Uniform distribution of Galois conjugates and beta-conjugates of a Parry number near the unit circle and dichotomy of Perron numbers

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Abstract. Concentration and equi-distribution, near the unit circle, in Solomyak's set, of the union of the Galois conjugates and the beta-conjugates of a Parry number β are characterized by means of the Erdős-Turán approach, and its improvements by Mignotte and Amoroso, applied to the analytical function $f_\beta(z) = -1 + \sum_{i \geq 1} t_i z^i$ associated with the Rényi β-expansion $d_\beta(1) = 0.t_1 t_2 \ldots$ of unity. Mignotte's discrepancy function requires the knowledge of the factorization of the Parry polynomial of β. This one is investigated using theorems of Cassels, Dobrowolski, Pinner and Vaaler, Smyth, Schinzel in terms of cyclotomic, reciprocal non-cyclotomic and non-reciprocal factors. An upper bound of Mignotte's discrepancy function which arises from the beta-conjugates of β which are roots of cyclotomic factors is linked to the Riemann hypothesis, following Amoroso. An equidistribution limit theorem, following Bilu's theorem, is formulated for the concentration phenomenon of conjugates of Parry numbers near the unit circle. Parry numbers are Perron numbers. Open problems on non-Parry Perron numbers are mentioned in the context of the existence of non-unique factorizations of elements of number fields into irreducible Perron numbers (Lind).

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Contents

1 Introduction

A Perron number is either 1 or a real number $\beta > 1$ which is an algebraic integer such that all its Galois conjugates $\beta^{(i)}$ satisfy: $|\beta^{(i)}| < \beta$ for all $i = 1, 2, \ldots, d-1$, if β is of degree $d \geq 1$ (with $\beta^{(0)} = \beta$). Let $\mathbb{P}$ be the set of Perron numbers. This set $\mathbb{P}$ is partitioned into two disjoint components whose frontier is badly known [V2]. This partitioning arises from the properties of the numeration in base β, i.e. of the β-shifts and the dynamical systems $([0,1], T_\beta)$ [Bl] [Fr1], where β runs over $(1, +\infty)$, and where $T_\beta(x) = \{\beta x\}$ is the beta-transformation ($\lfloor x \rfloor$, $\{x\}$ and $\lceil x \rceil$ denote the integer part, resp. the fractional part, resp. the smallest integer greater than or equal to a real number x). Let us recall this dichotomy and fix some notations.

Let $\beta > 1$ be a real number and assume throughout the paper that β is non-integer. The Rényi β-expansion of 1 is by definition denoted by

$$d_\beta(1) = 0.t_1 t_2 t_3 \ldots \qquad \text{and corresponds to} \qquad 1 = \sum_{i=1}^{+\infty} t_i \beta^{-i}\,, \tag{1.1}$$

where $t_1 = \lfloor \beta \rfloor, t_2 = \lfloor \beta\{\beta\} \rfloor = \lfloor \beta T_\beta(1) \rfloor, t_3 = \lfloor \beta\{\beta\{\beta\}\}\rfloor = \lfloor \beta T_\beta^2(1) \rfloor, \ldots$ The digits t_i belong to the finite alphabet $\mathcal{A}_\beta := \{0, 1, 2, \ldots, \lceil \beta - 1 \rceil\}$. β is said to be a Parry number if $d_\beta(1)$ is finite or ultimately periodic (i.e. eventually periodic); in particular, a Parry number β is said to be simple if $d_\beta(1)$ is finite.

A proof that Parry numbers are Perron numbers is given by Theorem 7.2.13 and Proposition 7.2.21 in Lothaire [Lo]. On the contrary a good proportion of Perron numbers are not Parry numbers, so that the dichotomy of $\mathbb{P}$ can be stated as

$$\mathbb{P} = \mathbb{P}_P \cup \mathbb{P}_a, \tag{1.2}$$

where $\mathbb{P}_P$ denotes the set of Parry numbers and $\mathbb{P}_a$ the set of Perron numbers which are not Parry numbers (a stands for aperiodic). The set $\mathbb{P}_P$ is dense in $(1, +\infty)$ [Pa], contains all Pisot numbers [B] [Bo2] [St] and Salem numbers of degree 4 [Bo1], at least [Bo3] [V2].

Following [V2] the present note continues the exploration of this dichotomy by the Erdős-Turán approach applied to the collection $\mathcal{B} := (f_\beta(z))_{\beta\in\mathbb{P}}$ of analytic functions

$$f_\beta(z) := \sum_{i=0}^{+\infty} t_i z^i \qquad \text{for } \beta \in \mathbb{P}, z \in \mathbb{C}, \tag{1.3}$$

with $t_0 = -1$, canonically associated with the Rényi expansions $d_\beta(1) = 0.t_1t_2t_3\ldots$, for which $f_\beta(z)$ is a rational fraction if and only if $\beta \in \mathbb{P}_P$ (Section 2); when $\beta \in \mathbb{P}_P$ the opposite of the reciprocal polynomial of the numerator of $f_\beta(z)$, namely $n^*_\beta(X)$, is the Parry polynomial of β, a multiple of the minimal polynomial of β.

Section 3 explores the factorization of $n^*_\beta(X)$, the geometry and the equidistribution of its roots near the unit circle when β is a Parry number. These roots are either Galois conjugates or beta-conjugates of β and comparison is made between these two collections of roots. The important points in this exploration are: (i) the discrepancy function as obtained by Mignotte [G] [Mt1] [Mt2] in a theorem which generalizes Erdős-Turán's Theorem, its splitting and its properties following Amoroso [A2] [AM], how the beta-conjugates of β which are roots of unity, as roots of the irreducible cyclotomic factors of $n^*_\beta(X)$, are linked to the Riemann hypothesis (R.H.), (ii) the number of positive real (Galois- or beta-) conjugates of β when the degree of β is large, by comparison with Kac's formula [K] [EK], (iii) upper bounds for the multiplicities of beta-conjugates, (iv) an equidistribution limit theorem in the same formulation as Bilu's Theorem [Bi] for convergent sequences of Parry numbers (or Parry polynomials) with Haar measure on the unit circle as limit measure.

Mignotte's discrepancy function allows a much better strategy than Erdős-Turán's discrepancy function in the Erdős-Turán approach of the Parry polynomial $n^*_\beta(X)$ since it is a subaddtive function on its factorizaton and enables to investigate the roles played by its factors, namely cyclotomic, reciprocal non-cyclotomic and non-reciprocal, term by term.

The factorization of the Parry polynomial $n^*_\beta(X)$ is itself a formidable challenge because of the difficulty of determining the types of its factors, their multiplicities and the way it is correlated to the Rauzy fractal (central tile) (Barat, Berthé, Liardet and Thuswaldner [B-T]). The problem of Lehmer of finding the smallest Mahler measure is essentially equivalent to the problem of estimating the number of irreducible non-cyclotomic factors of $n_{\beta}^{*}(X)$ (Pinner and Vaaler [PV3]).

Section 4 provides various examples on which Erdős-Turán's and Mignotte's discrepancy functions are computed. In Section 5 the case of Perron numbers which are not Parry numbers is evoked in the general context of the arithmetics of Perron numbers where non-unique factorizations on irreducible Perron numbers may occur.

Notations: $\mathrm{N}(\beta) = \prod_{i=0}^{d-1} \beta^{(i)}$ is the algebraic norm of the algebraic number $\beta(= \beta^{(0)})$, of degree $d \geq 1$; $P_{\beta}(X)$ is the minimal polynomial of the algebraic number $\beta > 1$, with positive leading coefficient; $R^{*}(X) = X^{m} R(1/X)$ is the reciprocal polynomial of the polynomial $R(X)$ (of degree m) and $R(X)$ is said reciprocal if $R(X) = R^{*}(X)$; $||Q||_2 = \left(\sum_{j=0}^{m} |\alpha_j|^2\right)^{1/2}$, resp. $||Q||_1 = \mathrm{L}(Q) = \sum_{j=0}^{m} |\alpha_j|$, $\mathrm{H}(Q) = \max_{0 \leq j \leq m} |\alpha_j|$, the 2-norm, resp. the 1-norm (or length), resp. the height, of the polynomial $Q(X) = \sum_{j=0}^{m} \alpha_j X^j$, $\alpha_j \in \mathbb{C}$; $\mathrm{M}(R) = |a_R| \prod_{j=0}^{m} \max\{1, |b_j|\}$ denotes the Mahler measure of the polynomial $R(X) = a_R \prod_{j=0}^{m} (X - b_j) \in \mathbb{C}[X]$ where a_R is the leading coefficient. $D(z_0, r)$ denotes the open disk centred at $z_0 \in \mathbb{C}$ of radius $r > 0$, $\overline{D(z_0, r)}$ its closure. $\mathrm{Log}^{+} x$ ($x > 0$) denotes $\max\{\mathrm{Log}\, x, 0\}$. The constants implied by the Vinogradov symbol '$\ll$' are absolute and computable; when written '$\ll_{\epsilon}$' for $\epsilon > 0$, they depend upon ϵ.

2 Szegő's Theorem and numeration

Every analytical function $f_{\beta}(z)$ with $\beta > 1$ ($\beta \notin \mathbb{N}$) obeys the dichotomy given by the following theorem ([Sg], [Di] p 324–7) since its coefficients belong to $\mathcal{A}_{\beta}$, which is finite.

Theorem 2.1 (Szegő). *A Taylor series $\sum_{n \geq 0} a_n z^n$ with coefficients in a finite subset S of $\mathbb{C}$ is either equal to*

(i) *a rational fraction $U(z) + z^{m+1} \dfrac{V(z)}{1 - z^{p+1}}$ where $U(z) = -1 + \sum_{i=1}^{m} b_i z^i$, $V(z) = \sum_{i=0}^{p} e_i z^i$ are polynomials with coefficients in S and $m \geq 1, p \geq 0$ integers, or*

(ii) *it is an analytic function defined on the open unit disk which is not continued beyond the unit circle (which is its natural boundary).*

Let us recall the Conditions of Parry [Bl] [Fr1] [Lo] [Pa]. Let $\beta > 1$ and $(c_i)_{i\geq 1}$ be the sequence of digits in $\mathcal{A}_\beta$ defined by

$$c_1c_2c_3\ldots := \begin{cases} t_1t_2t_3\ldots & \text{if } d_\beta(1) = 0.t_1t_2\ldots \text{ is infinite,} \\ (t_1t_2\ldots t_{q-1}(t_q-1))^\omega & \text{if } d_\beta(1) \text{ is finite, } = 0.t_1t_2\ldots t_q, \end{cases}$$

where $(\,)^\omega$ means that the word within $(\,)$ is indefinitely repeated. We say that $d_\beta(1)$ is finite if it ends in infinitely many zeros. A sequence $(y_i)_{i\geq 0}$ of elements of $\mathcal{A}_\beta$ (finite or not) is said admissible if and only if

$$(y_j, y_{j+1}, y_{j+2}, \ldots) <_{lex} (c_1c_2, c_3, \ldots) \qquad \text{for all } j \geq 0,$$

where $<_{lex}$ means *lexicographically smaller*. A polynomial $\sum_{j=0}^{n} y_j X^j$, resp. a formal series $\sum_{j=0}^{n} y_j X^j$, is said to be admissible if and only if its coeffciients vector (y_i) is admissible. Since $f_\beta(X) + 1$ is admissible by construction, the proof of Theorem 2.1 (i) by Dienes ([Di] pp 324–327) readily shows that both polynomials $U(X) + 1$ and $V(X)$ are admissible (we except the coefficient $t_0 = -1$ which does not belong to the alphabet $\mathcal{A}_\beta$) when β is a Parry number.

In both cases, every analytical function $f_\beta(z)$ admits $1/\beta$ as simple root since $f'_\beta(1/\beta) = \sum_{i\geq 1} i t_i \beta^{-i+1} > 0$.

The integers $m \geq 1$ and $p+1 \geq 1$ in Theorem 2.1 are respectively the preperiod length and the period length in $d_\beta(1)$ when β is a non-simple Parry number, and we take the natural convention $p+1 = 0$ when β is a simple Parry number (which corresponds to $V(X) \equiv 0$). The case $m = 0$ corresponds to a purely periodic Rényi expansion of unity in base β. Recall the β-transformation T_β: $T_\beta^1 = T_\beta : [0,1] \to [0,1],\ x \to \{\beta x\},\ T_\beta^{j+1}(x) = T_\beta(T_\beta^j(x)),\ j \geq 0$, and $T_\beta^0 = Id$. The sequence $(t_i)_{i\geq 1}$ is related to $(T_\beta^j(1))_{j\geq 0}$ [Fr1] [Lo] by:

$$T_\beta^0(1) = 1,\ T_\beta^j(1) = \beta^j - t_1\beta^{j-1} - t_2\beta^{j-2} - \ldots - t_j \text{ for } j \geq 1.$$

3 Parry polynomials, Galois- and beta-conjugates in Solomyak's set Ω for a Parry number β

3.1 Erdős-Turán's approach and Mignotte's discrepancy splitting

Assume that $\beta > 1$ is a Parry number (of degree $d \geq 2$). Using the notations of Theorem 2.1, the rational fraction $f_\beta(z)$ has

$$U(z), \qquad \text{resp.} \quad U(z)(1 - z^{p+1}) + z^{m+1}V(z),$$

as numerator, when β is a simple, resp. a non-simple, Parry number whose coefficients are in $\mathbb{Z}$ and are of moduli in $\mathcal{A}_\beta$. Since $f_\beta(1/\beta) = 0$, this numerator, say $-n_\beta(z)$, can be factored as

$$-n_\beta(z) = P_\beta^*(z) \times \prod_{j=1}^{s} \left(\Phi_{n_j}(z)\right)^{c_j} \times \prod_{j=1}^{q} \left(\kappa_j^*(z)\right)^{\gamma_j} \times \prod_{j=1}^{u} \left(g_j(z)\right)^{\delta_j} \tag{3.1}$$

where the polynomials $\Phi_{n_j}(X) \in \mathbb{Z}[X]$ are irreducible and cyclotomic, with $n_1 < n_2 < \ldots < n_s$, $\kappa_j(X) \in \mathbb{Z}[X]$ are irreducible and non-reciprocal, $g_j(X) \in \mathbb{Z}[X]$ are irreducible, reciprocal and non-cyclotomic, and $s, q, u \geq 0$, $c_j, \gamma_j, \delta_j \geq 1$, are integers. The Parry polynomial of β [Bo2] [Pa], with leading coefficient 1, is by definition

$$n_\beta^*(X) := P_\beta(X) \left(- \prod_{j=1}^{s} \left(\Phi_{n_j}(z)\right)^{c_j} \prod_{j=1}^{q} \left(\kappa_j(z)\right)^{\gamma_j} \prod_{j=1}^{u} \left(g_j(z)\right)^{\delta_j} \right) \tag{3.2}$$

Its degree, denoted by d_P, is $m+p+1$ with the notations of Theorem 2.1. In the case of non-simple Parry numbers, we have: $n_\beta^*(\beta) = 0 = T_\beta^{m+p+1}(1) - T_\beta^m(1)$ if $m \geq 1$ and $n_\beta^*(\beta) = 0 = T_\beta^{p+1}(1) - T_\beta^0(1) = T_\beta^{p+1}(1) - 1$ if $m = 0$ (pure periodicity) where $p+1$ is the period length. Hence, for non-simple Parry numbers:

$$n_\beta^*(X) = X^{m+p+1} - t_1 X^{m+p} - t_2 X^{m+p-1} - \ldots - t_{m+p} X - t_{m+p+1}$$

$$- X^m + t_1 X^{m-1} + t_2 X^{m-2} + \ldots + t_{m-1} X + t_m \tag{3.3}$$

in the first case and

$$n_\beta^*(X) = X^{p+1} - t_1 X^p - t_2 X^{p-1} - \ldots - t_p X - (1 + t_{p+1}) \tag{3.4}$$

in the case of pure periodicity. For simple Parry numbers, the Parry polynomial is

$$X^m - t_1 X^{m-1} - t_2 X^{m-2} - \ldots - t_{m-1} X - t_m \tag{3.5}$$

with $m \geq 1$. Since $t_1 = \lfloor \beta \rfloor$, the Parry polynomial is a polynomial of small height which has the basic property:

$$\lfloor \beta \rfloor \leq \mathrm{H}(n_\beta^*) \leq \lceil \beta \rceil \tag{3.6}$$

with all coefficients having a modulus $\leq \lfloor \beta \rfloor$ except possibly only one for which the modulus would be equal to $\lceil \beta \rceil$: the coefficient of the monomial of degree m in (3.3) and the constant term in (3.4). If β is a simple Parry number, then the equality

$$\mathrm{H}(n_\beta^*) \;=\; \lfloor \beta \rfloor \tag{3.7}$$

holds by (3.5). The polynomial

$$\frac{n_\beta^*(X)}{P_\beta(X)}$$

is called the complementary factor [Bo2]. It is a monic polynomial and some algebraic properties of $n^*_\beta(X)$ are known (Handelman [H]). A beta-conjugate of β is by definition the inverse ξ of a zero of $f_\beta(z)$, or in other words a zero of the Parry polynomial, which is not a Galois conjugate of β (i.e. $\xi \neq \beta^{(i)}$ for $i = 1, 2, \ldots, d-1$). Saying that the Parry polynomial of β is irreducible is equivalent to saying that β has no beta-conjugate.

Let $\mathcal{B} := \{f(z) = 1 + \sum_{j=1}^{\infty} a_j z^j \mid 0 \leq a_j \leq 1\}$ be the convex set of functions analytic in the open unit disk $D(0,1)$. Let

$$\mathcal{G} := \{\xi \in D(0,1) \mid f(\xi) = 0 \text{ for some } f \in \mathcal{B}\}$$

and $\mathcal{G}^{-1} := \{\xi^{-1} \mid \xi \in \mathcal{G}\}$. The external boundary $\partial\mathcal{G}^{-1}$ of $\mathcal{G}^{-1}$ is a curve which has a cusp at $z = 1$, a spike on the negative real axis, which is the segment $\left[-\frac{1+\sqrt{5}}{2}, -1\right]$, and is fractal at an infinite number of points (Figure 1; see Fig. 2 and §4 in [So]). It defines two components in the complex plane, and inverses of zeros of all $f \in \mathcal{B}$ are all necessarily within the bounded component $\Omega := \mathcal{G}^{-1} \cup \overline{D(0,1)}$.

Theorem 3.1 (Solomyak). *The Galois conjugates ($\neq \beta$) and the beta-conjugates of all Parry numbers β belong to Ω, occupy it densely, and*

$$\mathbb{P}_P \cap \Omega = \emptyset.$$

Proof. By the following identity

$$f_\beta(z) \;=\; -1 + \sum_{i=1}^{\infty} t_i z^i \;=\; (-1+\beta z)\big(1 + \sum_{j=1}^{\infty} T^j_\beta(1) z^j\big), \qquad |z| < 1, \quad (3.8)$$

the zeros $\neq \beta^{-1}$ of $f_\beta(z)$ are those of $1 + \sum_{j=1}^{\infty} T^j_\beta(1) z^j$; but $1 + \sum_{j=1}^{\infty} T^j_\beta(1) z^j$ is a Taylor series which belongs to $\mathcal{B}$. We deduce the claim. □

Let us show that a phenomenon of high concentration and equi-distribution of Galois conjugates ($\neq \beta$) and beta-conjugates of a Parry number β occur by clustering near the unit circle in Ω: from a "radial" viewpoint, using the 2-norm $\|\cdot\|_2$ and the 1-norm $\|\cdot\|_1$ of the Parry polynomial $n^*_\beta(X)$ (Theorem 3.2), and from an "angular" viewpoint by Mignotte's Theorem 3.4 [Mt2]. Though densely distributed in Ω the conjugates of Parry numbers reach a very high concentration close to the unit circle with maximality on the unit circle itself. Section 3.6 formulates limit theorems for this concentration phenomenon.

Theorem 3.2. *Let $\beta > 1$ be a Parry number. Let $\epsilon > 0$ and μ_ϵ the proportion of roots of the Parry polynomial $n^*_\beta(X)$ of β, with $d_P = \deg(n^*_\beta(X)) \geq 1$, which lie in Ω outside the annulus $\left(\overline{D(0,(1-\epsilon)^{-1})} \setminus D(0,(1-\epsilon))\right)$. Then*

$$(i) \quad \mu_\epsilon \;\leq\; \frac{2}{\epsilon\, d_P}\left(\mathrm{Log}\|n^*_\beta\|_2 - \frac{1}{2}\mathrm{Log}\beta\right), \qquad (3.9)$$

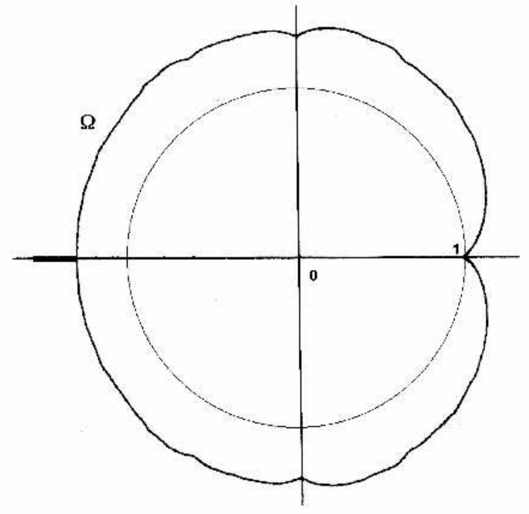

Figure 1. Solomyak's set Ω.

$$(ii) \quad \mu_\epsilon \;\leq\; \frac{2}{\epsilon\, d_P}\left(\mathrm{Log}\|n_\beta^*\|_1 - \frac{1}{2}\mathrm{Log}\big|n_\beta^*(0)\big|\right). \tag{3.10}$$

Proof. (i) Let $\mu_1 d_P$ the number of roots of $n_\beta^*(X)$ outside $\overline{D(0,(1-\epsilon)^{-1})}$ in Ω, except β (since $\beta \notin \Omega$). By Landau's inequality [La]

$$M(f) \leq \|f\|_2 \qquad \text{for } f(x) \in \mathbb{C}[X]$$

applied to $n_\beta^*(X)$ we deduce

$$\beta(1-\epsilon)^{-\mu_1 d_P} \;\leq\; M(n_\beta^*) \;\leq\; \|n_\beta^*\|_2.$$

Hence, since $-\mathrm{Log}(1-\epsilon) \geq \epsilon$,

$$\mu_1 \;\leq\; \frac{1}{\epsilon}\left(\frac{\mathrm{Log}\|n_\beta^*\|_2}{d_P} - \frac{\mathrm{Log}\beta}{d_P}\right).$$

Let $\mu_2 d_P$ the number of roots of $n_\beta^*(X)$ in $D(0,1-\epsilon)$. Then

$$(1-\epsilon)^{-\mu_2 d_P} \leq M(n_\beta) \leq \|n_\beta\|_2 = \|n_\beta^*\|_2$$

by Landau's inequality applied to $n_\beta(X)$. We deduce

$$\mu_2 \;\leq\; \frac{1}{\epsilon}\,\frac{\mathrm{Log}\|n_\beta^*\|_2}{d_P}.$$

Since $\mu_\epsilon = \mu_1 + \mu_2$, we deduce the inequality (3.9).

(ii) Applying Jensen's formula we deduce

$$\frac{1}{2\pi}\int_0^{2\pi} \mathrm{Log}\big|n_\beta^*(e^{i\phi})\big|d\phi - \mathrm{Log}\big|n_\beta^*(0)\big| = \sum_{|b_i|<1} \mathrm{Log}\frac{1}{|b_i|} \tag{3.11}$$

where (b_i) is the collection of zeros of $n_\beta^*(z)$. We have

$$\sum_{|b_i|<1} \mathrm{Log}\frac{1}{|b_i|} \geq \sum_{|b_i|<1-\epsilon} \mathrm{Log}\frac{1}{|b_i|} \geq \epsilon\, \mu_2 d_P.$$

From (3.11), since $\max_{\phi\in[0,2\pi]} \big|n_\beta^*(e^{i\phi})\big| \leq \|n_\beta^*\|_1$, we deduce

$$\mu_2 \leq \frac{1}{\epsilon\, d_P}\left(\mathrm{Log}\|n_\beta^*\|_1 - \mathrm{Log}\big|n_\beta^*(0)\big|\right).$$

Now the roots of $n_\beta(z)$ inside $D(0,1-\epsilon)$ are the roots of $n_\beta^*(z)$ outside the closed disk $\overline{D(0,(1-\epsilon)^{-1})}$, including possibly β, so that their number is $\mu_1 d_P$ or $\mu_1 d_P + 1$. Since $n_\beta^*(X)$ is monic, $|n_\beta(0)| = 1$. We apply Jensen's formula to $n_\beta(z)$ to deduce in a similar way

$$\mu_1 \leq \frac{1}{\epsilon\, d_P}\mathrm{Log}\|n_\beta\|_1$$

Since $\|n_\beta\|_1 = \|n_\beta^*\|_1$ and that $\mu_\epsilon = \mu_1 + \mu_2$, we deduce (3.10). □

Remark 3.3. The terminology "clustering near the unit circle" comes from the following fact: if (β_i) is a sequence of Parry numbers, of Parry polynomials of respective degree $d_{P,i}$ which satisfies

$$\lim_{i\to+\infty} d_{P,i} = +\infty \quad \text{and} \quad \lim_{i\to+\infty} \frac{\mathrm{Log}\,\beta_i}{d_{P,i}} = 0, \tag{3.12}$$

then, since $\|n_{\beta_i}^*\|_2 \leq (d_{P,i}+1)^{1/2}\lceil\beta_i\rceil$, the proportion $\mu_{\epsilon,i}$ relative to β_i satisfies

$$\mu_{\epsilon,i} \leq \frac{1}{\epsilon}\left(\frac{\mathrm{Log}(d_{P,i}+1)}{d_{P,i}} + \frac{\mathrm{Log}\lceil\beta_i\rceil}{d_{P,i}}\right)$$

by (3.9), what shows, for any real number $\epsilon > 0$, that

$$\mu_{\epsilon,i} \to 0, \qquad i \to +\infty. \tag{3.13}$$

The sufficient conditions (3.12) for having convergence of $(\mu_{\epsilon,i})_i$ to zero already cover a large range of examples [Bo3] [V2]. Let us notice that the conditions (3.12) do not imply that

- the corresponding sequence $(d_i)_i$ of the degrees of the minimal polynomials $P_{\beta_i}(X)$ tends to infinity; on the contrary, this sequence may remain bounded, even stationary (cf. the family of Bassino's cubic Pisot numbers in [V2]),

- the family of Parry numbers $(\beta_i)_i$ tends to infinity; it may remain bounded or not (cf. Boyd's family of Pisot numbers less than 2 in [V2]).

Define the radial operator $^{(r)} : \mathbb{Z}[X] \to \mathbb{R}[X]$,

$$R(X) = a_n \prod_{j=0}^{n} (X - b_j) \;\to\; R^{(r)}(X) = \prod_{j=0}^{n} \left(X - \frac{b_j}{|b_j|} \right).$$

All the polynomials in the image of this operator have their roots on the unit circle. This operator leaves invariant cyclotomic polynomials. It has the property: $P^{(r)} = (P^*)^{(r)}$ for all polynomials $P(X) \in \mathbb{Z}[X]$ and is multiplicative: $(P_1P_2)^{(r)} = P_1^{(r)} P_2^{(r)}$ for $P_1(X), P_2(X) \in \mathbb{Z}[X]$.

The (angular) discrepancy relative to the distribution of the (Galois- and beta-) conjugates of β near the unit circle in Ω is given by Erdős-Turán's Theorem [ET] improved by Ganelius [G], Mignotte [Mt2] and Amoroso [A2] [AM], as follows.

Theorem 3.4 (Mignotte). *Let*

$$R(X) = a_n X^n + a_{n-1} X^{n-1} + \ldots + a_1 X + a_0 = a_n \prod_{j=1}^{n} (X - \rho_j e^{i\phi_j}),$$

$$a_n \neq 0, \qquad \rho_1, \rho_2, \ldots, \rho_n > 0,$$

be a polynomial with complex coefficients, where $\phi_j \in [0, 2\pi)$ *for* $j = 1, \ldots, n$. *For* $0 \leq \alpha \leq \eta \leq 2\pi$, *put* $N(\alpha, \eta) = Card\{j \mid \phi_j \in [\alpha, \eta]\}$. *Let* $k = \sum_0^\infty \frac{(-1)^{m-1}}{(2m+1)^2} = 0.916\ldots$ *be Catalan's constant. Then*

$$\left| \frac{1}{n} N(\alpha, \eta) - \frac{\eta - \alpha}{2\pi} \right|^2 \leq \frac{2\pi}{k} \times \frac{\tilde{h}(R)}{n} \tag{3.14}$$

where

$$\tilde{h}(R) = \frac{1}{2\pi} \int_0^{2\pi} \mathrm{Log}^+ |R^{(r)}(e^{i\theta})| d\theta. \tag{3.15}$$

Denote $\mathrm{dis}(R) = \frac{\tilde{h}(R)}{n}$. Let us call Mignotte's discrepancy function the rhs of (3.14) so that

$$C \cdot \mathrm{dis}(R) \;=\; \frac{2\pi}{k} \times \frac{\tilde{h}(R)}{n}$$

with $C = \frac{2\pi}{k} = (2.619...)^2 = 6.859....$ The constant C, the same as in [G], is much smaller than $16^2 = 256$, computed in [ET]. Mignotte shows that C cannot be less than $(1.759...)^2 = 3.094...$ and that $\mathrm{dis}(R)$ gives much smaller numerical estimates (cf. example in Section 4) than the expression

$$\frac{1}{n}\,\mathrm{Log}\frac{L(R)}{\sqrt{|a_0\,a_n|}}$$

proposed by Erdős-Turán instead. In the following we investigate Mignotte's discrepancy function (as in [A2]) but applied to the new class of polynomials

$$\{R^{(r)}(X) \mid R(X) \in \mathcal{PP}\}$$

obtained by the radial operation $^{(r)}$ from the set of Parry polynomials

$$\mathcal{PP} := \{n_\beta^*(X) \mid \beta \in \mathbb{P}_P\}.$$

The angular control of the geometry of the beta-conjugates of a Parry number β with respect to the geometry of its Galois-conjugates, by rotating sectors of suitable opening angles in the unit disk [V2], is best for smallest possible estimates of the discrepancy function.

Mignotte's discrepancy function on $\mathbb{C}[X] \setminus \{0\}$ never takes the value zero for the simple reason that, in (3.14), the function $(\alpha, \eta) \to (\eta - \alpha)/2\pi$ is continuous and that $(\alpha, \eta) \to N(\alpha, \eta)/n$ takes discrete values, so that

$$\mathrm{dis}\bigl(n_\beta^{*\,(r)}\bigr) \;>\; 0 \tag{3.16}$$

for all $\beta \in \mathbb{P}_P$. In Section 3.5 we give a lower estimate of $\mathrm{dis}\bigl(n_\beta^{*\,(r)}\bigr)$.

Schinzel [Sz] has asked a certain number of questions, and reported some conjectures and theorems, on the number of cyclotomic factors, resp. non-cyclotomic non-reciprocal, resp. non-cyclotomic reciprocal factors in the factorization into irreducible factors of a general polynomial with integer coefficients. For giving an upper estimate of $\mathrm{dis}\bigl(n_\beta^{*\,(r)}\bigr)$ we will reformulate these questions in the particular context of Parry polynomials and state some theorems (Section 3.2). Then, as a consequence, we will separate out the contributions relative to the irreducible factors, since Mignotte's discrepancy function allows to do it, by the properties of $\mathrm{Log}^+ x$ and the multiplicativity of the radial operator $^{(r)}$: from (3.1) and (3.15), with $R(X) = n_\beta(X)$ or $R(X) = n_\beta^*(X)$, the splitting is as follows

$$\tilde{h}(n_\beta^*) \;=\; \tilde{h}(n_\beta) \;\le\; \tilde{h}(P_\beta) + \tilde{h}\Bigl(\prod_{j=0}^{s} \Phi_{n_j}^{c_j}\Bigr) + \tilde{h}\Bigl(\prod_{j=0}^{q} \kappa_j^{\gamma_j}\Bigr) + \tilde{h}\Bigl(\prod_{j=0}^{u} g_j^{\delta_j}\Bigr). \tag{3.17}$$

Section 3.3 is relative to the contributions of cyclotomic factors and Section 3.4 to those of non-cyclotomic factors. In Section 4 the numerical optimality of this splitting process is studied on examples.

3.2 Factorization and irreducible factors

Let $\beta > 1$ be a Perron number given by its minimal polynomial $P_\beta(X)$, for which we know that it is a Parry number. The factorization of its Parry polynomial $n^*_\beta(X)$ amounts to the knowledge of:

(i) its degree d_P, as a function of β, its Galois conjugates and d the degree of the minimal polynomial $P_\beta(X)$,

(ii) its distinct irreducible factors and their multiplicities.

The determination of degree $d_P = m + p + 1$ of $n^*_\beta(X)$ (with the notations of Theorem 2.1), which expresses the dynamics of the β-transformation, brings to light the need of the geometrical representation of the β-shift, the Rauzy fractal [PF] [B-T], above the set $\mathbb{Z}_\beta$ of β-integers, for providing upper estimates of it, as shown below for Pisot numbers. For a generic Parry number β, an explicit formula for d_P as a function of β, its Galois conjugates $\beta^{(i)}$ and the degree d of the minimal polynomial $P_\beta(X)$ is probably difficult to obtain in general, if it exists.

Assume that $\beta > 1$ is a Pisot number, of degree $d \geq 2$. We refer to [GV] p. 142 and Lemma 4.6 for details. Let $\{Z_0, Z_1, \ldots, Z_{d-1}\}$ be the canonical basis of $\mathbb{R}^d$ and $x \cdot y = \sum_{j=0}^{d-1} x_j y_j$ the standard scalar product in this basis. Let $B = {}^t(1\ \beta\ \beta^2 \ldots \beta^{d-1})$ and $u_B = \|B\|^{-1} B$, where ^{t} means transposition. Let ${}^tQ$ be the companion matrix of β, π_B is the orthogonal projection mapping onto $\mathbb{R}u_B$. F denotes generically a ${}^tQ$-invariant subspace of $\mathbb{R}^d$, of dimension 1 if the eigenvalue of ${}^tQ$ on F (i.e. one of the Galois conjugates $\beta^{(i)}$) is real, resp. of dimension 2 if it is complex, except the expanding line $\mathbb{R}u$ ([GV] Theorem 3.1 and Theorem 3.5). The projection mapping onto F, $\mathbb{R}^d \to F$, along its ${}^tQ$-invariant complementary space is denoted by π_F. Let $p_2 = \oplus_F \pi_F$ be the sum of the projection mappings. Let $\mathcal{C}' := \left\{\sum_{j=0}^{d-1} \alpha_j Z_j \mid \alpha_j \in [-1, +1] \text{ for all } j = 0, 1, \ldots, d-1\right\}$, $\delta'_F := \max_{x \in \mathcal{C}'} \|\pi_F(x)\|$, $\lambda_F :=$ the absolute value of the Galois conjugate of β associated with F, and $c'_F := \lfloor \beta \rfloor \frac{\delta'_F}{1 - \lambda_F^d}$. The canonical cut-and-project scheme associated with the set $\mathbb{Z}_\beta$ of the beta-integers is

$$\mathbb{R}u_B \xleftarrow{\pi_B} (\mathbb{R}u_B \times D \simeq \mathbb{R}^d, \mathbb{Z}^d) \xrightarrow{p_2} D = \oplus F.$$

Let Ω' be the $(d-1)$-dimensional window $\oplus_F \Omega'_F$ in D, direct sum of the windows Ω'_F on F defined by

$$\Omega'_F = \begin{cases} \text{closed interval centred at 0 in } F \text{ of length } 2c'_F \text{ if } \dim F = 1, \\ \text{closed disk centred at 0 in } F \text{ of radius } c'_F \text{ if } \dim F = 2. \end{cases}$$

This window Ω' is sized at its best on each 1-dim or 2-dim subspace F to contain the 1-dimensional, resp. the 2-dimensional, sections of the central tile of the Rauzy fractal.

Theorem 3.5. *Let $\beta > 1$ be a Pisot number of degree $d \geq 2$. Then*

$$d_P \leq \ \# \Big\{ x \in \mathbb{Z}^d \mid p_2(x) \ \in \ \frac{\mathrm{H}(n_\beta^*)}{\lfloor \beta \rfloor} \, \Omega', \quad \pi_B(x) \cdot u_B \ \in \Big[0, \frac{1}{\|B\|}\Big) \Big\}. \tag{3.18}$$

Proof. The polynomials $T_\beta^j(1)$, for $j = 1, 2, \ldots, d_P$, are all polynomials in β with coefficients having their modulus in $\{0, 1, \ldots, \mathrm{H}(n_\beta^*)\}$, of degree j, which are equal to their fractional part. The collection $(T_\beta^j(1))_{j=1,\ldots,d_P-1}$ is a family of linearly independent polynomials. Theorem 3.5 is then a consequence of Lemma 4.6 in [GV]. □

Remark 3.6. A much better upper bound of d_P would be given by the rhs of (3.18) in which the "box" Ω' is replaced by the true central tile of the Rauzy fractal [PF]. Indeed, this central tile may be disconnected, may contain lots of holes, and its topology is a prominent ingredient for counting points of the lattice $\mathbb{Z}^d$ which are projected by p_2 to this central tile.

Let us now turn to the factorization of $n_\beta^*(X)$ for β a Parry number.

Theorem 3.7. *Let β be a Parry number. If ξ is a beta-conjugate of β which is not a unit, then its multiplicity ν_ξ as root of the Parry polynomial $n_\beta^*(X)$ satisfies:*

$$\nu_\xi \ \leq \ \frac{1}{\log 2} \Big(\log\big(\mathrm{H}(n_\beta^*)\big) - \log |N(\beta)| \Big). \tag{3.19}$$

Moreover, if

$$|N(\beta)| \ \geq \ \frac{\mathrm{H}(n_\beta^*)}{3}, \tag{3.20}$$

then all beta-conjugates of β which are not units (if any) are simple roots of $n_\beta^*(X)$.

Proof. From (3.1), since $P_\beta(X)$ divides $n_\beta^*(X)$ and that $\mathrm{H}(n_\beta^*) \in \{\lfloor \beta \rfloor, \lceil \beta \rceil\}$ (cf. (3.6), (3.7)), we have

$$\Big|\prod_{j=1}^{q} \big(\kappa_j^*(0)\big)^{\gamma_j}\Big| \times \Big|\prod_{j=1}^{u} \big(g_j(0)\big)^{\delta_j}\Big| \ \leq \ \frac{\mathrm{H}(n_\beta^*)}{|N(\beta)|}.$$

As a consequence, if ξ is a beta-conjugate which is not a unit, then, since the irreducible factors of $n_\beta^*(X)$ are all monic, the inequality $|N(\xi)| \geq 2$ implies

$$2^{\nu_\xi} \ \leq \ \frac{\mathrm{H}(n_\beta^*)}{|N(\beta)|}.$$

Hence the claim. Now, if $|N(\beta)| \geq \frac{\mathrm{H}(n_\beta^*)}{3}$ then

$$|\prod_{j=1}^{q} \left(\kappa_j^*(0)\right)^{\gamma_j}| \times |\prod_{j=1}^{u} \left(g_j(0)\right)^{\delta_j}| \leq 3,$$

which necessarily implies that $\nu_\xi = 1$ for each beta-conjugate ξ of β which is not a unit. □

Corollary 3.8. *The beta-conjugates of a Parry number $\beta \in (1,3)$ which are not units are always simple roots of the Parry polynomial of β.*

Proof. Indeed, $\mathrm{H}(n_\beta^*) \in \{\lfloor\beta\rfloor, \lceil\beta\rceil\}$ and $\lceil\beta\rceil \leq 3$. Thus $\mathrm{H}(n_\beta^*)/3 \leq 1$. But $|N(\beta)| \geq 1$ so that (3.20) is satisfied. □

Let β be a Parry number, for which the Parry polynomial $n_\beta^*(X)$ is factored as in (3.2). We have

$1+s+q+u$ =	number of its distinct irreducible factors
$\sum_{j=1}^{s} c_j$ =	number of its cyclotomic (irreducible) factors counted with multiplicities
$1+\sum_{j=1}^{s} c_j + \sum_{j=1}^{q} \gamma_j + \sum_{j=1}^{u} \delta_j$ =	number of its irreducible factors counted with multiplicities
$1+\sum_{j=1}^{q} \gamma_j + \sum_{j=1}^{u} \delta_j$ =	number of its non-cyclotomic irreducible factors counted with multiplicities
$1+q+u$ =	number of its non-cyclotomic irreducible factors counted without multiplicities
$\gamma+\sum_{j=1}^{q} \gamma_j$ =	number of its non-reciprocal irreducible factors counted with multiplicites, with $\gamma = 1$ if $P_\beta(X)$ is non-reciprocal, and $\gamma = 0$ if $P_\beta(X)$ is reciprocal

The remarkable result of Smyth [Sy] implies easily

Theorem 3.9. *For every Parry number β, the inequality*

$$\gamma + \sum_{j=1}^{q} \gamma_j \ < \ \frac{\log \|n_\beta^*\|_2}{\log \theta_0} \tag{3.21}$$

holds where $\theta_0 = 1.3247...$ is the smallest Pisot number, dominant root of $X^3 - X - 1$, where $\gamma = 1$ if $P_\beta(X)$ is non-reciprocal and $\gamma = 0$ if $P_\beta(X)$ is reciprocal.

Let us remark that the upper bound in (3.21) for the number of non-reciprocal irreducible factors of the Parry polynomial of β depends upon its 2-norm, not of the degree d_P, meaning that it is strongly dependent upon the gappiness (lacunarity) of the Rényi β-expansion $d_\beta(1)$ of unity [V1], a strong gappiness leading to a small number of non-reciprocal factors in $n^*_\beta(X)$.

Corollary 3.10. *If β is a Parry number for which the minimal polynomial is non-reciprocal and $d_\beta(1) = 0.t_1t_2t_3\ldots$, of preperiod length $m \geq 0$ and period length $p+1$, satisfies (with $t_0 = -1$)*

$$\left.\begin{array}{ll} \textit{if } \beta \textit{ is simple} & \displaystyle\sum_{j=0}^{m} t_j^2 \\ \textit{if } \beta \textit{ is non-simple} & \displaystyle\sum_{j=0}^{p} t_j^2 + (1+t_{p+1})^2 + \sum_{j=1}^{m}(t_j - t_{p+j+1})^2 \end{array}\right\} \leq \theta_0^4 = 3.0794... \tag{3.22}$$

then the Parry polynomial of β has no non-reciprocal irreducible factor in it.

Proof. Indeed, in this case, $\gamma = 1$; (3.22) is equivalent to

$$\frac{\log \|n^*_\beta\|_2}{\log \theta_0} \leq 2 \quad \Leftrightarrow \quad \|n^*_\beta\|_2^2 \ \leq \ \theta_0^4$$

since $\|n^*_\beta\|_2^2$ is given by (3.3), (3.4) and (3.5). Therefore, from (3.21), we deduce $\sum_{j=1}^{q} \gamma_j = 0$ what implies the claim. □

Let us make explicit (3.22) in the "simple" case. We obtain Parry numbers β for which $d_\beta(1)$ has necessarily the form

$$d_\beta(1) = 0.1\underbrace{00\ldots0}_{\delta}1$$

for some integer $\delta \geq 0$, i.e. $\beta = \beta_\delta$ is the dominant root of the polynomial $X^{\delta+2} - X^{\delta+1} - 1$. The algebraic integers $(\beta_\delta)_{\delta\geq 3}$ are Perron numbers studied by Selmer [V2]. The case $\delta = 0$ corresponds to the golden mean $\tau = (1+\sqrt{5})/2$ since $d_\tau(1) = 0.11$.

The special sequence $(\Phi_{n_j})_{j=1,\ldots,s}$ of cyclotomic polynomials in the factorization of $n^*_\beta(X)$ is such that $\sum_{j=1}^{s} c_j \varphi(n_j) \leq d_P - d$, with $s \leq n_s$, where $\varphi(n)$ is the Euler function, and its determination is complemented by the remarkable Theorem 3 of Schinzel [Sz] which readily leads to

Theorem 3.11. *There exists an absolute constant $C_0 > 0$ such that, for every Parry number β, the number s of distinct cyclotomic irreducible factors of the Parry polynomial of β satisfies*

$$s \ \leq \ C_0 \sqrt{d_P}. \tag{3.23}$$

Concerning the non-cyclotomic factors of $n^*_\beta(X)$, the remarkable Theorem 2 of Dobrowolski [Do] implies

Theorem 3.12. *There exists an absolute constant $C_1 > 0$ such that for every Parry number β and $\epsilon > 0$ an arbitrary positive real number, then*

$$1 + \sum_{j=1}^{q} \gamma_j + \sum_{j=1}^{u} \delta_j \ \leq \ C_1 \Big((\frac{d_P}{\log \|n_\beta^*\|_2^2})^{\epsilon} \times \log \|n_\beta^*\|_2 \Big). \tag{3.24}$$

Both constants C_0, C_1 in (3.23) and in (3.24) are specific to the whole collection of Parry polynomials $\mathcal{PP}$; they are almost surely different from the constants relative to general polynomials with integral coefficients as in [Sz] and [Do] and are expected to be computable when $\mathcal{PP}$ will be characterized.

3.3 Cyclotomic factors

This section is relative to the beta-conjugates χ of β for which the minimal polynomial of χ is cyclotomic. The Möbius function μ is given by, for $n \geq 1$,

$$\mu(n) \ = \begin{cases} \ 0 & \text{if } n \text{ is divisible by a square,} \\ (-1)^k & \text{if } n \text{ is the product of } k \text{ prime numbers.} \end{cases}$$

For $n \geq 1$,

$$X^n - 1 = \prod_{j=1}^{n} \Phi_j(X).$$

By the Möbius inversion formula, the n-th cyclotomic polynomial is

$$\Phi_n(X) := \prod_{j=1}^{n} (X^d - 1)^{\mu(n/j)}.$$

Amoroso in [A1] proves the remarkable result that the assertion that the Riemann zeta function does not vanish for $\mathrm{Re} z \geq \sigma + \epsilon$ is equivalent to the inequality $\tilde{h}\Big(\prod_{n=1}^{N} \Phi_n\Big) \ \ll \ N^{\sigma+\epsilon}$, where σ is the supremum of the real parts of the non-trivial zeros of the Riemann zeta function, and where $\sigma = 1/2$ if the Riemann hypothesis (R.H.) is true. Here we reconsider the same arguments to adapt them to the particular products of cyclotomic polynomials which appear in the factorization of the Parry polynomial $n_\beta^*(X)$ as in (3.2).

Theorem 3.13. *Let $s \geq 1$. Let $\tilde{c} = (c_j)_{j=1,\ldots,s}$ be a collection of positive integers and $n_1 \leq n_2 \leq \ldots \leq n_s$ be an increasing sequence of positive integers. Let $\tilde{n} = (n_j)_{j=1,\ldots,s}$. Assume that the Riemann hypothesis is true. Then there exists a linear form $L_{\tilde{n}}(\tilde{c})$ with rational coefficients such that*

$$\tilde{h}\Big(\prod_{j=1}^{s} \Phi_{n_j}(X)^{c_j} \Big) \ \ll_\epsilon \ \Big| L_{\tilde{n}}(\tilde{c}) \Big| \, {n_s}^{\frac{1}{2}+\epsilon}. \tag{3.25}$$

Proof. Let $N = n_s$. Let

$$G(X) = \prod_{n=1}^{N} \Phi_n(x)^{\sigma_n}$$

with

$$\sigma_n \;=\; \begin{cases} 0 & \text{if } n \notin \{n_1, n_2, \ldots, n_s\} \\ c_j & \text{if } n = n_j \text{ for } j \in \{1, 2, \ldots, s\} \end{cases}$$

for $n \geq 0$. The remarkable Theorem 4.2 of Amoroso in [A2] implies

$$\tilde{h}(G) \;\leq\; \sqrt{\frac{\pi}{12} \sum_{m=1}^{N} \left(\sum_{j|m} \frac{\mu(j)}{j^2} \right) \left(\sum_{n \leq N/m} \sigma_{mn} \sum_{k|n} \frac{\mu(k)k}{n} \right)^2} \tag{3.26}$$

Since $0 \leq \sum_{j|m} \frac{\mu(j)}{j^2} \leq 1$ and that

$$\sum_{n \leq N/m} \sigma_{mn} \sum_{k|n} \frac{\mu(k)k}{n} \quad \text{can be written} \quad \sum_{k=1}^{N/m} L_{k,m}(\tilde{c})\mu(k)$$

where $(L_{k,m}(\tilde{c}))$ are a family of linear forms with rational coefficients, we deduce

$$\sum_{m=1}^{N} \left(\sum_{j|m} \frac{\mu(j)}{j^2} \right) \left(\sum_{n \leq N/m} \sigma_{mn} \sum_{k|n} \frac{\mu(k)k}{n} \right)^2 \;\leq\; \sum_{m=1}^{N} |L_m(\tilde{c})|^2 \left(\sum_{k=1}^{N/m} \mu(k) \right)^2$$

for some linear forms $L_m(\tilde{c})$ deduced from the family $(L_{k,m}(\tilde{c}))$ according to the signs of $\mu(k)$. Let us take $L_{\tilde{n}}(\tilde{c})$ such that $|L_{\tilde{n}}(\tilde{c})| = \sup_{m=1,\ldots,N} |L_m(\tilde{c})|$. The Riemann hypothesis is equivalent (Titchmarsh [T] 14.25C) to

$$\sum_{k \leq x} \mu(k) \;\ll\; x^{\frac{1}{2}+\epsilon} \quad \text{for all real } x \text{ and all } \epsilon > 0.$$

Then there exists a constant $A > 0$ such that, for $\epsilon > 0$,

$$\tilde{h}(G)^2 \;\leq\; \frac{\pi}{12} A \sum_{m=1}^{N} \left(\frac{N}{m} \right)^{2\,(\frac{1}{2}+\epsilon)} \times |L_{\tilde{n}}(\tilde{c})|^2 \;\leq\; \frac{\pi}{12} A \; |L_{\tilde{n}}(\tilde{c})|^2 \; N^{1+2\epsilon} \sum_{m=1}^{\infty} \frac{1}{m^{1+2\epsilon}}.$$

We deduce (3.25). □

Let us turn to making explicit upper bounds of the multiplicities of the primitive roots of unity involved in the product $\prod_{j=1}^{s} \Phi_{n_j}(X)^{c_j}$. Let $n \geq 1$ and ζ_n be a primitive nth-root of unity. Let us write the factorization of $\Phi_n(X)$

in $\mathbb{Q}(\zeta_n)[X]$ as

$$\Phi_n(X) = \prod_{m=1}^{\varphi(n)} \Phi_{n,m}(X).$$

The polynomial $\Phi_{n,m}(X)$ is $X - \xi_m$ for some primitive nth-root ξ_m of unity. Then

$$\prod_{j=1}^{s} \Phi_{n_j}(X)^{c_j} \;=\; \prod_{n=1}^{\infty} \prod_{m=1}^{\varphi(n)} \Phi_{n,m}(X)^{e(n,m)} \tag{3.27}$$

for some integers $e(n,m) \geq 0$. The total number of cyclotomic factors of the Parry polynomial $n_\beta^*(X)$ is then

$$\sum_{j=1}^{s} c_j \varphi(n_j) \;=\; \sum_{n=1}^{\infty} \sum_{m=1}^{\varphi(n)} e(n,m).$$

Let

$$\alpha(\mathbb{Q}) := \lim_{X\to\infty} \frac{1}{X} \sum_{\substack{n=1 \\ \varphi(n)\leq X}}^{+\infty} \varphi(n).$$

We have [PV1]: $\alpha(\mathbb{Q}) = \frac{\zeta(2)\zeta(3)}{\zeta(6)}$ where $\zeta(z)$ is the Riemann zeta function.

At each place v of the number field $\mathbb{K}$ we write $\mathbb{K}_v$ for the completion of $\mathbb{K}$ at v, $\overline{\mathbb{K}}$ for an algebraic closure of $\mathbb{K}_v$ and Ω_v for the completion of $\mathbb{K}_v$. The field Ω_v is complete as a metric space and algebraically closed. Two absolute values $| \; |_v$ and $\| \; \|_v$ are introduced on Ω_v. If $v|\infty$ then $\| \; \|_v$ restricted to $\mathbb{Q}$ is the usual Archimedean absolute value. If p is a prime number and $v|p$ then $\| \; \|_v$ restricted to $\mathbb{Q}$ is the usual p-adic value. They are related by

$$| \; |_v := \| \; \|_v^{[\mathbb{K}_v:\mathbb{Q}_v]/[\mathbb{K}:\mathbb{Q}]}.$$

Now let $F(X) \in \mathbb{Q}[X]$ and introduce the global measure of F as

$$\nu(F) := \prod_v \nu_v(F)$$

where

$$\nu_v(F) := \sup\{|F(z)|_v \;\mid\; z \in \Omega_v \text{ and } |z|_v = 1\}.$$

We can now introduce

$$\mathcal{R} := \max\left\{\frac{d_P}{\operatorname{Log}\nu(n_\beta^*)}, 3\right\}.$$

This quantity plays an important role in the multiplicites of the cyclotomic factors by Theorem 3.14. Its inverse $\mathcal{R}^{-1}$ is surprisingly deeply correlated to the convergence condition (3.12) relative to convergent sequences of Parry numbers by the following inequalities (Pinner and Vaaler [PV1], Lemma 2):

$$\operatorname{Log} \mathrm{H}(n_\beta^*) \ \leq \ \operatorname{Log} \nu(n_\beta^*) \ \leq \ 2\operatorname{Log} \mathrm{H}(n_\beta^*). \tag{3.28}$$

Indeed, since $\mathrm{H}(n_\beta^*) \in \{\lfloor \beta \rfloor, \lceil \beta \rceil\}$ we have the following estimate of the global measure of the Parry polynomial of β:

$$\nu(n_\beta^*) \ \in \ [\lfloor \beta \rfloor, \lceil \beta \rceil^2]$$

and $\mathcal{R}^{-1}$ is roughly equal to $\operatorname{Log} \beta / d_P$ when d_P is large enough compared to $\operatorname{Log} \mathrm{H}(n_\beta^*)$.

The remarkable Theorem 1 of Pinner and Vaaler [PV1] gives a system of four inequalities for sums containing the multiplicities $e(n,m)$, which readily leads to

Theorem 3.14. *Let β be a Parry number for which the Parry polynomial $n_\beta^*(X)$ factors into irreducible polynomials in $\mathbb{Q}[X]$ as in* (3.2) *and* (3.27). *Then*

(i) *for every $\epsilon > 0$ and $\mathcal{R} \ \geq \ \mathcal{R}_0(\epsilon)$,*

$$\sum_{n=1}^{\infty} \frac{1}{\varphi(n)} \sum_{m=1}^{\varphi(n)} e(n,m) \ \leq \ (1+\epsilon)\, d_P \left(\frac{\alpha(\mathbb{Q}) \operatorname{Log} \mathcal{R}}{\mathcal{R}} \right)^{1/2}, \tag{3.29}$$

(ii) *for every $\epsilon > 0$ and $\mathcal{R} \ \geq \ \mathcal{R}_1(\epsilon)$,*

$$\sum_{n=1}^{\infty} \sum_{m=1}^{\varphi(n)} e(n,m) \ \leq \ (1+\epsilon)\, d_P \left(\frac{\alpha(\mathbb{Q}) \operatorname{Log} \mathcal{R}}{\mathcal{R}} \right)^{1/2}, \tag{3.30}$$

(iii) *for each positive integer $n \leq \mathcal{R}$,*

$$\sum_{m \mid n} \sum_{h=1}^{\varphi(m)} e(m,h) \ \ll \ d_P \left(\frac{n}{\mathcal{R}} \right)^{1/2} \tag{3.31}$$

(iv) *for each integer n such that $\varphi(n) \leq \mathcal{R}$,*

$$\sum_{m=1}^{\varphi(n)} e(n,m) \ \ll \ d_P \left(\frac{\varphi(n)}{\mathcal{R}} \right)^{1/2} \left\{ 1 + \left(\frac{\operatorname{Log}\operatorname{Log} 20n}{\operatorname{Log}\left(\frac{\mathcal{R} \operatorname{Log}\operatorname{Log} 20n}{\varphi(n)} \right)} \right)^{1/2} \right\} \tag{3.32}$$

Let us investigate the role played by the gappiness (lacunarity) of the Rényi β-expansion of unity on the number s of distinct cyclotomic factors in $n_\beta^*(X)$.

With the notations of (3.27) we have:

$$s = \sum_{n=1}^{\infty} \sum_{m=1}^{\varphi(n)} \min\{1, e(n,m)\}.$$

Denote by $\tau(n)$ the number of positive divisors of n and define $\pi(m) := \#\{\text{prime number } p \mid p \leq m\}$. We now refer the reader to (3.3), (3.4) or (3.5) for the Parry polynomial of β as a sum of monomials. Define $N_m(n_\beta^*) :=$ the number of monomials in this sum. The remarkable Theorem 1 and Theorem 2 (i) in Pinner and Vaaler [PV2] yield

Theorem 3.15. *Let β be a Parry number for which the Parry polynomial $n_\beta^*(X)$ factors into irreducible polynomials in $\mathbb{Q}[X]$ as in* (3.2) *and* (3.27). *If*

$$n_\beta^*(X) = \sum_{i=1}^{N_m(n_\beta^*)} a_i X^{n_i}, \qquad \textit{with } a_i \neq 0,$$

then the number s of distinct cyclotomic factors of $n_\beta^*(X)$ satisfies

$$(i) \quad \textit{for every } \epsilon > 0, \qquad s \ll_\epsilon \left(d_P\right)^\epsilon N_m(n_\beta^*), \tag{3.33}$$

$$(ii) \quad s \;\leq\; \inf\left\{ \left(\sum_{i=1}^{N_m(n_\beta^*)} \tau(n_i - n_j) \right) 2^{\pi(N_m(n_\beta^*))} \;\middle|\; j \in \{1, \ldots, N_m(n_\beta^*)\} \right\}. \tag{3.34}$$

Theorem 3.15 (i) improves Theorem 3.11: it introduces in the upper bound (3.33) the term $N_m(n_\beta^*)$ for which the quantity $d_P + 1 - N_m(n_\beta^*)$ is an estimate of the gappiness of $d_\beta(1)$ by (3.3) and (3.4), and, by (3.5), of some possible identifications between the digits t_j. Theorem 3.15 (ii) bears an ingredient which does not seem to have been observed for Parry polynomials yet: *the number of monomials in $n_\beta^*(X)$*. Indeed, when it is small, the exponent $\pi(N_m(n_\beta^*))$ of 2 in the upper bound in (3.34) is small, and this may imply a small number of distinct cyclotomic factors by (3.34).

3.4 Non-cyclotomic factors

This section deals with the beta-conjugates χ of β for which the minimal polynomial (of χ) is non-cyclotomic. These minimal polynomials are irreducible factors in the factorization of the Parry polynomial of β: they are either a $g_j(X)$ or a $\kappa_j(X)$ in (3.2). In some cases, when a beta-conjugate χ lies, together with its Galois conjugates, very near the unit circle, then the form of the minimal polynomial of χ can be specified. The remarkable Theorems 1 and 2 in Cassels [C] imply

Theorem 3.16. *If χ is a beta-conjugate of a Parry number β such that the minimal polynomial $g(X)$ of χ is non-reciprocal, with $n = \deg(g)$, if $\chi_1, \ldots, \chi_{n-1}$ denote the Galois conjugates of $\chi = \chi_0$ (which are also beta-conjugates of β), then either*

(i) $|\chi_j| \ > \ 1 + \dfrac{0.1}{n}$ *for at least one $j \in \{0, 1, \ldots, n-1\}$, or*

(ii) $g(X) = -g^*(X)$ *if* $|\chi_j| \ \leq \ 1 + \dfrac{0.1}{n}$ *holds for all $j = 0, 1, \ldots, n-1$.*

In the second case, since $g(X) = \prod_{j=0}^{n-1}(X - \chi_j) = -\prod_{j=0}^{n-1}(1 - \chi_j X)$ is monic, all the beta-conjugates χ_j of β ($j = 0, 1, \ldots, n-1$) are algebraic units, i.e. $|N(\chi_j)| = 1$.

Theorem 3.17. *If χ is a beta-conjugate of a Parry number β such that the minimal polynomial (of degree n) of χ is non-cyclotomic and where $\chi_1, \ldots, \chi_{n-1}$ denote the Galois conjugates of χ ($= \chi_0$), if*

$$|\chi_j| \ \leq \ 1 + \frac{0.1}{n^2} \qquad \textit{for } j = 0, 1, \ldots, n-1, \tag{3.35}$$

then at least one of the beta-conjugates $\chi_0, \chi_1, \ldots, \chi_{n-1}$ of β has absolute value 1.

Theorem 3.16 and Theorem 3.17 are often applicable because beta-conjugates of Parry numbers are highly concentrated near the unit circle.

Theorem 3.18. *Let β be a Parry number with Parry polynomial $n_\beta^*(X)$ factored as in* (3.2)*. Then all its non-cyclotomic irreducible factors $\kappa_j(X)$ (with $j = 1, \ldots, q$) and $g_j(X)$ (with $j = 1, \ldots, u$) have at least one root of modulus* > 1.

Proof. By Kronecker's theorem [Kr], if χ is a beta-conjugate of β which lies in the closed unit disk with all its Galois conjugates, then it would be a root of unity, i.e. a root of one of the cyclotomic factors $\Phi_{n_j}(X)$ in (3.2), and never a root of one of the irreducible factors $\kappa_j(X)$ or $g_j(X)$. Hence if we assume the existence of non-reciprocal irreducible factors and of reciprocal non-cyclotomic factors in the factorization of the Parry polynomial of β, then these factors possess the mentioned property, hence the claim. □

Corollary 3.19. *Let β be a Parry number with Parry polynomial $n_\beta^*(X)$ factored as in* (3.2)*. Then*

$$\#\{\textit{beta-conjugate } \chi \textit{ of } \beta \mid |\chi| > 1\}$$

(counted with multiplicities) is

$$\begin{cases} \geq \ \ \sum_{j=1}^{q} \gamma_j + \sum_{j=1}^{u} \delta_j & \textit{if} \quad q + u > 0, \\ = \ \ 0 & \textit{if} \quad q = u = 0. \end{cases}$$

The computation of an upper bound of Mignotte's discrepancy function on the non-cyclotomic irreducible factors of the Parry polynomial of β will be reported elsewhere.

3.5 Real positive conjugates of a Parry number

For $R(X) \in \mathbb{R}[X]$ let $\mathrm{rp}(R) :=$ be the number of real positive roots of $R(X)$ (counted with multiplicities).

Proposition 3.20. *Let β be a Parry number. We have:* $\mathrm{rp}(n_\beta^*) \geq 1$*, and, if $\chi \in (0,1)$ is a Galois- or a beta-conjugate of β, then either*

(i) $P_\beta(\chi) = 0$*, and if $P_\beta(X)$ is reciprocal then $\chi = \beta^{-1}$ is the only real positive Galois conjugate of β, or*

(ii) $\kappa_j(\chi) = 0$ *for some $j \in \{1, 2, \ldots, q\}$ where $\kappa_j(X)$ is one of the non-reciprocal irreducible factors in the factorization* (3.2) *of $n_\beta^*(X)$.*

Proof. The reciprocal irreducible factors in (3.2) which are cyclotomic polynomials have no root outside $|z| = 1$, therefore cannot cancel at χ. Assume now that an irreducible factor in (3.2) is reciprocal, non-cyclotomic, and cancels at $\chi \in (0,1)$. Let us show that it is impossible. Indeed, it would also have χ^{-1} as conjugate root of χ, hence χ and χ^{-1} would be beta-conjugates of β. Since $\Omega \cap (1, +\infty) = \emptyset$ and that Ω contains all beta-conjugates of all Parry numbers, it would implies the existence of a beta-conjugate of β outside Ω, which is a contradiction. □

Proposition 3.21. *Let β be a Parry number. Then*

$$\left| \frac{\mathrm{rp}(n_\beta^*)}{d_P} \right|^2 \leq \frac{2\pi}{k} \cdot \mathrm{dis}(n_\beta^*). \tag{3.36}$$

Proof. Let us consider the angular sector $0 \leq \alpha \leq 2\pi - \alpha \leq 2\pi$ with $\alpha > 0$ small enough so that its complementary sector only contains β and the real positive conjugates of β. From (3.14), with $\alpha \to 0^+$, we deduce the lower bound (3.36) of Mignotte's discrepancy function. □

There are two natural questions:

(i) given $n \geq 1$ an integer what is the average value of $\mathrm{rp}(n_\beta^*)$ over all Parry polynomials n_β^* of degree $d_P = n$? Denote by $\mathbb{E}_P(n)$ this average value;

(ii) how behaves $\mathbb{E}_P(n)/n$ when n tends to infinity?

The general context of such questions is Kac's formula [K] and its recent improvements [EK]. Let us recall it first. The expected number of real zeros $\mathbb{E}(n)$ of a random polynomial of degree n with real coefficients is given by

$$\mathbb{E}(n) = \frac{2}{\pi} \log(n) + 0.6257358072\ldots + \frac{2}{n\pi} + O(1/n^2) \qquad \text{for } n \to \infty, \tag{3.37}$$

assuming coefficients are following independent standard normal laws. The $\frac{2}{\pi}\log n$ term was obtained by Kac in 1943 and the other terms require integral formulas and their asymptotic series from curves drawn on spheres [EK]. For each $n \geq 1$ these averaging techniques could be adapted to the smaller set of Parry polynomials of degree n, instead of the whole set of polynomials of $\mathbb{R}[X]$ of degree n, and to real positive roots only, to compute $\mathbb{E}_P(n)$, but the set $\mathcal{PP}$ is badly described and is not suitable for this type of computation. At least, since $\lim_{n\to\infty} \mathbb{E}(n)/n = 0$ by (3.37) we could expect

$$\lim_{n\to\infty} \mathbb{E}_P(n)/n = 0. \tag{3.38}$$

It seems that we cannot prove (3.38) yet. However, the remarkable Theorem 4.1 of Borwein, Erdélyi and Kós [B-K] readily implies, for a large class of Parry numbers $\beta \in (1,2)$ which are algebraic units,

Theorem 3.22. *There is an absolute constant $C_3 > 0$ such that, for all Parry numbers β such that*

$$\mathrm{H}(n_\beta^*) = 1 \qquad \textit{and} \qquad |N(\beta)| = 1,$$

the inequality

$$\frac{\mathrm{rp}(n_\beta^*)}{d_P} \;\leq\; C_3 \,\frac{1}{\sqrt{d_P}} \tag{3.39}$$

holds.

When the Parry polynomial is irreducible, the remarkable Theorem in Mignotte [Mt1] p. 83 leads to a result of the same type: it readily implies

Theorem 3.23. *For every Parry number β such that the Parry polynomial $n_\beta^*(X)$ is irreducible, then the inequality*

$$\frac{\mathrm{rp}(n_\beta^*)}{d_P} \;\leq\; \sqrt{\frac{2\pi}{k}}\,\sqrt{3\,\mathrm{Log}(2d_P) + 4\,\mathrm{Log}\,\mathrm{M}(P_\beta)}\,\frac{1}{\sqrt{d_P}} \tag{3.40}$$

holds.

Let us now show that Mignotte's Theorem 3.4 gives in a simple way an upper bound to the multiplicity of a beta-conjugate of a Parry number, valid for any beta-conjugate, complementing then Theorem 3.7, Corollary 3.8 and Theorem 3.14.

Let β be a Parry number with Parry polynomial $n_\beta^*(X)$ of degree d_P. Then the multiplicity q of a beta-conjugate of β satisfies:

$$q \;\leq\; \sqrt{\frac{2\pi}{k}}\sqrt{\mathrm{dis}(n_\beta^*)}\,d_P\,, \tag{3.41}$$

where k is Catalan's constant.

This inequality (3.41) is obtained as follows: let q be the multiplicity of a beta-conjugate $\chi \in \Omega$ of β in the Parry polynomial $n_\beta^*(X)$. Let us write $\chi = re^{i\phi}$ with $r > 0$. Let us take $\eta = \phi + \epsilon/2$, $\alpha = \phi - \epsilon/2$, for $\epsilon > 0$ small enough so that (i) the angular sector

$$\mathcal{S}_{\phi,\epsilon} := \{z \mid \arg(z) \in [\alpha, \eta]\}$$

contains χ, with eventually other beta-conjugates or Galois-conjugates having the same argument ϕ, but does not contain other roots of $n_\beta^*(X)$ having an argument $\neq \phi$, (ii) the angular sector $\mathcal{S}_{0,\epsilon}$ only contains the real positive conjugates of β, including β. By rotating $\mathcal{S}_{0,\epsilon}$ of an angle ϕ and allowing ϵ to tend to 0^+, we obtain from Mignotte's Theorem 3.4

$$\left| \frac{q}{d_P} \right|^2 \leq \frac{2\pi}{k} \operatorname{dis}(n_\beta^*),$$

from which (3.41) is deduced. This proof uses the relation $e^{i\phi}\mathcal{S}_{0,\epsilon} = \mathcal{S}_{\phi,\epsilon}$ with ϵ very small, and the counting processes of the roots of the Parry polynomial at 0 and at ϕ. They are correlated: a large number of real positive conjugates of β leads to large Mignotte's discrepancies as the inequality (3.36) shows it, and this means that the *common* upper bound (3.41) of the multiplicities of the beta-conjugates is probably not very good in this case. This is likely to occur when the number of non-reciprocal irreducible factors in the factorization of the Parry polynomial of β is large, from Proposition 3.20 (ii). However, this type of upper bound is not good from a numerical viewpoint as shown in Section 4, what suggests that Mignotte's approach could be improved.

3.6 An equidistribution limit theorem

Theorem 3.2, Theorem 3.4 and Conditions (3.12) express the "speed of convergence" and the "angular equidistributed character" of the conjugates of a Parry number, towards the unit circle, or of the collection of Galois conjugates and beta-conjugates of a sequence of Parry numbers. So far, the limit of this equidistribution phenomenon is not yet formulated. In which terms should it be done? What is the natural framework for considering at the same time all the conjugates of a Parry number and what is the topology which can be invoked?

In this respect we will follow Bilu's equidistibution limit theorem in Bilu [Bi] [Gr], though the conditions of convergence of Parry numbers are here much more general than those considered by Bilu.

Let β be a Parry number for which all beta-conjugates are simple roots of the Parry polynomial $n_\beta^*(X)$. Let $\mathbb{K}$ be the algebraic number field generated by β, its Galois conjugates and its beta-conjugates over $\mathbb{Q}$. We have the following field extension: $\mathbb{K} \supset \mathbb{Q}(\beta)$ and $\mathbb{K}/\mathbb{Q}$ is Galois. We denote by $\mathbb{K}_v$ the completion of $\mathbb{K}$ for the Archimedean or non-Archimedean place v of $\mathbb{K}$. The absolute logarithmic height of β is defined as:

$$h(\beta) := \frac{1}{[\mathbb{K} : \mathbb{Q}]} \sum_v [\mathbb{K}_v : \mathbb{Q}_v] \max\{0, \mathrm{Log}|\beta|_v\}.$$

Let us now consider the whole set of Galois conjugates $(\beta^{(i)})$ and beta-conjugates (ξ_j) of β. Denote

$$\Delta_\beta := \frac{1}{[\mathbb{K} : \mathbb{Q}]} \sum_{\sigma:\mathbb{K}\to\mathbb{C}} \delta_{\{\sigma(\beta)\}}$$

the weighted sum of the Dirac measures at all the conjugates $\sigma(\beta)$ of β, where σ runs over the d_P $\mathbb{Q}$-automorphisms of $\mathbb{K}$

$$\sigma : \beta \to \beta^{(i)}, \qquad \text{or} \qquad \sigma : \beta \to \xi_j$$

which send β either to one of its Galois conjugates or to one of its beta-conjugates.

Let us recall that a sequence (α_k) of points in $\overline{\mathbb{Q}}^*$ is strict if any proper algebraic subgroup of $\overline{\mathbb{Q}}^*$ contains α_k for only finitely many values of k.

The topology which is used is the following: a sequence of probability measures $\{\mu_k\}$ on a metric space S weakly converges to μ if for any bounded continuous function $f : S \to \mathbb{R}$ we have

$$(f, \mu_k) \to (f, \mu) \qquad \text{as } k \to +\infty.$$

The remarkable Theorem 1.1 in [Bi] readily implies

Theorem 3.24. *Let $(\beta_i)_{i\geq 1}$ be a strict sequence of Parry numbers whose Parry polynomials have all simple roots, and which satisfies*

$$\lim_{i\to\infty} h(\beta_i) \;=\; 0. \tag{3.42}$$

Then

$$\lim_{i\to\infty} \Delta_{\beta_i} \;=\; \nu_{\{|z|=1\}} \qquad \textit{weakly} \tag{3.43}$$

where $\nu_{\{|z|=1\}}$ is the Haar measure on the unit circle.

In the proof of his theorem Bilu uses the Erdős-Turán theorem as basic ingredient and the fact that the minimal polynomials of the β_is' have distinct roots. Here working with non-irreducible Parry polynomials for which all the roots are distinct and Mignotte's theorem suffices to give the same conclusion.

Let us observe that the convergence condition (3.42) means in particular that $\beta_i \to 1^+, i \to \infty$, while convergence conditions (3.12) cover many other cases for general sequences of Parry numbers. In the convergence condition

(3.42) is also included some conditions on the p-dic valuations of the beta-conjugates of the β_is'. These aspects will be reconsidered elsewhere by the author.

4 Examples

Table 1 gives Mignotte's discrepancy function $\frac{2\pi}{k}\frac{\tilde{h}(n_\beta^*)}{d_P}$ (k is Catalan's constant) relative to the following four Pisot numbers:

- the confluent Parry number $\beta = 9.999\ldots$ dominant root of $X^{40} - 9\sum_{i=1}^{39} X^i - 4 = P_\beta(X) = n_\beta^*(X)$. It is a Pisot number which has no beta-conjugate [V2], for which $d_\beta(1) = 0.k_1^{d-1}k_2$, with $k_1 = 9, k_2 = 4$ and $d = d_P = 40$. The height of the Parry polynomial of β is 9,
- Bassino's cubic Pisot number $\beta = 30.0356\ldots$ dominant root of $X^3-(k+2)X^2 + 2kX - k = P_\beta(X)$, with $k = 30$, for which the complementary factor is the product $\Phi_2\Phi_3\Phi_5\Phi_6\Phi_{10}\Phi_{15}\Phi_{30}\Phi_{31}$ of cyclotomic factors. The height of the Parry polynomial of β is 30 and $d_P = 62$,
- the smallest Pisot number $\beta = 1.767...$ for which the complementary factor is (NC) reciprocal and non-cyclotomic (Boyd [Bo2] p. 850): it is the dominant root of $P_\beta(X) = X^{12} - X^{10} - 2X^9 - 2X^8 - X^7 - X^6 - X^5 - X^4 + X^2 + X + 1$ and has $\Phi_4\Phi_6\Phi_{12}\Phi_{30}L(-X)$ as complementary factor where $L(X) = X^{10} + X^9 - X^7 - X^6 - X^5 - X^4 - X^3 + X + 1$ is Lehmer's polynomial. The Rényi β-expansion of 1 has preperiod length 4 and period length 34. The Parry polynomial of β has degree d_P equal to 38 and height 1,
- the second-smallest Pisot number $\beta = 1.764\ldots$ for which the complementary factor is (NR) non-reciprocal (Boyd [Bo2] p. 850). We have $P_\beta(X) = X^{11} - 2X^9 - 2X^8 - X^7 + 2X^5 + 2X^4 + X^3 - X - 1$ and $\Phi_6 G(X)$ as complementary factor, where $G(X) = X^{22} + X^{15} + X^8 - X^7 - 1$ is non-reciprocal. The Rényi β-expansion of 1 has preperiod length 30, period length 5, and $\mathrm{H}(n_\beta^*) = 1$.

On each line, in the column "Parry", is reported Mignotte's discrepancy function with the value (ET) of the discrepancy function $16^2 \times \frac{1}{d_P}\mathrm{Log}\left(\frac{\|n_\beta^*\|_1}{\sqrt{|n_\beta^*(0)|}}\right)$ of Erdős-Turán for comparison.

In the column "Mini" is reported the value $\frac{2\pi}{k}\frac{\tilde{h}(P_\beta)}{d_P}$, resp. in the column "cycl." the value $\frac{2\pi}{k}\frac{\tilde{h}(\prod_{j=1}^{s}\Phi_{n_j}^{c_j})}{d_P}$, resp. in the column "rec. non-cycl." the value $\frac{2\pi}{k}\frac{\tilde{h}(\prod_{j=1}^{q} g_j^{\delta_j})}{d_P}$, resp. in the column "non-rec." the value $\frac{2\pi}{k}\frac{\tilde{h}(\prod_{j=1}^{u}\kappa_j^{\gamma_j})}{d_P}$, with the notations of (3.2).

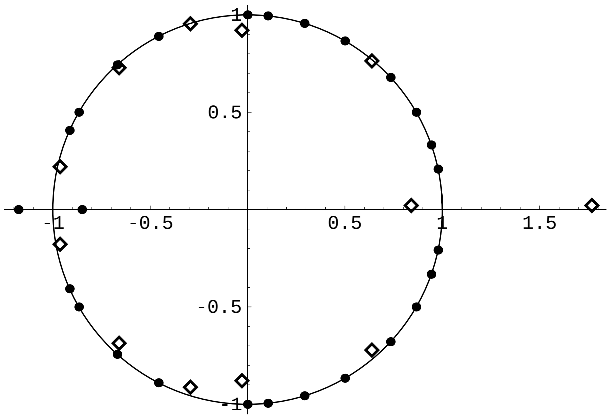

Figure 2. Concentration and equi-distribution of the Galois conjugates ($\diamond$) $\neq \beta$ and the beta-conjugates ($\bullet$) of the smallest NC Pisot number $\beta = 1.767\ldots$ near the unit circle.

The sharpness of the splitting (3.17) is not too bad in these examples: the ratio between the sum of Mignotte's discrepancies of the irreducible factors of the Parry polynomial and Mignotte's discrepancy applied to the Parry polynomial itself is always less than 4, and the sum of Mignotte's discrepancies of the factors is always much lower than the value ET.

β	"Parry"	"Mini."	"cycl."	"rec. non-cycl."	"non-rec."
Confluent $k_1 = 9, k_2 = 4$	$0.0695\ldots$ (ET$= 33.16\ldots$)	$0.0695\ldots$	/	/	/
Bassino $k = 30$	$0.0631\ldots$ (ET$= 21.20\ldots$)	$0.106\ldots$	$0.0893\ldots$	/	/
$1.767\ldots$ smallest NC Pisot	$0.0979\ldots$ (ET$= 17.77\ldots$)	$0.0927\ldots$	$0.0946\ldots$	$0.100\ldots$	
$1.764\ldots$ second-smallest NR Pisot	$0.107\ldots$ (ET$= 22.26\ldots$)	$0.124\ldots$	$0.0761\ldots$	/	$0.0840\ldots$

Table 1.

Ganelius, Mignotte and Amoroso [A1] [A2] [G] [Mt2] have already mentioned the (numerically) bad discrepancy function given by Erdős-Turán and Table 1 shows it as well: there exists a factor greater than 180 between ET and Mignotte's discrepancy applied to the Parry polynomial, even much larger in some other cases. The upper bound of the multiplicities of the beta-conjugates, computed from Mignotte's discrepancy function, according to (3.41), for the

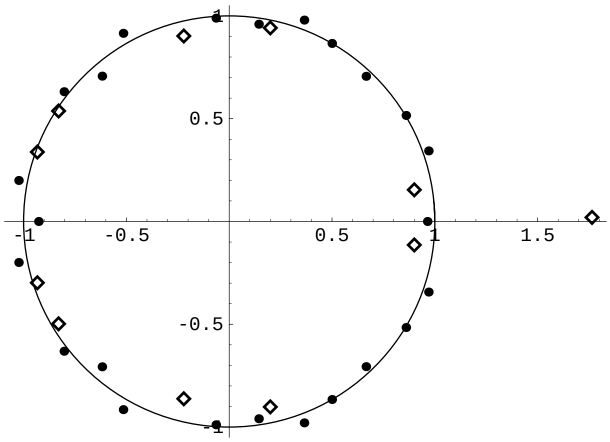

Figure 3. Concentration and equi-distribution of the Galois conjugates ($\diamond$) $\neq \beta$ and the beta-conjugates ($\bullet$) of the second-smallest NR Pisot number $\beta = 1.764\ldots$ near the unit circle.

four cases of Table 1, is respectively: 27, 40, 31, 30. These values are much higher than the true one: 1 in each case.

Figure 3 and Figure 6 in Verger-Gaugry [V2] show the equi-distribution of the conjugates of the first two examples of Pisot numbers near the unit circle, and are not reported here. The last two cases are illustrated in Figure 2 and Figure 3.

5 Arithmetics of Perron numbers and non-Parry case

If β is a Perron number which is not a Parry number the analytical function $f_\beta(z)$ has the unit circle as natural boundary, by Szegő's Theorem 2.1. It is such that $f_\beta(1/\beta) = 0$ and satisfies

$$\overline{f_\beta(D(0,1))} \;=\; \mathbb{C}$$

by Theorem 1 in Salem ([Sa] p. 161). Questions on the number and the type of the other zeros of this analytical function in the open unit disk, in particular beta-conjugates, can be found in [V2].

The set of Perron numbers $\mathbb{P}$ admits an arithmetic structure which does not take into account whether a Perron number is a Parry number or not (Lind, Section 5 in [Li]). First let us recall two theorems.

Theorem 5.1. *Every real algebraic number field $\mathbb{K}$ is generable by a Pisot number. If $d = [\mathbb{K} : \mathbb{Q}]$ denotes the degree of the field extension $\mathbb{K}/\mathbb{Q}$, the number field $\mathbb{K}$ contains infinitely many Pisot numbers of degree d, some of which being algebraic units.*

Proof. It is a tradition to call Pisot numbers S-numbers ([B-S] p. 84). This is Theorem 5.2.2 in [B-S]. □

Let S be the set of Pisot numbers and T the set of Salem numbers ([B-S] p. 84). We have: $S \subset \mathbb{P}_P$ [B] [St] and

$$T \cap \mathbb{P}_P \neq \emptyset, \qquad T \cap \mathbb{P}_a \neq \emptyset.$$

This dichotomy of Salem numbers is still obscure.

Theorem 5.2. *Let $\beta \in T$. The algebraic number field $\mathbb{K} = \mathbb{Q}(\beta)$ is a real quadratic extension of a totally real field. There exists $\tau_0 \in \mathbb{K} \cap T$ such that*

(i) $\mathbb{K} = \mathbb{Q}(\tau_0)$,

(ii) $\mathbb{K} \cap T = \{\tau_0^n \mid n = 1, 2, \ldots\}$.

Every number in $\mathbb{K} \cap T$ is quotient of two numbers in $\mathbb{K} \cap S$.

Proof. Theorem 5.2.3 in [B-S], or [Ld]. □

Lind has introduced the notion of irreducible Perron number in [Li] Section 5, based on the fact that $\mathbb{P}$ is closed under multiplication and addition ([Li] Proposition 1) and on the following ([Li] Proposition 5)

Proposition 5.3. *If $\lambda = \alpha\beta$ with $\lambda, \alpha, \beta \in \mathbb{P}$ then $\alpha, \beta \in \mathbb{Q}(\lambda)$.*

Let us observe that $1 \in \mathbb{P}$ (by convention) and that $1 \notin S, 1 \notin T$.

Definition 5.4. A Perron number $\lambda \in \mathbb{P}$ is said irreducible if $\lambda > 1$ and if it cannot be written as $\alpha\beta$ with $\alpha, \beta \in \mathbb{P}$ and $\alpha, \beta > 1$.

Theorem 5.5. *Every Perron number $\beta > 1$ can be factored into a finite number of irreducible Perron numbers λ_i :*

$$\beta = \lambda_1 \lambda_2 \ldots \lambda_s.$$

There is only a finite number of such factorizations of β, and unique factorization of β may occur (two factorizations of β are the same when they differ only by the order of the terms).

Proof. Theorem 4 in [Li]. □

In Theorem 5.5 the irreducible Perron numbers λ_i all belong to the number field $\mathbb{Q}(\beta) \subset \mathbb{R}$ by Proposition 5.3. The existence of non-unique factorizations in a number field is implied by the following remarkable theorem of Lind ([Li] Theorem 5).

Theorem 5.6. *Let $\mathbb{K}$ be an algebraic number field. The following are equivalent:*

(i) $\mathbb{K} \cap \mathbb{P}$ *contains elements which have non-unique factorizations,*

(ii) $\mathbb{K} \cap \mathbb{P}$ *contains non-rational Perron numbers,*

(iii) $\mathbb{K} \cap \mathbb{R} \neq \mathbb{Q}$.

A basic question is about the nature and the dispatching of the irreducible Perron numbers in the two classes $\mathbb{P}_P$ and $\mathbb{P}_a$. The characterization of the family of irreducible Perron numbers in a given number field is obscure, in particular when the number field is generated by a non-Parry Perron number. By Theorem 5.1 non-unique factorizations occur in every real number field $\mathbb{K}$ since such number fields are generated by Pisot numbers, but though Pisot numbers are always Parry numbers, this does mean that the irreducible Perron numbers in $\mathbb{K}$ are necessarily Parry numbers.

Corollary 5.7. *For every Salem number β the number field $\mathbb{Q}(\beta)$ contains elements which have non-unique factorizations into irreducible Perron numbers of $\mathbb{Q}(\beta)$.*

Proof. It is a consequence of Theorem 5.2 (ii) and of Theorem 5.6 (iii). □

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