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# On the Calderón-Zygmund lemma for Sobolev functions

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**Abstract**

We correct an inaccuracy in the proof of a result in [Aus1].

2000 MSC: 42B20, 46E35
Key words: Calderón-Zygmund decomposition; Sobolev spaces.

We recall the lemma.

**Lemma 0.1.** *Let $n \geq 1$, $1 \leq p \leq \infty$ and $f \in \mathcal{D}'(\mathbb{R}^n)$ be such that $\|\nabla f\|_p < \infty$. Let $\alpha > 0$. Then, one can find a collection of cubes $(Q_i)$, functions $g$ and $b_i$ such that*

$$f = g + \sum_i b_i \tag{0.1}$$

*and the following properties hold:*

$$\|\nabla g\|_\infty \leq C\alpha, \tag{0.2}$$

$$b_i \in W_0^{1,p}(Q_i) \text{ and } \int_{Q_i} |\nabla b_i|^p \leq C\alpha^p |Q_i|, \tag{0.3}$$

$$\sum_i |Q_i| \leq C\alpha^{-p} \int_{\mathbb{R}^n} |\nabla f|^p, \tag{0.4}$$

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$$\sum_i \mathbf{1}_{Q_i} \leq N, \tag{0.5}$$

*where $C$ and $N$ depend only on dimension and $p$.*

This lemma was first stated in [Aus1] in $\mathbb{R}^n$. Then it appears in various forms and extensions in [Aus2] (same proof), [AC] (same proof on manifolds), [AM] (on $\mathbb{R}^n$ but with a doubling weight), B. Ben Ali's PhD thesis [Be] and [AB] (The Sobolev space is modified to adapt to Schrödinger operators), N. Badr's PhD thesis [Ba] and [Ba1, Ba2] (used toward interpolation of Sobolev spaces on manifolds and measured metric spaces) and in [BR] (Sobolev spaces on graphs). The same inaccuracy can be corrected everywhere as below. The proof of the generalisation to higher order Sobolev spaces in [Aus1] can also be corrected with similar ideas.

The second equation tells that $g$ is in fact Lipschitz continuous. There is a direct proof of this fact in N. Badr's thesis [Ba]. The proof proposed in [Aus1] is as follows:

Define $b_i = (f - c_i)\mathcal{X}_i$ where $c_i$ are appropriate numbers and $(\mathcal{X}_i)$ forms a smooth partition of unity of $\Omega = \cup Q_i$ subordinated to the cubes $(\frac{1}{2}Q_i)$ with support of $\mathcal{X}_i$ contained in $Q_i$. For example, the choice $c_i = f(x_i)$ for some well chosen $x_i$ or $c_i = m_{Q_i} f$, the mean of $f$ over the cube $Q_i$, ensures that $\sum_i |b_i|\ell_i^{-1}$ is locally integrable ($\ell_i$ is the length of $Q_i$) and that $\sum_i b_i$ is a distribution on $\mathbb{R}^n$. Then $g$ defined as $g = f - \sum_i b_i$ is a distribution on $\mathbb{R}^n$. Its gradient $\nabla g$ can be calculated as $\nabla g = (\nabla f)\mathbf{1}_F + h$ in the sense of distributions (on $\mathbb{R}^n$) with $h = \sum_i c_i \nabla \mathcal{X}_i$. It is then a consequence of the construction of the set $F = \Omega^c$ that $|\nabla f|$ is bounded on $F$ by $\alpha$ and then it is shown that $|h|$ is bounded by $C\alpha$ , which implies the boundedness of $|\nabla g|$.

Everything is correct in the argument above BUT the representation of $h$. The series $\sum_i c_i \nabla \mathcal{X}_i$, viewed as the distributional derivative on $\mathbb{R}^n$ of $\sum_i c_i \mathcal{X}_i$, may not be a measurable function (section) on $\mathbb{R}^n$. For example, if $c_i = 1$ for all $i$, then $\sum_i \nabla \mathcal{X}_i = \nabla \mathbf{1}_\Omega$ is a non-zero distribution supported on the boundary of $\Omega$ (a measure if $\Omega$ has locally finite perimeter). One needs to renormalize the series to make it converge in the distribution sense.

Here we give correct renormalizations of $h$. A first one is obtained right away from differentiation of $g$:

$$h = -\sum_i (f - c_i)\nabla \mathcal{X}_i.$$

The convergence in the distributional sense in $\mathbb{R}^n$ is in fact hidden of [Aus1].

A second one is

$$h = -\sum_m \left( \sum_i (c_m - c_i) \nabla \mathcal{X}_i \right) \mathcal{X}_m.$$

This representation converges in the distributional sense in $\mathbb{R}^n$ and can be shown to be a bounded function.

Let us show how to obtain the second representation in the sense of distributions. Then the proof of boundedness is as in [Aus1]. Take a test function $\phi$ in $\mathbb{R}^n$. Then by definition the distribution $\sum_i \nabla b_i$ tested against $\phi$ is given by

$$\sum_i \int \nabla b_i \ \phi.$$

To compute this, we take a finite subset $J$ of the set $I$ of indices $i$ and we have to pass to the limit in the sum restricted to $J$ as $J \uparrow I$. Because now the sum is finite, and all functions have support in the set $\Omega$, we can introduce $\sum_m \mathcal{X}_m = \mathbf{1_\Omega}$. We have

$$\sum_{i \in J} \int \nabla b_i \ \phi = \sum_m \sum_{i \in J} \int \nabla b_i \ \mathcal{X}_m \ \phi.$$

Now recall that $b_i = (f - c_i)\mathcal{X}_i$ . Call $I_m$ the set of indices such that the support of $\mathcal{X}_i$ meets the support of $\mathcal{X}_m$. By property of the Whitney cubes, $I_m$ is a finite set with bounded cardinal. Hence we can write

$$\sum_m \sum_{i \in J} \int \nabla b_i \ \mathcal{X}_m \ \phi = \sum_m \sum_{i \in J \cap I_m} \int \nabla f \ \mathcal{X}_i \ \mathcal{X}_m \ \phi + \sum_m \sum_{i \in J \cap I_m} \int (f - c_i) \nabla \mathcal{X}_i \ \mathcal{X}_m \ \phi.$$

It is clear that the first sum in the RHS converges to $\int_\Omega \nabla f \ \phi$ as $J \uparrow I$. As for the second it is equal to

$$\sum_m \sum_{i \in J \cap I_m} \int (c_m - c_i) \nabla \mathcal{X}_i \ \mathcal{X}_m \ \phi + \sum_m \sum_{i \in J \cap I_m} \int (f - c_m) \nabla \mathcal{X}_i \ \mathcal{X}_m \ \phi$$

As one can show (with the argument in [Aus1]) that

$$\sum_m \sum_i |c_m - c_i| |\nabla \mathcal{X}_i| |\mathcal{X}_m| \leq C\alpha$$

one has

$$\lim_{J\uparrow I}\sum_m\sum_{i\in J\cap I_m}\int(c_m-c_i)\nabla\mathcal{X}_i\ \mathcal{X}_m\ \phi=-\int h\ \phi$$

where $h$ is defined above.

Finally, write

$$\sum_m\sum_{i\in J\cap I_m}\int(f-c_m)\nabla\mathcal{X}_i\ \mathcal{X}_m\ \phi=\sum_m\int b_mR_{m,J}\ \phi$$

with

$$R_{m,J}=\sum_{i\in J\cap I_m}\nabla\mathcal{X}_i.$$

By construction of the $\mathcal{X}_i$ and properties of Whitney cubes,

$$\sum_{i\in I_m}|\nabla\mathcal{X}_i|\le C\ell_m^{-1}$$

where $\ell_m$ is the length of $Q_m$, and on the support of $\mathcal{X}_m$

$$\sum_{i\in I_m}\nabla\mathcal{X}_i=\sum_{i\in I}\nabla\mathcal{X}_i=0.$$

As $\sum_i|b_m|\ell_m^{-1}$ has been shown to be locally integrable, one can conclude by the Lebesgue dominated convergence theorem that

$$\lim_{J\uparrow I}\sum_m\int b_mR_{m,J}\ \phi=0.$$

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