mathscr{E}$ was redefined by equation (2) above. With these changes made, all of the equations other than (6) in Paper I remain valid, in particular the definition (18) of the quantities a_{ij} which are the kernels of the integral equations (24) and (25):

$$\alpha_{ij}\ (\omega,\ \boldsymbol{\kappa}) \equiv \frac{e_i\ e_j^*}{\rho}. \qquad (3)$$

On the right-hand side, e_i and e_j are components of the vector **e**, the asterisk denotes the complex conjugate, and ρ is the energy density of the elementary plane wave. Evidently, as a consequence of the redefinition of **e**, these kernels now have ε_0 times their former values.

Broader changes are needed in Paper II, where expressions are obtained for the kernels $a_{ij}\ (\omega, \boldsymbol{\kappa})$ in terms of the plasma parameters, albeit through the intermediary of another set of kernels named b_{ij}. These are the kernels of two integral equations analogous to (24) and (25) of Paper I, but which apply if the electric and magnetic wave fields are represented, not in the Cartesian coordinate system used in that paper, but in a *complex principal axis* coordinate system. The latter system is described in the appendix to Paper II, where equations are given relating the components of the same vector in the two systems. The vectors $\mathscr{E}$ and **e** in the Cartesian system have, as their respective counterparts in the complex principal axis system, two vectors named $\mathscr{F}$ and **f**, which are defined by equations (7) and (9) of Paper II. To begin with, these vectors must be redefined in the same way, i.e., by giving them $\varepsilon_0^{1/2}$ times their former values. The kernels b_{ij} are then defined by equation (10) of Paper II, which is similar to equation (18) of Paper I – reproduced as equation (3) above – except that the product $e_i e_j^*$ is replaced by $f_i\ f_j^*$ where f_i and f_j are components of **f**:

$$b_{ij}\ (\omega,\ \boldsymbol{\kappa}) \equiv \frac{f_i\ f_j^*}{\rho}. \qquad (4)$$

This equation is unchanged, but with the new definition of the vector **f** the kernels b_{ij}, like the a_{ij}, have ε_0 times their former values.

Because of the change in the definition of **f**, some equations need to be modified in section 5, where an expression is derived for ρ, the wave energy density. Thus, in the intermediate equations (39) and (40), the quantity Z_0 should be replaced by $\varepsilon_0 Z_0 = c^{-1}$, where c is the speed of light; in other words, Z_0 in the denominator should be replaced by c in the numerator. The same modification should be made in the final expression (45), but here one can go further and replace the product $Z_0 V_g$ in the denominator by n_g in the numerator, where V_g is the modulus of the component of the group velocity in the direction of the wave normal and n_g is the corresponding refractive index, given by equation (31) of Paper II.

Likewise, in equation (46) at the outset of section 6, the factor $Z_0 V_g$ should be replaced by n_g^{-1}. This equation is the general expression for the kernels b_{ij} in the complex principal axis coordinate system.

In the rest of section 6, expressions are derived for these 36 kernels individually, in the case of a cold magnetoplasma with no collisions. Here the most convenient place

at which to take account of the above changes is in the definition of the quantity ξ, which is equation (49). The new definition is

$$\xi \equiv 8\,[\nu_g\,(\lambda+\mu)]^{-1} \tag{5}$$

which makes ξ dimensionless, whereas previously it had the dimensions of ε_0^{-1}. The individual kernels are given by the expressions (50a) through (50u), in all of which ξ appears as a multiplier. An error in the expression for b_{55} has been corrected by Lefeuvre et al. (1986). With the new definitions of the 6-component field vector, of the spectral matrix, and of ξ, these expressions remain valid and the kernels all become dimensionless also.

In section 7, the expressions (55a) through (56u) are approximations to the kernels b_{ij} for the whistler wave mode. As they stand, they all contain ε_0^{-1} as a factor, but this goes away when the new definitions are used.

None of the other equations in Paper II need to be altered. In particular, equations (12a) through (12f), which relate the set of the kernels a_{ij} to the set of the b_{ij}, still apply.

Besides being more elegant than the originals, the revised equations are preferable for numerical calculations. Given that $\varepsilon_0 \approx 10^{-11}$ F m^{-1}, the disappearance of the factor ε_0^{-1} from the expressions for the kernels generally brings their values closer to unity. Thus the equations may no longer need to be scaled as a precaution against overflow or underflow, in which case the calculations would run slightly faster. For these reasons I intend always to use the revised equations in the future, and I urge others to do the same.

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References

Lefeuvre, F., Y. Marouan, M. Parrot, and J. L. Rauch, Rapid determination of the sense of polarization and propagation for random electromagnetic wave fields. Application to GEOS-1 and AUREOL-3 data, *Ann. Geophys., 4A,* 457–468, 1986.

Priestley, M. B., *Spectral Analysis and Time Series***,** Academic Press, London, 1981.

Storey, L. R. O., and F. Lefeuvre, The analysis of 6-component measurements of a random electromagnetic field in a magnetoplasma – 1. The direct problem, *Geophys. J. R. astr. Soc., 56,* 255–270, 1979.

Storey. L. R. O., and F. Lefeuvre, The analysis of 6-component measurements of a random electromagnetic field in a magnetoplasma – II. The integration kernels, *Geophys. J. R. astr. Soc., 62,* 173–194, 1980.

Suchy, K., and C. Altman, The Maxwell field, its adjoint field, and the 'conjugate' field in anisotropic absorbing media, *J. Plasma Phys., 13,* 299–316, 1975.