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Abstract

We introduce a notion of Kripke model for classical logic for which we constructively prove soundness and cut-free completeness. We discuss the meaning of the new notion and its applications to call-by-name proof normalisation.

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1. Introduction

Kripke models have been introduced as means of giving semantics to modal logics and were later used to give semantics for intuitionistic logic as well. [Kri59, Kri63]

The purpose of the present paper is to show that Kripke models can also be used as a semantics for *classical* logic. Of course, a Kripke semantics can be indirectly assigned to classical logic by means of some appropriate double-negation translation. Our purpose here is however to provide with a direct presentation of a notion of Kripke semantics for classical logic. Concretely, and because we are ultimately interested in the computational contents of classical logic, we will use the $LK_{\mu\tilde{\mu}}$ sequent calculus of [CH00] to represent proofs. However, the conclusions given here apply to any complete formal system for classical logic.

This paper is organised as follows. Section 2 introduces the notion of classical Kripke model, based on two modifications to the traditional notion, and discusses the relationship between the traditional and our notion. Section 3 introduces the sequent calculus $LK_{\mu\tilde{\mu}}$ and gives a soundness theorem for it. Section 4 proves a completeness theorem for a universal model constructed from the deduction system itself. Section 5 discusses the implications to proof normalisation (or $\bar{\lambda}\mu\tilde{\mu}$-term reduction) and anticipates that we actually get two notions of classical Kripke models, one corresponding to call-by-name and the other to call-by-value reduction from the theory of functional programming languages. Section 6 gives a summary of results and discusses related and future work. All statements and proofs are constructive.

2. Classical Kripke Models

Kripke models can be considered as the "most classical" of all the semantics for intuitionistic logic, for two reasons: first, each of the 'possible worlds' that define a Kripke model is a classical world in itself (where either an atom or its negation are true); second, it is the single of the semantics for intuitionistic logic which has only a classical proof of completeness, when disjunction and existential quantification are considered.[1]

In the last two decades, the Curry-Howard correspondence between intuitionistic proof systems and typed lambda-calculi has been extended to classical proof systems [Gri90, Par92, CH00]. The motivation for introducing Kripke models for classical logic comes from their usefulness in providing normalisation-by-evaluation for intuitionistic proof systems [Coq93, Coq02]. To account for a classical proof system we modify the traditional notion of Kripke model in the following two ways.

Not taking the forcing relation as primitive. We take as primitive the notion of "strong refutation", and define forcing in terms of it.[2] The forcing definition we get in this way partially coincides with the traditional definition of forcing, as explained in subsection 2.2.

Allowing certain nodes to validate absurdity. We allow certain possible worlds to be marked as "fallible", or "exploding". This approach has been taken for Kripke models in [Vel76], for Beth models by Friedman [TvD88] and seems necessary in order to have a constructive proof of completeness, in the view of the meta-mathematical results from [Kre62, McC94, McC02], which preclude constructive proofs of completeness in case one wants to retain that absurdity must never be valid in a possible world. (extending the class of Boolean models with inconsistent models is also the key to the constructive proof of the classical completeness theorem in [Kri96])

Definition 1. *A* classical Kripke model *is given by a poset* $(K, \leq)$ *of "possible worlds", together with a binary relation* $w : X \Vdash_s$ *of "strong refutation" between worlds w and atomic formulae X (monotone with respect to* $\leq$*), a unary relation on worlds* $w :\Vdash_\perp$ *labelling a world as "exploding", and a domain of quantification* $D(w)$ *for each world (monotone with respect to* $\leq$*). We also require that strong refutation be stable under substitution i.e. if* $w : A(x) \Vdash_s$ *then for any* $t \in D(w), w : A(t) \Vdash_s$.

The strong refutation relation is extended from atomic to composite formulae inductively and by mutually defining the relations of *forcing* and (non-strong) *refutation*.

Definition 2. *The relation* $(-) : (-) \Vdash_s$ *of* strong refutation is extended to composite formulae, *inductively, together with two new relations:*

- *A formula A is* forced *in the world w (notation* $w :\Vdash A$*) if any world* $w' \geq w$*, which strongly refutes A, is exploding;*

[1] an attempt to give a constructive proof has been made in [Vel76], but it makes use of the fan theorem which is not universally recognised as constructive

[2] but, see end of section 5 for another possibility

- *A formula A is* refuted *in the world w (notation $w : A \Vdash$) if any world $w' \geq w$, which forces A, is exploding;*

- $w : A \wedge B \Vdash_s$ *if* $w : A \Vdash$ *or* $w : B \Vdash$*;*

- $w : A \vee B \Vdash_s$ *if* $w : A \Vdash$ *and* $w : B \Vdash$*;*

- $w : A \rightarrow B \Vdash_s$ *if* $w :\Vdash A$ *and* $w : B \Vdash$*;*

- $w : \forall x.A(x) \Vdash_s$ *if* $w : A(t) \Vdash$ *for some* $t \in D(w)$*;*

- $w : \exists x.A(x) \Vdash_s$ *if* $w : A(t) \Vdash$ *for all* $t \in D(w)$*;*

- $\bot$ *is always strongly refuted;*

- $\top$ *is never strongly refuted.*

2.1. Properties

We list some properties satisfied by the newly defined relations.

Lemma 3. *In all worlds w and for all formulae A, if $w : A \Vdash_s$, then $w : A \Vdash$.*

Proof. Immediate, from unfolding the definition of refutation. □

Lemma 4. *Strong refutation, forcing and refutation are monotone in any classical Kripke model.*

Proof. All three statements proved separately by induction on the formula in question. □

2.2. Relation to Traditional Forcing

It is natural to ask which properties of traditional forcing carry over to our non-primitive forcing.

Proposition 5. *The following statements hold.*

$$w :\Vdash A \rightarrow B \iff \text{for all } w' \geq w, w' :\Vdash A \Rightarrow w' :\Vdash B \tag{1}$$
$$w :\Vdash \forall x.A(x) \iff \text{for all } w' \geq w \text{ and } t \in D(w'), w' :\Vdash A(t) \tag{2}$$
$$w :\Vdash \bot \iff w :\Vdash_{\bot} \tag{3}$$
$$w :\Vdash \top \iff \text{true} \tag{4}$$
$$w :\Vdash A \wedge B \iff w :\Vdash A \text{ and } w :\Vdash B \tag{5}$$
$$w :\Vdash A \vee B \Longleftarrow w :\Vdash A \text{ or } w :\Vdash B \tag{6}$$
$$w :\Vdash \exists x.A(x) \Longleftarrow \text{for some } t \in D(w), w :\Vdash A(t) \tag{7}$$

Proof. *(1)* Suppose $w :\Vdash A \to B, w' \geq w$ and $w' :\Vdash A$. To show $w' :\Vdash B$ we let $w'' \geq w'$ and $w'' : B \Vdash_{\mathfrak{s}}$ and have to show that w'' is exploding. Since then $w'' : A \to B \Vdash_{\mathfrak{s}}$ holds by monotonicity and Lemma 3, the claim follows from the definition of $w :\Vdash A \to B$. For the other direction, suppose a world $w' \geq w$ in which $A \to B$ is strongly refuted, i.e. $w' :\Vdash A$ and $w' : B \Vdash$, and we have to show w' is exploding. But, this is immediate, since B is also forced by hypothesis (the right-hand side of the equivalence).

(2) By definition, $w :\Vdash \forall x.A(x)$ iff $\forall w' \geq w, (\exists s \in D(w').w' : A(s) \Vdash) \Rightarrow w' :\Vdash_{\perp}$, which is equivalent to the right-hand side of the equivalence thanks to Lemma 3 and refutation being defined in terms of forcing. (We used quantifier symbols at meta-level.)

(5) Assume $w :\Vdash A$, $w :\Vdash B$, $w \leq w'$, and $w' : A \wedge B \Vdash_{\mathfrak{s}}$. Therefore we have $w' : A \Vdash$ or $w' : B \Vdash$. Each case leads to $w' :\Vdash_{\perp}$ since $w' :\Vdash A$, $w' :\Vdash B$ with monotonicity.

The rest of the cases follow from the definitions and the monotonicities of "$\Vdash$" and $D(-)$. □

The previous propositions give us that our notion of forcing constructively coincides with the traditional one only for implication, conjunction, and the universal quantifier. We can also say that, constructively, forcing of $\perp$ and $\top$ behaves like expected with respect to exploding nodes [Vel76, Kri96]. The coincidence of our classical Kripke semantics with intuitionistic Kripke semantics on the $\to \wedge \forall$ fragment can be emphasised by showing that our notion of model explicitly characterises the semantics of classical logic obtained by interpretation into intuitionistic Kripke semantics through the following double-negation translation:

$$\begin{array}{lcl} X^* & = & \neg X \\ (A \vee B)^* & = & \neg(\neg A \wedge \neg B) \\ (\exists x.A(x))^* & = & \neg(\forall x.\neg A(x)) \end{array}$$

where $\neg A$ is $A \to \perp$ and all other cases are compositional. Indeed, we have the following characterisation:

Proposition 6. *Any classical Kripke model on $(K, \leq)$ with strong refutation on atoms $w : X \Vdash_{\mathfrak{s}}$, exploding worlds $w \Vdash_{\perp}$ and domain $D(w)$ is the image of an intuitionistic Kripke model with exploding worlds which is defined on the same poset $(K, \leq)$, with the same exploding worlds $w \Vdash_{\perp}$, the same domain $D(w)$ and with forcing on atoms defined by $w \Vdash' X$ iff $w : \neg X \Vdash_{\mathfrak{s}}$. Indeed, for such an intuitionistic model we have $w :\Vdash A$ in the classical model iff $w \Vdash A^*$ in the intuitionistic model.*

Proof. It is enough to observe that we have the following facts in the classical model:

- $w :\Vdash X$ iff $w : \neg X \Vdash_{\mathfrak{s}}$
- $w :\Vdash A \vee B$ iff $w :\Vdash \neg(\neg A \wedge \neg B)$
- $w :\Vdash \exists x.A(x)$ iff $w :\Vdash \neg\forall x.\neg A(x)$

□

Remark 7. *The following are false, even if reasoning classically.*

$\dfrac{}{\Gamma \mid A \vdash A, \Delta}\ (\mathrm{Ax}_L)$	$\dfrac{}{A, \Gamma \vdash A \mid \Delta}\ (\mathrm{Ax}_R)$
$\dfrac{\Gamma, A \vdash \Delta}{\Gamma \mid A \vdash \Delta}\ (\tilde{\mu})$	$\dfrac{\Gamma \vdash A, \Delta}{\Gamma \vdash A \mid \Delta}\ (\mu)$
$\dfrac{\Gamma \vdash A \mid \Delta \qquad \Gamma \mid B \vdash \Delta}{\Gamma \mid A \to B \vdash \Delta}\ (\to_L)$	$\dfrac{\Gamma, A \vdash B \mid \Delta}{\Gamma \vdash A \to B \mid \Delta}\ (\to_R)$
$\dfrac{\Gamma \mid A \vdash \Delta \qquad \Gamma \mid B \vdash \Delta}{\Gamma \mid A \vee B \vdash \Delta}\ (\vee_L)$	$\dfrac{\Gamma \vdash A \mid \Delta}{\Gamma \vdash A \vee B \mid \Delta}\ (\vee_R^1) \qquad \dfrac{\Gamma \vdash B \mid \Delta}{\Gamma \vdash A \vee B \mid \Delta}\ (\vee_R^2)$
$\dfrac{\Gamma \mid A \vdash \Delta}{\Gamma \mid A \wedge B \vdash \Delta}\ (\wedge_L^1) \qquad \dfrac{\Gamma \mid B \vdash \Delta}{\Gamma \mid A \wedge B \vdash \Delta}\ (\wedge_L^2)$	$\dfrac{\Gamma \vdash A \mid \Delta \qquad \Gamma \vdash B \mid \Delta}{\Gamma \vdash A \wedge B \mid \Delta}\ (\wedge_R)$
$\dfrac{\Gamma \mid A(x) \vdash \Delta \qquad x \text{ fresh}}{\Gamma \mid \exists x A(x) \vdash \Delta}\ (\exists_L)$	$\dfrac{\Gamma \vdash A(t) \mid \Delta}{\Gamma \vdash \exists x.A(x) \mid \Delta}\ (\exists_R)$
$\dfrac{\Gamma \mid A(t) \vdash \Delta}{\Gamma \mid \forall x.A(x) \vdash \Delta}\ (\forall_L)$	$\dfrac{\Gamma \vdash A(x) \mid \Delta \qquad x \text{ fresh}}{\Gamma \vdash \forall x A(x) \mid \Delta}\ (\forall_R)$
$\dfrac{}{\Gamma \mid \bot \vdash \Delta}\ (\bot_L)$	$\dfrac{}{\Gamma \vdash \top \mid \Delta}\ (\top_R)$
$\dfrac{\Gamma \vdash A \mid \Delta \qquad \Gamma \mid A \vdash \Delta}{\Gamma \vdash \Delta}\ (\mathrm{Cut})$	

Table 1: The sequent calculus $\mathrm{LK}_{\mu\tilde{\mu}}$

- $w :\Vdash A \vee B \Longrightarrow w :\Vdash A$ *or* $w :\Vdash B$.
- $w :\Vdash \exists x.A(x) \Longrightarrow$ *for some* $t \in D(w), w :\Vdash A(t)$.

The reason why they are in general false can be explained as follows. Below we are going to prove cut-free completeness using the universal Kripke model based on the deduction system itself. In the universal Kripke model, $(\Gamma, \emptyset) :\Vdash A$ *is equivalent to* $\Gamma \vdash A$*. On the other hand,* $\Gamma \vdash B \vee C$ *is in general not equivalent to* $\Gamma \vdash B$ *or* $\Gamma \vdash C$*. A similar argument can be done against the case with existential quantification.*

3. $\mathrm{LK}_{\mu\tilde{\mu}}$ and Soundness

Because we are interested in the symmetry of classical logic, we chose to formalise classical logic using a Gentzen's LK-style sequent calculus. Moreover, since we are eventually interested in using our Kripke semantics to perform proof normalisation, we decided to rely on Curien and Herbelin's $\mathrm{LK}_{\mu\tilde{\mu}}$ variant of LK for the sobriety and expressivity of its underlying core calculus of proof-terms (so-called $\mu\tilde{\mu}$ subsystem [Her05]). $LK_{\mu\tilde{\mu}}$ is presented on Table 1. It differs from LK in the following points:

- Sequents come with an explicitly distinguished formula on the right or on the left, or no distinguished formula at all, resulting in three kinds of sequents: "$\Gamma \vdash \Delta$", "$\Gamma|A \vdash \Delta$" and "$\Gamma \vdash A|\Delta$". Especially, the distinguished formula plays an "active" rôle in the rules.
- Accordingly, the axiom rule splits into two variants (Ax_L) and (Ax_R) depending on whether the left active formula or the right active formula is distinguished. There are also two new rules, (μ) and ($\tilde{\mu}$), for making a formula active.
- There are no explicit contraction rules: contractions are derivable from a cut against an axiom as follows:

$$\dfrac{\dfrac{}{\Gamma, A \vdash A|\Delta}\,(Ax_R) \qquad \Gamma, A|A \vdash \Delta}{\Gamma, A \vdash \Delta}\,(Cut) \qquad \dfrac{\Gamma \vdash A|A, \Delta \qquad \dfrac{}{\Gamma|A \vdash A, \Delta}\,(Ax_L)}{\Gamma \vdash A, \Delta}\,(Cut)$$

- Consequently, the notion of normal proof is slightly different from the notion of cut-freeness in LK: a normal proof is a proof whose only cuts are of the form of a cut between an axiom and an introduction rule. ((μ) and ($\tilde{\mu}$) are not introduction rules)

The correspondence between LK and $LK_{\mu\tilde{\mu}}$ is direct. If we present LK with weakening rules attached to the axiom rules *à la* Kleene's G_3, we obtain an LK proof from an $LK_{\mu\tilde{\mu}}$ proof by erasing the bars serving to distinguish active formulae, and by removing the trivial inferences coming from the rules (μ) and ($\tilde{\mu}$). In the other way round, every introduction rule of LK can be derived in $LK_{\mu\tilde{\mu}}$ by applying the rules (μ) and ($\tilde{\mu}$) on the premises and a (possibly dummy) contraction (i.e. a cut against an axiom) on the conclusion of the rule. Similarly for the axiom rule (for which there are two possible derivations) and the cut rule. Especially, cut-free proofs of LK maps to normal proofs of $LK_{\mu\tilde{\mu}}$ and vice-versa.

Theorem 8 (Soundness)**.** *In any classical Kripke model the following hold:*

$$\begin{aligned}
\Gamma \vdash \Delta &\Longrightarrow \textit{ for any } w \textit{ such that } w :\Vdash \Gamma \textit{ and } w : \Delta \Vdash, w :\Vdash_{\bot} \\
\Gamma \vdash A|\Delta &\Longrightarrow \textit{ for any } w \textit{ such that } w :\Vdash \Gamma \textit{ and } w : \Delta \Vdash, w :\Vdash A \\
\Gamma|A \vdash \Delta &\Longrightarrow \textit{ for any } w \textit{ such that } w :\Vdash \Gamma \textit{ and } w : \Delta \Vdash, w : A \Vdash
\end{aligned}$$

Proof. One proves easily the three statements simultaneously, by induction on the derivations. □

4. Completeness

As usual when constructively proving completeness of Kripke semantics for a fragment[3] of intuitionistic logic [Coq93, HL], we define a special purpose model, called the

[3]as previously remarked, there is no constructive proof for full intuitionistic predicate logic

universal model, built from the deduction system itself. Once we show completeness for this special model, completeness for any model follows (Corollary 15).

Definition 9. *The* Universal classical Kripke model $\mathcal{U}$ *is obtained by setting:*

- *K to the set of pairs* (Γ, Δ) *of contexts of* $LK_{\mu\tilde{\mu}}$ *;*
- $(\Gamma, \Delta) \leq (\Gamma', \Delta')$ *iff both* $\Gamma \subseteq \Gamma'$ *and* $\Delta \subseteq \Delta'$*;*
- $(\Gamma, \Delta) : X \Vdash_{s}$ *iff the sequent* $\Gamma|X \vdash \Delta$ *is provable in* $LK_{\mu\tilde{\mu}}$ *;*
- $(\Gamma, \Delta) : \Vdash_{\perp}$ *iff the sequent* $\Gamma \vdash \Delta$ *is provable without a cut in* $LK_{\mu\tilde{\mu}}$ *;*
- *for any w,* $D(w)$ *is the set of individuals of* $LK_{\mu\tilde{\mu}}$ *(that is,* $D(-)$ *is constant)*

To show that $\mathcal{U}$ is indeed a model, we have to prove two things, monotonicity of strong refutation on atoms and its stability under substitutions. These are provided by the following two lemmas, proved by induction on the derivations.

Lemma 10 (Weakening)**.** *The following hold in* $\mathrm{LK}_{\mu\tilde{\mu}}$ *:*

$$\begin{aligned}\Gamma \vdash \Delta &\Longrightarrow \textit{ for all } (\Gamma', \Delta') \geq (\Gamma, \Delta),\ \Gamma' \vdash \Delta' \\ \Gamma \vdash A|\Delta &\Longrightarrow \textit{ for all } (\Gamma', \Delta') \geq (\Gamma, \Delta),\ \Gamma' \vdash A|\Delta' \\ \Gamma|A \vdash \Delta &\Longrightarrow \textit{ for all } (\Gamma', \Delta') \geq (\Gamma, \Delta),\ \Gamma'|A \vdash \Delta'\end{aligned}$$

We emphasise that the proof of Lemma 10 needs not introduce any new cuts in the derivations generated on the right-hand side of the implications. This will be important for the proof of cut-free completeness.

Lemma 11 (Substitutivity)**.** *The following hold in* $\mathrm{LK}_{\mu\tilde{\mu}}$ *, whenever x does not appear in* Γ *nor* Δ*:*

$$\begin{aligned}\Gamma \vdash A(x)|\Delta &\Longrightarrow \textit{ for all } t, \Gamma \vdash A(t)|\Delta \\ \Gamma|A(x) \vdash \Delta &\Longrightarrow \textit{ for all } t, \Gamma|A(t)|\Delta\end{aligned}$$

To carry out the proof of *cut-free* completeness, we also need to say when a formula is "neutral" with respect to all derivations that comprehend a context (Γ, Δ).

Definition 12 ($\mathrm{NT}(-)$)**.** *A formula A is said to be* neutral with respect to provability *in the context* (Γ, Δ) *(notation* $\mathrm{NT}(A, \Gamma, \Delta)$*), if for any extension* (Γ', Δ') *of the context we have*

$$\Gamma'|A \vdash \Delta' \Longrightarrow \Gamma' \vdash \Delta'$$

where the derivation on the right of the implication is cut-free.

Definition 13 ($\mathrm{NE}(-)$)**.** *A formula A is said to be* neutral with respect to refutability *in the context* (Γ, Δ) *(notation* $\mathrm{NE}(A, \Gamma, \Delta)$*), if for any extension* (Γ', Δ') *of the context we have*

$$\Gamma' \vdash A|\Delta' \Longrightarrow \Gamma' \vdash \Delta'$$

where the derivation on the right of the implication is cut-free.

Theorem 14 (Cut-Free Completeness for $\mathcal{U}$)**.** *For any A, Γ and Δ, the following hold in $\mathcal{U}$:*

$$(\Gamma, \Delta) :\Vdash A \Longrightarrow \Gamma \vdash A|\Delta \tag{1}$$
$$\mathrm{NT}(A, \Gamma, \Delta) \Longrightarrow (\Gamma, \Delta) :\Vdash A \tag{2}$$
$$(\Gamma, \Delta) : A \Vdash \Longrightarrow \Gamma|A \vdash \Delta \tag{3}$$
$$\mathrm{NE}(A, \Gamma, \Delta) \Longrightarrow (\Gamma, \Delta) : A \Vdash \tag{4}$$

Moreover, the derivations on the right-hand side of (1) and (3) are cut-free.

Proof. We proceed by simultaneously proving all four statements by induction on the formula A.

Base case. In the base case we have forcing and refutation on atomic formulae, which by definition reduce to strong refutation on atomic formulae, which by definition reduces just to statements about the deductions in $\mathrm{LK}_{\mu\tilde{\mu}}$

(1) Suppose that for all $(\Gamma', \Delta') \geq (\Gamma, \Delta)$,

$$\Gamma'|X \vdash \Delta' \Rightarrow \Gamma' \vdash \Delta' \ldots \tag{*}$$

Then:

$$\dfrac{\dfrac{\dfrac{}{\Gamma|X \vdash X, \Delta}\,(\mathrm{Ax}_L)}{\Gamma \vdash X, \Delta}\,(*)}{\Gamma \vdash X|\Delta}\,(\mu)$$

(2) The hypothesis is $\mathrm{NT}(X, \Gamma, \Delta)$. Given $(\Gamma', \Delta') \geq (\Gamma, \Delta)$ such that $\Gamma'|X \vdash \Delta'$, we have:

$$\dfrac{\Gamma'|X \vdash \Delta'}{\Gamma' \vdash \Delta'}\,\mathrm{NT}(X, \Gamma, \Delta)$$

(3) We have $(\Gamma, \Delta) : X \Vdash$, i.e.,

$$\forall(\Gamma', \Delta') \geq (\Gamma, \Delta), \{\forall(\Gamma'', \Delta'') \geq (\Gamma', \Delta'), \Gamma''|X \vdash \Delta'' \Rightarrow \Gamma'' \vdash \Delta''\} \Rightarrow \Gamma' \vdash \Delta' \tag{*}$$

We can show $\Gamma|X \vdash \Delta$ by applying the $(\tilde{\mu})$-rule and (*), but we also have to show the sub-statement in curly brackets of (*):

$$\dfrac{\dfrac{\text{because } X \in (X, \Gamma) \subseteq \Gamma''}{\Gamma'' \vdash X|\Delta''}\,(\mathrm{Ax}_R) \qquad \Gamma''|X \vdash \Delta''}{\Gamma'' \vdash \Delta''}\,(\mathrm{Cut})$$

(4) Suppose $\mathrm{NE}(X, \Gamma, \Delta)$ and suppose $(\Gamma', \Delta') \geq (\Gamma, \Delta)$ such that

$$\forall(\Gamma'', \Delta'') \geq (\Gamma', \Delta'), \Gamma''|X \vdash \Delta'' \Rightarrow \Gamma'' \vdash \Delta'' \tag{\#}$$

Then:

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma'|X \vdash X, \Delta'}\,(\mathrm{Ax}_L)}{\Gamma' \vdash X, \Delta'}\,(\#)}{\Gamma' \vdash X|\Delta'}\,(\mu)}{\Gamma' \vdash \Delta'}\,\mathrm{NE}(X, \Gamma, \Delta)$$

Induction case for implication.

(1) We can strengthen the hypothesis $(\Gamma,\Delta) :\Vdash A_1 \to A_2$ using the induction hypotheses to obtain:

$$\forall(\Gamma',\Delta') \geq (\Gamma,\Delta), \mathrm{NT}(A_1,\Gamma',\Delta') \Rightarrow \mathrm{NE}(A_2,\Gamma',\Delta') \Rightarrow \Gamma' \vdash \Delta' \qquad (\#)$$

Now we have:

$$\dfrac{\dfrac{\dfrac{}{A_1,\Gamma \vdash A_2,\Delta}\,(\#)}{A_1,\Gamma \vdash A_2|\Delta}\,(\mu)}{\Gamma \vdash A_1 \to A_2|\Delta}\,(\to_R)$$

And we have to show $\mathrm{NT}(A_1,(A_1,\Gamma),(A_2,\Delta))$ and $\mathrm{NE}(A_2,(A_1,\Gamma),(A_2,\Delta))$, which is easy using weakening because the neutral formulae already appear in the contexts.

(2) Suppose $\mathrm{NT}(A_1 \to A_2,\Gamma,\Delta)$ and suppose $(\Gamma',\Delta') \geq (\Gamma,\Delta)$ such that $(\Gamma',\Delta') :\Vdash A_1$ and $(\Gamma',\Delta') : A_2 \Vdash$. The induction hypotheses give us that $\Gamma' \vdash A_1|\Delta'$ and $\Gamma'|A_2 \vdash \Delta'$. Now we have:

$$\dfrac{\dfrac{\Gamma' \vdash A_1|\Delta' \qquad \Gamma'|A_2 \vdash \Delta'}{\Gamma'|A_1 \to A_2 \vdash \Delta'}\,(\to_L)}{\Gamma' \vdash \Delta'}\,\mathrm{NT}(A_1 \to A_2,\Gamma,\Delta)$$

(3) We have $(\Gamma,\Delta) : A_1 \to A_2 \Vdash$, i.e.,

$$\begin{aligned}\forall(\Gamma',\Delta') \geq (\Gamma,\Delta), \{&\forall(\Gamma'',\Delta'') \geq (\Gamma',\Delta'), \mathrm{NT}(A_1,\Gamma'',\Delta'') \Rightarrow \\ &\mathrm{NE}(A_2,\Gamma'',\Delta'') \Rightarrow \Gamma'' \vdash \Delta''\} \Rightarrow \Gamma' \vdash \Delta'\end{aligned} \qquad (*)$$

$$\dfrac{\dfrac{\dfrac{}{\Gamma \vdash A_1,\Delta}\,(*)}{\Gamma \vdash A_1|\Delta}\,(\mu) \qquad \dfrac{\dfrac{}{\Gamma, A_2 \vdash \Delta}\,(*)}{\Gamma|A_2 \vdash \Delta}\,(\tilde{\mu})}{\Gamma|A_1 \to A_2 \vdash \Delta}\,(\to_L)$$

Due to the use of (*) we have to show the sub-expression in curly brackets. Let us show only one case, the other is symmetric:

$$\dfrac{\dfrac{}{\Gamma''|A_1 \vdash \Delta''}\,(\mathrm{Ax}_L)\text{, since } (A_1,\Delta) \subseteq \Delta''}{\Gamma'' \vdash \Delta''}\,\mathrm{NT}(A_1,\Gamma,(A_1,\Delta))$$

(4) Let $\mathrm{NE}(A_1 \to A_2,\Gamma,\Delta)$ and let $(\Gamma',\Delta') \geq (\Gamma,\Delta)$ be given such that:

$$\forall(\Gamma'',\Delta'') \geq (\Gamma',\Delta'), \mathrm{NT}(A_1,\Gamma'',\Delta'') \Rightarrow \mathrm{NE}(A_2,\Gamma'',\Delta'') \Rightarrow \Gamma'' \vdash \Delta'' \qquad (\#)$$

We show $\Gamma' \vdash \Delta'$:

$$\dfrac{\dfrac{\dfrac{\dfrac{}{A_1,\Gamma' \vdash A_2,\Delta'}\,(\#)}{A_1,\Gamma' \vdash A_2|\Delta'}\,(\mu)}{\Gamma' \vdash A_1 \to A_2|\Delta'}\,(\to_R)}{\Gamma' \vdash \Delta'}\,\mathrm{NE}(A_1 \to A_2,\Gamma,\Delta)$$

For the application of (#) we have to show the corresponding $\mathrm{NT}(A_1,(A_1,\Gamma'),(A_2,\Delta'))$ and $\mathrm{NE}(A_2,(A_1,\Gamma'),(A_2,\Delta'))$, but this is easy since the formulae are already inside the corresponding contexts.

Induction case for $\vee$.

(1) Suppose $(\Gamma, \Delta) :\Vdash A_1 \vee A_2$, which can be strengthened using the induction hypotheses to:

$$\forall(\Gamma', \Delta') \geq (\Gamma, \Delta), \mathrm{NE}(A_1, \Gamma', \Delta') \Rightarrow \mathrm{NE}(A_2, \Gamma', \Delta') \Rightarrow \Gamma' \vdash \Delta' \qquad (*)$$

Here is a derivation of $\Gamma \vdash A_1 \vee A_2 | \Delta$:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma \vdash A_2, A_1, A_1 \vee A_2, \Delta}\,(*)}{\Gamma \vdash A_2 | A_1, A_1 \vee A_2, \Delta}\,(\mu)}{\Gamma \vdash A_1 \vee A_2 | A_1, A_1 \vee A_2, \Delta}\,(\vee_L^2) \quad \dfrac{}{\Gamma | A_1 \vee A_2 \vdash A_1, A_1 \vee A_2, \Delta}\,(\mathrm{Ax}_L)}{\Gamma \vdash A_1, A_1 \vee A_2, \Delta}\,(\mathrm{Cut})}{\Gamma \vdash A_1 | A_1 \vee A_2, \Delta}\,(\mu)}{\Gamma \vdash A_1 \vee A_2 | A_1 \vee A_2, \Delta}\,(\vee_L^1) \quad \dfrac{}{\Gamma | A_1 \vee A_2 \vdash A_1 \vee A_2, \Delta}\,(\mathrm{Ax}_L)}{\dfrac{\Gamma \vdash A_1 \vee A_2, \Delta}{\Gamma \vdash A_1 \vee A_2 | \Delta}\,(\mu)}\,(\mathrm{Cut})$$

It is only left to prove that $\mathrm{NE}(A_1, \Gamma, (A_2, A_1, A_1 \vee A_2, \Delta))$ and $\mathrm{NE}(A_2, \Gamma, (A_2, A_1, A_1 \vee A_2, \Delta))$, but that is trivial because the neutral formulae are already in the context.

(2) Let $\mathrm{NT}(A_1 \vee A_2, \Gamma, \Delta)$ and suppose given a $(\Gamma', \Delta') \geq (\Gamma, \Delta)$ such that (by induction hypotheses) $\mathrm{NE}(A_1, \Gamma', \Delta')$ and $\mathrm{NE}(A_2, \Gamma', \Delta')$.

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma', A_1 \vdash A_1 | \Delta'}\,(\mathrm{Ax}_R)}{\Gamma', A_1 \vdash \Delta'}\,\mathrm{NE}(A_1, \Gamma', \Delta')}{\Gamma' | A_1 \vdash \Delta'}\,(\tilde{\mu}) \quad \dfrac{\dfrac{\dfrac{}{\Gamma', A_2 \vdash A_2 | \Delta'}\,(\mathrm{Ax}_R)}{\Gamma', A_2 \vdash \Delta'}\,\mathrm{NE}(A_2, \Gamma', \Delta')}{\Gamma' | A_2 \vdash \Delta'}\,(\tilde{\mu})}{\Gamma' | A_1 \vee A_2 \vdash \Delta'}\,(\vee_L)}{\Gamma' \vdash \Delta'}\,\mathrm{NT}(A_1 \vee A_2, \Gamma, \Delta)$$

(3) Using the induction hypotheses we get from $(\Gamma, \Delta) : A_1 \vee A_2 \Vdash$:

$$\begin{array}{c}\forall(\Gamma',\Delta') \geq (\Gamma, \Delta), \{\forall(\Gamma'', \Delta'') \geq (\Gamma', \Delta'), \mathrm{NE}(A_1, \Gamma'', \Delta'') \Rightarrow \\ \mathrm{NE}(A_2, \Gamma'', \Delta'') \Rightarrow \Gamma'' \vdash \Delta''\} \Rightarrow \Gamma' \vdash \Delta'\end{array} \qquad (*)$$

We can have the following derivation

$$\dfrac{\dfrac{\dfrac{}{\Gamma, A_1 \vdash \Delta}\,(*)}{\Gamma | A_1 \vdash \Delta}\,(\tilde{\mu}) \quad \dfrac{\dfrac{}{\Gamma, A_2 \vdash \Delta}\,(*)}{\Gamma | A_2 \vdash \Delta}\,(\tilde{\mu})}{\Gamma | A_1 \vee A_2 \vdash \Delta}\,(\vee_L)$$

but, we have to prove that the sub-statement in curly brackets from (*) holds for both the context A_1, Γ and the context A_2, Γ. Here is one of them: (the other is analogous)

$$\dfrac{\dfrac{}{\Gamma'' \vdash A_1 | \Delta''}\,(\mathrm{Ax}_R), \text{ since } (A_1, \Gamma) \subseteq \Gamma''}{\Gamma'' \vdash \Delta''}\,\mathrm{NE}(A_1, \Gamma'', \Delta'')$$

(4) Suppose NE($A1 \vee A_2, \Gamma, \Delta$), $(\Gamma', \Delta') \geq (\Gamma, \Delta)$ and, using the induction hypothesis, suppose:

$$\forall(\Gamma'', \Delta'') \geq (\Gamma', \Delta'), \mathrm{NE}(A_1, \Gamma'', \Delta'') \Rightarrow \mathrm{NE}(A_2, \Gamma'', \Delta'') \Rightarrow \Gamma'' \vdash \Delta'' \qquad (*)$$

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma' \vdash A_2, A_1, \Delta'}\,(*)}{\Gamma' \vdash A_2|A_1, \Delta'}\,(\mu)}{\Gamma' \vdash A_1 \vee A_2|A_1, \Delta'}\,(\vee_L^2)}{\Gamma' \vdash A_1, \Delta'}\,\mathrm{NE}(A_1 \vee A_2, \Gamma, \Delta)}{\Gamma' \vdash A_1|\Delta'}\,(\mu)}{\Gamma' \vdash A_1 \vee A_2|\Delta'}\,(\vee_L^1)}{\Gamma' \vdash \Delta'}\,\mathrm{NE}(A_1 \vee A_2, \Gamma, \Delta)$$

This constitutes a derivation as required, given that it is easy to prove NE($A_1, \Gamma', (A_2, A_1, \Delta')$) and NE($A_2, \Gamma', (A_2, A_1, \Delta')$) which arise from the use of (*).

Induction case for $\wedge$.

(1) Let

$$\forall(\Gamma', \Delta') \geq (\Gamma, \Delta), \mathrm{NE}(A_1, \Gamma', \Delta') \text{ or } \mathrm{NE}(A_2, \Gamma', \Delta') \Rightarrow \Gamma' \vdash \Delta' \qquad (*)$$

Here is the required derivation:

$$\dfrac{\dfrac{\dfrac{}{\Gamma \vdash A_1, \Delta}\,(*)}{\Gamma \vdash A_1|\Delta}\,(\mu) \qquad \dfrac{\dfrac{}{\Gamma \vdash A_2, \Delta}\,(*)}{\Gamma \vdash A_2|\Delta}\,(\mu)}{\Gamma \vdash A_1 \wedge A_2|\Delta}\,(\wedge_R)$$

Where it is easy to show that A_1 and A_2 are neutral in the two cases arising from the use of (*).

(2) Suppose NT($A_1 \wedge A_2, \Gamma, \Delta$). To show $(\Gamma, \Delta) :\Vdash A_1 \wedge A_2$, let $(\Gamma', \Delta') \geq (\Gamma, \Delta)$ and (NE(A_1, Γ', Δ') or NE(A_2, Γ', Δ')) be true. Without loss of generality, let NE(A_1, Γ', Δ') be true. Then:

$$\dfrac{\dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma', A_1 \vdash A_1|\Delta'}\,(\mathrm{Ax}_R)}{\Gamma', A_1 \vdash \Delta'}\,\mathrm{NE}(A_1, \Gamma', \Delta')}{\Gamma'|A_1 \vdash \Delta'}\,(\tilde{\mu})}{\Gamma'|A_1 \wedge A_2 \vdash \Delta'}\,(\wedge_L^1)}{\Gamma' \vdash \Delta'}\,\mathrm{NT}(A_1 \wedge A_2, \Gamma', \Delta')$$

(3) Suppose $(\Gamma, \Delta) : A_1 \wedge A_2 \Vdash$, which can be strengthened using the induction hypotheses to:

$$\begin{aligned}\forall(\Gamma',\Delta') \geq (\Gamma, \Delta), \{&\forall(\Gamma'', \Delta'') \geq (\Gamma', \Delta'), \mathrm{NE}(A_1, \Gamma'', \Delta'') \text{ or} \\ &\mathrm{NE}(A_2, \Gamma'', \Delta'') \Rightarrow \Gamma'' \vdash \Delta''\} \Rightarrow \Gamma' \vdash \Delta'\end{aligned} \qquad (*)$$

Now we have:

$$\dfrac{\dfrac{}{\Gamma, A_1 \wedge A_2}\,(*)}{\Gamma|A_1 \wedge A_2}\,(\tilde{\mu})$$

But we have to show the hypothesis of (*). Let $(\Gamma'', \Delta'') \geq ((A_1 \wedge A_2, \Gamma), \Delta)$ and suppose, without loss of generality, that the left disjunct is true, i.e., we have $\mathrm{NE}(A_1, \Gamma'', \Delta'')$. Then:

$$\dfrac{\dfrac{}{\Gamma'' \vdash A_1 \wedge A_2 | \Delta''}(\mathrm{Ax}_R) \quad \dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma'', A_1 \vdash A_1 | \Delta''}(\mathrm{Ax}_R)}{\Gamma'', A_1 \vdash \Delta''}\mathrm{NE}(A_1, \Gamma'', \Delta'')}{\Gamma'' | A_1 \vdash \Delta''}(\tilde{\mu})}{\Gamma'' | A_1 \wedge A_2 \vdash \Delta''}(\wedge_L^1)}{\Gamma'' \vdash \Delta''}(\mathrm{Cut})$$

(4) Suppose $\mathrm{NE}(A_1 \wedge A_2, \Gamma, \Delta)$, $(\Gamma', \Delta') \geq (\Gamma, \Delta)$ and, using the induction hypotheses, suppose:

$$\forall (\Gamma'', \Delta'') \geq (\Gamma', \Delta'), \mathrm{NE}(A_1, \Gamma'', \Delta'') \text{ or } \mathrm{NE}(A_2, \Gamma'', \Delta'') \Rightarrow \Gamma'' \vdash \Delta'' \qquad (*)$$

We have:

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma' \vdash A_1, \Delta'}(\#)}{\Gamma' \vdash A_1 | \Delta'}(\mu) \quad \dfrac{\dfrac{}{\Gamma' \vdash A_2, \Delta'}(\#)}{\Gamma' \vdash A_2 | \Delta'}(\mu)}{\Gamma' \vdash A_1 \wedge A_2 | \Delta'}(\wedge_R)}{\Gamma' \vdash \Delta'}\mathrm{NE}(A_1 \wedge A_2, \Gamma, \Delta)$$

where $\mathrm{NE}(A_1, \Gamma', (A_1, \Delta'))$ and $\mathrm{NE}(A_2, \Gamma', (A_2, \Delta'))$ arising from the use of (#) are easy to prove, because the formulae are already inside the contexts.

Induction case for $\forall$. In the induction cases for $\forall$ and $\exists$ we leave out the membership in the domain of individuals $D(-)$ since in $\mathcal{U}$ we have a constant domain and we use the quantifier symbols also at the meta-level, to shorten the notation.

(1) Let $(\Gamma, \Delta) :\Vdash \forall x.A(x)$. Using the induction hypotheses we get:

$$\forall (\Gamma', \Delta') \geq (\Gamma, \Delta), (\exists t, \mathrm{NE}(A(t), \Gamma', \Delta')) \Rightarrow \Gamma' \vdash \Delta' \qquad (*)$$

Here follows the required derivation:

$$\dfrac{\dfrac{\dfrac{}{\Gamma \vdash A(x), \Delta}(*)}{\Gamma \vdash A(x) | \Delta}(\mu)}{\Gamma \vdash \forall x.A(x) | \Delta}(\forall_R),\ x\text{-fresh}$$

One easily shows that $\mathrm{NE}(A(x), \Gamma, (A(x), \Delta))$.

(2) Suppose $\mathrm{NT}(\forall x.A(x), \Gamma, \Delta)$, $(\Gamma', \Delta') \geq (\Gamma, \Delta)$ and we have t such that, by induction hypothesis, $\Gamma' | A(t) \vdash \Delta'$. Here is a derivation of $\Gamma' \vdash \Delta'$:

$$\dfrac{\dfrac{\Gamma' | A(t) \vdash \Delta'}{\Gamma' | \forall x.A(x) \vdash \Delta'}(\forall_L)}{\Gamma' \vdash \Delta'}\mathrm{NT}(\forall x.A(x), \Gamma, \Delta)$$

(3) Suppose $(\Gamma, \Delta) : \forall x.A(x) \Vdash$. We strengthen this using the induction hypothesis to:

$$\begin{aligned} &\forall (\Gamma', \Delta') \geq (\Gamma, \Delta), \{\forall (\Gamma'', \Delta'') \geq (\Gamma', \Delta'), \\ &\quad (\exists t, \mathrm{NE}(A(t), \Gamma'', \Delta'')) \Rightarrow \Gamma'' \vdash \Delta''\} \Rightarrow \Gamma' \vdash \Delta' \end{aligned} \qquad (*)$$

This

$$\dfrac{\dfrac{}{\Gamma, \forall x.A(x) \vdash \Delta}\,{\scriptstyle (*)}}{\Gamma|\forall x.A(x) \vdash \Delta}\,{\scriptstyle (\tilde{\mu})}$$

is the derivation we need, in case we prove the sub-expression in curly brackets of (*). Therefore, suppose $(\Gamma'', \Delta'') \geq ((\forall x.A(x), \Gamma), \Delta)$ and suppose a t with $\mathrm{NE}(A(t), \Gamma'', \Delta'')$. We have to prove $\Gamma'' \vdash \Delta''$:

$$\dfrac{\dfrac{}{\Gamma'' \vdash \forall x.A(x)|\Delta''}\,{\scriptstyle (\mathrm{Ax}_R)} \qquad \dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma'', A(t) \vdash A(t)|\Delta''}\,{\scriptstyle (\mathrm{Ax}_R)}}{\Gamma'', A(t) \vdash \Delta''}\,{\scriptstyle \mathrm{NE}(A(t), \Gamma'', \Delta'')}}{\Gamma''|A(t) \vdash \Delta''}\,{\scriptstyle (\tilde{\mu})}}{\Gamma''|\forall x.A(x) \vdash \Delta''}\,{\scriptstyle (\forall_L)}}{\Gamma'' \vdash \Delta''}\,{\scriptstyle (\mathrm{Cut})}$$

(4) Suppose $\mathrm{NE}(\forall x.A(x), \Gamma, \Delta)$. To show $(\Gamma, \Delta) : \forall x.A(x) \Vdash$, let $(\Gamma', \Delta') \geq (\Gamma, \Delta)$ and let

$$\forall (\Gamma'', \Delta'') \geq (\Gamma', \Delta'), (\exists t, \mathrm{NE}(A(t), \Gamma'', \Delta'')) \Rightarrow \Gamma'' \vdash \Delta'' \qquad (*)$$

Here is the required derivation:

$$\dfrac{\dfrac{\dfrac{\dfrac{}{\Gamma' \vdash A(x), \Delta'}\,{\scriptstyle (*)}}{\Gamma' \vdash A(x)|\Delta'}\,{\scriptstyle (\mu)}}{\Gamma' \vdash \forall x.A(x)|\Delta'}\,{\scriptstyle (\forall_R),\ x\text{-fresh}}}{\Gamma' \vdash \Delta'}\,{\scriptstyle \mathrm{NE}(\forall x.A(x), \Gamma, \Delta)}$$

For the application of (*) one can easily show that $\mathrm{NE}(A(x), \Gamma', (A(x), \Delta'))$.

Induction case for $\exists$.

(1) Suppose $(\Gamma, \Delta) :\Vdash \exists x.A(x)$, which using the induction hypothesis can be strengthened to:

$$\forall (\Gamma', \Delta') \geq (\Gamma, \Delta), (\forall t, \mathrm{NE}(A(t), \Gamma', \Delta')) \Rightarrow \Gamma' \vdash \Delta' \qquad (*)$$

The following is a good derivation

$$\dfrac{\dfrac{}{\Gamma \vdash \exists x.A(x), \Delta}\,{\scriptstyle (*)}}{\Gamma \vdash \exists x.A(x)|\Delta}\,{\scriptstyle (\mu)}$$

if we manage to show the hypothesis from applying (*). For that, let $t, \Gamma' \supseteq \Gamma, \Delta' \supseteq (\exists x.A(x), \Delta)$ be given such that $\Gamma' \vdash A(t)|\Delta'$.

$$\dfrac{\dfrac{\Gamma' \vdash A(t)|\Delta'}{\Gamma' \vdash \exists x.A(x)|\Delta'}\,{\scriptstyle (\exists_R)} \qquad \dfrac{}{\Gamma'|\exists x.A(x) \vdash \Delta'}\,{\scriptstyle (\mathrm{Ax}_L)}}{\Gamma' \vdash \Delta'}\,{\scriptstyle (\mathrm{Cut})}$$

(2) Let $\mathrm{NT}(\exists x.A(x), \Gamma, \Delta)$ and $(\Gamma', \Delta') \geq (\Gamma, \Delta)$ be given and suppose

$$\forall t, \Gamma'|A(t) \vdash \Delta' \qquad (\#)$$

The required derivation is:

$$\dfrac{\dfrac{\dfrac{}{\Gamma'|A(x) \vdash \Delta'}\,{\scriptstyle (\#)}}{\Gamma'|\exists x.A(x) \vdash \Delta'}\,{\scriptstyle (\exists_L),\ x\text{-fresh}}}{\Gamma' \vdash \Delta'}\,{\scriptstyle \mathrm{NT}(\exists x.A(x), \Gamma, \Delta)}$$

(3) Suppose $(\Gamma, \Delta) : \exists x.A(x) \Vdash$ which gives, thanks to the induction hypothesis:

$$\forall(\Gamma',\Delta') \geq (\Gamma, \Delta), \{\forall(\Gamma'', \Delta'') \geq (\Gamma', \Delta'), \\ (\forall t, \mathrm{NE}(A(t), \Gamma'', \Delta'')) \Rightarrow \Gamma'' \vdash \Delta''\} \Rightarrow \Gamma' \vdash \Delta' \tag{*}$$

The required derivation is

$$\dfrac{\dfrac{\dfrac{}{\Gamma, A(x) \vdash \Delta}\,(*)}{\Gamma|A(x) \vdash \Delta}\,(\tilde{\mu})}{\Gamma|\exists x.A(x) \vdash \Delta}\,(\exists_L),\ x\text{-fresh}$$

but we also have to show the statement in curly brackets arising from the use of (*). Therefore, suppose $(\Gamma'', \Delta'') \geq ((A(x), \Gamma), \Delta)$ and suppose, using the induction hypothesis, that $\forall t, \Gamma''|A(t) \vdash \Delta''$. We have to prove $\Gamma'' \vdash \Delta''$:

$$\dfrac{\dfrac{}{\Gamma'' \vdash A(x)|\Delta''}\,(\mathrm{Ax}_R) \qquad \dfrac{}{\Gamma''|A(x) \vdash \Delta''}\,(\#)}{\Gamma'' \vdash \Delta''}\,(\mathrm{Cut})$$

(4) Suppose $\mathrm{NE}(\exists x.A(x), \Gamma, \Delta)$ and let $(\Gamma', \Delta') \geq (\Gamma, \Delta)$ such that $(\Gamma', \Delta') :\Vdash \exists x.A(x)$. Using the induction hypothesis, this last thing strengthens to:

$$\forall(\Gamma'', \Delta'') \geq (\Gamma', \Delta'), (\forall t, \mathrm{NE}(A(t), \Gamma'', \Delta'')) \Rightarrow \Gamma'' \vdash \Delta'' \tag{*}$$

To show $\Gamma' \vdash \Delta'$, we immediately apply (*) and then have to show the hypothesis: let t, Γ_3, Δ_3 be such that $\Gamma_3 \vdash A(t)|\Delta_3$ and $(\Gamma_3, \Delta_3) \geq (\Gamma'', \Delta'')$. Then this is what we are looking for:

$$\dfrac{\dfrac{\Gamma_3 \vdash A(t)|\Delta_3}{\Gamma_3 \vdash \exists x.A(x)|\Delta_3}\,(\exists_R)}{\Gamma_3 \vdash \Delta_3}\,\mathrm{NE}(\exists x.A(x), \Gamma, \Delta)$$

Induction case for $\top$.

(1) Immediate, from $(\top_R)$.
(2) Easy, using Proposition 5.
(3) Easy, using Proposition 5.
(4) Easy, a (Cut) with $\top$ and then $(\top_R)$ and (Ax_L).

Induction case for $\bot$.

(1) Easy, using Proposition 5.
(2) Easy, a (Cut) with $\bot$ and then $(\bot_L)$ and (Ax_R).
(3) Immediate, from $(\bot_L)$.
(4) Immediate, by applying the hypothesis.

All the given derivations are cut-free. By inspection of the proof trees and having in mind that NT, NE, weakening and the derivations arising from exploding nodes, do not introduce cuts, we convince ourselves that indeed the completeness theorem produces only cut-free derivations. □

Corollary 15 (Completeness of Classical Logic)**.** *If in every Kripke model, at every possible world, the formula A is forced whenever all the formulae of* Γ *are forced and all the formulae of* Δ *are refuted, then there exists a derivation in* $\mathrm{LK}_{\mu\tilde{\mu}}$ *of the sequent* $\Gamma \vdash A|\Delta$.

Proof. If the hypothesis holds for any Kripke model, so does it hold for $\mathcal{U}$. To show $\Gamma \vdash A|\Delta$, by Theorem 14, it is enough to show that $(\Gamma, \Delta) :\Vdash B$ for all $B \in \Gamma$ and $(\Gamma, \Delta) : C \Vdash$ for all $C \in \Delta$, something that is immediate from the definitions of NT and NE, after applying (2) and (4) of the preceeding theorem. □

5. Normalisation and the Call-by-Value Variant

A constructive cut-free completeness theorem can be used for proof normalisation.

Corollary 16 (Normalisation-by-Evaluation)**.** *For all contexts* Γ, Δ*, if there is a derivation* $\Gamma \vdash \Delta$*, then there is a cut-free derivation* $\Gamma \vdash_{nf} \Delta$.

Proof. From the hypothesis $\Gamma \vdash \Delta$, the soundness theorem applied to $\mathcal{U}$ gives us that there is indeed $\Gamma \vdash_{nf} \Delta$, if we manage to show that the world (Γ, Δ) forces all formulae of Γ and refutes all formulae of Δ. This is done as in the proof of the preceeding corollary. □

The sequent calculus $\mathrm{LK}_{\mu\tilde{\mu}}$ introduced in section 3 is in proofs-as-programs correspondence with the calculus $\bar{\lambda}\mu\tilde{\mu}$ of [CH00] and has actually never been presented separately from it. We chose to present only the "logical" side of the two in order to decrease the level of details, because our main aim was to introduce the new notion of model.

Therefore, the last corollary can be also seen as a result about computation i.e. reduction to normal form of $\bar{\lambda}\mu\tilde{\mu}$-commands. Since in $\bar{\lambda}\mu\tilde{\mu}$-calculus there are two choices for a reduction strategy, call-by-name and call-by-value, the question is which one did we obtain. Our experiments with using the Coq formalisation[4] of the presented proofs, as an algorithm, confirm that we have obtained a call-by-name evaluation strategy, as was actually anticipated by the third author in the beginnings of this work.

More precisely, for the $\bar{\lambda}\mu\tilde{\mu}$ critical pair $\langle \mu\alpha.c \| \tilde{\mu}x.c' \rangle$ the algorithm gives priority to the $(\tilde{\mu})$ reduction rule and the reduction for connectives $\rightarrow, \wedge, \vee$ appears to work correctly. (see [Her05] for the reduction rules) In the formalisation we did not implement the connectives $\forall, \exists, \bot, \top$ in particular due to the additional time required for formal handling of binders.

[4] available at `http://www.lix.polytechnique.fr/~danko/lbmmt_kripke.v`

Call-by-Value Models. $\mathrm{LK}_{\mu\tilde{\mu}}$ and $\overline{\lambda}\mu\tilde{\mu}$ enjoy a strong duality between call-by-name and call-by-value. This gives us the right to conjecture that the proofs presented in this paper would work, essentially unmodified, for a call-by-value notion of classical Kripke model. In that notion we would have "strong *forcing*" on atoms as primitive and "refutation" and non-strong "forcing" defined through it by orthogonality, by analogy to the call-by-name case.

6. Conclusion

Sambin has already remarked in [Sam95] that having richer semantics, gives a simpler completeness proof and that this applies in particular to classical logic as well. While his semantics are based on phase spaces, or pre-topologies, the notion of semantics presented here does not fall into that category. Our completeness proof might be slightly more involved than Sambin's, but it is certainly much less complex than a traditional Gödel-Henkin style proof, which is due to our use of a more "descriptive" semantics than Boolean semantics.

In future we hope to be able to compare the computational content of the presented completeness theorem with the computational content of *constructive* Gödel-Henkin style proofs. [Kri96, BV04]

We also plan to prove that the normalisation-by-evaluation algorithm we got is correct with respect to the reduction relation of $\overline{\lambda}\mu\tilde{\mu}$. Note that Corollary 16 only tells us that we get a derivation in normal form, but it does not guarantee that we get the right normal form derivations – the ones matching the behaviour of $\overline{\lambda}\mu\tilde{\mu}$ reduction.

We would like to verify if our conjectured call-by-value models exist and, also, to generalise the results to a version of $\mathrm{LK}_{\mu\tilde{\mu}}$ with generic treatment of connectives. In order to get a more "regular" behaviour of disjunction, we will consider defining strong refutation of it in terms of *strong* refutation and might also consider adopting a different attitude to constructive disjunction at the meta-level, i.e. one not necessarily satisfying the BHK-interpretation of logical constants.

Finally, we would like to understand better the new notion of model.

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