On rough differential equations*

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January 8, 2009

Abstract

We prove that the Itô map, that is the map that gives the solution of a differential equation controlled by a rough path of finite p-variation with $p \in [2, 3)$ is locally Lipschitz continuous in all its arguments and we give some sufficient conditions for global existence for non-bounded vector fields.

Keywords: Rough paths, Itô map, controlled differential equations.

AMS classification: 60H10 (34F05, 34G05).

1 Introduction

The theory of rough paths allows one to properly define solutions of controlled differential equations driven by irregular paths and to assert the continuity — in the right topology — of the map that send the control to the solution [17, 16, 14, 11, 12]. As this theory was initially designed to overcome the problem of defining stochastic differential equations in a pathwise way, this map is called the *Itô map*. As Brownian paths are α-Hölder continuous for any $\alpha < 1/2$, the theory of rough paths asserts that one can define in a pathwise way solution of stochastic differential equations as soon as one shows that the iterated integrals $t \mapsto \int_0^t (B_s - B_0) \otimes \circ \, \mathrm{d}B_s$ have the right regularity property, and then, that the Brownian path can be embedded in

*This article was started while the author was staying at the Mittag-Leffler Institute and the author wishes to express his thanks to the organizers of the semester in "Stochastic Partial Differential Equations" for their invitation. The author is also indebted to Laure Coutin and Shigeki Aida for some interesting discussions about this article.

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an enhanced Brownian path with values in a non-commutative Lie group given by a truncated tensor algebra. The theory of rough path can indeed be used for many stochastic processes (See for example the book [7]).

The goal of this article is then to bring some precisions on the Itô map, which is that map that transform a rough path x of finite p-variation with $p < 3$ into the solution z of

$$z_t = z_0 + \int_0^t f(z_s)\,\mathrm{d}x_s$$

for a vector field f which is smooth enough. In particular, we show that the assumptions on the boundedness on the vector field can be dropped to assume that f has a linear growth to get local existence up to a given horizon, and sometimes global existence with some global control on the growth of f for a function f with a bounded, γ-Hölder continuous derivative with $2 + \gamma > p$ (indeed, we may even relax the conditions to consider the case where f has only a locally γ-Hölder continuous derivative). Also, we give a sufficient condition for global existence, but which is not a necessary condition. In [5], A.M. Davie dealt with the rough differential equations using an approach that relies on Euler scheme and also gave a sufficient condition for non-explosion. He also showed that explosion may happens by constructing a *ad hoc* counter-example. In [7], P. Friz and N. Victoir assert that no explosion occurs for rough differential equations driven by geometric rough paths (the ones that may be approximated by smooth rough paths) under a linear growth condition when the underlying paths live in $\mathbb{R}^d$. Constructing sub-Riemaniann geodesics play an important role there, so that extension to more general setting as not an easy issue. Yet in most of the case we do not know whether genuine explosion may happen, and stochastic differential equations provides us an example where there is no explosion even with conditions weaker than those in [7]. However, in this case, it is possible to take profit from the fact that we have an extra-information on the solution, which is the fact that it is a semi-martingale and then have finite p-variation.

The special case of a linear vector field f can be studied independently and global existence occurs. Note that linear rough differential equations have already been studied in [15, 1, 7, 8].

In addition, we show that the Itô map $(z_0, f, x) \mapsto z$ is locally Lipschitz in all its arguments (a slightly weaker statement on the continuity with respect to the vector field was proved in [3, 9] and [13], and on the Lipschitz continuity with respect to x in [16, 1]) under the more stringent assumption that f is twice differentiable with γ-Hölder continuous second-order derivative with $2 + \gamma > p$. The proof relies on some estimates on the distance between two solutions of rough differential equations, which can be also used as *a priori* estimates. From this, we deduce an alternative proof of the uniqueness of the solution of a differential equation controlled by a rough paths of finite p-variation with $p < 3$.

The Lipschitz continuity of the Itô map can be important for applications, since it allows one to deduce for example rates of convergence of approximate solutions of differential equations from the approximations of the driving signal (see for example [18] for a practical application to functional quantization).

In this article, our main tool is to use the approach that allows one to pass from an almost rough path to a rough path. Another approach to deal with rough differential equations consists in studying convergence of Euler scheme in the way initiated by A.M. Davie [5] (see also [6, 7]) where estimates are obtained on smooth paths. However, our approach can be used in a very general context, including infinite dimensional ones and we are not bound in using geometric rough paths (although any rough path may be interpreted as a geometric rough path, but at the price of a higher complexity [10]). Yet we also believe that all the results of this article may be extended to deal with (p, q)-rough paths [10], although it leads to much more complicated computations.

2 Notations

Let U and V be some finite-dimensional Banach spaces (we have to note however that the dimension does not play a role here when we consider functions that are $(2 + \gamma)$-Lipschitz continuous, but not $(1 + \gamma)$-Lipschitz continuous, where the solutions of the rough differential equations are defined using a non-contractive fixed point theorem, which is not always possible to apply in an infinite dimensional setting). The tensor product between two such spaces will be denoted by $\mathrm{U} \otimes \mathrm{V}$, and such a space is equipped with a norm $|\cdot|$ such that $|x \otimes y|_{\mathrm{U}\otimes\mathrm{V}} \leq |x|_{\mathrm{U}} \times |y|_{\mathrm{V}}$. To simplify the notations, the norm on a Banach space X is then denoted by $|\cdot|$ instead of $|\cdot|_{\mathrm{X}}$, as there is no ambiguity. For a Banach space X, we choose a norm on the tensor space $\mathrm{X} \otimes \mathrm{X}$ such that $|a \otimes b| \leq |a| \cdot |b|$ for a, b in X. For a Banach space $\mathrm{X} = \mathrm{Y} \oplus \mathrm{Z}$ with a sub-space Y, we denote by $\pi_{\mathrm{Y}}(x)$ the projection of x onto the sub-space Y. In addition, we define on $\mathrm{X} \oplus (\mathrm{X} \otimes \mathrm{X})$ a norm $|\cdot|$ by $|a| = \max\{|\,\pi_{\mathrm{X}}(a)|, |\,\pi_{\mathrm{X}\otimes\mathrm{X}}(a)|\}$.

A *control* is a $\mathbb{R}_+$-valued function ω defined on

$$\Delta^2 \stackrel{\text{def}}{=} \{(s, t) \in [0, T]^2 \,|\, 0 \leq s \leq t \leq T\}$$

which is continuous near the diagonal with $\omega(s,s) = 0$ for $s \in [0,T]$ and which is super-additive, that is

$$\omega(s,r) + \omega(r,t) \leq \omega(s,t),\ 0 \leq s \leq r \leq t \leq T.$$

For a Banach space U, let us define the space $T^2(\mathrm{U}) = \{1\} \oplus \mathrm{U} \oplus (\mathrm{U} \otimes \mathrm{U})$, which is a subspace of the vector space $\mathbb{R} \oplus \mathrm{U} \oplus (\mathrm{U} \otimes \mathrm{U})$. Let us note that $(T^2(\mathrm{U}), \otimes)$ is a Lie group.

For $2 \leq p < 3$, a *rough path* x from $[0,T]$ to $T^2(\mathrm{U})$ of finite p-variation controlled by ω is x is a function from $[0,T]$ to $(T^2(\mathrm{U}), \otimes)$ such that

$$\|x\|_{p,\omega} \stackrel{\mathrm{def}}{=} \sup_{\substack{(s,t)\in\Delta^2 \\ s\neq t}} \max\left\{ \frac{|\,\pi_{\mathrm{U}}(x_{s,t})|}{\omega(s,t)^{1/p}}, \frac{|\,\pi_{\mathrm{U}\otimes\mathrm{U}}(x_{s,t})|}{\omega(s,t)^{2/p}} \right\}$$

is finite, where $x_{s,t} \stackrel{\mathrm{def}}{=} x_s^{-1} \otimes x_t$ for $(s,t) \in \Delta^2$ and x_s^{-1} is the inverse of x_s in the Lie group $(T^2(U), \otimes)$.

Let $L(\mathrm{U}, \mathrm{V})$ be the space of linear maps from U to V.

For $\gamma \in (0,1]$, we denote by $\mathrm{Lip}_{\mathrm{LG}}(1+\gamma)$ the class of continuous functions $f : \mathrm{V} \to L(\mathrm{U}, \mathrm{V})$ for which there exists a bounded, continuous function $\nabla f : \mathrm{V} \to L(\mathrm{V} \otimes \mathrm{U}, \mathrm{V})$ such that

$$f(z)x - f(y)x = \int_0^1 \nabla f(y + \tau(z-y))(z-y) \otimes x \,\mathrm{d}\tau$$

and ∇f is γ-Hölder continuous: for some constant $C > 0$,

$$\|\nabla f(z) - \nabla f(y)\|_{L(\mathrm{V}\otimes\mathrm{U},\mathrm{V})} \leq C|z-y|^\gamma,\ \forall z, y \in \mathrm{V}.$$

In other words, if $\mathrm{U} = \mathbb{R}^d$ and $\mathrm{V} = \mathbb{R}^m$, this means that $f = (f_1, \dots, f_d)$ has bounded derivatives ∇f_i which are γ-Hölder continuous from $\mathbb{R}^m$ to $\mathbb{R}$. Note that here, f is not necessarily bounded, but it growths at most linearly.

We denote by $\mathrm{Lip}(1+\gamma)$ the class of functions in $\mathrm{Lip}_{\mathrm{LG}}(1+\gamma)$ which are bounded.

We also denote by $\mathrm{Lip}_{\mathrm{LG}}(2+\gamma)$ the class of continuous functions $f : \mathrm{V} \to L(\mathrm{U}, \mathrm{V})$ with bounded, continuous functions $\nabla f : \mathrm{V} \to L(\mathrm{V} \otimes \mathrm{U}, \mathrm{V})$ that are $\mathrm{Lip}(1+\gamma)$.

We denote by $\mathrm{Lip}(2+\gamma)$ the class of functions in $\mathrm{Lip}_{\mathrm{LG}}(2+\gamma)$ which are bounded.

For a γ-Hölder continuous function, we denote by $H_\gamma(f)$ its Hölder norm. We also set $N_\gamma(f) \stackrel{\mathrm{def}}{=} H_\gamma(f) + \|f\|_\infty$, which is a norm on the space of γ-Hölder continuous functions.

3 The main results

3.1 The case of bounded vector fields

For $z_0 \in \mathrm{V}$, let z be a solution to the rough differential equation

$$z_t = z_0 + \int_0^t f(z_s)\,\mathrm{d}x_s. \tag{1}$$

By this, we mean a rough path in $T^2(\mathrm{U} \oplus \mathrm{V})$ such that $\pi_{T^2(\mathrm{U})}(z) = x$ and which is such that $z_t = z_0 + \int_0^t \mathfrak{D}(f)(z_s)\,\mathrm{d}z_s$ where $\mathfrak{D}(f)$ is the differential form on $\mathrm{U} \oplus \mathrm{V}$ defined by $\mathfrak{D}(f)(x,z) = 1\,\mathrm{d}x + f(z)\,\mathrm{d}x$, where 1 is the map from $\mathrm{U} \oplus \mathrm{V}$ to $L(\mathrm{U},\mathrm{U})$ defined by $1(x,z)a = a$ for all $(x,z,a) \in \mathrm{U} \times \mathrm{V} \times \mathrm{U}$.

Here, we consider that x is of finite p-variation controlled by ω, and we then consider only solutions that are of finite p-variation controlled by ω.

We recall one of the main result in the theory of rough paths.

Theorem 1 ([17, 16, 10])**.** *Let $\gamma \in (0,1]$ and $p \in [2,3)$ such that $2+\gamma > p$. If f belongs to $\mathrm{Lip}(1+\gamma)$ and x is of p-finite variation controlled by ω with $p \in [2,3)$ then there exists a solution to (1) (this solution is not necessarily unique). If moreover, f belongs to $\mathrm{Lip}(2+\gamma)$, then the solution is unique. In this case, $x \mapsto z$ is continuous from the space of p-rough paths from $[0,T]$ to $T^2(\mathrm{U})$ controlled by ω to the space of p-rough paths from $[0,T]$ to $T^2(\mathrm{U} \oplus \mathrm{V})$ controlled by ω.*

The map $(z_0, x, f) \mapsto z$ is called the *Itô map*.

This result can be shown using a contractive fixed point theorem if f belongs to $\mathrm{Lip}(2+\gamma)$ and by a Schauder fixed point theorem if f belongs to $\mathrm{Lip}(1+\gamma)$. It can also be deduced from the following result, the existence of a unique solution if x is smooth and the continuity result, provided that one has uniform estimates on the p-variation of the solution. Of course, it is a consequence of Theorem 3 below, which is itself an adaptation of the proof presented in [10].

3.2 The case of non-bounded vector fields

Our goal is to extend Theorem 1 to the case of functions with a linear growth, as in the case of ordinary differential equations. Indeed, we were only able to prove local existence in the general case. However, the linear case is a special case where global existence holds.

Theorem 2. *Let x be a rough path of finite p-variation controlled by ω with $p \in [1,3)$ on T and f be the vector field $f(y) = Ay$ for some linear application A from V to $L(\mathrm{U},\mathrm{V})$. Then there exists a unique solution to the rough* *differential equation* $z_t = z_0 + \int_0^t f(z_s)\,\mathrm{d}x_s$*. In addition, for some universal constant* K_1*,*

$$\sup_{t\in[0,T]} |\pi_{\mathrm{V}}(z_t - z_0)| \leq |z_0| \exp(K_2\omega(0,T)\|A\|^p \max\{1, \|x\|_{p,\omega}^p\}) \tag{2}$$

for some universal constant K_2*.*

Remark 1. For practical application, for example for dealing with derivatives of rough differential equations with respect to some parameters, then one needs to deal with equations of type

$$z_t = z_0 + \int_0^t A(x_s) z_s \,\mathrm{d}x_s. \tag{3}$$

Indeed, if one consider the rough path $X_t = \int_0^t A(x_s)\,\mathrm{d}x_s$ with values in $T^2(L(\mathrm{U},\mathrm{V}))$, it is easily shown that the solution of (3) is also the solution to $z_t = z_0 + \int_0^t \mathrm{d}X_s z_s$ for which the results of Theorem 2 may be applied.

To simplify we assume that ω is continuous on Δ^2 and $t \mapsto \omega(0,t)$ is increasing (this are very weak hypotheses, since using a time change it is possible to consider that x is indeed Hölder continuous and then to consider that $\omega(s,t) = t-s$). For a $\mathrm{Lip}_{\mathrm{LG}}(1+\gamma)$-vector field f, provided that $H_\gamma(f) > 0$ (otherwise, ∇f is constant and then it corresponds to the linear case covered by Theorem 2), set

$$\mu \stackrel{\mathrm{def}}{=} \left(\frac{\|\nabla f\|_\infty}{H_\gamma(\nabla f)}\right)^{1/\gamma}.$$

Theorem 3. *If* x *is a rough path of finite* p*-variation controlled by* ω*,* f *is a* $\mathrm{Lip}_{\mathrm{LG}}(1+\gamma)$*-vector field with* $3 > 2+\gamma > p \geq 2$*, then for* τ *such that*

$$\omega(0,\tau)^{1/p} G(z_0)\|x\|_{p,\omega} \leq K_3\mu \text{ with } G(z_0) = \sup_{a\in\mathrm{W},\ |a-z_0|\leq\mu} |f(a)|, \tag{4}$$

for some universal constant K_3 *(depending only on* γ *and* p*), there exists a solution* z *to* (1) *in the sense of rough paths which is such that for some universal constant* K_4 *(depending only on* γ *and* p*),*

$$|\pi_{\mathrm{V}}(z_{s,t})| \leq 2\|x\|_{p,\omega} G(z_0)\omega(s,t)^{1/p}, \tag{5a}$$

$$|\pi_{\mathrm{V}\otimes\mathrm{U}}(z_{s,t})| \leq 2\|x\|_{p,\omega} G(z_0)\omega(s,t)^{2/p}, \tag{5b}$$

$$|\pi_{\mathrm{U}\otimes\mathrm{V}}(z_{s,t})| \leq 2\|x\|_{p,\omega} G(z_0)\omega(s,t)^{2/p}, \tag{5c}$$

$$|\pi_{\mathrm{V}\otimes\mathrm{V}}(z_{s,t})| \leq K_4\|x\|_{p,\omega}(1 + \|x\|_{p,\omega} + \|\nabla f\|_\infty) G(z_0)^2\omega(s,t)^{2/p}, \tag{5d}$$

$$\text{and } \sup_{t\in[0,\tau]} |\pi_{\mathrm{V}}(z_t - z_0)| \leq \mu \tag{5e}$$

for all $0 \leq s \leq t \leq \tau$*. If* f *is a vector field in* $\mathrm{Lip}_{\mathrm{LG}}(2+\gamma)$ *with* $3 \geq 2+\gamma > p$*, then the solution is unique.*

Remark 2. The idea of the proof is to show that if z satisfies (5a)–(5d), then $\widetilde{z}_t = z_0 + \int_0^t f(z_s)\,\mathrm{d}x_s$ also satisfies (5a)–(5d), and then to apply a Schauder fixed point theorem. The results are proved under the assumption that ∇f is bounded and Hölder continuous. However, we may assume that ∇f is only locally Hölder continuous. In which case, we may consider a compact K as well as $\mu(K) = (\|\nabla f\|_{\infty;K}/H_\gamma(\nabla f;K))^{1/\gamma}$ where $\|\nabla f\|_{\infty;K}$ (resp. $H_\gamma(\nabla f;K)$) are the uniform (resp. Hölder norm) of ∇f. In this case, $\sup_{s\in[0,t]}|z_{0,s}| \leq \mu\omega(0,t)^{1/p}/\omega(0,\tau)^{1/p}$ for the choice of τ given by (4) and one has also to impose that τ is also such that $z_{0,t}$ belongs to K for $t \in [0,\tau]$.

To simplify the description, we assume that the control ω is defined on $0 \leq s \leq t$ for $s,t \in \mathbb{R}_+$ and that the rough path x is defined as a rough path on $\mathbb{R}_+$ with $\|x\|_{p,\omega} < +\infty$.

Once the local existence established, it is then possible to construct recursively a sequence $\{\tau_i\}$ of times and a sequence $\{\xi_i\}$ of points of V such that

$$\omega(\tau_i,\tau_{i+1})^{1/p}G(\xi_i)\|x\|_{p,\omega} = K_3\mu,$$

where $\xi_0 = z_0$ and $\xi_i = \pi_{\mathrm{V}}(z_{\tau_i})$ for a solution z_t to $z_t = \xi_{i-1} + \int_0^t f(z_s)\,\mathrm{d}x_s$ on $[\tau_{i-1},\tau_i]$, $i \geq 1$.

This way, it is possible to paste the solutions on the time intervals $[\tau_i,\tau_{i+1}]$. Given a time $T > 0$, our question is to know whether or not there exists a solution up to the time T, which means that $\lim_{i\to\infty}\tau_i > T$.

Of course, it is possible to relate the explosion time, if any, to the explosion of the p-variation of z or of the uniform norm of z.

Lemma 1. *It holds that* $S = \lim_{i\to\infty}\tau_i < +\infty$ *if and only if* $\lim_{t\nearrow S}\|z\|_{p,\omega,t} = +\infty$ *and* $\lim_{t\nearrow S}\|z\|_{\infty,t} = +\infty$*, where* $\|z\|_{p,\omega,t}$ *(resp.* $\|z\|_{\infty,t}$*) is the* p*-variation (resp. uniform) norm of* z *(resp.* $\pi_{\mathrm{V}}(z)$*) on* $[0,t]$.

Proof. Let us assume that for all $S > 0$, $\sup_{t\leq S}\|z\|_{p,\omega,t} < +\infty$ (which implies also that $\sup_{t\leq S}\|z\|_{\infty,\tau_i} < +\infty$) or $\sup_{t\geq 0}\|z\|_{\infty,\tau_i} < +\infty$. Then $G(\xi_i)$ is bounded and then

$$\sum_{i=0}^{n}\omega(\tau_i,\tau_{i+1}) = \sum_{i=0}^{n}\frac{K_3^p\mu^p}{G(\xi_i)^p\|x\|_{p,\omega}^p} \xrightarrow[n\to\infty]{} +\infty.$$

As $\sum_{i=0}^{n-1}\omega(\tau_i,\tau_{i+1}) \leq \omega(0,\tau_n)$, the time τ_n converges to $+\infty$ as $n \to \infty$.

Conversely, if $\lim_{i\to+\infty}\tau_i = +\infty$, then there exists n such that $\tau_n > S$ for any $S > 0$ and then, since the p-variation of a path on $[0,S]$ may deduced from the p-variation of the path restricted to $[\tau_i,\tau_{i+1}]$, $\|z\|_{p,\omega,S} < +\infty$ and $\|z\|_{\infty,S} < +\infty$. □

In practical, the following criterion is useful to determine whether or not a rough differential equation has a global solution or a solution up to time T. It has to be compared with the one given by A.M. Davie in [5].

Lemma 2. *It is possible to solve $z_t = z_0 + \int_0^t f(z_s)\,\mathrm{d}x_s$ up to time T if and only if*

$$\sum_{i=0}^{+\infty} \omega(\tau_i, \tau_{i+1}) > \omega(0, T). \tag{6}$$

Proof. Assume (6). As $\sum_{i=0}^{n-1} \omega(\tau_i, \tau_{i+1}) \leq \omega(0, \tau_n)$ and $t \mapsto \omega(0, t)$ is increasing then for n large enough $\omega(0, \tau_n) > \omega(0, T)$ and $\tau_n > T$, which means that at most n intervals are sufficient to solve (1) on $[0, T]$. Conversely, if $\sum_{i=0}^{+\infty} \omega(\tau_i, \tau_{i+1}) < \omega(0, T)$, then $S = \lim_{i\to\infty} \tau_i < T$ and the solution explodes at time S. □

As

$$\sum_{i=0}^{n} \omega(\tau_i, \tau_{i+1}) = \sum_{i=0}^{n} \frac{K_3^p \mu^p}{G(\xi_i)^p \|x\|_{p,\omega}^p}$$

it is then possible to consider several situations. First, let us note that for all integer i,

$$G(\xi_i) \leq h((i+1)\mu) \text{ with } h(r) \stackrel{\text{def}}{=} \sup_{a \in \mathrm{W},\ |a - z_0| \leq r} |f(a)|.$$

Thus (assuming for simplicity that $h(0) > 0$),

$$\sum_{i=0}^{+\infty} \omega(\tau_i, \tau_{i+1}) \geq \frac{K_3^p \mu^p}{\|x\|_{p,\omega}^p} \Theta \text{ with } \Theta \stackrel{\text{def}}{=} \sum_{i=0}^{+\infty} \frac{1}{h((i+1)\mu)^p}.$$

The next lemma is a direct consequence of Lemma 2. Let us note that Θ depends only on the vector field f.

Lemma 3. *If $\Theta = +\infty$, then there exists a solution to* (1) *with a vector field f in $\mathrm{Lip}_{\mathrm{LG}}(1+\gamma)$, $2 + \gamma > p$, up to any time $T > 0$.*

We should now detail some cases.

Influence of μ. Of course, the favourable cases are those for which $\omega(\tau_i, \tau_{i+1})$ is big, but this does not mean that μ itself shall be big, as $G(\xi_i)$ also may depend on ∇f. For example, consider the linear growth case, in which case $G(\xi_i)$ is of order $|f(\xi_i)| + \mu\|\nabla f\|_\infty$ and $\omega(\tau_i, \tau_{i+1})$ is of order $\mu/G(\xi_i)\|x\|_{p,\omega}$. In this case, if μ is big, then $\omega(\tau_i, \tau_{i+1})$ is of order $1/\|\nabla f\|_\infty^p$. Thus, if μ is big because $H_\gamma(\nabla f)$ is small, then the value of $\omega(\tau_i,\tau_{i+1})$ will be insensitive to the position ξ_i and a solution may exists at least for for a large time, if not for any time. For $\gamma=1$ and in the case of the linear growth, one obtains easily that

$$\sum_{i=0}^{n}\omega(\tau_i,\tau_{i+1})\geq\sum_{i=0}^{n}\left(\frac{1}{1+|f(x_0)|\|\nabla^2 f\|_\infty/\|\nabla f\|_\infty+(i+1)\|\nabla^2 f\|_\infty}\right)^p.$$

The linear case may be deduced as a limit of this case by considering sequence of functions f such that $\|\nabla^2 f\|_\infty$ decreases to 0. On the other hand, if μ is big because $\|\nabla f\|_\infty$ is big, then $\omega(\tau_i,\tau_{i+1})^{1/p}$ will be small. On the other hand, if μ is small, then $\mu/G(\xi_i)$ will be equivalent to $\mu/|f(\xi_i)|$ and then $\omega(\tau_i,\tau_{i+1})^{1/p}$ will be small.

The case $p=1$. If $p=1$, which means that x is a path of bounded variation, then $h(i\mu)\leq|f(z_0)|+i\mu\|\nabla f\|_\infty$ and Θ is a divergent series. Here, we recover the usual result that Lipschitz continuity is sufficient to solve a controlled differential equation. In addition

$$\begin{aligned}\omega(0,\tau_n)&\geq\sum_{i=0}^{n-1}\omega(\tau_i,\tau_{i+1})\\&\geq\sum_{i=1}^{n}\frac{\mu K_3/\|x\|_{1,\omega}}{|f(z_0)|+i\mu\|\nabla f\|_\infty}\geq\int_1^{n+1}\frac{\mu K_3/\|x\|_{1,\omega}}{|f(z_0)|+\sigma\mu\|\nabla f\|_\infty}\,\mathrm{d}\sigma\end{aligned}$$

and the last term in the inequality is equal to

$$\frac{K_3/\|x\|_{1,\omega}}{\|\nabla f\|_\infty}\left(\log(|f(z_0)|+(n+1)\mu\|\nabla f\|_\infty)-\log(|f(z_0)|+\mu\|\nabla f\|_\infty)\right).$$

Then, the number of steps n required to get $\tau_n>T$ is such that

$$\begin{aligned}&\log(|f(z_0)|+n\mu\|\nabla f\|_\infty)\\&\qquad\leq\omega(0,T)\|x\|_{1,\omega}\|\nabla f\|_\infty/K_3+\log(|f(z_0)|+\mu\|\nabla f\|_\infty)\\&\qquad\qquad\leq\log(|f(z_0)|+(n+1)\mu\|\nabla f\|_\infty)\end{aligned}$$

and it follows that

$$\sup_{t\in[0,T]}|\pi_{\mathrm{V}}(z_t-z_0)|\leq n\mu\leq\frac{|f(z_0)|+\mu\|\nabla f\|_\infty}{\|\nabla f\|_\infty}\exp\Big(\omega(0,T)\|x\|_{1,\omega}\|\nabla f\|_\infty/K_3\Big).$$

Note that this expression is similar to (2). However, one cannot compare the two expressions, because formally, $\mu=+\infty$ in the linear case.

Conditions on the growth of f. If f is bounded, then h is bounded and Θ is also a divergent series. Of course, there are other cases where Θ is divergent, for example if $h(a) \sim_{a\to\infty} a^\delta$ with $0 < \delta < 1/p$ or $h(a) \sim_{a\to\infty} \log(a)$. The boundedness of f is not a necessary condition to ensure global existence.

The condition $\Theta < +\infty$ does not mean explosion. Yet it is not sufficient to deduce from the fact that $\Theta < +\infty$ that there is no global existence. Indeed, we have constructed Θ from a rough estimate. The real estimate depends on where the path z has passed through. The linear case illustrates this point, but here is another counter-example.

Set $\mathrm{U} = \mathbb{R}^m$, $\mathrm{V} = \mathbb{R}^d$, consider a $\mathrm{Lip}_{\mathrm{LG}}(2+\gamma)$-vector field f for some $\gamma > 0$ and solve the stochastic differential equation

$$Z_t = z_0 + \int_0^t f(Z_s) \circ \mathrm{d}B_s$$

for a Brownian motion B in $\mathbb{R}^m$. This equation has a unique solution on any time interval $[0,T]$. This solution is a semi-martingale, which has then a finite p-variation for any $p > 2$ (see [4] for example). Then, it is possible to replace f by a bounded vector field g and then to solve $z_t = z_0 + \int_0^t g(z_s)\,\mathrm{d}\mathbf{B}_s$, where $\mathbf{B}$ is the rough path associated to B by $\mathbf{B}_t = 1 + B_t - B_0 + \int_0^t (B_s - B_0) \otimes \circ\, \mathrm{d}B_s$. One knows that $\pi_{\mathrm{V}}(z) = Z$ so that z has a finite p-variation on $[0,T]$. With Lemma 1 and Lemma 8 below, this shows that $y_t = z_0 + \int_0^t f(y_s)\,\mathrm{d}x_s$ has a solution on $[0,T]$, which is z. On the other hand, our criteria just give the existence of a solution up to a finite time.

This case is covered by Exercise 10.61 in [7]. However, it is still valid in our context if one replace $\mathbf{B}$ by the non-geometric rough path $1 + B_t - B_0 + \int_0^t (B_s - B_0) \otimes \mathrm{d}B_s$ in which case Z is the solution to the Itô stochastic differential equation $Z_t = Z_0 + \int_0^t f(Z_s)\,\mathrm{d}B_s$. In addition if f is only a $\mathrm{Lip}_{\mathrm{LG}}(1+\gamma)$-vector field, then this still holds thanks to a result in [2] which asserts that the solution of the stochastic differential equation may be interpreted as a solution of a rough differential equation.

3.3 A continuity result

We now state a continuity result, which improves the results on [17, 16, 10] for the continuity with respect to the signal, and the results from [3, 13] on the continuity with respect to the vector fields.

For two elements z and $\widehat{z}$ in V, we set $\delta(z, \widehat{z}) \stackrel{\mathrm{def}}{=} |\widehat{z} - z|$. For two p-rough paths x and $\widehat{x}$ of finite p-variation controlled by ω, we set

$$\delta(x,\widehat{x}) \stackrel{\text{def}}{=} \sup_{(s,t)\in\Delta^2} \max\left\{\frac{|\,\pi_{\mathrm{U}}(x_{s,t}-\widehat{x}_{s,t})|}{\omega(s,t)^{1/p}}, \frac{|\,\pi_{\mathrm{U}\otimes\mathrm{U}}(x_{s,t}-\widehat{x}_{s,t})|}{\omega(s,t)^{2/p}}\right\}.$$

Finally, for f and $\widehat{f}$ in $\mathrm{Lip}_{\mathrm{LG}}(2+\kappa)$ and ρ fixed, we set

$$\delta_\rho(f,\widehat{f}) \stackrel{\text{def}}{=} \sup_{z\in B_{\mathrm{V}}(\rho)} |f(z)-\widehat{f}(z)|_{L(\mathrm{U},\mathrm{V})}$$
$$\text{and } \delta_\rho(\nabla f,\nabla\widehat{f}) \stackrel{\text{def}}{=} \sup_{z\in B_{\mathrm{V}}(\rho)} |\nabla f(z)-\nabla\widehat{f}(z)|_{L(\mathrm{V}\otimes\mathrm{U},\mathrm{V})},$$

where $B_{\mathrm{W}}(\rho)=\{z\in\mathrm{W}\,|\,|z|\leq\rho\}$ for a Banach space W.

Theorem 4. *Let f and $\widehat{f}$ be two $\mathrm{Lip}_{\mathrm{LG}}(2+\kappa)$-vector fields and x, $\widehat{x}$ be two paths of finite p-variation controlled by ω, with $2\leq p<2+\kappa\leq 3$. Denote by z and $\widehat{z}$ the solutions to $z=z_0+\int_0^\cdot f(z_s)\,\mathrm{d}x_s$ and $\widehat{z}=\widehat{z}_0+\int_0^\cdot \widehat{f}(\widehat{z}_s)\,\mathrm{d}\widehat{x}_s$. Assume that both z and $\widehat{z}$ belong to $B_{T^2(\mathrm{U}\oplus\mathrm{V})}(\rho)$ and $\max\{\|z\|_{p,\omega},\|\widehat{z}\|_{p,\omega}\}\leq\rho$. Then*

$$\delta(z,\widehat{z})\leq C\big(\delta_\rho(f,\widehat{f})+\delta_\rho(\nabla f,\nabla\widehat{f})+\delta(z_0,\widehat{z}_0)+\delta(x,\widehat{x})\big), \tag{7}$$

where C depends only on ρ, T, ω, p, κ, $\|\nabla f\|_\infty$, $N_\kappa(\nabla^2 f)$, $\|\nabla\widehat{f}\|_\infty$ and $N_\kappa(\nabla^2\widehat{f})$.

Remark 3. Let us note that this theorem implies also the uniqueness of the solution to (1) for a vector field in $\mathrm{Lip}_{\mathrm{LG}}(2+\gamma)$.

Remark 4. Of course, (7) allows one to control $\|z-\widehat{z}\|_\infty$, since $\|z-\widehat{z}\|_\infty\leq\delta(z,\widehat{z})\omega(0,T)^{1/p}+\delta(z_0,\widehat{z}_0)$.

In the previous theorem, we are not forced to assume that z and $\widehat{z}$ belong to $B_{T^2(\mathrm{U}\oplus\mathrm{V})}(\rho)$ but one may assume that, by properly changing the definition of $\delta_\rho(f,\widehat{f})$, they belong to the shifted ball $a+B_{T^2(\mathrm{U}\oplus\mathrm{V})}(\rho)$ for any $a\in\mathrm{V}$ without changing the constants. This is a consequence of the next lemma.

Lemma 4. *For f in $\mathrm{Lip}(2+\gamma)$ and for $a\in\mathrm{U}$, let z be the rough solution to $z_t=a+\int_0^t f(z_s)\,\mathrm{d}x_s$ and y be the rough solution to $y_t=\int_0^t g(y_s)\,\mathrm{d}x_s$ where $g(y)=f(a+y)$. Then $z=a+y$.*

Proof. Let us set $u_t=a+y_{0,t}$ for $t\in[0,T]$ and then $u_{s,t}\stackrel{\text{def}}{=}u_s^{-1}\otimes u_t=y_{s,t}$. Thus, the almost rough path associated to $\int_0^t f(u_s)\,\mathrm{d}x_s$ is

$$\begin{aligned} h_{s,t}=1+x_{s,t}+f(u_s)x^1_{s,t}+\nabla f(u_s)\pi_{W\otimes V}(u_{s,t})& \\ +f(u_s)\otimes 1\cdot x^2_{s,t}+1\otimes f(u_s)\cdot x^2_{s,t}+f(u_s)\otimes f(u_s)\cdot x^2_{s,t}& \end{aligned}$$

and is then equal to the almost rough path associated to $\int_0^t g(y_s)\,\mathrm{d}x_s$. Hence,

$$\int_0^t f(u_s)\,\mathrm{d}x_s = \int_0^t g(y_s)\,\mathrm{d}x_s = y_t = a^{-1}\otimes u_t.$$

Then, u is solution to $u_t = a\otimes\int_0^t f(u_s)\,\mathrm{d}x_s$ and by uniqueness, the result is proved. □

Remark 5. One may be willing to solve $z_t^a = a\otimes\int_0^t g(z_s^a)\,\mathrm{d}x_s$ for $a\in T^2(\mathrm{U}\oplus\mathrm{V})$ with $\pi_{T^2(\mathrm{U})}(a)=1$, which is a more natural statement when one deals with tensor spaces. However we note that $a^{-1}\otimes z^a = \widehat{a}^{-1}\otimes z^{\widehat{a}}$ if $\pi_V(a)=\pi_V(\widehat{a})$ and then z^a is easily deduced from $z^{\pi_{\mathrm{V}}(a)}$. This is why, for the sake of simplicity, we only deal with starting points in V.

4 Preliminary computations

We fix $T>0$, $p\in(2,3]$ and we define $\Delta^3 \stackrel{\text{def}}{=} \{(s,r,t)\in[0,T]^3 \mid s\leq r\leq t\}$.

For $(y_{s,t})_{(s,t)\in\Delta^2}$ with $y_{s,t}$ in $T^2(\mathrm{U}\oplus\mathrm{V})$ define

$$\|y\|_{p,\omega} = \sup_{\substack{(s,t)\in\Delta^2\\ s\neq t}} \max\left\{\frac{|y^1_{s,t}|}{\omega(s,t)^{1/p}}, \frac{|y^2_{s,t}|}{\omega(s,t)^{2/p}}\right\}$$

when this quantity is finite. We have already seen that a rough path of finite p-variation controlled by ω is by definition a function $(x_s)_{s\in\Delta^1}$ with values in the Lie group $(T^2(\mathrm{U}\oplus\mathrm{V}),\otimes)$ to which one can associate a family $(x_{s,t})_{(s,t)\in\Delta^2}$ by $x_{s,t} = x_s^{-1}\otimes x_t$ such that $\|x\|_{p,\omega}$ is finite.

We set $y_{s,r,t} \stackrel{\text{def}}{=} y_{s,t} - y_{s,r}\otimes y_{r,t}$. By definition, a rough path is a path y such that $y_{s,r,t}=0$. An *almost rough path* is a family $(y_{s,t})_{(s,t)\in\Delta^2}$ such that $\|y\|_{p,\omega}$ is finite and for some $\theta>1$ and some $C>0$

$$|y_{s,r,t}| \leq C\omega(s,t)^\theta. \tag{8}$$

Let us recall the following results on the construction of a rough path from an almost rough path (see for example [17, 16, 11, 12]).

Lemma 5. *If y is an almost rough path of finite p-variation and satisfying* (8)*, then there exists a rough path x of finite p-variation such that for $K_5 = \sum_{n\geq 1} 1/n^\theta$,*

$$|y^1_{s,t} - x^1_{s,t}| \leq CK_5\omega(s,t)^\theta,$$
$$|y^2_{s,t} - x^2_{s,t}| \leq CK_5(1+2K_5(\|y\|_{p,\omega} + K_5C\omega(0,T)^\theta)\omega(0,T)^{1/p})\omega(s,t)^\theta.$$

The rough path is unique in the sense that if z is another rough path of p-variation controlled by ω and such that $|y_{s,t} - z_{s,t}| \leq C'\omega(s,t)^{\theta'}$ for some $C' > 0$ and $\theta' > 1$, then $z = x$.

Lemma 6. *Let y and $\widehat{y}$ be two almost rough paths such that for some $\theta > 1$ and some constant C,*

$$\|y\|_{p,\omega} \leq C,\ \|\widehat{y}\|_{p,\omega} \leq C,\ |y_{s,r,t}| \leq C\omega(s,t)^{\theta},\ |\widehat{y}_{s,r,t}| \leq C\omega(s,t)^{\theta}$$

and for some $\epsilon > 0$,

$$\|y - \widehat{y}\|_{p,\omega} \leq \epsilon \text{ and } |y_{s,r,t} - \widehat{y}_{s,r,t}| \leq \epsilon\omega(s,t)^{\theta}.$$

Then the rough paths x and $\widehat{x}$ associated respectively to y and $\widehat{y}$ satisfy

$$|x_{s,t} - \widehat{x}_{s,t}| \leq \epsilon K\omega(s,t)^{\theta}$$

for some constant K that depends only on $\omega(0,T)$, θ, p and C.

Given a solution z of (1) for a vector field f which is $\mathrm{Lip}(1+\gamma)$ with $2+\gamma > 2$ (we know that (1) may have a solution, but which is not necessarily unique), set

$$y_{s,t} = 1 + x_{s,t} + f(z_s)x^1_{s,t} + \nabla f(z_s)z^{\times}_{s,t} + f(z_s)\otimes 1\cdot x^2_{s,t} \\ + 1\otimes f(z_s)\cdot x^2_{s,t} + f(z_s)\otimes f(z_s)\cdot x^2_{s,t}, \quad (9)$$

where $z^{\times} = \pi_{\mathrm{V}\otimes\mathrm{U}}(z)$, $x^1 = \pi_{\mathrm{U}}(x)$ and $x^2 = \pi_{\mathrm{U}\otimes\mathrm{U}}(x)$. In (9), if a (resp. b) is a linear forms from a Banach space X to a Banach space X′ (resp. from Y to Y′), and x belongs to X (resp. Y), then we denote by $a\otimes b$ the bilinear form from $\mathrm{X}\otimes\mathrm{Y}$ to $\mathrm{X}'\otimes\mathrm{Y}'$ defined by $a\otimes b\cdot x\otimes y = a(x)\otimes b(y)$. In the previous expression, $f(z_s)x^1_{s,t} + \nabla f(z_s)z^{\times}_{s,t}$ projects onto V, $f(z_s)\otimes f(z_s)\cdot x^2_{s,t}$ projects onto $\mathrm{V}\otimes\mathrm{V}$, $1\otimes f(z_s)\cdot x^2_{s,t}$ projects onto $\mathrm{U}\otimes\mathrm{V}$ while $f(z_s)\otimes 1\cdot x^2_{s,t}$ projects onto $\mathrm{V}\otimes\mathrm{U}$.

The result in the next lemma is a direct consequence of the definition of the iterated integrals. However, we gives its proof, since some of the computations will be used later, and y is the main object we will work with.

Lemma 7. *For a rough path x of finite p-variation controlled by ω and f in $\mathrm{Lip}(1+\gamma)$ with $2+\gamma > p$, the family $(y_{s,t})_{(s,t)\in\Delta^2}$ defined by (9) for a solution z to (1) is an almost rough path whose associated rough path is z.*

Proof. For a function g, we set $\|g\circ z\|_\infty=\sup_{t\in[0,T]}|g(z_t)|$. Since $x_{s,t}=x_{s,r}\otimes x_{r,t}$,

$$
\begin{aligned}
y_{s,r,t} \stackrel{\text{def}}{=} & y_{s,t}-y_{s,r}\otimes y_{r,t}\\
=&(\nabla f(z_s)-\nabla f(z_r))z^\times_{r,t} && \text{(10a)}\\
&+(f(z_s)-f(z_r))x^1_{r,t}+\nabla f(z_s)(z^\times_{s,t}-z^\times_{s,r}-z^\times_{r,t}) && \text{(10b)}\\
&+(f(z_s)-f(z_r))\otimes 1\cdot x^2_{r,t}+1\otimes(f(z_s)-f(z_r))\cdot x^2_{r,t} && \text{(10c)}\\
&+(f(z_s)\otimes f(z_s)-f(z_r)\otimes f(z_r))\cdot x^2_{r,t} && \text{(10d)}\\
&+(f(z_s)\otimes(f(z_r)-f(z_s)))\cdot x^1_{s,r}\otimes x^1_{r,t} && \text{(10e)}\\
&+1\otimes(f(z_s)-f(z_r))\cdot x^1_{s,r}\otimes x^1_{r,t} && \text{(10f)}\\
&+\Upsilon_{s,r,t} && \text{(10g)}
\end{aligned}
$$

with

$$
\begin{aligned}
\Upsilon_{s,r,t}=\nabla f(z_s)z^\times_{s,r}\otimes(x^1_{r,t}+f(z_r)x^1_{r,t}+\nabla f(z_r)z^\times_{r,t}&\\
+(x^1_{s,r}+f(z_r)x^1_{s,r}+\nabla f(z_s)z^\times_{s,r})\otimes\nabla f(z_s)z^\times_{r,t}.& \quad (11)
\end{aligned}
$$

We denote by $L_{(i)}$ the quantity of Line (i).

Since $z_{s,t}=z_{s,r}\otimes z_{r,t}$, we get that $z^\times_{s,t}=z^\times_{s,r}+z^\times_{r,t}+z^1_{s,r}\otimes x^1_{r,t}$ and then that Line (10b) is equal to

$$L_{\text{(10b)}}=(f(z_s)-f(z_r))x^1_{r,t}+\nabla f(z_s)z^1_{s,r}\otimes x^1_{r,t}.$$

Since f is in $\mathrm{Lip}(1+\gamma)$,

$$f(z_r)-f(z_s)=\int_0^1\nabla f(z^1_s+\tau z^1_{s,r})z^1_{s,r}\,\mathrm{d}\tau$$

and then

$$|L_{\text{(10b)}}|\leq\left|\int_0^1\nabla f(z_s+\tau z^1_{s,r})z^1_{s,r}\,\mathrm{d}\tau-\nabla f(z_s)z^1_{s,r}\right|\cdot|x^1_{r,t}|\leq C_1\omega(s,t)^{(2+\gamma)/p}$$

with $C_1\leq H_\gamma(\nabla f)\|z\|^{2+\gamma}_{p,\omega}$ (note that since x is a part of z, $\|x\|_{p,\omega}\leq\|z\|_{p,\omega}$). Similarly, since $|x^1_{s,r}\otimes x^1_{r,t}|\leq\omega(s,t)^{2/p}$,

$$
\begin{gathered}
|L_{\text{(10a)}}|\leq C_2\omega(s,t)^{(2+\gamma)/p},\ |L_{\text{(10c)}}|\leq C_3\omega(s,t)^{3/p}, |L_{\text{(10d)}}|\leq C_4\omega(s,t)^{3/p}\\
|L_{\text{(10e)}}|\leq C_5\omega(s,t)^{3/p}\text{ and }|L_{\text{(10f)}}|\leq C_6\omega(s,t)^{3/p},
\end{gathered}
$$

with $C_2\leq H_\gamma(\nabla f)$, $C_3\leq\|\nabla f\circ z\|_\infty$, $C_4\leq 2\|f\circ z\|_\infty\|\nabla f\circ z\|_\infty$, $C_5\leq\|f\circ z\|_\infty\|\nabla f\circ z\|_\infty$ and $C_6\leq\|\nabla f\circ z\|_\infty$.

Finally,

$$|\Upsilon_{s,r,t}| \leq C_7\omega(s,t)^{3/p}$$

where C_7 depends only on $\|f \circ z\|_\infty$, $\|\nabla f \circ z\|_\infty$ and $\|z\|_{p,\omega}$.

To summarize all the inequalities, for $(s,r,t) \in \Delta_3$,

$$|y_{s,r,t}| \leq C_8\omega(s,t)^{(2+\gamma)/p},$$

for some constant C_8 that depends only on $N_\gamma(\nabla f)$, $\|f \circ z\|_\infty$ ω, T, γ, p and $\|z\|_{p,\omega}$. In addition, on gets easily that

$$\|y\|_{p.\omega} \leq \max\{\|f \circ z\|_\infty + \|\nabla f \circ z\|_\infty \omega(0,T)^{1/p}, \|f \circ z\|_\infty^2\}$$

and then that y is an almost rough path. Of course, the rough path associated to y is z from the very definition of the integral of a differential form along the rough path z. $\square$

The proof of the following lemma is immediate and will be used to localize.

Lemma 8. *Let us assume that z is a rough path of finite p-variations in $T^2(\mathrm{U} \oplus \mathrm{V})$ such for some $\rho > 0$, $|z_t| \leq \rho$ for $t \in [0,T]$ and let us consider two vector fields f and g in $\mathrm{Lip}(1+\gamma)$ with $f = g$ for $|x| \leq \rho$. Then $\int_0^t \mathfrak{D}(f)(z_s)\,\mathrm{d}z_s = \int_0^t \mathfrak{D}(g)(z_s)\,\mathrm{d}z_s$ for all $t \in [0,T]$.*

5 Proof of Theorems 2 and 3 on existence of solutions

We prove first Theorem 3 and then Theorem 2 whose proof is much more simpler.

5.1 The non-linear case: proof of Theorem 3

Let f be a function in $\mathrm{Lip}(1+\gamma)$ with $\gamma \in [0,1]$, $2+\gamma > p$. Let us consider a rough path z in $T^2(\mathrm{U}\oplus\mathrm{V})$ whose projection on $T^2(\mathrm{U})$ is x. We set $\zeta \stackrel{\text{def}}{=} \|x\|_{p,\omega}$. Let us set $z^1 = \pi_{\mathrm{V}}(z)$, $z^\times = \pi_{\mathrm{V}\otimes\mathrm{U}}(z)$, $x^1 = \pi_{\mathrm{U}}(x)$, $x^2 = \pi_{\mathrm{U}\otimes\mathrm{U}}(x)$. We assume that for some $R \geq 1$

$$|z^1_{s,t}| \leq R\omega(s,t)^{1/p} \text{ and } |z^\times_{s,t}| \leq R\omega(s,t)^{2/p}$$

for all $(s,t) \in \Delta^2$. Let us also set $\|f \circ z\|_{\infty,t} = \sup_{s\in[0,t]} |f(\pi_{\mathrm{V}}(z_s))|$.

Let $\widehat{z}$ be rough path defined by

$$\widehat{z}_t = z_0 + \int_0^t f(z_s)\,\mathrm{d}x_s.$$

Indeed, to define $\widehat{z}$, we only need to know x, z^1 and $z^\times$. This is why we only get some control over the terms $\widehat{z}^1 = \pi_{\mathrm{V}}(\widehat{z})$ and $\widehat{z}^\times = \pi_{\mathrm{V}\otimes\mathrm{U}}(\widehat{z})$. Let us consider first

$$y^1_{s,t} = f(z_s)x^1_{s,t} + \nabla f(z_s) z^\times_{s,t},$$

so that

$$y^1_{s,r,t} \stackrel{\text{def}}{=} y^1_{s,t} - y^1_{s,r} - y^1_{r,t} = \int_0^1 (\nabla f(z^1_s + \tau z^1_{s,r}) - \nabla f(z^1_s)) z^1_{s,r} x^1_{r,t} \,\mathrm{d}\tau \\ + (\nabla f(z^1_r) - \nabla f(z^1_s)) z^\times_{r,t}$$

and then

$$|y^1_{s,r,t}| \leq (1+\zeta) H_\gamma(f) R^{1+\gamma} \omega(s,t)^{(2+\gamma)/p}.$$

It follows that for some universal constant K_5,

$$|\widehat{z}^1_{s,t} - y^1_{s,t}| \leq (1+\zeta) K_5 H_\gamma(f) R^{1+\gamma} \omega(s,t)^{(2+\gamma)/p}. \tag{12}$$

On the other hand

$$|y^1_{s,t}| \leq \|f\circ z\|_{\infty,t} \omega(s,t)^{1/p} \zeta + R \|\nabla f\|_\infty \omega(s,t)^{2/p}. \tag{13}$$

Let us consider also

$$y^2_{s,t} = x^1_{s,t} + \widehat{z}^1_{s,t} + f(y_s) \otimes 1 \cdot x^2_{s,t} \in T^2(\mathrm{U}) \oplus \mathrm{V} \oplus (\mathrm{V}\otimes\mathrm{U}).$$

This way, if $y^2_{s,r,t} \stackrel{\text{def}}{=} \pi_{T_1(\mathrm{U})\oplus\mathrm{V}\oplus\mathrm{V}\otimes\mathrm{U}}(y^2_{s,t} - y^2_{s,r} \otimes y^2_{r,t})$, then for a partition $\{t_i\}_{i=1,\dots,n}$ of $[0,T]$ and $0 \leq k \leq n-2$,

$$\pi_{T_1(\mathrm{U})\oplus\mathrm{V}\oplus\mathrm{V}\otimes\mathrm{U}}(y^2_{t_0,t_1} \otimes \dots \otimes y^2_{t_k,t_{k+1}} \otimes y^2_{t_{k+1},t_{k+1}} \otimes \dots \otimes y^2_{t_{n-1},t_n}) \\ - \pi_{T_1(\mathrm{U})\oplus\mathrm{V}\oplus\mathrm{V}\otimes\mathrm{U}}(y^2_{t_0,t_1} \otimes \dots \otimes y^2_{t_k,t_{k+2}} \otimes \dots \otimes y^2_{t_{n-1},t_n}) = y^2_{t_k,t_{k+1},t_{k+2}}. \tag{14}$$

Since for all $(s,r,t) \in \Delta^3$,

$$y^2_{s,r,t} = (f(y_s)x^1_{s,r} - \widehat{z}^1_{s,r}) \otimes x^1_{r,t} - (f(z_r) - f(z_s)) \otimes 1 \cdot x^2_{r,t}$$

and

$$|f(y_s)x^1_{s,r} - \widehat{z}^1_{s,r}| \leq (1+\zeta) K_5 H_\gamma(f) R^{1+\gamma} \omega(s,t)^{(2+\gamma)/p} + \|\nabla f\|_\infty R \omega(s,t)^{2/p},$$

it follows that

$$|\pi_{\mathrm{V}\otimes\mathrm{U}}(y^2_{s,r,t})| \leq 2\zeta \|\nabla f\|_\infty R \omega(s,t)^{3/p} + (1+\zeta)\zeta K_5 H_\gamma(f) R^{1+\gamma} \omega(s,t)^{(3+\gamma)/p}.$$

From (14), since $\widehat{z}^{\times}_{s,t}$ is the limit of $\pi_{\mathrm{V}\otimes\mathrm{U}}(\bigotimes_{i=1}^{n-1} y^2_{t^n_i,t^n_{i+1}})$ over a family of partitions $\{t^n_i\}_{i=1,\dots,n}$ whose meshes decrease to 0 as n goes to infinity, it follows from standard argument that for the universal constant K_5,

$$|\widehat{z}^{\times}_{s,t} - \pi_{V\otimes U}(y^2_{s,t})| \leq 2\zeta K_5 R \|\nabla f\|_\infty \omega(s,t)^{3/p} \\ + (1+\zeta)K_5^2 H_\gamma(f) R^{1+\gamma}\omega(s,t)^{(3+\gamma)/p}. \quad (15)$$

On the other hand for all $(s,t)\in\Delta^2$,

$$\|\pi_{V\otimes U}(y^2_{s,t})\| \leq \zeta \|f\circ z\|_{\infty,t}\omega(s,t)^{2/p}. \quad (16)$$

For $\tau\in(0,T]$, consider a positive quantity $s_\tau(f)$ such that

$$s_\tau(f) \geq \|f\circ z\|_{\infty,\tau},\ \tau\in[0,T].$$

and we choose λ such that $R=\lambda s_\tau(f)$. From (12) and (13), for $0\leq s\leq t\leq \tau$,

$$|\widehat{z}^1_{s,t}| \leq s_\tau(f)\omega(s,t)^{1/p}\big(\zeta + \lambda\|\nabla f\|_\infty \omega(0,\tau)^{1/p} \\ + (1+\zeta)K_5 H_\gamma(f) s_\tau(f)^\gamma \lambda^{1+\gamma}\omega(0,\tau)^{(1+\gamma)/p}\big). \quad (17)$$

and from (15) and (16),

$$|\widehat{z}^{\times}_{s,t}| \leq s_\tau(f)\omega(s,t)^{2/p}\big(\zeta + \lambda 2\zeta K_5\|\nabla f\|_\infty \omega(0,\tau)^{1/p} \\ + (1+\zeta)K_5^2 H_\gamma(f) s_\tau(f)^\gamma \lambda^{1+\gamma}\omega(0,\tau)^{(1+\gamma)/p}\big). \quad (18)$$

If

$$\|\widehat{z}\|_{p,\omega,\tau} \stackrel{\text{def}}{=} \sup_{0\leq s\leq t\leq\tau} \max\left\{\frac{|z^1_{s,t}|}{\omega(s,t)^{1/p}}, \frac{|z^{\times}_{s,t}|}{\omega(s,t)^{1/p}}\right\}$$

and

$$C_9 \stackrel{\text{def}}{=} \max\{1, 2\zeta K_5\}\|\nabla f\|_\infty \text{ and } C_{10} \stackrel{\text{def}}{=} (1+\zeta)K_5^2 H_\gamma(f)$$

one deduce from (17) and (18) that

$$\|\widehat{z}\|_{p,\omega,\tau} \leq s_\tau(f)(\zeta + \lambda C_9\omega(0,\tau)^{1/p} + \lambda^{1+\gamma}C_{10}s_\tau(f)^\gamma\omega(0,\tau)^{(1+\gamma)/p}).$$

Let us assume that τ is such that $C_9\omega(0,\tau)^{1/p} < 1/4$ and set

$$\lambda \stackrel{\text{def}}{=} \frac{\zeta}{1-2C_9\omega(0,\tau)^{1/p}} \leq 2\zeta.$$

Now, we assume that τ is such that

$$\zeta + \lambda C_9\omega(0,\tau)^{1/p} + \lambda^{1+\gamma}C_{10}s_\tau(f)^\gamma\omega(0,\tau)^{(1+\gamma)/p} \leq \zeta + 2\lambda C_9\omega(0,\tau)^{1/p} = \lambda$$

which means that

$$\lambda^{1+\gamma}\omega(0,\tau)^{(1+\gamma)/p}s_\tau(f)^\gamma C_{10} \leq \lambda C_9\omega(0,\tau)^{1/p}$$

and then that

$$\lambda^\gamma\omega(0,\tau)^{\gamma/p}s_\tau(f)^\gamma C_{10} \leq C_9. \tag{19}$$

If $C_{10} > 0$ (the case $C_{10} = 0$ corresponds to $H_\gamma(\nabla f) = 0$ and then to the linear case), (19) is true if

$$\alpha_{0,\tau}s_\tau(f) \leq \rho \tag{20}$$

with

$$\alpha_{s,t} \stackrel{\text{def}}{=} \frac{\zeta\omega(s,t)^{1/p}}{1-2C_9\omega(s,t)^{1/p}} \leq 2\zeta\omega(s,t)^{1/p}$$

and

$$\rho \stackrel{\text{def}}{=} \left(\frac{C_9}{C_{10}}\right)^\gamma \text{ with } \frac{C_9}{C_{10}} \in \left[\frac{1}{K_5^2}\frac{\|\nabla f\|_\infty}{H_\gamma(\nabla f)}, \frac{2}{K_5}\frac{\|\nabla f\|_\infty}{H_\gamma(\nabla f)}\right].$$

Such a choice is possible, as $\omega(0,t)$ decreases to 0 when t decreases to 0.

This choice implies that

$$\|\widehat{z}\|_{p,\omega,\tau} \leq \lambda s_\tau(f) = R$$

and owing to (19),

$$|\widehat{z}_{0,t}| \leq \|\widehat{z}\|_{p,\omega,\tau}\omega(0,\tau)^{1/p} \leq \rho.$$

for $t \in [0,\tau]$. The constant ρ depends only on ζ, $H_\gamma(\nabla f)$ and $\|\nabla f\|_\infty$.

To summarize, if for τ small enough (with τ such that $C_9\omega(0,\tau)^{1/p} < 1/4$),

$$\|f\circ z\|_{\infty,\tau} \leq s_\tau(f) \text{ and } \|z\|_{p,\omega,\tau} \leq \lambda s_\tau(f)$$

and (20) holds then $\widehat{z}$ satisfies

$$\|\widehat{z}\|_{p,\omega,\tau} \leq \lambda s_\tau(f),\ \sup_{t\in[0,\tau]}|\widehat{z}_{0,t}| \leq \rho \text{ and } \|f\circ\widehat{z}\|_{\infty,\tau} \leq G(z_0)$$

with

$$G(z_0) \stackrel{\text{def}}{=} \sup_{a\in\mathrm{V}\text{ s.t. }|a-z_0|\leq\rho} |f(a)|.$$

Consequently, if τ is such that

$$\alpha_{0,\tau}G(z_0) \leq \rho$$

and z is such that

$$|z^1_{0,t}| \leq \rho,\ t\in[0,\tau] \text{ and } \|z\|_{p,\omega,\tau} \leq \lambda G(z_0), \tag{21}$$

then, since $\|f\circ z\|_{\infty,\tau}\leq G(z_0)$, the path $\widehat{z}$ also satisfies (21).

Let us consider the set of paths with values in $\mathrm{V}\oplus(\mathrm{V}\otimes\mathrm{U})$ starting from z_0 and such that $z_{s,t}=z_{s,r}+z_{r,t}+z_{s,r}\otimes x_{r,t}$ for all $(s,r,t)\in\Delta^3$ and that satisfies $\|z\|_{p,\omega,\tau}\leq\lambda G(z_0)$ and $|z_{0,t}|\leq\rho$ for all $t\in[0,\tau]$. Clearly, this is a closed, convex ball. By the Ascoli-Arzelà theorem, it is easily checked that this set relatively compact in the topology generated by the norm $\|\cdot\|_{q,\omega}$ for any $q>p$. And any function in this set is such that $\sup_{t\in[0,t]}|f(z_t)|\leq G(z_0)$.

By the Schauder fixed point theorem, there exists a solution to

$$z_t=z_0+\int_0^t f(z_s)\,\mathrm{d}x_s \tag{22}$$

living in $T^2(\mathrm{U})\oplus\mathrm{V}\oplus(\mathrm{V}\otimes\mathrm{U})$. This solution may be lifted as a genuine rough path $\widetilde{z}$ with values in $T^2(\mathrm{U}\oplus\mathrm{V})$ associated to the almost rough path

$$\widetilde{y}_{s,t}=z_{s,t}+1\otimes f(z_s)\cdot x^2_{s,t}+f(z_s)\otimes f(z_s)\cdot x^2_{s,t}.$$

The arguments are similar to those in [10]. The uniqueness for a $\mathrm{Lip}(2+\gamma)$-vector field f follows from the uniqueness of the solution of a rough differential equation in the case of a bounded vector field, as thanks to Lemma 8, one may assume that f is bounded. We may also use Theorem 4.

Remark 6. If f is a $\mathrm{Lip}(2+\gamma)$, then from the computations used in the proof of Theorem 4, one may proved that the Picard scheme will converge (see Remark 7 below). This can be used in the infinite dimensional setting where a ball is not compact and then the Schauder theorem cannot be used because the set of paths with a p-variation smaller than a given value is no longer relatively compact.

We have solved (22) on the time interval $[0,\tau]$ in order to have $\|z\|_{p,\omega,\tau}\leq 2\zeta G(z_0)$ (if $C_9\omega(0,\tau)^{1/p}<1/4$) and $|z_{0,t}|\leq\rho$ for $t\in[0,\tau]$.

It remains to estimate the p-variation norm of $\widetilde{z}$ in order to complete the proof. In this case, the computations are similar for $\pi_{\mathrm{U}\otimes\mathrm{V}}(\widetilde{z}_{s,t})$ as for $\pi_{\mathrm{V}\otimes\mathrm{U}}(\widetilde{z}_{s,t})=\pi_{\mathrm{V}\otimes\mathrm{U}}(z_{s,t})$ since

$$|\pi_{\mathrm{U}\otimes\mathrm{V}}(\widetilde{z}_{s,t})|\leq\lambda G(z_0)\leq 2\zeta G(z_0)\omega(s,t)^{2/p}.$$

Since $\max\{1,2\zeta K_5\}\|\nabla f\|_\infty\omega(0,\tau)^{1/p}\leq 1/4$, $\lambda\leq 2\zeta$ and using (19),

$$\begin{aligned}|f(z_s)x^1_{s,r}-z^1_{s,r}|&\leq\|\nabla f\|_\infty\|z\|_{p,\omega}\omega(s,t)^{2/p}\\&\quad+\lambda\frac{\max\{1,2\zeta K_5\}}{K_5}\|\nabla f\|_\infty G(z_0)\omega(0,\tau)^{1/p}\omega(s,t)^{2/p}\\&\leq 2\zeta G(z_0)\omega(s,t)^{2/p}\left(\|\nabla f\|_\infty+\max\{K_5^{-1},2\zeta\}\omega(0,\tau)^{1/p}\|\nabla f\|_\infty\right)\\&\leq 2\zeta G(z_0)\omega(s,t)^{2/p}\left(\|\nabla f\|_\infty+K_6\right)\end{aligned}$$

for some universal constant K_6. On the other hand

$$|f(z_s) - f(z_r)| \leq \|\nabla f\|_\infty \lambda G(z_0)\omega(s,t)^{1/p} \leq 2\zeta\omega(s,t)^{1/p}G(z_0).$$

Hence

$$|\pi_{\mathrm{V}\otimes\mathrm{V}}(\widetilde{y}_{s,r,t})| \leq 4\zeta^2\omega(s,t)^{3/p}G(z_0)^2 + 4\zeta G(z_0)^2\omega(s,t)^{3/p}(\|\nabla f\|_\infty + K_6)$$

and then

$$|\pi_{\mathrm{V}\otimes\mathrm{V}}(\widetilde{z}_{s,t}) - f(z_s)\otimes f(z_s)x^2_{s,t}| \leq 4K_5\zeta G(z_0)^2\omega(s,t)^{3/p}(\zeta + \|\nabla f\|_\infty + K_6).$$

It follows that

$$|\pi_{\mathrm{V}\otimes\mathrm{V}}(\widetilde{z}_{s,t})| \leq K_7\zeta G(z_0)^2\omega(s,t)^{2/p}(1 + \|\nabla f\|_\infty + \zeta)$$

for some universal constant K_7.

5.2 The linear case: proof of Theorem 2

The linear case is simpler. Let us write $f(z_t) = Az_t$ and let us set

$$y^1_{s,t} = Az_s \cdot x^1_{s,t} + Az^\times_{s,t}.$$

Since $z^\times_{s,t} = z^\times_{s,r} + z^\times_{r,t} + z^1_{s,r}\otimes x^1_{r,t}$, one gets that $y^1_{s,r,t} = 0$ for any $(s,r,t)\in\Delta^3$. Again, let us set $\zeta \stackrel{\text{def}}{=} \|x\|_{p,\omega}$.

This way, $\widehat{z}_t = z_0 + \int_0^t Az_s\,\mathrm{d}x_s$ satisfies

$$\widehat{z}^1_{s,t} = y^1_{s,t},\ (s,t)\in\Delta^2.$$

Also, for

$$y^2_{s,t} = x_{s,t} + \widehat{z}^2_{s,t} + Az_s\otimes 1\cdot x^2_{s,t},$$

one obtains that

$$y^2_{s,r,t} = A(z_s - z_r)\otimes 1\cdot x^2_{r,t} - Az^\times_{s,r}\otimes x^1_{r,t}.$$

Hence for all $0\leq s\leq t\leq\tau$ for a given $\tau < T$

$$|\widehat{z}^\times_{s,t} - \pi_{\mathrm{V}\otimes\mathrm{U}}(y^2_{s,t})| \leq 2K_5\|A\|\|z\|_{p,\omega,\tau}\zeta\omega(s,t)^{3/p}$$

and thus

$$|\widehat{z}^\times_{s,t}| \leq \|A\|\omega(s,t)^{2/p}(\zeta|z_s| + K_5\zeta\|z\|_{p,\omega,\tau}\omega(s,t)^{1/p}). \tag{23}$$

On the other hand,

$$|\widehat{z}^1_{s,t}| \leq \|A\|\omega(s,t)^{1/p}(\zeta|z_s| + \|z\|_{p,\omega,\tau}\omega(s,t)^{1/p}). \tag{24}$$

As $|z_s| \leq |z_0| + \zeta\omega(0,s)^{1/p}\|z^1\|_{p,\omega,\tau}$, it follows from (24) and (23) that

$$\|\widehat{z}\|_{p,\omega,\tau} \leq \|A\| \max\{1,\zeta\}(|z_0| + K_8\|z\|_{p,\omega,\tau}\omega(0,\tau)^{1/p})$$

for some universal constant K_8.

Set $\beta = \|A\| \max\{1,\zeta\}$. Assume that $\|z\|_{p,\omega,\tau} \leq L|z_0|$ for some $L > 0$. Then

$$\|\widehat{z}\|_{p,\omega,\tau} \leq \beta(1 + LK_8\omega(0,\tau)^{1/p})|z_0|.$$

Fix $0 < \eta < 1$ and choose τ such that

$$\beta K_8\omega(0,\tau)^{1/p} \leq \eta$$

then set

$$L \stackrel{\text{def}}{=} \frac{\beta}{1 - \beta K_8\omega(0,\tau)^{1/p}} \leq \frac{\beta}{1-\eta}$$

so that

$$\beta(1 + LK_8\omega(0,\tau)^{1/p}) \leq L$$

is satisfied and then

$$\|\widehat{z}\|_{p,\omega,\tau} \leq L|z_0| \text{ and } |\widehat{z}_{0,t}| \leq L|z_0|\omega(0,\tau)^{1/p}. \tag{25}$$

By the Schauder fixed point theorem, there exists a solution in the space of paths z with values in $\mathrm{V}\oplus(\mathrm{V}\otimes\mathrm{U})$ starting from z and satisfying (25) as well as $\widehat{z}_{s,t} = \widehat{z}_{s,r} + \widehat{z}_{r,t} + \widehat{z}_{s,r}\otimes x^1_{r,t}$. We do not prove uniqueness of the solution, which follows from the computations of Section 6.

This way, it is possible to solve globally $z_t = z_0 + \int_0^t Az_s\,\mathrm{d}x_s$ by solving this equation on time intervals $[\tau_i,\tau_{i+1}]$ with $\omega(\tau_i,\tau_{i+1}) = (\eta/\beta K_8)^p$, assuming that ω is continuous. The number N of such intervals is the smallest integer for which $N(\eta/\beta K_8)^p \geq \omega(0,T)$. Thus, since $|z_{\tau_i,\tau_{i+1}}| \leq L|z_{\tau_i}|\omega(\tau_i,\tau_{i+1})^{1/p}$, we get that for $i = 0,1,\dots,N-1$,

$$\begin{aligned}\sup_{s\in[\tau_i,\tau_{i+1}]} |z_{\tau_i,t}| &\leq |z_{\tau_i}|\left(1 + \frac{\eta}{1-\eta}\frac{1}{K_8}\right) \leq |z_0|\left(1 + \frac{\eta}{1-\eta}\frac{1}{K_8}\right)^{N-1}\\ &\leq |z_0|\left(1 + \frac{\eta}{1-\eta}\frac{1}{K_8}\right)^{\omega(0,T)(\eta/\beta K_8)^{-p}}\end{aligned}$$

which leads to (2).

6 Proof of Theorem 4 on the Lipschitz continuity of the Itô map

In order to understand how we get the Lipschitz continuity under the assumption that f and $\widehat{f}$ belongs to $\mathrm{Lip}_{\mathrm{LG}}(2+\gamma)$ with $2+\gamma > p$, we evaluate first the distance between to almost rough paths associated to the solutions of controlled differential equations when the vector fields only belong to $\mathrm{Lip}_{\mathrm{LG}}(1+\gamma)$. Without loss of generality, we assume that indeed f and $\widehat{f}$ are bounded and belong to $\mathrm{Lip}(1+\gamma)$.

6.1 On the distance between the almost rough paths associated to controlled differential equations

For a rough path $\widehat{x}$ of finite p-variation, and let $\widehat{z}$ be the solution to $\widehat{z}_t = \widehat{z}_0 + \int_0^t \widehat{f}(\widehat{z}_s)\,\mathrm{d}\widehat{x}_s$. Define $\widehat{y}$ with the same formula as in (9) by replacing z (resp. f, x) by $\widehat{z}$ (resp. $\widehat{f}$, $\widehat{x}$).

We set $z^1 = \pi_{\mathrm{V}}(z)$ and $\widehat{z}^1 = \pi_{\mathrm{V}}(\widehat{z})$, and we set $\|z_s\|_{\infty,[s,t]} \stackrel{\mathrm{def}}{=} \sup_{r\in[s,t]} |z_r|$. In addition, we assume that z and $\widehat{z}$ are such that $\max\{\|z\|_\infty, \|z\|_{p,\omega}\} \leq \rho$ and $\max\{\|\widehat{z}\|_\infty, \|\widehat{z}\|_{p,\omega}\} \leq \rho$.

In the following, the constants C_i depend on $\|f\|_\infty$, $N_\gamma(\nabla f)$, ω, T, p, κ, $\|x\|_{p,\omega}$ and ρ.

Let us note that

$$\begin{aligned}
|y^1_{s,t} - \widehat{y}^1_{s,t}| \leq |(f(z_s) - f(\widehat{z}_s))x^1_{s,t}| + |(f(\widehat{z}_s) - \widehat{f}(\widehat{z}_s))x^1_{s,t}| + |\widehat{f}(\widehat{z}_s)(x^1_{s,t} - \widehat{x}^1_{s,t})| \\
+ |(\nabla f(z_s) - \nabla f(\widehat{z}_s))z^\times_{s,t}| + |(\nabla f(\widehat{z}_s) - \nabla \widehat{f}(\widehat{z}_s))z^\times_{s,t}| \\
+ |\nabla \widehat{f}(\widehat{z}_s)(z^\times_{s,t} - \widehat{z}^\times_{s,t})| + |x^1_{s,t} - \widehat{x}^1_{s,t}|.
\end{aligned}$$

It follows that

$$\begin{aligned}
|y^1_{s,t} - \widehat{y}^1_{s,t}| \leq C_{11}(\|z^1 - \widehat{z}^1\|_{\infty,[s,t]} + \delta_\rho(f,\widehat{f}) + \delta(x,\widehat{x}))\omega(s,t)^{1/p} \\
+ H_\gamma(\nabla f)\big(\|z^1 - \widehat{z}^1\|^\gamma_{\infty,[s,t]} + \delta_\rho(\nabla f, \nabla \widehat{f})\big)\|z\|_{p,\omega}\|x\|_{p,\omega}\omega(s,t)^{2/p} \\
+ \|\nabla \widehat{f}\|_\infty \delta(z,\widehat{z})\|x\|_{p,\omega}\omega(s,t)^{2/p}
\end{aligned}$$

with

$$C_{11} \leq \max\{\|x\|_{p,\omega}, \|\widehat{x}\|_{p,\omega}\}\max\{1, \|\widehat{f}\|_\infty, \|\nabla f\|_\infty\}.$$

With similar computations,

$$|y^2_{s,t} - \widehat{y}^2_{s,t}| \leq C_{12}(\delta_\rho(f,\widehat{f}) + \delta(x,\widehat{x}) + \|z^1 - \widehat{z}^1\|_{\infty,[s,t]})\omega(s,t)^{2/p}.$$

We are now willing to estimate $|y_{s,r,t}-\widehat{y}_{s,r,t}|$. By watching at the expressions (10a)–(10d), we get

$$
\begin{aligned}
|y_{s,r,t}-\widehat{y}_{s,r,t}| \leq & C_{13}(\delta_\rho(f,\widehat{f})+\delta_\rho(\nabla f,\nabla\widehat{f})+\delta(x,\widehat{x}))\omega(s,t)^{(2+\gamma)/p} && \text{(26a)}\\
&+|(f(z_s)-f(z_r))x^1_{r,t}+\nabla f(z_s)z^1_{s,r}\otimes x^1_{r,t} && \text{(26b)}\\
&\quad -(f(\widehat{z}_s)-f(\widehat{z}_r))x^1_{r,t}-\nabla f(\widehat{z}_s)\widehat{z}^1_{s,r}\otimes x^1_{r,t}| && \text{(26c)}\\
&+|(f(z_s)-f(z_r))\otimes 1\cdot x^2_{r,t}+1\otimes(f(z_s)-f(z_r))\cdot x^2_{r,t} && \text{(26d)}\\
&\quad -(f(\widehat{z}_s)-f(\widehat{z}_r))\otimes 1\cdot x^2_{r,t}+1\otimes(f(\widehat{z}_s)-f(\widehat{z}_r))\cdot x^2_{r,t}| && \text{(26e)}\\
&+|(f(z_s)\otimes f(z_s)-f(z_r)\otimes f(z_r))\cdot x^2_{r,t} && \text{(26f)}\\
&\quad -(f(\widehat{z}_s)\otimes f(\widehat{z}_s)-f(\widehat{z}_r)\otimes f(\widehat{z}_r))\cdot x^2_{r,t}| && \text{(26g)}\\
&+|f(z_s)\otimes(f(z_s)-f(z_r))\cdot x^1_{s,r}\otimes x^1_{r,t} && \text{(26h)}\\
&\quad -f(\widehat{z}_r)\otimes(f(\widehat{z}_s)-f(\widehat{z}_r))\cdot x^1_{s,r}\otimes x^1_{r,t}| && \text{(26i)}\\
&+|1\otimes(f(z_s)-f(z_r)-f(\widehat{z}_s)+f(\widehat{z}_r))\cdot x^1_{s,r}\otimes x^1_{r,t}| && \text{(26j)}\\
&+|\Upsilon_{s,r,t}-\widehat{\Upsilon}_{s,r,t}|, && \text{(26k)}
\end{aligned}
$$

where $\widehat{\Upsilon}_{s,r,t}$ is similar to $\Upsilon_{s,r,t}$ defined by (11) with f, x and z replaced by $\widehat{f}$, $\widehat{x}$ and $\widehat{z}$. First we consider Lines (26d)-(26j). For this, let us note that

$$
\begin{aligned}
|(f(z_r)-f(z_s))-(f(\widehat{z}_r)-f(\widehat{z}_s))| &= \Big|\int_0^1 \nabla f(z_s+\tau(\widehat{z}_s-z_s))(\widehat{z}_s-z_s)\,\mathrm{d}\tau\\
&\qquad -\int_0^1 \nabla f(z_r+\tau(\widehat{z}_r-z_r))(\widehat{z}_r-z_r)\,\mathrm{d}\tau\Big|\\
&\leq \left|\int_0^1 (\nabla f(z_s+\tau(\widehat{z}_s-z_s))-\nabla f(z_r+\tau(\widehat{z}_r-z_r)))(\widehat{z}_r-z_r)\,\mathrm{d}\tau\right|\\
&\qquad +\left|\int_0^1 \nabla f(z_r+\tau(\widehat{z}_r-z_r))(\widehat{z}^1_{s,r}-z^1_{s,r})\,\mathrm{d}\tau\right|\\
&\leq \|z^1-\widehat{z}^1\|_{\infty,[s,t]}H_\gamma(\nabla f)\int_0^1 |\tau(z^1_s-z^1_r)+(1-\tau)(\widehat{z}^1_s-\widehat{z}^1_r)|^\gamma\,\mathrm{d}\tau\\
&\qquad +\|\nabla f\|_\infty\delta(z,\widehat{z})\omega(s,t)^{1/p}\\
&\leq C_{14}\omega(s,t)^{\gamma/p}\|z^1-\widehat{z}^1\|_{\infty,[s,t]}(\|z\|^\gamma_{p,\omega}+\|\widehat{z}\|^\gamma_{p,\omega})\\
&\qquad +C_{15}\delta(z,\widehat{z})\omega(s,t)^{1/p}. \qquad (27)
\end{aligned}
$$

With this, we get a control over Lines (26d)-(26e), (26f)-(26g), (26h)-(26i) and (26j) of type

$$
\begin{aligned}
|L_{(26\mathrm{d})-(26\mathrm{e})} + L_{(26\mathrm{f})-(26\mathrm{g})} + L_{(26\mathrm{h})-(26\mathrm{i})} + L_{(26\mathrm{j})}|& \\
\leq C_{16}\omega(s,t)^{(2+\gamma)/p}\|z^1-\widehat{z}^1\|_{\infty,[s,t]}(\|z\|_{p,\omega}^\gamma + \|\widehat{z}\|_{p,\omega}^\gamma)& \\
+ C_{17}\delta(z,\widehat{z})\omega(s,t)^{3/p}.&
\end{aligned}
$$

Similarly, we have

$$
\begin{aligned}
|L_{(26\mathrm{k})}| \leq C_{18}\big(\|z^1-\widehat{z}^1\|_{\infty,[s,t]}^\gamma + \|z^1-\widehat{z}^1\|_{\infty,[s,t]}& \\
+ \delta(z,\widehat{z}) + \delta(x,\widehat{x}) + \delta_\rho(f,\widehat{f}) + \delta_\rho(\nabla f,\nabla\widehat{f})\big).&
\end{aligned}
$$

For (26b)-(26c), we use the same kind of computations as we did for $L_{(10\mathrm{b})}$ and then,

$$
\begin{aligned}
|L_{(26\mathrm{b})-(26\mathrm{c})}| \leq \omega(s,t)^{1/p}\|x\|_{p,\omega}\Big|\int_0^1(\nabla f(z_s+\tau z^1_{s,r}) - \nabla f(z_s))z^1_{s,r}\,\mathrm{d}\tau& \\
- \int_0^1(\nabla f(\widehat{z}_s+\tau\widehat{z}^1_{s,r}) - \nabla f(\widehat{z}_s))\widehat{z}^1_{s,r}\,\mathrm{d}\tau\Big|& \\
\leq \omega(s,t)^{1/p}\|x\|_{p,\omega}\Big|\int_0^1((\nabla f(z_s+\tau z^1_{s,r}) - \nabla f(z_s)) - (\nabla f(\widehat{z}_s+\tau\widehat{z}^1_{s,r}) - \nabla f(\widehat{z}_s)))z^1_{s,r}\,\mathrm{d}\tau\Big|& \\
+ \omega(s,t)^{1/p}\|x\|_{p,\omega}\Big|\int_0^1(\nabla f(\widehat{z}_s+\tau\widehat{z}^1_{s,r}) - \nabla f(\widehat{z}_s))(\widehat{z}^1_{s,r}-z^1_{s,r})\,\mathrm{d}\tau\Big|.&
\end{aligned}
$$

The last term in the previous inequality is bounded by the quantity

$$
H_\gamma(\nabla f)\|x\|_{p,\omega}\omega(s,t)^{(2+\gamma)/p}\|\widehat{z}\|_{p,\omega}^\gamma\delta(z,\widehat{z}). \tag{28}
$$

On the other hand, for $\tau\in[0,1]$, if

$$
\Delta \stackrel{\mathrm{def}}{=} |((\nabla f(z_s+\tau z^1_{s,r}) - \nabla f(z_s)) - (\nabla f(\widehat{z}_s+\tau\widehat{z}^1_{s,r}) - \nabla f(\widehat{z}_s)))|
$$

then

$$
\Delta \leq \begin{cases} 2H_\gamma(\nabla f)\|z^1-\widehat{z}^1\|_{\infty,[s,t]}^\gamma \\ 2^{1-\gamma}H_\gamma(\nabla f)(\|z\|_{p,\omega}+\|\widehat{z}\|_{p,\omega})^\gamma\omega(s,t)^{\gamma/p}. \end{cases} \tag{29}
$$

Thus, we may choose $\eta\in(0,1)$ and combine the two terms in (29) to obtain

$$
\Delta \leq C_{19}\omega(s,t)^{\eta\gamma/p}(\|z\|_{p,\omega}+\|\widehat{z}\|_{p,\omega})^{\gamma\eta}\|z^1-\widehat{z}^1\|_{\infty,[s,t]}^{(1-\eta)\gamma}.
$$

It follows that with (28),

$$
\begin{aligned}
|L_{(26\mathrm{b})-(26\mathrm{c})}| \leq C_{20}\omega(s,t)^{(2+\gamma)/p}\delta(z,\widehat{z})& \\
+ C_{21}(\|z\|_{p,\omega}+\|\widehat{z}\|_{p,\omega})^{\gamma\eta}\|z^1-\widehat{z}^1\|_{\infty,[s,t]}^{(1-\eta)\gamma}\omega(s,t)^{(2+\eta\gamma)/p}&
\end{aligned} \tag{30}
$$

where C_{20} and C_{21} depend on $H_\gamma(\nabla f)$.

The case of a $\mathrm{Lip}(2+\kappa)$**-vector field.** If f belongs to $\mathrm{Lip}(2+\kappa)$ with $2+\kappa > p$, then one can set $\gamma = 1$ in the previous equations. In this case, one can give a better estimate on $L_{(26b)-(26c)}$, which we use below to prove the Lipschitz continuity of the Itô map. In this case, we use the same computation as in (27) by replacing f by ∇f to get

$$|L_{(26b)-(26c)}| \leq C_{22}\omega(s,t)^{3/p}\delta(z,\widehat{z}) + C_{23}\omega(s,t)^{(2+\kappa)/p}\|z^1 - \widehat{z}^1\|_{\infty,[s,t]} \\ + C_{24}\delta(z,\widehat{z})\omega(s,t)^{(2+\kappa)/p}, \quad (31)$$

where C_{22} and C_{23} depend on $H_\gamma(\nabla^2 f)$ and $\|\nabla^2 f\|_\infty$ and C_{24} depends on $\|\nabla f\|_\infty$.

On the difference between the two cases. In order to prove the Lipschitz continuity of the Itô map, which implies uniqueness using a regularization procedure, we will see that we will get an inequality of type $\delta(z,\widehat{z}) \leq A(1+C_T\delta(z,\widehat{z}))$ with C_T converges to 0, and then one can compare $\delta(z,\widehat{z})$ with A. This is possible thanks to (31). With (30), we get an inequality of type $\delta(z,\widehat{z}) \leq A(1+C_T\delta(z,\widehat{z})^\lambda)$ with $\lambda < 1$ and then it is impossible to deduce an inequality on $\delta(z,\widehat{z})$ in function of A when $\delta(z,\widehat{z}) \leq 1$. Anyway, in [5], A.M. Davie showed that there could exist several solutions to a rough differential equation for f in $\mathrm{Lip}(1+\gamma)$ but not in $\mathrm{Lip}(2+\gamma)$.

6.2 Proof of Theorem 4

Since the solutions of rough differential equations remains bounded, we may assume that f belongs to $\mathrm{Lip}(2+\kappa)$ instead of $\mathrm{Lip}_{\mathrm{LG}}(2+\kappa)$ with $2 \leq p \leq 2+\kappa < 3$.

Let us note that

$$\|z - \widehat{z}\|_\infty \leq \delta(z_0,\widehat{z}_0) + \alpha_{0,T}\delta(z,\widehat{z}) \quad (32)$$

with $\alpha_{s,t}$ defined by $\alpha_{s,t} \stackrel{\text{def}}{=} \max\{\omega(s,t)^{1/p}, \omega(s,t)^{2/p}\}$. Note that $\alpha_{0,T}$ decreases to 0 as T converges to 0.

Since $\gamma = 1$, we have obtained from (31) and (32) that for all $(s,r,t) \in \Delta^3$, we get that

$$|y_{s,r,t} - \widehat{y}_{s,r,t}| \leq C_{25}(\delta_\rho(f,\widehat{f}) + \delta(x,\widehat{x}) + \delta(z_0,\widehat{z}_0) \\ + \omega(s,t)^{(1-\kappa)/p}\delta(z,\widehat{z}) + \alpha_{0,T}\delta(z,\widehat{z}))\omega(s,t)^{(2+\kappa)/p}, \quad (33a)$$

that

$$|y^1_{s,t} - \widehat{y}^1_{s,t}| \leq C_{26}(\alpha_{0,T}\delta(z,\widehat{z}) + \delta(z_0,\widehat{z}_0) + \delta_\rho(f,\widehat{f}) + \delta(x,\widehat{x}))\omega(s,t)^{1/p} \\ + C_{27}((1+\alpha_{0,T})\delta(z,\widehat{z}) + \delta(z_0,\widehat{z}_0) + \delta_\rho(\nabla f,\nabla\widehat{f}))\omega(s,t)^{2/p}, \quad (33b)$$

that

$$|y^2_{s,t} - \widehat{y}^2_{s,t}| \leq C_{28}(\delta_\rho(f, \widehat{f}) + \delta(x, \widehat{x}) + \alpha_{0,T}\delta(z, \widehat{z}) + \delta(z_0, \widehat{z}_0))\omega(s,t)^{2/p}, \quad (33c)$$

and that

$$|y_{s,r,t}| \leq C_{29}\omega(s,t)^{3/p} \text{ and } |\widehat{y}_{s,r,t}| \leq C_{30}\omega(s,t)^{3/p}. \quad (33d)$$

For convenience, we assume that $\kappa < 1$.

Since z (resp. $\widehat{z}$) is the rough path associated to y (resp. z), it follows from (33a)-(33d) and Lemma 5 that

$$|y_{s,t} - z_{s,t}| \leq C_{31}\omega(s,t)^{3/p},\ |\widehat{y}_{s,t} - \widehat{z}_{s,t}| \leq C_{31}\omega(s,t)^{3/p}$$

and from Lemma 6 that

$$|z^1_{s,t} - \widehat{z}^1_{s,t}| \leq \beta(T)\omega(s,t)^{1/p} \text{ and } |z^2_{s,t} - \widehat{z}^2_{s,t}| \leq \beta(T)\omega(s,t)^{2/p} \quad (34)$$

with

$$\begin{aligned}\beta(T) \leq C_{32}\max\{&\alpha_{0,T}\delta(z, \widehat{z}) + \delta(z_0, \widehat{z}_0) + \delta_\rho(f, \widehat{f}) + \delta(x, \widehat{x})\alpha_{0,T},\\ &((1+\alpha_{0,T})\delta(z, \widehat{z}) + \delta(z_0, \widehat{z}_0) + \delta_\rho(\nabla f, \nabla\widehat{f}))\omega(0,T)^{1/p},\\ &(\delta_\rho(f, \widehat{f}) + \delta(x, \widehat{x})\alpha_{0,T} + \delta(z_0, \widehat{z}_0)\\ &+ \omega(0,T)^{(1-\kappa)/p}\delta(z, \widehat{z}) + \alpha_{0,T}\delta(z, \widehat{z}))\}.\end{aligned}$$

In particular, (34) means that $\delta(z, \widehat{z}) \leq \beta(T)$, but $\beta(T)$ also depends on $\delta(z, \widehat{z})$. More precisely, the constant $\beta(T)$ satisfies

$$\begin{aligned}\beta(T) \leq\ &C_{32}\delta(z, \widehat{z})(\alpha_{0,T} + (1+\alpha_{0,T})\omega(0,T)^{1/p} + \omega(0,T)^{(1-\kappa)/p})\\ &+ C_{32}(\delta_\rho(f, \widehat{f}) + \delta_\rho(\nabla f, \nabla\widehat{f}) + \delta(x, \widehat{x}) + \delta(z_0, \widehat{z}_0)).\end{aligned}$$

Remark 7. The same computations can be used in order to prove the convergence of the Picard scheme $z^{n+1} = z_0 + \int_0^t f(z^n_s)\,\mathrm{d}x_s$ of the Picard scheme, in which case

$$\delta(z^{n+1}, z^n) \leq C_{33}\max\{\alpha_{0,T}, \alpha^2_{0,T}, \omega(0,T)^{(1-\kappa)/p}\}\delta(z^n, z^{n-1})$$

and it is then possible to choose T small enough to get $\delta(z^{n+1}, z^n) \leq k\delta(z^n, z^{n-1})$ with $k < 1$.

Now, choose τ small enough so that for all $s \in [0, T-\tau]$,

$$C_{32}\left(\alpha_{s,s+\tau} + (1+\alpha_{s,s+\tau})\omega(s,s+\tau)^{1/p} + \omega(s,s+\tau)^{(1-\kappa)/p}\right) \leq \frac{1}{2}.$$

This is possible since by definition, $(s,t)\in\Delta^2\mapsto\omega(s,t)$ is continuous close to its diagonal.

Hence, one may choose τ small enough in order to get

$$\delta(z,\widehat{z})\leq 2C_{32}(\delta_\rho(f,\widehat{f})+\delta_\rho(\nabla f,\nabla\widehat{f})+\delta(x,\widehat{x})+\delta(z_0,\widehat{z}_0))$$

for $T\leq\tau$. By a standard stacking argument where z_0 (resp. $\widehat{z}_0$) is replaced recursively by $z_{k\tau}$ (resp. $\widehat{z}_{k\tau}$) for $k=1,2,\dots$, we can then obtain that

$$\delta(z,\widehat{z})\leq C_{34}(\delta_\rho(f,\widehat{f})+\delta_\rho(\nabla f,\nabla\widehat{f})+\delta(x,\widehat{x})+\delta(z_0,\widehat{z}_0)),$$

since the choice of τ does not depend on z_0.

In other words, the Itô map is locally Lipschitz continuous in all its arguments.

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