A concise characterization of 3d simple points

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Abstract

We recall a possible definition of a simple point which uses the digital fundamental group introduced by T.Y. Kong in [1]. Then, we prove that a more concise but not less restrictive definition can be given. Indeed, we prove that there is no need to consider the fundamental group of the complement of an object in order to characterize its simple points. In order to prove this result, we do not use the fact that “the number of tunnels of X is equal to the number of tunnels in $\overline{X}$” but we use the linking number defined in [2]. In so doing, we formalize the proofs of several results stated without proof in the literature (Bertrand, Kong, Morgenthaler).

Key words: Simple point, topology preservation, thinning, fundamental group, linking number.

Introduction

The definition of a *simple point* is the key notion in the context of thinning algorithms. Indeed, this definition leads to the most commonly admitted criterion for checking that a given thinning algorithm preserves the topology of a digital image. Usually, one says that an image $\mathcal{I}_1$ is *topologically equivalent* to an image $\mathcal{I}_2$ if $\mathcal{I}_1$ can be obtained from $\mathcal{I}_2$ by sequential additions or deletions of simple points. Thus, we obtain a convenient definition of topological equivalence [2] in the digital context as soon as we have defined the meaning of

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[2] *topological equivalence* is used here with a different meaning than the one of homeomorphism as in classical topology.

"preserving the topology by removing or adding a single point" (a so called *simple point*). One problem with topology preservation in 3D is that taking care not to change the number of connected components in the image as well as in its background is not sufficient as in the 2D case. In 3D, one must also take care not to change the number and the location of the *tunnels* as donuts have. Thus, we say that a simple closed curve in $\mathbb{Z}^3$ has a tunnel, and this cannot be stated using only connectivity considerations like in the 2D case. Now, different characterizations have been proposed by several authors which all lead to equivalent local characterizations. A first set of characterizations makes use of the Euler characteristic in order to count the number of tunnels, but even if this kind of characterization leads to a *good* local characterization it is limited by the fact that no information about the localization of the tunnels is provided by the Euler characteristic (see Figure 4). Another definition for the words *topology preservation* in the digital case has been proposed by Kong in [3] which is based on remarks made by Morgenthaler in [4], with a new formalism which involves the digital fundamental group [1]. In this latter definition, topology preservation is expressed in terms of the existence of a canonical isomorphism between the fundamental group of the object and the fundamental group of the object without the point to be removed; a similar isomorphism being required for the background of the image. In this paper, we prove that this second condition is in fact implied by the first one. In other words, we show that preserving the tunnels of an object will imply the preservation of the tunnels in its background. In order to prove that such a more concise characterization can be given, we use the linking number between paths of voxels as defined in [2] which provides an efficient way to prove that a given path cannot be homotopic to a degenerated path. Theorem 12 (Section 2) comes with its complete justification and some parts of the proofs given here are the direct answers to some open questions left by Morgenthaler in [4] such as: do any two paths which can be be *continuously* deformed one into each other in an object X keep this property after removal of a simple point of X?

1 Definitions

1.1 Digital image, paths, connectivity

In this paper, we consider objects as subsets of the 3 dimensional space $\mathbb{Z}^3$. The set of points which do not belong to an object $O \subset \mathbb{Z}^3$ constitutes the complement of the object and is denoted by $\overline{O}$. Any point $v = (i, j, k) \in \mathbb{Z}^3$ is identified with a unit cube in $\mathbb{R}^3$ centered at this point: a *voxel* (short for "volume elements"). Now, we can define some binary symmetric and anti-reflexive relations between points by analogy with the following relations between voxels. Two voxels are said $6-$*adjacent* if they share a face, $18-$*adjacent* if they share an edge and $26-$*adjacent* if they share a vertex. By transitive closure of these adjacency relations, we can define another one: connectivity between points. We first define an *n-path* π with a length l from a point a to a point b in $O \subset \mathbb{Z}^3$ as a sequence of points $(y_i)_{i=0\dots l}$ such that for $0 \leq i < l$ the point y_i is n-adjacent or equal to y_{i+1}, with $y_0 = a$ and $y_l = b$. The path π is a *closed path* if $y_0 = y_l$ and is called a *simple path* if $y_i \neq y_j$ when $i \neq j$ (except for y_0 and y_l if the path is closed). The points y_0 and y_l are called the *extremities* of π even in the case when the path is closed and we denote by π^* the set of points of π. An $n-$connected set of points C such that any point $x \in C$ has exactly two $n-$neighbors in C is called a *simple closed $n-$curve*. An simple closed path π such that π^* is a simple closed $n-$curve is called a *parameterized simple closed $n-$curve*. A closed path (x, x) with a length 1 for $x \in \mathbb{Z}^3$ is called a *trivial path*. If x is a point of $\mathbb{Z}^3$ and $n \in \{6, 18, 26\}$ then we denote by $N_n(x)$ the set of points of $\mathbb{Z}^3$ which are $n-$adjacent to x. We call $N_n(x)$ the $n-$neighborhood of x.

Given a path $\pi = (y_k)_{k=0,\dots,l}$, we denote by π^{-1} the sequence $(y'_k)_{k=0,\dots,l}$ such that $y_k = y'_{l-k}$ for $k \in \{0, \dots, l\}$.

Now we can define the connectivity relation: two points a and b are said *n-connected* in an object O if there exists an n-path π from a to b in O. This is an equivalence relation between points of O, and the *$n-$connected components* of an object O are the equivalence classes of points according to this relation. Using this relation on the complement of an object we can define a *background component* of O as an $\overline{n}-$connected component of $\overline{O}$.

In order to avoid topological paradoxes, we always study the topology of an object using an $n-$adjacency for the object and a complementary adjacency $\overline{n}$ for its complement. We sum up this by the use of a pair $(n, \overline{n}) \in \{(6, 26), (6+, 18), (18, 6+), (26, 6)\}$. The notation $6+$ is used in order to distinguish the $6-$connectivity associated to the $26-$connectivity from the $(6+)-$connectivity associated to the $18-$connectivity.

If $\pi = (y_i)_{i=0,\dots,p}$ and $\pi' = (y'_k)_{k=0,\dots,p'}$ are two $n-$paths such that $y_p = y'_0$ then we denote by $\pi.\pi'$ the path $(y_0, \dots, y_{p-1}, y'_0, \dots, y'_{p'})$ which is the concatenation of the two paths π and π'.

1.2 Geodesic neighborhoods and topological numbers

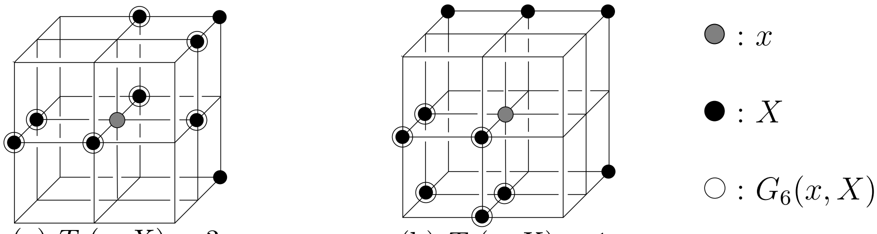
The geodesic neighborhoods have been introduced by Bertrand ([5]) in order to formulate a local characterization of simple points.

Definition 1 (geodesic neighborhood) *Let $x \in X \subset \mathbb{Z}^3$. The* geodesic neighborhood of x in X, *denoted by $G_n(x, X)$, is defined as follows:*

- $G_6(x, X) = (N_6(x) \cap X) \cup \{y \in N_{18}(x) |\ y$ *is* $6-$*adjacent to a point of* $N_6(x) \cap X\}$.
- $G_{26}(x, X) = N_{26}(x) \cap X$.

Definition 2 (topological numbers) *Let $X \subset \mathbb{Z}^3$ and $x \in \mathbb{Z}^3$. The* topological number associated to x and X, *denoted by $T_n(x, X)$ for*

$(n, \overline{n}) \in \{(6, 26), (26, 6)\}$, *is defined as the number of* $n-$*connected components of* $G_n(x, X)$ *(see Figure 1).*



(a) $T_6(x, X) = 3$ (b) $T_6(x, X) = 1$

Fig. 1. Two examples of geodesic neighborhoods and topological numbers.

1.3 Digital fundamental group

In this section, we define the digital fundamental group of a subset X of $\mathbb{Z}^3$ following the definition of Kong in [1] and [6].

First, we need to introduce the $n-$*homotopy relation* between $n-$paths in X. Intuitively, a path π is homotopic to a path π' if π can be "continuously deformed" into π'. Let us consider $X \subset \mathbb{Z}^3$. We introduce the notion of an *elementary* $n-$*deformation*: two closed $n-$paths π and π' in X having the same extremities are *the same up to an elementary* $n-$*deformation (with fixed extremities) in* X, and we denote $\pi \sim_n \pi'$, if they are of the form $\pi = \pi_1.\gamma.\pi_2$ and $\pi = \pi_1.\gamma'.\pi_2$, the $n-$paths γ and γ' having the same extremities and being both included in a 2×2×2 cube if $(n, \overline{n}) = (26, 6)$, in a 2×2 square if $(n, \overline{n}) = (6, 26)$. Then, the two $n-$paths π and π' are said to be $n-$*homotopic (with fixed extremities) in* X if there exists a finite sequence of $n-$paths $\pi = \pi_0, \ldots, \pi_m = \pi'$ such that for $i = 0, \ldots, m-1$ the $n-$paths π_i and π_{i+1} are the same up to an elementary $n-$deformation (with fixed extremities). In this case, we denote $\pi \simeq_n \pi'$. A closed $n-$path $\pi = (x_0, \ldots, x_q = x_0)$ in X is said to be $n-$reducible in X if $\pi \simeq_n (x_0, x_0)$ in X.

Let $B \in X$ be a fixed point of X called the *base point*. We denote by $A^n_B(X)$ the set of all closed $n-$paths $\pi = (x_0, \ldots, x_p)$ which are included in X and such that $x_0 = x_p = B$. The $n-$homotopy relation is an equivalence relation on $A^n_B(X)$, and we denote by $\Pi^n_1(X, B)$ the set of the equivalence classes of this equivalence relation. If $\pi \in A^n_B(X)$, we denote by $[\pi]_{\Pi^n_1(X,B)}$ the equivalence class of π under this relation.

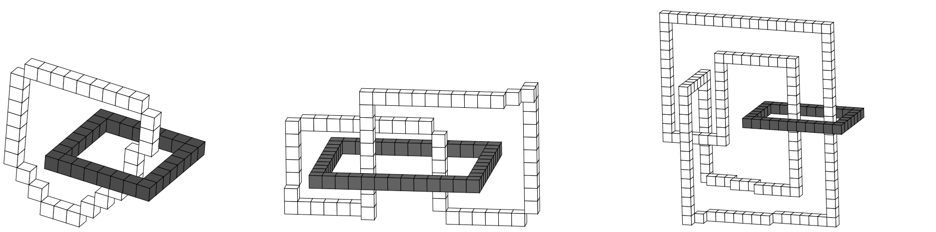
The concatenation of closed $n-$paths is compatible with the $n-$homotopy relation, hence it defines an operation on $\Pi^n_1(X, B)$, which to the class of π_1 and the class of π_2 associates the class of $\pi_1.\pi_2$. This operation provides $\Pi^n_1(X, B)$ with a group structure. We call this group the $n-$*fundamental group of* X *with base point* B. The $n-$fundamental group defined using a point $B' \in X$ as the base point is isomorphic to the $n-$fundamental group defined using another point $B \in X$ as the base point if X is $n-$connected.

Now, let X and Y be such that $Y \subset X \subset \mathbb{Z}^3$ and let $B \in Y$ a the base point.

A closed $n-$path in Y is a particular case of a closed $n-$path in X. In particular, if two closed $n-$paths of Y are $n-$homotopic (with fixed extremities) in Y, then they are $n-$homotopic (with fixed extremities) in X. These two properties enable us to define a canonical morphism $i_* : \Pi_1^n(Y, B) \longrightarrow \Pi_1^n(X, B)$, which we call the morphism induced by the inclusion map $i : Y \longrightarrow X$. To the class of a closed $n-$path $\alpha \in A_B^n(Y)$ in $\Pi_1^n(Y, B)$ the morphism i_* associates the class of the same $n-$path in $\Pi_1^n(X, B)$.

1.4 The digital linking number

The *digital linking number*, denoted by $L_{\pi,\kappa}$, has been defined in [2] for a couple (π, κ) of closed paths of $\mathbb{Z}^3$ which do not intersect each other. It is the digital analogue of the linking number defined in knot theory (see for example [7]) and it is immediately computable (see [2]) for any couple (π, κ) of disjoint paths such that π is an $n-$path and κ is an $\overline{n}-$path with $(n, \overline{n}) \in \{(6, 26), (26, 6), (6+, 18), (18, 6+)\}$ (following the terminology used in knot theory, we call such a couple of paths a *link*). This number counts the number of times two digital closed paths are interlaced one in the other, as illustrated in Figure 2. In this subsection, we recall both the definition of the linking number and the two main theorems from [2].



(a) $L_{\pi,\kappa} = \pm 1$. (b) $L_{\pi,\kappa} = \pm 2$. (c) The Whitehead's link: $L_{\pi,\kappa} = 0$.

Fig. 2. Three different links between a $6-$path π in black and a $18-$path κ in white.

Notation 1 *We will denote by $\mathcal{P}$ the following map:*

$$\begin{aligned} \mathcal{P} \ : \quad \mathbb{Z}^3 &\longrightarrow \mathbb{Z}^2 \\ (i, j, k) &\longmapsto (i, j) \end{aligned}$$

Definition 3 (*Pred* **and** *Succ***)** *Let $\kappa = (x_i)_{i=0,\dots,q}$ be a closed $n-$path and x_i be a point of κ for $i \in \{0, \dots, q\}$. Then, $Succ_\kappa(i)$ is the smallest integer l greater than i such that $\mathcal{P}(x_i) \neq \mathcal{P}(x_l)$; if such an integer l does not exist then $Succ_\kappa(i)$ is the smallest $l < i$ such that $\mathcal{P}(x_i) \neq \mathcal{P}(x_l)$. If in turn such an l does not exist then, clearly $\mathcal{P}(x_i) = \mathcal{P}(x_l)$ for all $l \in \{0, \dots, q\}$ and we define $Succ_\kappa(i) = i$.*
Similarly, $Pred_\kappa(i)$ is the subscript l which precedes i in the cyclic parameterization of κ and such that $\mathcal{P}(x_i) \neq \mathcal{P}(x_l)$, or $Pred_\kappa(i) = i$ if $\mathcal{P}(x_i) = \mathcal{P}(x_l)$ for all $l \in \{0, \dots, q\}$.

Definition 4 (projective movement) *Let $\kappa = (x_i)_{i=0,\ldots,q}$ be an $n-$path and $i \in \{0,\ldots,q\}$. Let V be the $8-$neighborhood of $(0,0)$ in the digital plane, i.e. $V = (\{-1,0,1\}\times\{-1,0,1\})\setminus\{(0,0)\}$. We define the* projective movement *$P_\kappa(i) \in V \times V$ associated to the subscript i of κ by:*

$$P_\kappa(i) \;=\; ((x^1_{Pred_\pi(i)} - x^1_i, x^2_{Pred_\pi(i)} - x^2_i),(x^1_{Succ_\pi(i)} - x^1_i, x^2_{Succ_\pi(i)} - x^2_i))$$

We also denote: $P_\kappa(i)^{Pred} = (x^1_{Pred_\pi(i)} - x^1_i, x^2_{Pred_\pi(i)} - x^2_i)$ *and*

$$P_\kappa(i)^{Succ} = (x^1_{Succ_\pi(i)} - x^1_i, x^2_{Succ_\pi(i)} - x^2_i).$$

The projective movement represents the position of the preceding and the following points of x_i in κ whose projections do not coincide with the projection of x_i. These positions are normalized in a 3×3 grid centered at the point $(0,0)$ which is associated to the projection of x_i. Note that this projective movement will be essentially used when $Pred_\kappa(i) = i-1$.

Definition 5 (left and right) *Let $\kappa = (x_i)_{i=0,\ldots,q}$ be an $n-$path and V be the set introduced in Definition 4. One can parameterize the points of V using the counterclockwise order around the point $(0,0)$. Then, given a projective movement $\mathcal{P} = P_\kappa(i)$, we define the two sets Left($\mathcal{P}$) and Right($\mathcal{P}$) as follows:*

Right($\mathcal{P}$) is the set of points met when looking after points of V from $\mathcal{P}^{Pred}$ to $\mathcal{P}^{Succ}$ following the counterclockwise order on V, excluding $\mathcal{P}^{Succ}$ and $\mathcal{P}^{Pred}$.

Left($\mathcal{P}$) is the set of points met when looking after points of V from $\mathcal{P}^{Succ}$ to $\mathcal{P}^{Pred}$ following the counterclockwise order on V, excluding $\mathcal{P}^{Succ}$ and $\mathcal{P}^{Pred}$.

Example: *If $\mathcal{P} = ((-1,0),(1,-1))$ then Right($\mathcal{P}$) $= \{(-1,-1), (0,-1)\}$ and Left($\mathcal{P}$) $= \{(1,0),(1,1),(0,1),(-1,1)\}$.*

Notation 2 *In the sequel we say that two paths π and κ satisfy the property $\mathcal{H}(\pi,\kappa)$ if π is a closed $n-$path for $n \in \{6, 6+\}$ and κ is closed $\overline{n}-$path such that $\kappa^* \cap \pi^* = \emptyset$.*

Definition 6 (contribution to the linking number) *Let $\pi = (y_k)_{k=0,\ldots,p}$ and $\kappa = (x_i)_{i=0,\ldots,q}$ be two closed paths such that $\mathcal{H}(\pi,\kappa)$ holds. We define as follows $W_{\pi,\kappa}(k,i)$, the* contribution to the linking number *of a couple (k,i), where $0 \le k \le p$ and $0 \le i \le q$.*

- *If the third coordinate of y_k is greater than the third coordinate of x_i, or if $\mathcal{P}(y_k) \neq \mathcal{P}(x_i)$ or $\mathcal{P}(y_k) = \mathcal{P}(y_{k-1})$ or $\mathcal{P}(x_i) = \mathcal{P}(x_{i-1})$ then $W_{\pi,\kappa}(k,i) = 0$,*
- *otherwise, let $\mathcal{P}_\pi = P_\pi(k)$ and $\mathcal{P}_\kappa = P_\kappa(i)$ be the projective movements associated to the subscripts i and k (note that in this case $Pred_\pi(k) = k-1$ and $Pred_\kappa(i) = i-1$):*

– *If $\mathcal{P}^{Pred}_\pi = \mathcal{P}^{Succ}_\pi$ then $W_{\pi,\kappa}(k,i) = 0$,*

– *otherwise $W_{\pi,\kappa}(k,i) = W^-_{\pi,\kappa}(k,i) + W^+_{\pi,\kappa}(k,i)$ where*

$W^-_{\pi,\kappa}(k,i) = -0.5$ *if* $\mathcal{P}^{Pred}_\kappa \in$ *Left*$(\mathcal{P}_\pi)$, $\quad W^+_{\pi,\kappa}(k,i) = 0.5$ *if* $\mathcal{P}^{Succ}_\kappa \in$ *Left*$(\mathcal{P}_\pi)$,

$W^-_{\pi,\kappa}(k,i) = 0.5$ *if* $\mathcal{P}^{Pred}_\kappa \in$ *Right*$(\mathcal{P}_\pi)$, $\quad W^+_{\pi,\kappa}(k,i) = -0.5$ *if* $\mathcal{P}^{Succ}_\kappa \in$ *Right*$(\mathcal{P}_\pi)$,

$W^-_{\pi,\kappa}(k,i) = 0$ *otherwise.* $\quad W^+_{\pi,\kappa}(k,i) = 0$ *otherwise.*

An illustration of the "otherwise" part of the latter definition is given by Figure 3.

Definition 7 (linking number) *Let $\pi = (y_k)_{k=0,\ldots,p}$ and $\kappa = (x_i)_{i=0,\ldots,q}$ be*

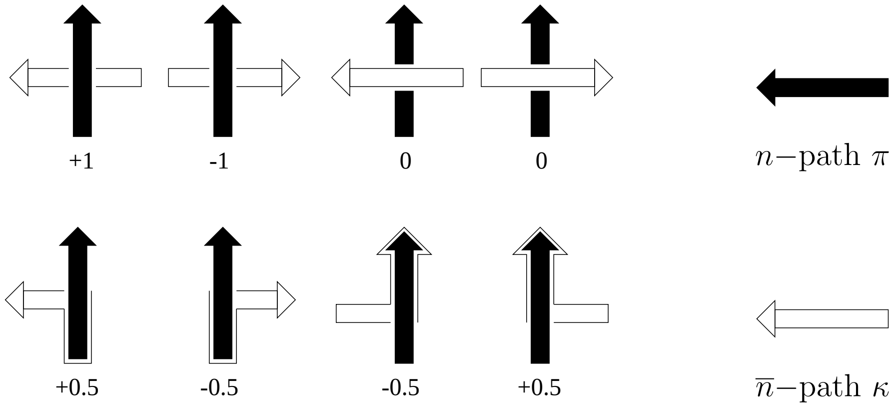


Fig. 3. Contributions associated with points where the two paths of a link overlap in a 2D projection of the link.

two closed paths such that $\mathcal{H}(\pi, \kappa)$ holds. We define the digital linking number *of π and κ (denoted by $L_{\pi,\kappa}$) by:*

$$L_{\pi,\kappa} = \sum_{k=0}^{p-1} \sum_{i=0}^{q-1} W_{\pi,\kappa}(k, i)$$

The two following theorems have been proved in [2] and allows to say that the linking number is a new topological invariant in the field of digital topology.

Theorem 8 *Let π and π' be two closed $n-$path ($n \in \{6, 6+\}$) and κ be a closed $\overline{n}-$path of $\mathbb{Z}^3$ such that $\pi^* \cap \kappa^* = \emptyset$ and $\pi'^* \cap \kappa^* = \emptyset$. If π is $n-$homotopic to π' in $\mathbb{Z}^3 \setminus \kappa^*$ then $L_{\pi,\kappa} = L_{\pi',\kappa}$.*

Theorem 9 *Let π be a closed $n-$path ($n \in \{6, 6+\}$), let κ and κ' be two closed $\overline{n}-$path of $\mathbb{Z}^3$ such that $\pi^* \cap \kappa^* = \emptyset$ and $\pi^* \cap \kappa'^* = \emptyset$. If κ is $\overline{n}-$homotopic to κ' in $\mathbb{Z}^3 \setminus \pi^*$ then $L_{\pi,\kappa'} = L_{\pi,\kappa'}$.*

Remark 1 *It is clear that the linking number can be defined using any 2D projection of a digital link in $\mathbb{Z}^3$. Now, even if the equality (up to the sign) between two linking numbers of a given link computed using two distinct projections has not been proved, it is obvious that the invariance of the linking number can be proved for any projection plane orthogonal to a coordinate axis which one could consider.*

The latter remark allows us to treat configurations of points up to rotations and symmetries

Remark 2 *If κ is a trivial path, then $L_{\kappa,\pi} = 0$ for any closed $n-$path such that $\kappa^* \cap \pi^* = \emptyset$. It follows that if a closed $n-$path κ in $X \subset \mathbb{Z}^3$ is $n-$reducible in X, then $L_{\kappa,\pi} = 0$ for all closed $\overline{n}-$path π in $\overline{\kappa^*}$.*

1.5 Characterization of simple points

A *simple point* for $X \subset \mathbb{Z}^3$ is a point the deletion of which does not change the topology of X. Now, topology preservation in 3D is not as simple to express as in the 2D case because of the existence of *tunnels*. A few authors have used two main tools to study topology preservation: the Euler characteristic which allows to count the number of tunnels of an object (see [8]), and the digital fundamental group ([1]) which allows to "localize" the tunnels. Indeed, as depicted in Figure 4, counting the number of tunnels is not sufficient to characterize the fact that the topology is preserved. In this paper, we are interested by a definition of simple points which uses the digital fundamental group and which avoids the problem previously mentioned. The following definition appears as one of the most convenient for the property "the deletion of x preserves topology of X". It comes from the criterion given in [1] for saying that a thinning algorithm preserves the topology.

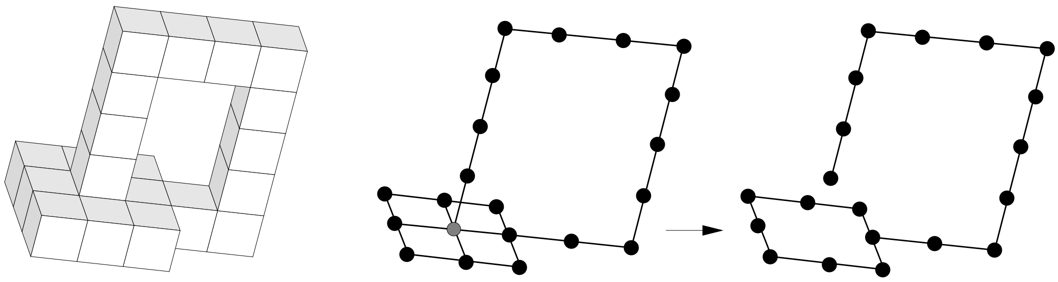


Fig. 4. When $(n, \overline{n}) \in \{(6, 18), (6, 26)\}$, the gray point can be removed without changing the number of tunnels in the object which is equal to 1 in both sets. However, this point is obviously not simple.

Definition 10 *Let $X \subset \mathbb{Z}^3$ and $x \in X$. The point x is said to be $n-$*simple *if:*

i) X and $X \setminus \{x\}$ have the same number of $n-$connected components.

ii) $\overline{X}$ and $\overline{X} \cup \{x\}$ have the same number of $\overline{n}-$connected components.

iii) For each point B in $X \setminus \{x\}$, the group morphism $i_* : \Pi_1^n(X \setminus \{x\}, B) \longrightarrow \Pi_1^n(X, B)$ induced by the inclusion map $i : X \setminus \{x\} \longrightarrow X$ is an isomorphism.

iv) For each point B' in $\overline{X}$, the group morphism $i'_* : \Pi_1^{\overline{n}}(X, B')\overline{X} \longrightarrow \Pi_1^{\overline{n}}(\overline{X} \cup \{x\}, B')$ induced by the inclusion map $i' : \overline{X} \longrightarrow \overline{X} \cup \{x\}$ is an isomorphism.

Bertrand, in [9], gave a local characterization for 3D simple points in term of the number of connected components in geodesic neighborhoods. However, the definition of simple point given in [9] differs from the definition used here since it does not consider any morphism between digital fundamental groups but just require the preservation of cavities and "tunnels". An intermediate purpose of this paper is to prove that the local characterization given by Bertrand is a consequence of the three first conditions of Definition 10 and conversely that the four conditions of this definition are themselves consequences of the local characterization by the topological numbers.

We recall here the characterization given by Bertrand in [9]. Note that the definition of *simple points* used in this proposition slightly differs from Definition 10.

Proposition 11 ([9]) *Let $x \in X$ and $(n, \overline{n}) \in \{(6,26),(26,6)\}$. The point x is a $n-$simple point if and only if $T_n(x, X) = 1$ and $T_{\overline{n}}(x, \overline{X}) = 1$.*

2 A new characterization of 3D simple points

In the sequel of this paper $(n, \overline{n}) \in \{(6,26),(26,6)\}$.
In this section, we state the main result of this paper which is that a not less restrictive criterion for topology preservation is obtained using the only conditions *i)*, *ii)* and *iii)* of Definition 10. In other words, we prove the following theorem:

Theorem 12 *Let $X \subset \mathbb{Z}^3$ and $x \in X$. The point x is $n-$simple if and only if:*

i) *X and $X \setminus \{x\}$ have the same number of connected components.*
ii) *$\overline{X}$ and $\overline{X} \cup \{x\}$ have the same number of connected components.*
iii) *For each point B in $X \setminus \{x\}$, the group morphism $i_* : \Pi_1^n(X \setminus \{x\}, B) \longrightarrow \Pi_1^n(X, B)$ induced by the inclusion map $i : X \setminus \{x\} \longrightarrow X$ is an isomorphism.*

In order to prove this theorem, we first prove (Subsection 2.1) that a point which satisfies the three conditions of Theorem 12 also satisfies the local characterization given by Proposition 11 and then, we show (Subsection 2.2) that this characterization itself implies that the four conditions of Definition 10 are satisfied.

In the sequel, we may suppose without loss of generality that X is an $n-$connected subset of $\mathbb{Z}^3$; and that x and B are two distinct points of X whereas B' is a point of $\overline{X}$. Furthermore, $i_* : \Pi_1^n(X \setminus \{x\}, B) \longrightarrow \Pi_1^n(X, B)$ is the group morphism induced by the inclusion of $X \setminus \{x\}$ in X; and $i'_* : \Pi_1^{\overline{n}}(\overline{X}, B') \longrightarrow \Pi_1^{\overline{n}}(\overline{X} \cup \{x\}, B')$ is the group morphism induced by the inclusion of $\overline{X}$ in $\overline{X} \cup \{x\}$.

Remark 3 *We shall admit the basic property that, if $Y \subset X$ are $n-$connected subsets of $\mathbb{Z}^3$, the group morphism from $\Pi_1^n(Y, B)$ to $\Pi_1^n(X, B)$ induced by the inclusion of Y in X for a base point $B \in Y$ is an isomorphism if and only if the group morphism between $\Pi_1^n(Y, B')$ and $\Pi_1^n(X, B')$ is an isomorphism for any base point $B' \in Y$.*

2.1 First step of the proof of Theorem 12

The purpose of this section is to prove the following proposition.

Proposition 13 *If the conditions i), ii) and iii) of Definition 10 are satisfied, then $T_n(x, X) = 1$ and $T_{\overline{n}}(x, \overline{X}) = 1$.*

In order to prove this proposition, we introduce several other propositions and lemmas. The proof of the following proposition is adapted from [9] to the formalism used here which involves the digital fundamental group.

Proposition 14 *If $T_n(x,X) \geq 2$, then either an $n-$connected component of X is created by deletion of x, or the morphism i_* is not onto.*

The proof of Proposition 14 will use the following number ν.

Definition 15 *Let C be an $n-$connected component of $G_n(x,X)$ and let α be an $n-$path in X. We define $\nu_n(x,\alpha,C)$ as the number of times α goes from C to x minus the number of time α goes from x to C.*

Lemma 16 *Let C be an $n-$connected component of $G_n(x,X)$ and let α and α' be two closed $n-$paths from p to p in X where $p \in X \setminus \{x\}$. If $\alpha \simeq_n \alpha'$ then $\nu_n(x,\alpha,C) = \nu_n(x,\alpha',C)$.*

Proof of Lemma 16. It is sufficient to prove this lemma when α and α' are the same up to an elementary $n-$deformation. Then, we have $\alpha = \pi_1.\gamma.\pi_2$ and $\alpha' = \pi_1.\gamma'.\pi_2$ where γ and γ' have the same extremities and are included in a common $2{\times}2{\times}2$ cube $\mathcal{C}$ if $(n,\overline{n}) = (26,6)$, in a $2{\times}2$ square if $(n,\overline{n}) = (6,26)$. It is obvious that $\nu_n(x,\alpha,C) - \nu_n(x,\alpha',C) = \nu_n(x,\gamma,C) - \nu_n(x,\gamma',C)$.

• Case $(6,26)$: In this case, $\mathcal{C}$ is a $2{\times}2$ square. If $x \notin \mathcal{C}$ then it is clear that $\nu_6(x,\gamma,C) = \nu_6(x,\gamma',C) = 0$. If $x \in \mathcal{C}$ and $\mathcal{C} \cap C = \emptyset$ then $\nu_6(x,\gamma,C) = \nu_6(x,\gamma',C) = 0$.

Now, if $x \in \mathcal{C}$ and $\mathcal{C} \cap C \neq \emptyset$ then let a and b be the two extremities of γ and γ'.

If one point of $X \cap \mathcal{C}$ is $6-$adjacent to x, then since $\mathcal{C} \cap C \neq \emptyset$ it follows that this point belongs to C. In this case, $\nu_6(x,\gamma,C) = \nu_6(x,\gamma',C) = 0$ if $a = b = x$ or $\{a,b\} \subset C$; $\nu_6(x,\gamma,C) = \nu_6(x,\gamma',C) = +1$ if $a \in C$ and $b = x$; and $\nu_6(x,\gamma,C) = \nu_6(x,\gamma',C) = -1$ if $a = x$ and $b \in C$.

If two points of $X \cap \mathcal{C}$ are $6-$adjacent to x and these two points belong to C then $\nu_6(x,\gamma,C) = \nu_6(x,\gamma',C) = -1$ if $a \in C$ and $b = x$; $\nu_6(x,\gamma,C) = \nu_6(x,\gamma',C) = +1$ if $a = x$ and $b \in C$; $\nu_6(x,\gamma,C) = \nu_6(x,\gamma',C) = 0$ if $a = b = x$ or $\{a,b\} \in C$.

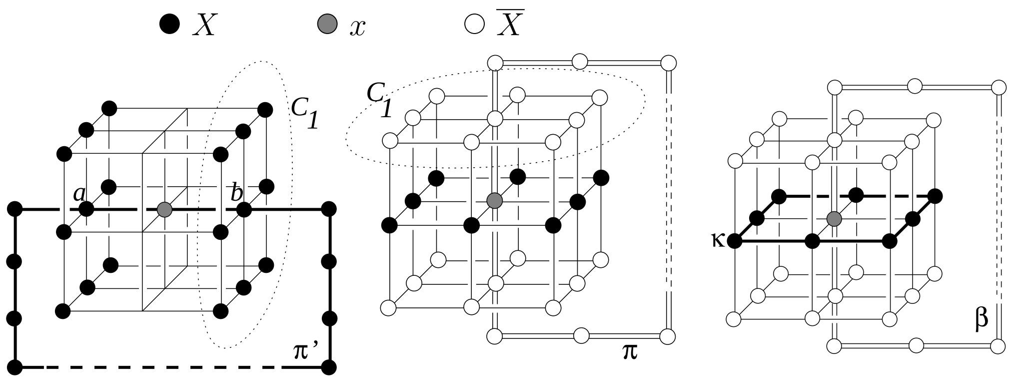
If two points of $X \cap \mathcal{C}$ are $6-$adjacent to x and only one of these points, say d, belongs to C, then the remaining point r of $\mathcal{C}$ which is $18-$adjacent but not $6-$adjacent to x cannot be in X and so nor in C. It follows that γ and γ' are both included in $\{x,d,r\}$ and that $\nu_6(x,\gamma,C) = \nu_6(x,\gamma',C)$. Finally, in all case we have $\nu_6(x,\gamma,C) = \nu_6(x,\gamma',C)$ so that $\nu_6(x,\alpha,C) = \nu_6(x,\alpha',C)$.

• Case $(26,6)$ If $x \notin \mathcal{C}$ then it is clear that $\nu_{26}(x,\gamma,C) = \nu_{26}(x,\gamma',C) = 0$. If $x \in \mathcal{C}$ and $\mathcal{C} \cap C = \emptyset$ then $\nu_{26}(x,\gamma,C) = \nu_{26}(x,\gamma',C) = 0$. Now, if $x \in \mathcal{C}$ and $\mathcal{C} \cap C \neq \emptyset$ then $(\mathcal{C} \cap X) \subset C$ so γ,γ' are contained in $C \cup \{x\}$. Let a and b be the two extremities of γ and γ'.

If $a = b = x$ then $\nu_{26}(x,\gamma,C) = \nu_{26}(x,\gamma',C) = 0$. In the case when $a = x$ and $b \in C$ we have $\nu_{26}(x,\gamma,C) = \nu_{26}(x,\gamma',C) = -1$. If $a \in C$ and $b = x$ then $\nu_{26}(x,\gamma,C) = \nu_{26}(x,\gamma',C) = +1$. And, if $\{a,b\} \subset C$ we have $\nu_{26}(x,\gamma,C) = \nu_{26}(x,\gamma',C) = 0$. Eventually, in all case we have $\nu_{26}(x,\gamma,C) = \nu_{26}(x,\gamma',C)$ so that $\nu_{26}(x,\alpha,C) = \nu_{26}(x,\alpha',C)$. $\square$

Proof of Proposition 14. Let C_1 and C_2 be two $n-$connected components of $G_n(x, X)$ which are $n-$adjacent to x. If C_1 and C_2 are not $n-$connected in $X \setminus \{x\}$, since they are $n-$connected in X then a new $n-$connected component is created by deletion of x.

Now, suppose that C_1 and C_2 are $n-$connected in $X \setminus \{x\}$. Let a and b be two points of X which are $n-$adjacent to x and such that $a \in C_1$ and $b \in C_2$. Thus, there exists an $n-$path π from a to b in $X \setminus \{x\}$. Now, let π' be the closed $n-$path $(a).\pi.(b, x, a)$ (see an example of such a path π' and component C_1 in Figure 5(a)). It is clear that $\nu_n(x, \pi', C_1) = +1$ since $x \notin \pi^*$. Suppose that there exists in $A_n^a(X \setminus \{x\})$ a closed $n-$path α such that $i_*([\alpha]_{\Pi_1^n(X\setminus\{x\},a)}) = [\alpha]_{\Pi_1^n(X,a)} = [\pi']_{\Pi_1^n(X,a)}$. Then, α would be $n-$homotopic to π' in X, but since $\alpha \in A_n^a(X \setminus \{x\})$ it follows that $\nu_n(x, \alpha, C_1) = 0$ whereas $\nu_n(x, \pi', C_1) = +1$ from the very construction of the path π'. From Lemma 16 it follows that α cannot be $n-$homotopic to π' and then the morphism j_* : $\Pi_1^n(X \setminus \{x\}, a) \longrightarrow \Pi_1^n(X, a)$ induced by the inclusion of $X \setminus \{x\}$ in X is not onto. Finally, following Remark 3, the morphism i_* cannot be onto. □



(a) $\nu(x, \pi', C_1) = \pm 1$ (b) $\nu(x, \pi', C_1) = \pm 1$

Fig. 5. Illustrations of the proofs of Propositions 14 and 19

$L_{\kappa,\beta} = \pm 1$

Fig. 6. Idea of the proof of Proposition 18

Proposition 17 *If $T_{\overline{n}}(x, \overline{X}) = 0$ then an $\overline{n}-$connected component of $\overline{X}$ is created by deletion of x.*

Proof. If $T_{\overline{n}}(x, \overline{X}) = 0$, then no point of $\overline{X}$ is $\overline{n}-$adjacent to x so that x becomes an $\overline{n}-$connected component of $\overline{X \setminus \{x\}}$. □

Proposition 18 *If $T_n(x, X) = 1$ and $T_{\overline{n}}(x, \overline{X}) \geq 2$ then two $\overline{n}-$connected component of $\overline{X}$ are merged by deletion of x or i_* is not one to one.*

The main idea of this paper is to use the linking number in order to prove Proposition 18. Indeed, until this paper and the possible use of the linking number, one could prove that when $T_n(x, X) = 1$ and $T_{\overline{n}}(x, \overline{X}) \geq 2$ and no $\overline{n}-$connected component of $\overline{X}$ are merged by deletion of x then the morphism i_*' is not onto. In other words, "a tunnel is created in $\overline{X} \cup \{x\}$". Indeed, a similar proof to Proposition 14 leads to the following proposition (see Figure 5(b)).

Proposition 19 *If $T_n(x,X) = 1$ and $T_{\overline{n}}(x,\overline{X}) \geq 2$ then two $\overline{n}-$connected component of $\overline{X}$ are merged by deletion of x or the morphism i'_* is not onto.*

In this paper, we show that in this case "a tunnel is created in $X \setminus \{x\}$" or more formally, i_* is not one to one. This is proved using the linking number as illustrated in Figure 6. In this figure, the closed path κ is reducible in X (Lemma 26 below) whereas it is not reducible in $X \setminus \{x\}$ since $L_{\kappa,\beta} = \pm 1$ (Remark 2). This shows that a condition on the preservation of tunnels in the object (Condition *iii* of Definition 10) is sufficient to ensure that tunnels of the complement are also left unchanged. And the proof of this result is obtained with the only formalism provided by the use of the digital fundamental group. Before proving Proposition 18, we must state several lemmas.

Definition 20 ($6-$extremity point) *Let x be a point of $Z \subset \mathbb{Z}^3$, then x is called* a $6-$extremity point of Z *if x has exactly one $6-$neighbor in Z.*

Definition 21 (set $K_6(y, X, C)$) *Let $y \in X$ such that $T_6(y,X) = 1$ and $T_{26}(y,\overline{X}) \geq 2$. Let $A = G_6(y,X)$, which is $6-$connected, and C be one of the $26-$connected components of $G_{26}(y,\overline{X})$. Then, $K_6^0(y,X,C)$ is the set of points of A which are $26-$adjacent to a point of C. We define $K_6(y,X,C)$ as the set obtained after recursive deletions of $6-$extremity points in K_6^0.*

Definition 22 ($26-$bold point) *Let y be a point of X, then y is a* $26-$bold point in X *if all the points of X which are $26-$adjacent to y are included in a common 2×2×2 cube.*

Definition 23 (set $K_{26}(y, X, C)$) *Let $y \in X$ such that $T_{26}(y,X) = 1$ and $T_6(y,\overline{X}) \geq 2$. Let $A = G_{26}(y,X)$, which is $26-$connected, and C be one of the $6-$connected components of $G_6(y,\overline{X})$. Then, $K_{26}^0(y,X,C)$ is the set of points of A which are $6-$adjacent to a point of C. We define $K_{26}(y,X,C)$ as the set obtained after iterative deletions of $26-$bold points in K_{26}^0.*

Lemma 24 *If $T_n(x,X) = 1$ and $T_{\overline{n}}(x,\overline{X}) \geq 2$, then there exists an $\overline{n}-$connected component C of $G_{\overline{n}}(x,\overline{X})$ such that $K_n(x,X,C)$ is a simple closed $n-$curve.*

Proof. In order to prove this Lemma, we have investigated using a computer all the 2^{26} possible configurations of $N_{26}(x)$. For each configuration such that $T_n(x,X) = 1$ and $T_{\overline{n}}(x,\overline{X}) \geq 2$ (there are 34653792 such configurations if $(n,\overline{n}) = (26,6)$ and 4398983 for the case $(n,\overline{n}) = (6,26)$), we have computed the different $\overline{n}-$connected components C_i of $G_{\overline{n}}(x,\overline{X})$ and checked that for at least one of them, the set $K_n(x,X,C_i)$, which can be computed following Definition 21 or Definition 23, was a simple closed $n-$curve. □

Lemma 25 *Let $x \in X$ such that $T_n(x,X) = 1$ and $T_{\overline{n}}(x,\overline{X}) \geq 2$ and let $A = G_n(x,X)$. Then there exists a parameterized simple closed $n-$curve κ in A and a closed $\overline{n}-$path $\beta = (a).\beta'.(b,x,a)$ such that:*

- $\beta^* \subset \overline{N_{26}(x) \cap X}$,
- *a and b are the only points of β' in $N_{26}(x)$,*
- *If $(n,\overline{n}) = (6,26)$ then $L_{\kappa,\beta} = \pm 1$ and if $(n,\overline{n}) = (26,6)$ then $L_{\beta,\kappa} = \pm 1$.*

Proof of Lemma 25 in the case $(6, 26)$**.** From Lemma 24, if $T_n(x, X) = 1$ and $T_{\overline{n}}(x, \overline{X}) \geq 2$ one can find a simple closed $6-$curve $\kappa = K_6(x, X, C)$ in $G_6(x, X)$ for come C. Furthermore, from the very definition of the set $K_6(x, X, C)$, each point of this curve is $26-$adjacent the $26-$connected component C of $G_{26}(x, \overline{X})$. In Figure 7, we have depicted up to rotations and symmetries all the possible simple closed $6-$curve κ in the $26-$neighborhood of a point x. We should investigate here each kind of curve and show that for each one a convenient simple closed $6-$curve can be found in $G_6(x, X)$ together with a closed $26-$path β in $\overline{X} \cup \{x\}$ which satisfy the properties of Lemma 25. However, due to length considerations, we only give here the way to find such a path β in the case of Figure 7(a). The remaining cases are left to the reader which can check them easily using similar considerations.

Case of Figure 7(a)

From the definition of $K_6(x, X, C)$, each point of κ must be $26-$adjacent to C. Then, two cases may occur: either C is constituted by the unique point z or not. If C is reduced to the point z, then since $T_{\overline{n}}(x, \overline{X}) \geq 2$, at least one of the remaining "not black" points must belong to some connected component of $G_{26}(x, \overline{X}) = N_{26}(x) \cap X$ different from $C = \{z\}$. Let u be such a point, then it is clear that u and z can be connected by a $26-$path β' in $\overline{N_{26}(x) \cap X}$ such that $L_{\kappa,(x,u).\beta'.(z,x)} = \pm 1$ as depicted in Figure 8(a) where κ is the set of black points of Figure 7(a). In this figure, it is clear that the only couple of subscripts of κ and $\beta = (x, u).\beta'.(z, x)$ which have a contribution (see Definition 6) different from 0 is the couple corresponding to the point x in β and a in κ (the projection plane being orthogonal to the vector $a - x$). Now, from the definition of this contribution, we have $L_{\kappa,\beta} = \pm 1$.

If $z \notin C$ and $z \notin X$ then z constitutes a $26-$connected component of $G_{26}(x, \overline{X})$ and in this case it can be linked to any point of C by a path β' such that the path $\beta = (x, u).\beta'.(z, x)$ satisfies the properties of Lemma 25 with the simple closed $6-$curve κ made of the black points of Figure 7(a).

Finally the case when $z \notin C$ and $z \in X$ remains. In this case, since any point of $K_6(x, X, C)$ must be $26-$adjacent to C, and from the fact that $G_{26}(x, \overline{X})$ must have two connected components, one of the connected components must be reduced to the point t of Figure 7(a). Now, it follows that all the points of $N_{26}(x) \cap N_{18}(t)$ must belong to X. Otherwise, it is clear that t would be $26-$adjacent to C; indeed for any point v of $N_{26}(x) \cap N_{18}(t)$ it is possible to find a point w of the $6-$path κ such that any point of $N_{26}(x) \setminus (\kappa^* \cup \{z\})$ which is $26-$adjacent to w is also $26-$adjacent to v. We obtain the configuration depicted in Figure 8(b). Now, let κ' be the simple closed $6-$curve constituted by the $18-$neighbors of t, this curve is included in C since C is connected and all its points belong to $G_6(x, X)$. Furthermore, some of the points represented in dotted lines in Figure 8(b) must not be in X (otherwise, $T_{26}(x, X)$ would be equal to 1). Let u be one of these points; similarly with the previous case, one can construct a $26-$path β' between t and u such that the path $\beta = (x, t).\beta'.(u, x)$ satisfies the properties of Lemma 25 with κ'.

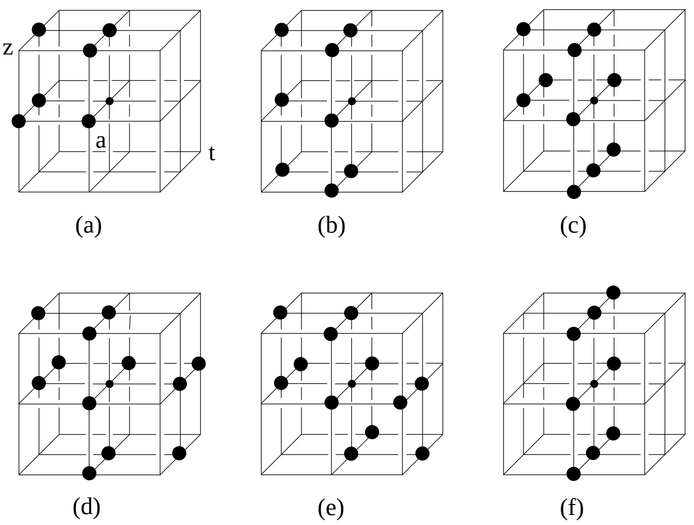


Fig. 7. The possible simple closed 6−curves (in black points) in $N_{18}(x)$ up to rotations and symmetries.

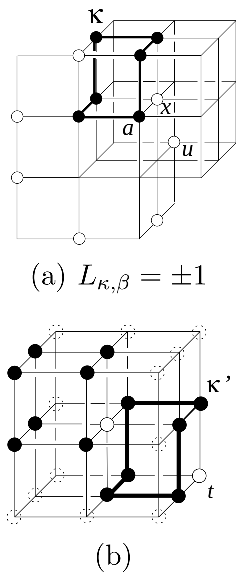


Fig. 8.

□

Proof of Lemma 25 in the case $(26, 6)$**.** From Lemma 24, if $T_{26}(x, X) = 1$ and $T_6(x, \overline{X}) \geq 2$ one can find a simple closed 26−curve $\kappa = K_{26}(x, X, C)$ in $G_{26}(x, X)$. In fact, from the very definition of $K_{26}(x, X, C)$, it is clear that the curve $K_{26}(x, X, C)$ is included in $N_{18}(x)$. Indeed, κ cannot contain any point of $N_{26}(x) \setminus N_{18}(x)$ since obviously such a point would be a bold 26−point which cannot occur in $K_{26}(x, X, C)$.

Furthermore, from the very definition of the set $K_{26}(x, X, C)$, each point of this curve is 6−adjacent to some 6−connected component of $G_6(x, \overline{X})$. In Figure 9, we have depicted up to rotations and symmetries all the possible simple closed 26−curve κ in the 18−neighborhood of a point x. Like in the case $(6, 26)$, we should investigate each kind of curve and show that for each one a convenient simple closed 26−curve can be found in $G_{26}(x, X)$ together with a closed 6−path β in $\overline{X} \cup \{x\}$ which satisfy the properties of Lemma 25. The proof is then similar to the case $(6, 26)$. □

Lemma 26 *Let x be a point of X such that $T_n(x, X) = 1$. Any closed n−path κ in $G_n(x, X)$ is n−reducible in X.*

Proof. Let $\kappa = (y_0, \ldots, y_p)$ with $y_0 = y_p$. If $(n, \overline{n}) = (26, 6)$, then let κ' be the closed path obtained after insertion of the point x in κ between any two consecutive points of κ. It is clear that $\kappa \simeq_{26} \kappa'$ in X since for any two consecutive points of κ, x belongs to a 2×2×2 cube which contains these two points. Now, κ' is of the following form: $\kappa' = (y_0, x, y_1, x, \ldots, x, y_n)$. In κ', each sequence of the form (x, y_i, x) can be reduced to (x) by an elementary 26−deformation. It follows that $\kappa \simeq_{26} \kappa' \simeq_{26} (y_0, x, y_n) \simeq_{26} (y_0, y_n)$.

If $(n, \overline{n}) = (6, 26)$, we first observe that any closed 6−path in $N_{18}(x)$ can be deformed in X into a path which only contains multiple occurrences of the

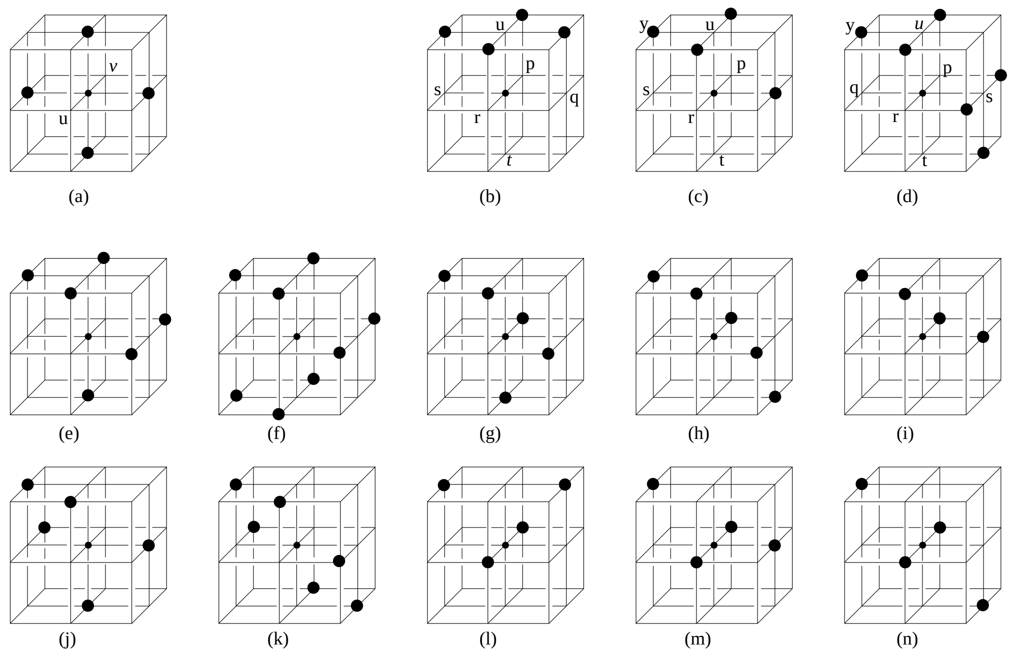


Fig. 9. The possible simple closed $26-$curves in $N_{26}(x)$ up to rotations and symmetries.

point x and $6-$neighbors of x in X. Indeed, any point z of κ which belongs to $N_{18} \setminus N_6(x)$ occurs in a sub-sequence (u, z, v) (note that κ can also be made of a single point of $N_{18}(x) \cap X$). Then, u and v are $6-$neighbors of x and the points u, z, v and x are included in a $2{\times}2$ square. It follows that the sequence (u, z, v) can be replaced by the sequence (u, x, v) in κ by an elementary $6-$deformation. Repeating this deformation for any such point z in κ will lead to a path κ' such that $\kappa'^* \subset \{x\} \cup (N_6(x) \cap X)$ and it is then immediate that $\kappa' \simeq_6 (y_0, y_p)$ in X. $\square$

Proof of Proposition 18. Let x be a point of X such that $T_n(x, X) = 1$ and $T_{\overline{n}}(x, \overline{X}) \geq 2$. Let κ, β' and β be the paths of Lemma 25 and let a and b be the extremity points of β' which are the only two points of β' in $N_{26}(x)$ which are $\overline{n}-$adjacent to x. If a and b are not $\overline{n}-$connected in $\overline{X}$ then it is clear that they are $\overline{n}-$connected in $\overline{X} \cup \{x\}$ so that two $\overline{n}-$connected components of $\overline{X}$ are merged by deletion of x from X.
In the case when a and b are connected by an $\overline{n}-$path α in $\overline{X}$, it is obvious that the two $\overline{n}-$paths β' and α are $\overline{n}-$homotopic with fixed extremities in $\overline{(N_{26}(x) \cap X) \cup \{x\}}$. It follows that β is $\overline{n}-$homotopic to the path $\alpha' = (a).\alpha.(b, x, a)$ in $\overline{(N_{26}(x) \cap X)}$. Since $\overline{(N_{26}(x) \cap X)} \subset \overline{\kappa^*}$ and from Theorem 9 then $L_{\kappa,\beta} = L_{\kappa,\alpha'} = \pm 1$.
From Theorem 8 (and Remark 2), it follows that the path κ is not $n-$reducible in $\overline{\alpha'^*}$ and since $\alpha'^* \subset \overline{X} \cup \{x\}$ then $X \setminus \{x\} \subset \overline{\alpha'^*}$ so that *a fortiori* α' cannot be $n-$reducible in $X \setminus \{x\}$. Formally, if B is the point of $X \setminus \{x\}$ such that κ is a closed $n-$path from B to B, we have $[\kappa]_{\Pi_1^n(X \setminus \{x\}, B)} \neq [(B, B)]_{\Pi_1^n(X \setminus \{x\}, B)}$.

Now, from Lemma 26, $\kappa \simeq_n (B, B)$ in X so that $i_*([\kappa]_{\Pi_1^n(X\setminus\{x\},B)}) = [\kappa]_{\Pi_1^n(X,B)}$ $= [(B,B)]_{\Pi_1^n(X,B)} = i_*([(B,B)]_{\Pi_1^n(X\setminus\{x\},B)})$, i.e., i_* is not one to one. $\square$

Proof of Proposition 13. Suppose that properties *i*), *ii*) and *iii*) of Definition 10 are satisfied.

From Proposition 14 we deduce that if i_* is onto for any point B in $X\setminus\{x\}$, and no $n-$connected component of X is created by deletion of x then $T_n(x, X) < 2$. Furthermore, if no connected component of X is removed then $T_n(x, X) \neq 0$ (indeed, $T_n(x, X) = 0$ means that x constitutes an $n-$connected component of X since no other point of X is $n-$adjacent to x). Finally, $T_n(x, X) = 1$.

From Proposition 18 we deduce that if i_* is one to one and no $\overline{n}-$connected components of $\overline{X}$ are merged by addition of x in $\overline{X}$ then $T_{\overline{n}}(x, \overline{X}) < 2$. Furthermore, if no connected component of $\overline{X}$ is created then $T_{\overline{n}}(x, \overline{X}) \neq 0$ (indeed, $T_{\overline{n}}(x, \overline{X})$ means that no point of $\overline{X}$ is $\overline{n}-$adjacent to x so that x constitutes an $\overline{n}-$connected component of $\overline{X} \cup \{x\}$). Eventually, $T_{\overline{n}}(x, \overline{X}) = 1$. $\square$

2.2 Second step of the proof of Theorem 12

In this section, we prove that properties *i*), *ii*), *iii*) and *iv*) of Definition 10 are satisfied when $T_n(x, X) = T_{\overline{n}}(x, \overline{X}) = 1$.

Proposition 27 *If $T_n(x, X) = 1$ and $T_{\overline{n}}(x, \overline{X}) = 1$, then conditions i), ii), iii) and iv) of Definition 10 are satified.*

In order to prove Proposition 27 we will state several propositions.

Proposition 28 *If X has more $n-$connected components than $X \setminus \{x\}$, then $T_n(x, X) = 0$.*

Proof. If X has more $n-$connected components than $X \setminus \{x\}$, then a connected component of X is removed by deletion of x. It follows that no other point of X can belong to this component. Thus, x has no $n-$neighbor in X and $T_n(x, X) = 0$. $\square$

Proposition 29 *If $X \setminus \{x\}$ has more $n-$connected components than X then $T_n(x, X) \geq 2$.*

Proof. If $X \setminus \{x\}$ has more $n-$connected components than X, at least one connected component of $X \setminus \{x\}$ has been created by deletion of x. In other words, there exist two points a and b in X such that a and b are connected in X but not in $X \setminus \{x\}$. That is to say, every $n-$path between a and b in X contains the point x. Now, suppose that $T_n(x, X) < 2$, then $T_n(x, X)$ cannot be equal to zero since in this case no path between a and b in X could contain x. So, $T_n(x, X) = 1$. In this case, for any $n-$path κ between a and b in X, one can find a path κ' from a to b in $X \setminus \{x\}$. Indeed, for any sequence of the form (y, x, z) in κ, the points y and z both belong to $G_n(x, X)$ which is $n-$connected, so there is an $n-$path in $X \setminus \{x\}$ between y and z. Then, any such sequence (y, x, z) in κ can be replaced by an $n-$path which does not contain x. Finally, a and b are $n-$connected in $X\setminus\{x\}$ which is a contradiction.

Eventually, $T_n(x, X)$ must be greater or equal to 2. $\Box$

Proposition 30 *If $\overline{X}$ has more $\overline{n}-$connected components than $\overline{X} \cup \{x\}$, then $T_{\overline{n}}(x, \overline{X}) \geq 2$.*

Proof. This proof is similar to the proof of Proposition 29. $\Box$

Proposition 31 *If $\overline{X} \cup \{x\}$ has more $\overline{n}-$connected components than $\overline{X}$, then $T_{\overline{n}}(x, \overline{X}) = 0$.*

Proof. This proof is similar to the proof of Proposition 28. $\Box$

Proposition 32 *If $T_n(x, X) = 1$ and $T_{\overline{n}}(x, \overline{X}) = 1$ then i_* is an isomorphism.*

Corollary 33 *If $T_n(x, X) = 1$ and $T_{\overline{n}}(x, \overline{X}) = 1$ then i'_* is an isomorphism.*

Proof of Corollary 33. Let $Y = \overline{X} \cup \{x\}$ and $(m, \overline{m}) = (\overline{n}, n)$. Furthermore, let B' be a point of $\overline{X}$. Then $T_m(x, Y) = 1$, $T_{\overline{m}}(x, \overline{Y}) = 1$ and $B' \in Y \setminus \{x\}$. From Proposition 32, the morphism $j_* : \ \Pi_1^m(Y \setminus \{x\}, B') \longrightarrow \Pi_1^m(Y, B')$ induced by the inclusion map $j : \ Y \setminus \{x\} \longrightarrow Y$ is an isomorphism. But, $Y \setminus \{x\} = \overline{X}$ and $Y = \overline{X} \cup \{x\}$ so $j_* = i_*$ is the morphism induced by the inclusion of $\overline{X}$ in $\overline{X} \cup \{x\}$. $\Box$

In order to prove Proposition 32 we will first state that i_* is onto by proving Lemma 35 and then sate Lemma 40 which will allow us two prove that i_* is one to one.

Lemma 34 *Let a and b be two points of $N_n(x) \cap X$ and suppose that $T_n(x, X)$ $= 1$. Then there exists a simple $n-$path γ between a and b in $G_n(x, X)$ such that $(a, x, b) \sim_n \gamma$ in X.*

Proof. Since $G_n(x, X)$ is $n-$connected, there exists a simple $n-$path $\gamma = (y_0, \ldots, y_k)$ in $G_n(x, X)$ such that $y_0 = a$ and $y_k = b$.

If $(n, \overline{n}) = (26, 6)$, it is clear that the points $a = y_0$, x and y_1 are included in a 2×2×2 cube. Then $(a, x, b) \sim_{26} (a, y_1, x, b)$ and we can repeat this process since two consecutive points y_i and y_{i+1} in γ are always included in a common 2×2×2 cube with x. We obtain that $(a, x, b) \sim_{26} (a, y_1, x, b) \sim_{26} \ldots \sim_{26} (a, y_1, \ldots, y_{k-1}, x, b)$. Finally, $(a, y_1, \ldots, y_{k-1}, x, b) \sim_{26} (a = y_0, y_1, \ldots, y_{k-1}, b = y_k)$.

If $(n, \overline{n}) = (6, 26)$ then we first observe that k is necessarily even. Now, $a = y_0 \in N_6(x) \cap X$ so that $y_1 \in (N_{18}(x) \setminus N_6(x)) \cap X$ and $y_2 \in N_6(x) \cap X$. Then the points y_0, x, y_1 and y_2 are included in a 2×2 square so that $(a = y_0, x) \sim_6 (y_0, y_1, y_2, x)$. This process can be iterated to obtain that $(a, x) \sim_6 (y_0, \ldots y_k, x)$ so that $(a, x, b) \simeq_6 (y_0, \ldots, y_k, x, y_k) \sim_6 (y_0, \ldots, y_k)$. $\Box$

Lemma 35 *If $T_n(x, X) = 1$ and $T_{\overline{n}}(x, \overline{X}) = 1$ then for all $n-$path κ of $A_n^B(X)$, there exists a path κ'in $A_n^B(X \setminus \{x\})$ such that $\kappa \simeq_n \kappa'$ in X.*

Proof. Let $\kappa = (y_0, \ldots, y_q)$ be a closed $n-$path from B to B in X ($B = y_0 = y_l$). For any maximal sequence $(y_i, \ldots, y_j)$ such that $y_{i-1} \neq x$, $y_{j+1} \neq x$ and $y_k = x$ for $k = i, \ldots, j$ it is obvious that $\kappa \simeq_n (y_0, \ldots, y_{i-1}, x, y_{j+1}, \ldots, y_q)$ (observe that $0 < i \leq j < l$). Now, from Lemma 34 and since $\{y_{i-1}, y_{j+1}\} \subset$

$N_n(x)$, then $(y_{i-1}, x, y_{j+1}) \simeq_n \gamma$ in X where γ is a path from y_{i-1} to y_{j+1} in $G_n(x,X)$ so that $x \notin \gamma^*$. Finally, $\kappa \simeq_n (y_0, \ldots, y_{i-1}).\gamma.(y_{j+1}, \ldots, y_q)$. By repeating such an $n-$homotopic deformation for any similar maximal sequence $(y_i, \ldots, y_j)$ in κ, it is clear that κ is $n-$homotopic in X to a closed $n-$path κ' such that $x \notin \kappa'^*$ (i.e., $\kappa' \in A_n^B(X \setminus \{x\})$). $\square$

Lemma 36 *If $T_n(x,X) = 1$ and $T_{\overline{n}}(x, \overline{X}) = 1$ then two paths π_1 and π_2 which have the same extremities and are included in $G_n(x,X)$ are $n-$homotopic with fixed extremities in $N_{26}(x) \cap X$.*

In order to prove Lemma 36 we will use the following lemma.

Lemma 37 *If $T_n(x,X) = 1$ and $T_{\overline{n}}(x, \overline{X}) = 1$, then any simple closed $n-$path in $G_n(x,X)$ is $n-$reducible in $N_{26}(x) \cap X$.*

Corollary 38 *If $T_n(x,X) = 1$ and $T_{\overline{n}}(x, \overline{X}) = 1$ then any closed $n-$path in $G_n(x,X)$ is $n-$reducible in $N_{26}(x) \cap X$.*

Proof of Lemma 37 in the $(6, 26)$ case. In this case, any simple closed $6-$path in $G_6(x,X) \subset N_{18}(x) \cap X$ is in fact a simple closed $6-$curve. In Figure 7 we have depicted up to rotations and symmetries all the possible simple closed $6-$curves in $N_{18}(x)$.

Case of Figure 7(a) Let κ be the set of black points of Figure 7(a). In this case, either $z \in X$ or all points of $N_{26}(x) \setminus (\kappa^* \cup \{z\})$ must belong to X. Indeed, the case when $z \in \overline{X}$ and some point of $N_{26}(x) \setminus (\kappa^* \cup \{z\})$ belongs to $\overline{X}$ contradicts the fact that $T_{26}(x, \overline{X}) = 1$.

Now, if $z \in X$ it is clear that κ is $6-$homotopic in $N_{26}(x) \cap X$ to a path reduced to any of its points, similarly when $z \notin X$ then $N_{26}(x) \setminus (\kappa^* \cup \{z\}) \subset X$ and κ is obviously $6-$homotopic to a path reduced to any of its points.

Case of Figure 7(b) In this case, either $\{r, s, t\} \subset X$ or $N_{26}(x) \setminus (\kappa^* \cup \{r, s, t\}) \subset X$. In both case, we can conclude as in the previous case.

Cases of Figure 7(c),...,(f) are similar to the previous ones. $\square$

Lemma 39 *Let $x \in X$ such that $T_{26}(x,X) = 1$ and $T_6(x, \overline{X}) = 1$ and let κ be the parameterization of a simple closed $26-$curve in $G_{26}(x,X)$. Then κ is $26-$reducible in $G_{26}(x,X)$.*

Proof. In Figure 9 are depicted up to rotations and symmetries all the possible simple closed $26-$curves in $N_{26}(x)$. Now, we must investigate each of them and prove that, under the hypothesis $T_{26}(x,X) = 1$ and $T_6(x, \overline{X}) = 1$, a parameterization of each simple closed curve is $26-$reducible in $G_{26}(x,X)$.

Case of Figure 9(a) In this case, exactly one point of $\{u, v\}$ must belong to $\overline{X}$, indeed $\{u, v\} \subset X$ contradicts the fact that $T_6(x, \overline{X}) = 1$ whereas $\{u, v\} \subset X$ implies that $T_6(x, \overline{X}) = 0$. If $u \in X$ [resp. $v \in X$], it is then obvious that κ is $26-$reducible in $G_{26}(x,X)$.

Case of Figure 9(b) If $u \in X$ then it is clear that the curve κ is $26-$reducible in $G_{26}(x,X)$. If $u \notin X$ then $\{p, q, r, s, t\} \subset X$. Indeed, otherwise $G_6(x, \overline{X})$ would not be $6-$connected. As an example, Figure 10 shows a sequence of elementary $26-$deformations in $G_{26}(x,X)$ which leads from κ to the path reduced to its extremities when κ is the parameterization of the curve which starts en ends at this latter point.

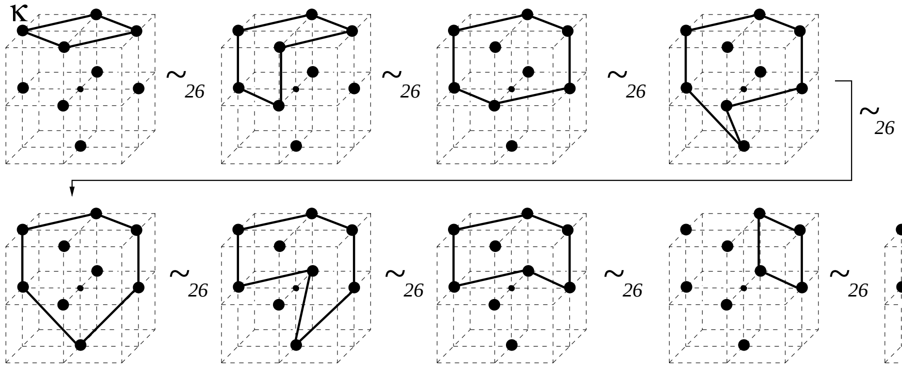


Fig. 10. A 26−homotopic deformation of the closed path κ.

Case of Figure 9(c) Since $T_6(x, \overline{X}) = 1$ we deduce that either $u \in X$ or $\{p, s, r, t\} \subset X$. In both cases, any parameterization κ of the curve is 26−reducible in $G_{26}(x, X)$.

Case of Figure 9(d) In this case, either $\{u, s\} \subset X$ or $\{p, q, r, t\} \subset X$ and we can conclude in both cases that any parameterization κ of the simple closed curve is 26−reducible in $G_{26}(x, X)$.

Cases of Figures 9(e),...,(n) In all these case, we can separate the set $N_6(x) \setminus \kappa^*$ into two sets A and B such that either $A \subset X$ or $B \subset X$. In any case, the inclusion of one of these sets in X allows the 26−deformation of κ into a trivial path in $G_{26}(x, X)$. □

Proof of Lemma 37 in the (26, 6) case. We prove this Lemma by induction on the length of κ. Let $\kappa^0 = \kappa$ and suppose that κ^i is a simple closed 26−path with a length l in $G_{26}(x, X)$ which is n−connected.

First, suppose that there exists in κ^i three consecutive points which are included in a 2×2×2 cube $\mathcal{C}$. In other words, $\kappa^i = \kappa_1.(y, z, t).\kappa_2$ where y, z and t belong to $\mathcal{C}$. Then, $\kappa^i \sim_{26} \kappa^{i+1} = \kappa_1^i.(y, t).\kappa_2^i$ which has a length of $l - 1$.

Now, we suppose that for any sequence (y, z, t) in κ^i, the two points y and t are not 26−adjacent. Furthermore, suppose that there exists in κ^i a point y such that y has more than two 26−adjacent points in ${\kappa^i}^*$. In other words, there exists another point z in κ^i which is neither the successor nor the predecessor of y in κ^i but which is 26−adjacent to y. Then, $\kappa^i = \kappa_1^i.(y).\kappa_2^i.(z).\kappa_3^i$ with $l(\kappa_2^i) > 3$ (indeed, if $l(\kappa_2^i) = 3$ then $\kappa_2^i = (y, u, z)$ where y is 26−adjacent to z). We may suppose that the path κ_2^i is a shortest such subpath of κ^i which can be found satisfying the 26−adjacency property of its extremities. Then, it follows that any point of κ_2^i distinct from y and z has exactly two neighbors in ${\kappa_2^i}^*$: its predecessor and its successor in κ_2. Indeed, the existence of a point of κ_2^i which has more then two 26−adjacent points in ${\kappa_2^i}^*$ would contradict the fact that κ_2^i is a shortest subpath of κ^i whose extremities are 26−adjacent. Furthermore, y [resp. z] has exactly two neighbors in ${\kappa_2^i}^*$: its successor in

κ_2^i and z [resp. its predecessor in κ_2^i and y]. Then, ${\kappa_2^i}^*$ is a simple closed 26−curve and $\kappa_2^i.(z,y)$ is a parameterization of this curve. From Lemma 39, we have $\kappa_2^i.(z,y) \simeq_{26} (y,y)$ in $G_{26}(x,X)$. On the other hand, it is obvious that $\kappa^i \simeq_{26} \kappa_1^i.(y).\kappa_2^i.(z,y,z).\kappa_3^i$ in $G_{26}(x,X)$. Finally $\kappa^i \simeq_{26} \kappa_1^i.(y,z).\kappa_3^i = \kappa^{i+1}$ in $G_{26}(x,X)$ and κ^{i+1} is a simple closed 26−path such that $l(\kappa^{i+1}) < l(\kappa^i)$.

In the remaining case, any point of κ^i has exactly two 26−adjacent points in κ^{i*}. Then, κ^i is a parametrization of a simple closed n−curve and from Lemma 39 is 26−homotopic to a path κ^{i+1} reduced to a single point.

In all cases, κ^i is 26−homotopic to a simple closed 26−path κ^{i+1} such that $l(\kappa^{i+1}) < l(\kappa^i)$. By induction, there exists a path κ^j such that $l(\kappa^j) = 1$ and $\kappa^0 \simeq_{26} \kappa^j$. □

Proof of Corollary 38. If κ is not simple there must exist a simple closed n−path γ from a point $y \in \kappa^*$ to y such that $\kappa = \kappa_1.\gamma.\kappa_2$. Then, from Lemma 37, we have $\gamma \simeq_n (y,y)$ in $G_n(x,X)$ so that $\kappa \simeq \kappa_1.\kappa_2$ in $G_n(x,X)$. Now, we can iterate this process to obtain that κ is n−homotopic to a simple closed path, itself n−homotopic to a path reduced to one point in $G_n(x,X)$. □

Proof of Lemma 36. Let π and π' be two n−paths from a point a to a point b in $G_n(x,X)$. From Corollary 38, $(b,b) \simeq_n \pi_1^{-1}.\pi_2$ so that $\pi_1 \simeq_n \pi_1.\pi_1^{-1}.\pi_2$. Now, it is clear that $\pi_1.\pi_1^{-1} \simeq_n (a,a)$, then $\pi_1.\pi_1^{-1}.\pi_2 \simeq_n \pi_2$. Finally, $\pi_1 \simeq_n \pi_2$ in $G_n(x,X)$. □

Lemma 40 *If $T_n(x,X) = 1$ and $T_{\overline{n}}(x,\overline{X}) = 1$ then two closed n−paths κ and κ' of $A_n^B(X \setminus \{x\})$ which are n−homotopic in X are n−homotopic in $X \setminus \{x\}$.*

Proof. Given a closed n−path κ in $A_n^B(X)$, we denote by $\sigma(\kappa)$ the n−path of $A_n^B(X\setminus\{x\})$ which is n−homotopic to κ in X following the proof of Lemma 35. It is sufficient to prove that if κ and κ' are the same up to an elementary n−deformation in X then the two paths $\sigma(\kappa)$ and $\sigma(\kappa')$ are n−homotopic in $X \setminus \{x\}$. We suppose that $\kappa = \kappa_1.\gamma.\kappa_2$ and $\kappa' = \kappa_1.\gamma'.\kappa_2$ where γ and γ' are two n−path with the same extremities and included in a 2×2×2 cube if $(n,\overline{n}) = (26,6)$, in a 2×2 square if $(n,\overline{n}) = (6,26)$.

If $x \notin \gamma^* \cup \gamma'^*$ we observe that $\sigma(\kappa) = \sigma(\kappa_1).\gamma.\sigma(\kappa_2)$ and $\sigma(\kappa') = \sigma(\kappa_1).\gamma'.\sigma(\kappa_2)$ and then $\kappa \sim_n \kappa'$ in $X \setminus \{x\}$.

If $x \in \gamma^* \cup \gamma'^*$ let a be the last point of κ_1 distinct from x and let b be the first point of κ_2 distinct from x. Then, let δ be the sub-path of κ from a to b and δ' be the sub-path of κ' between a and b. We denote by π_1 the sub-path of κ from its first point to a and by π_2 the sub-path of κ from b to its last point. Finally, we have $\kappa = \pi_1.\delta.\pi_2$ and $\kappa' = \pi_1.\delta'.\pi_2$. Since a and b, the two extremities of δ and δ', are distinct from x, it follows that: $\sigma(\kappa) = \sigma(\pi_1).\sigma(\delta).\sigma(\pi_2)$ and $\sigma(\kappa') = \sigma(\pi_1).\sigma(\delta').\sigma(\pi_2)$.

Now, since $x \in \gamma^* \cup \gamma'^*$ and since γ and γ' are 6−paths [resp. 26−paths] included in a 2×2 square which contains x [resp. a 2×2×2 cube], it is straightforward that γ and γ' are paths included in $G_6(x,X) \cup \{x\}$ [resp. $G_{26}(x,X)$]

and from their construction so are δ and δ'. Now, from the very definition of $\sigma(\delta)$ and $\sigma(\delta')$ (see the proof of Lemma 35) it is straightforward that $\sigma(\delta)$ and $\sigma(\delta')$ are two $n-$paths in $G_n(x, X)$ with same extremities. From Lemma 36, we conclude that $\sigma(\delta) \simeq_n \sigma(\delta')$ in $N_{26}(x) \cap X \subset X \setminus \{x\}$. Finally, $\sigma(\kappa) \simeq_n \sigma(\kappa')$ in $X \setminus \{x\}$. $\square$

Proof of Proposition 32. Let B be a point of $X \setminus \{x\}$. From Lemma 35, for any closed path $\kappa' \in A_n^B(X)$ (thus for any homotopic class of path $[\kappa']_{\Pi_1^n(X,B)}$) there exists a path $\kappa \in A_n^B(X \setminus \{x\})$ such that $\kappa \simeq_n \kappa'$ in X, so that $i_*([\kappa]_{\Pi_1^n(X\setminus\{x\},B)}) = [\kappa]_{\Pi_1^n(X,B)} = [\kappa']_{\Pi_1^n(X,B)}$, i.e, the morphism i_* is onto.
Now, suppose that κ_1 and κ_2 are two closed paths of $A_n^B(X \setminus \{x\})$ such that $[\kappa_1]_{\Pi_1^n(X,B)} = [\kappa_2]_{\Pi_1^n(X,B)}$, where $[\kappa_1]_{\Pi_1^n(X,B)} = i_*([\kappa_1]_{\Pi_1^n(X\setminus\{x\},B)})$ and $[\kappa_2]_{\Pi_1^n(X,B)} = i_*([\kappa_2]_{\Pi_1^n(X\setminus\{x\},B)})$. Then, $\kappa_1 \simeq_n \kappa_2$ in X and from Lemma 40 it follows that $\kappa_1 \simeq_n \kappa_2$ in $X \setminus \{x\}$. Finally, we have $[\kappa_1]_{\Pi_1^n(X\setminus\{x\},B)} = [\kappa_2]_{\Pi_1^n(X\setminus\{x\},B)}$ so that i_* is one to one. $\square$

Proof of Proposition 27. Suppose that $T_n(x, X)$ and $T_{\overline{n}}(x, \overline{X}) = 1$. Following Proposition 28 and Proposition 29, $T_n(x, X) = 1$ implies that condition $i)$ of Definition 10 is satisfied. Furthermore, from Proposition 30 and Proposition 31, $T_{\overline{n}}(x, \overline{X}) = 1$ implies the condition $ii)$ of Definition 10. Finally, from Proposition 32 and Corollary 33, we have $T_n(x, X) = 1$ and $T_{\overline{n}}(x, \overline{X}) = 1 \Rightarrow$ $iii)$ and $iv)$. $\square$

Proof of Theorem 12. Following Definition 10, a simple point obviously satisfies the three conditions of Theorem 12. Now, from Proposition 13, a point which satisfies the three conditions of Theorem 12 is such that $T_n(x, X) = T_{\overline{n}}(x, \overline{X}) = 1$. Finally, from Proposition 27, if $T_n(x, X) = T_{\overline{n}}(x, \overline{X}) = 1$ then x satisfies the four conditions of Definition 10. $\square$

Conclusion

The digital linking number allowed us to formalize in a comprehensive way the characterization of 3D simple points for the complementary adjacency couples $(6, 26)$ and $(26, 6)$. The new theorem which was proved here shows the usefulness of the linking number in order to prove new theorems which involve the digital fundamental group in $\mathbb{Z}^3$.
Now, even if the linking number is well defined for $(n, \overline{n}) \in \{(6+, 18), (18, 6+)\}$, it has not been used yet to provide a characterization of 3D simple points, similar to Theorem 12, for the latter couples of adjacency relations. This, because an open question remains about the existence of a simple closed curve, analogue to the curves $K_6(x, X, C)$ and $K_{26}(x, X, C)$ (Definitions 21 and 23), in this case. Nevertheless, further investigations should allow us to provide a simple process (such as "recursive deletion of $26-$bold points") which leads to the construction of the convenient curve, given a local configuration.

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