

# Position and Force Control of Nonsmooth Lagrangian Dynamical Systems without Friction

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**Abstract.** Analyses of position and force control laws in the case of perfectly rigid bodies have been made so far with strong assumptions on the state of the contacts such as supposing that they are permanent. We're interested here in having a look at what happens when no such assumptions is made: we are led therefore to propose a Lyapunov stability analysis of a position and force control law in the mathematical framework of nonsmooth Lagrangian dynamical systems, a typical example of hybrid dynamical systems.

## 1 Introduction

Many applications of robot manipulators require contact phases between the robots and their environments, and a regulation of both the position of the robots and the reaction forces at the contact points is usually demanded in this case. So far, analyses of the corresponding position and force control laws have been either focusing on robot manipulators and environments with finite stiffnesses [1] or they have been made in the case of perfectly rigid bodies with strong assumptions on the state of the contacts [2] such as supposing that they are permanent [3]. Note that with such assumptions, the dynamical system solutions are ensured to be continuous, so that they fit into the classical framework of control theory [4]. In the case of perfectly rigid bodies, when the assumption of permanent contact is relaxed, the analysis of dynamical systems turns out to be greatly complicated, we're interested in this paper in having a look at what happens in this case, and more precisely what happens with the propositions of [3].

As we will see in section 2, the dynamical behaviour of mechanical systems with non permanent contact may present impacts, discrete events that intertwine with the continuous dynamics, the landmark of hybrid dynamical system [5]. More precisely, the frictionless contact law that we will consider fits into the smaller subclass of complementarity systems [5], [6]. Now, instead of considering piecewise smooth solutions as is usually done for hybrid systems [7, 8], we will consider here the wider class of nonsmooth solutions that has been proposed in [9], [10], [11], [12], [13]. These solutions are introduced through an extensive use of convex and nonsmooth analysis, and since these mathematical tools are still unusual in control theory, we are going to spend some time in section 2 to present

them, and to see how this relates to the usual framework of hybrid systems. Section 3 is then entirely devoted to analysing in this nonsmooth dynamics framework the Lyapunov stability of the position and force control law proposed in [3] without the strong assumption of permanent contacts.

## 2 Nonsmooth Lagrangian dynamical systems

### 2.1 Systems with non-permanent contacts

With  $n$  the number of degrees of freedom of the dynamical system, let us consider a time-variation of generalized coordinates  $\mathbf{q} : \mathbb{R} \rightarrow \mathbb{R}^n$  and the related velocity  $\dot{\mathbf{q}} : \mathbb{R} \rightarrow \mathbb{R}^n$ :

$$\forall t, t_0 \in \mathbb{R}, \mathbf{q}(t) = \mathbf{q}(t_0) + \int_{t_0}^t \dot{\mathbf{q}}(\tau) d\tau.$$

We're interested here with Lagrangian dynamical systems which may experience non-permanent contacts of perfectly rigid bodies. Geometrically speaking, the non-overlapping of rigid bodies can be expressed as a constraint on the position of the corresponding dynamical system, a constraint that will take the form here of a closed set  $\Phi \subset \mathbb{R}^n$  in which the generalized coordinates are bound to stay [11]:

$$\forall t \in \mathbb{R}, \mathbf{q}(t) \in \Phi.$$

This way, contact phases correspond to phases when  $\mathbf{q}(t)$  lies on the boundary of  $\Phi$ , and non-contact phases to phases when  $\mathbf{q}(t)$  lies in the interior of  $\Phi$ . We will suppose that this closed set is time-invariant, and we will have to suppose that it is convex for the stability analysis of section 3.

We can define then for all  $\mathbf{q} \in \Phi$  the tangent cone [14]

$$\mathcal{T}(\mathbf{q}) = \left\{ \mathbf{v} \in \mathbb{R}^n : \begin{array}{l} \exists \tau_k \rightarrow 0, \tau_k > 0, \\ \exists \mathbf{q}_k \rightarrow \mathbf{q}, \mathbf{q}_k \in \Phi \\ \text{with } \frac{\mathbf{q}_k - \mathbf{q}}{\tau_k} \rightarrow \mathbf{v} \end{array} \right\},$$

and we can readily observe that if the velocity  $\dot{\mathbf{q}}(t)$  has a left and right limit at an instant  $t$ , then obviously  $-\dot{\mathbf{q}}^-(t) \in \mathcal{T}(\mathbf{q}(t))$  and  $\dot{\mathbf{q}}^+(t) \in \mathcal{T}(\mathbf{q}(t))$ .

Now, note that  $\mathcal{T}(\mathbf{q}) = \mathbb{R}^n$  in the interior of the domain  $\Phi$ , but it reduces to a half-space or even less on its boundary (Fig. 1): if the system reaches this boundary with a velocity  $\dot{\mathbf{q}}^- \notin \mathcal{T}(\mathbf{q})$ , it won't be able to continue its movement with a velocity  $\dot{\mathbf{q}}^+ = \dot{\mathbf{q}}^-$  and still stay in  $\Phi$  (Fig. 1). A discontinuity of the velocity will have to occur then, corresponding to an impact between contacting rigid bodies, the landmark of *nonsmooth* dynamical systems.

We can also define for all  $\mathbf{q} \in \Phi$  the normal cone [14]

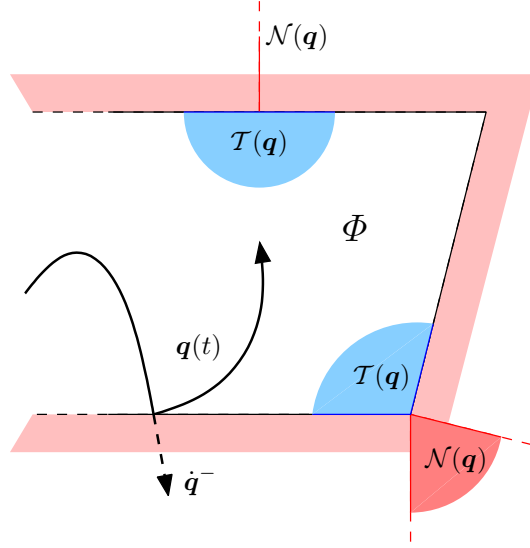
$$\mathcal{N}(\mathbf{q}) = \{ \mathbf{v} \in \mathbb{R}^n : \forall \mathbf{q}' \in \Phi, \mathbf{v}^T(\mathbf{q}' - \mathbf{q}) \leq 0 \},$$

and we will see in the inclusion (4) of section 2.3 that it is directly related to the reaction forces arising from the contacts between rigid bodies.

Now, note that  $\mathcal{N}(\mathbf{q}) = \{0\}$  in the interior of the domain  $\Phi$ , and it contains at least a half-line of  $\mathbb{R}^n$  on its boundary (Fig. 1): this will imply the obvious observation that non-zero contact forces may be experienced only on the boundary of the domain  $\Phi$ , precisely when there is a contact. Discontinuities of the contact forces might be induced because of that, what will be discussed later.

In the end, note that with these definitions, the state  $(\mathbf{q}(t), \dot{\mathbf{q}}(t))$  appears now to stay inside the set

$$\Omega = \{(\mathbf{q}, \dot{\mathbf{q}}) : \mathbf{q} \in \Phi, \dot{\mathbf{q}} \in \mathcal{T}(\mathbf{q})\}.$$



**Fig. 1.** Examples of tangent cones  $\mathcal{T}(\mathbf{q})$  and normal cones  $\mathcal{N}(\mathbf{q})$  on the boundary of the domain  $\Phi$ , and example of a trajectory  $\mathbf{q}(t) \in \Phi$  that reaches this boundary with a velocity  $\dot{\mathbf{q}}^- \notin \mathcal{T}(\mathbf{q})$ .

## 2.2 Nonsmooth Lagrangian dynamics

The dynamics of Lagrangian systems subject to Lebesgues-integrable forces are usually expressed as differential equations,

$$\mathbf{M}(\mathbf{q}) \frac{d\dot{\mathbf{q}}}{dt} + \mathbf{N}(\mathbf{q}, \dot{\mathbf{q}}) \dot{\mathbf{q}} = \mathbf{f},$$

with  $\mathbf{M}(\mathbf{q})$  the symmetric positive definite inertia matrix that we will suppose to be a  $C^1$  function of  $\mathbf{q}$ ,  $\mathbf{N}(\mathbf{q}, \dot{\mathbf{q}}) \dot{\mathbf{q}}$  the corresponding nonlinear effects and  $\mathbf{f}$  the Lebesgues-integrable forces. Classical solutions to these differential equations lead to smooth motions, with a locally absolutely continuous velocity  $\dot{\mathbf{q}}(t)$ .

But we have seen that discontinuities of the velocity may have to occur in the case of Lagrangian systems experiencing non-permanent contacts between rigid bodies. A mathematically rigorous way to allow such discontinuities in the dynamics of Lagrangian system has been proposed through measure differential equations [11], [15],

$$\mathbf{M}(\mathbf{q}) d\dot{\mathbf{q}} + \mathbf{N}(\mathbf{q}, \dot{\mathbf{q}}) \dot{\mathbf{q}} dt = \mathbf{f} dt + d\mathbf{r}, \quad (1)$$

with  $dt$  the Lebesgues measure and  $d\mathbf{r}$  the reaction forces arising from the contacts between rigid bodies, an abstract measure which may not be Lebesgues-integrable. This way, the measure acceleration  $d\dot{\mathbf{q}}$  may not be Lebesgues-integrable either so that the velocity may not be locally absolutely continuous anymore but only with locally bounded variation (see the remark below)  $\dot{\mathbf{q}} \in \text{lbv}(\mathbb{R}, \mathbb{R}^n)$  [11] [15]. Functions with locally bounded variation have left and right limits at every instant, and we have for every compact subinterval  $[\sigma, \tau] \subset \mathbb{R}$

$$\int_{[\sigma, \tau]} d\dot{\mathbf{q}} = \dot{\mathbf{q}}^+(\tau) - \dot{\mathbf{q}}^-(\sigma).$$

Considering then the integral of the measure differential equations (1) over a singleton  $\{\tau\}$ , we have

$$\begin{aligned} \int_{\{\tau\}} \mathbf{M}(\mathbf{q}) d\dot{\mathbf{q}} &= \mathbf{M}(\mathbf{q}) \int_{\{\tau\}} d\dot{\mathbf{q}} = \mathbf{M}(\mathbf{q})(\dot{\mathbf{q}}^+(\tau) - \dot{\mathbf{q}}^-(\tau)), \\ \int_{\{\tau\}} (\mathbf{N}(\mathbf{q}, \dot{\mathbf{q}}) \dot{\mathbf{q}} - \mathbf{f}) dt &= (\mathbf{N}(\mathbf{q}, \dot{\mathbf{q}}) \dot{\mathbf{q}} - \mathbf{f}) \int_{\{\tau\}} dt = 0, \end{aligned}$$

leading to the following relationship between possible discontinuities of the velocities and possible atoms of the contact forces,

$$\mathbf{M}(\mathbf{q}) (\dot{\mathbf{q}}^+(\tau) - \dot{\mathbf{q}}^-(\tau)) = \int_{\{\tau\}} d\mathbf{r},$$

or,  $\mathbf{M}(\mathbf{q})$  being invertible,

$$\dot{\mathbf{q}}^+(\tau) = \dot{\mathbf{q}}^-(\tau) + \mathbf{M}(\mathbf{q})^{-1} \int_{\{\tau\}} d\mathbf{r}. \quad (2)$$

**Remark 1** A function  $f$  is of locally bounded variation on  $\mathbb{R}$  if its variation on any compact subinterval  $I$  of  $\mathbb{R}$  is finite:

$$\text{Var}(f; I) = \sup_{\substack{t_0 \leq t_1 \leq \dots \leq t_n \\ \forall t_i \in I}} \sum_{i=1}^n \|f(t_i) - f(t_{i-1})\| < +\infty.$$

Note that in the framework of nonsmooth analysis, properties of functions with bounded variation are far more important than their definition, we are particularly interested in the following two properties. First as we have already seen,

functions with locally bounded variation have left and right limits at every instant, and if  $d\dot{\mathbf{q}}$  denotes their differential measure (an abstract measure that may not be Lebesgues-integrable), we have for every compact subinterval  $[\sigma, \tau] \subset \mathbb{R}$

$$\int_{[\sigma, \tau]} d\dot{\mathbf{q}} = \dot{\mathbf{q}}^+(\tau) - \dot{\mathbf{q}}^-(\sigma).$$

Then, functions with locally bounded variation can be decomposed into the sum of a continuous function and a countable set of discontinuous step functions [16]. In specific cases, as when data are piecewise analytic [12], the solution to the dynamics (1) can be shown to be piecewise continuous with possibly infinitely (countably) many discontinuities. In this case, it is possible to focus distinctly on each continuous piece and each discontinuity as in the framework of hybrid systems [8],[17]. But this is usually done through an ordering of the discontinuities strictly increasing with time, which in particular when having to go through accumulations of impacts. The framework of nonsmooth analysis appears therefore more appropriate for obtaining results in the case of impacting systems, even though the calculus rules for functions with bounded variation require some care as shown in the following Proposition that will be used in section 3.2,

**Proposition 1.** *If  $x \in lbv(I, \mathbb{R})$  and  $y \in lbv(I, \mathbb{R})$ , then the continuous function  $\Phi(x, y)$ ,  $t \rightarrow \Phi(x(t), y(t))$  is an element of  $lbv(I, \mathbb{R})$  whose differential measure equals*

$$d\Phi(x, y) = \Phi(dx, \frac{y^+ + y^-}{2}) + \Phi(dy, \frac{x^+ + x^-}{2}).$$

### 2.3 Inelastic frictionless unilateral contacts

Following [11], we will consider that the non-permanent contacts that may be experienced by our Lagrangian systems are perfectly unilateral, frictionless and inelastic. Expressing the  $\mathbb{R}^n$  valued measure  $d\mathbf{r}$  as the product of a non-negative real measure  $d\mu$  and a  $\mathbb{R}^n$  valued function  $\mathbf{r}'_\mu \in L^1_{loc}(\mathbb{R}, d\mu; \mathbb{R}^n)$ ,

$$d\mathbf{r} = \mathbf{r}'_\mu d\mu, \quad (3)$$

the unilaterality of the contacts (no adhesive forces) together with the absence of friction corresponds to the inclusion

$$\forall t \in \mathbb{R}, -\mathbf{r}'_\mu(t) \in \mathcal{N}(\mathbf{q}(t)), \quad (4)$$

and the inelasticity of the contacts corresponds to the complementarity condition

$$\forall t \in \mathbb{R}, \dot{\mathbf{q}}^+(t)^T \mathbf{r}'_\mu(t) = 0. \quad (5)$$

One can note that on top of this complementarity condition, the inclusion (4) also induces a complementarity between  $\mathbf{r}'_\mu(t)$  and  $\mathbf{q}(t)$ , but for a more in-depth presentation of these concepts and equations which are quite subtle, the interested reader should definitely refer to [11].

## 2.4 Some Lyapunov stability theory

The Lyapunov stability theory is usually presented for dynamical systems with states that vary continuously with time [4], [17], [18], Filippov systems for example [19], [20], but we have seen that in the case of nonsmooth mechanics, the velocity and thus the state  $x(t) = (\mathbf{q}(t), \dot{\mathbf{q}}(t))$  may present discontinuities. Lyapunov stability theory is hopefully not strictly bound to continuity properties, some results for discontinuous dynamical systems can still be derived both in the usual framework of hybrid systems [21] and in the framework of nonsmooth analysis [22]. In the following we will prefer this latter for the reasons mentioned in Remark 1, which provides for example the following theorem.

**Theorem 1.** *A closed invariant set  $\mathcal{S} \subset \Omega$  is globally stable if and only if there exists a function  $V : \Omega \rightarrow \mathbb{R}$  such that*

- (i) *there exist two continuous strictly increasing functions  $\alpha(\cdot)$  and  $\beta(\cdot)$  satisfying  $\alpha(0) = 0, \beta(0) = 0$  such that*

$$\forall x \in \Omega, \alpha(d(x, \mathcal{S})) \leq V(x) \leq \beta(d(x, \mathcal{S})),$$

*with  $d(x, \mathcal{S})$  the distance between the state  $x$  and the set  $\mathcal{S}$ , and*

- (ii) *for all solutions  $x(t)$  to the nonsmooth dynamics (1), the function  $V(x(t))$  is non-increasing with time.*

*Such a function is called a Lyapunov function with respect to the stable set  $\mathcal{S}$ .*

Note now that the position and force control law that we are going to study in the next section is proved to be asymptotically stable in [3] through the use of LaSalle's invariance theorem. This latter is tightly bound to the continuity of trajectories of the systems with respect to initial conditions, a property of nonsmooth dynamical systems that holds only in some specific cases [13],[23]. For dynamical systems satisfying this continuity property, it is still possible to propose a theorem equivalent to LaSalle's [22],[24]. However in this paper, we will only consider the global stability proposed in the previous theorem.

## 3 Lyapunov stability analysis of a position and force control law

This section aims at analysing the Lyapunov stability of the position and force control law proposed in [3] without the assumption of permanent contacts between the end-effector and its environment. Based on the nonsmooth analysis framework and in particular in Theorem 1, we will be able to conclude on the stability of this control law with no need for any assumptions concerning the state of the contact experienced by the systems. Note that [3] proved that both the desired position and the desired contact forces were stable, but non-permanent contacts unfortunately won't allow to conclude on the stability of the contact forces.

### 3.1 A position and force control law

Let us consider now that the Lebesgues-integrable forces  $\mathbf{f}$  acting on the dynamics (1) consist of some external forces  $\mathbf{e}_f$  and a control  $\mathbf{u}$ ,

$$\mathbf{f} = \mathbf{e}_f + \mathbf{u}.$$

With the help of this control  $\mathbf{u}$ , we would like to stabilize both the position  $\mathbf{q}$  of the dynamical systems and the reaction forces  $d\mathbf{r}$  to some desired constant values  $\mathbf{q}_d$  and  $\mathbf{r}_d dt$  (following (3), the desired contact forces are defined through the product of the Lebesgues measure  $dt$  and a constant vector  $\mathbf{r}_d \in \mathbb{R}^n$ ). First of all, these desired position and reaction forces have to be consistent with the contact model (4),

$$-\mathbf{r}_d \in \mathcal{N}(\mathbf{q}_d). \quad (6)$$

Following then the proposition of [3], we define the control  $\mathbf{u}$  through the derivative of a strictly convex  $C^1$  potential function  $P(\mathbf{q})$ , a dissipative term  $\mathbf{C} \dot{\mathbf{q}}$  with  $\mathbf{C}$  a positive definite matrix, and a compensation of the external forces,

$$\mathbf{u} = -\frac{dP}{d\mathbf{q}}(\mathbf{q}) - \mathbf{C} \dot{\mathbf{q}} - \mathbf{e}_f. \quad (7)$$

With this control law, the dynamics (1) becomes

$$\mathbf{M}(\mathbf{q}) d\dot{\mathbf{q}} + \mathbf{N}(\mathbf{q}, \dot{\mathbf{q}}) \dot{\mathbf{q}} dt = -\frac{dP}{d\mathbf{q}}(\mathbf{q}) dt - \mathbf{C} \dot{\mathbf{q}} dt + d\mathbf{r}, \quad (8)$$

the equilibria of which, with  $\dot{\mathbf{q}} = 0$ , are positions for which

$$0 = -\frac{dP}{d\mathbf{q}}(\mathbf{q}) dt + \mathbf{r}'_{\mu} d\mu.$$

This equation of measures is satisfied if and only if  $d\mu = dt$  and

$$0 = -\frac{dP}{d\mathbf{q}}(\mathbf{q}) + \mathbf{r}'_{\mu}, \quad (9)$$

and through theorem VII.1.1.1 of [14], this corresponds together with (4) to the specification of the minima of  $P(\mathbf{q})$  over the domain  $\Phi$ : the equilibria of the closed loop dynamics correspond to the minima of the potential function. More precisely, since  $\Phi$  is assumed to be convex and  $P(\mathbf{q})$  strictly convex, if there is such a minimum then it is reached at a unique position: if there is an equilibrium position of the closed loop dynamics, then it is unique.

If we assume now that the potential function satisfies explicitly

$$\frac{dP}{d\mathbf{q}}(\mathbf{q}_d) = \mathbf{r}_d,$$

then there is such a minimum through (6) and the same theorem of [14]: this minimum is  $P(\mathbf{q}_d)$ , reached at the position  $\mathbf{q}_d$ , and equation (9) becomes

$$0 = -\mathbf{r}_d + \mathbf{r}'_{\mu},$$

so that the contact forces will be as desired at this equilibrium,

$$d\mathbf{r} = \mathbf{r}_d dt.$$

### 3.2 Lyapunov stability analysis

Since  $P(\mathbf{q})$  has a global minimum reached at the unique position  $\mathbf{q}_d$ , with

$$K(\mathbf{q}, \dot{\mathbf{q}}) = \frac{1}{2} \dot{\mathbf{q}}^T \mathbf{M}(\mathbf{q}) \dot{\mathbf{q}}$$

the kinetic energy of the dynamical system, the function

$$V(\mathbf{q}, \dot{\mathbf{q}}) = K(\mathbf{q}, \dot{\mathbf{q}}) + P(\mathbf{q}) - P(\mathbf{q}_d)$$

has 0 as a global minimum, reached at the unique state  $(\mathbf{q}_d, 0)$ .

Since it is convex with a minimum reached at a unique position, we know from proposition IV.3.2.5 and definition IV.3.2.6 of [14] that the function  $P(\mathbf{q})$  is radially unbounded. Excluding pathological behaviours of the inertia matrix, we can suppose quite directly then that the function  $V(\mathbf{q}, \dot{\mathbf{q}})$  is also radially unbounded. Lemma 3.5 of [4] allows then to conclude that it satisfies condition (i) of the theorem of section 2.4 with respect to the set  $\mathcal{S} = \{(\mathbf{q}_d, 0)\}$ , appearing therefore as a possible Lyapunov function.

Indeed, through Proposition 1 given in Remark 1, classical differentiation rules of lbv allow to compute the time-derivative of the kinetic energy,

$$dK = \frac{1}{2} \dot{\mathbf{q}}^T \dot{\mathbf{M}}(\mathbf{q}, \dot{\mathbf{q}}) \dot{\mathbf{q}} dt + \frac{(\dot{\mathbf{q}}^+ + \dot{\mathbf{q}}^-)^T}{2} \mathbf{M}(\mathbf{q}) d\dot{\mathbf{q}}.$$

For the closed loop dynamics (8), this time-derivative is

$$\begin{aligned} dK &= \frac{1}{2} \dot{\mathbf{q}}^T (\dot{\mathbf{M}}(\mathbf{q}, \dot{\mathbf{q}}) - 2 \mathbf{N}(\mathbf{q}, \dot{\mathbf{q}})) \dot{\mathbf{q}} dt - \dot{\mathbf{q}}^T \frac{dP}{d\mathbf{q}}(\mathbf{q}) dt \\ &\quad - \dot{\mathbf{q}}^T \mathbf{C} \dot{\mathbf{q}} dt + \frac{(\dot{\mathbf{q}}^+ + \dot{\mathbf{q}}^-)^T}{2} d\mathbf{r}, \end{aligned}$$

(note that  $\dot{\mathbf{q}}^+ dt = \dot{\mathbf{q}}^- dt = \dot{\mathbf{q}} dt$ ) where the first term is identically 0 since  $\dot{\mathbf{M}}(\mathbf{q}, \dot{\mathbf{q}}) - 2 \mathbf{N}(\mathbf{q}, \dot{\mathbf{q}})$  is an antisymmetric matrix and  $-\dot{\mathbf{q}}^T \mathbf{C} \dot{\mathbf{q}}$  is non-positive since  $\mathbf{C}$  is a positive matrix. Recalling then relations (2) and (3), we have

$$\begin{aligned} \frac{(\dot{\mathbf{q}}^+ + \dot{\mathbf{q}}^-)^T}{2} d\mathbf{r} &= \dot{\mathbf{q}}^{+T} d\mathbf{r} - \frac{1}{2} \left[ \int_{\{\tau\}} d\mathbf{r} \right]^T \mathbf{M}(\mathbf{q})^{-1} d\mathbf{r} \\ &= \dot{\mathbf{q}}^{+T} \mathbf{r}'_{\mu} d\mu \\ &\quad - \frac{1}{2} \left[ \int_{\{\tau\}} d\mu \right] \mathbf{r}'_{\mu}{}^T \mathbf{M}(\mathbf{q})^{-1} \mathbf{r}'_{\mu} d\mu \end{aligned}$$

where the first term is identically 0 because of the complementarity condition (5) and the second term is non-positive since the inertia matrix is positive and  $d\mu \geq 0$ . All this ends up with

$$dK \leq -\dot{\mathbf{q}}^T \frac{dP}{d\mathbf{q}}(\mathbf{q}) dt,$$

and since the time-derivative of the potential function is precisely

$$dP = \dot{\mathbf{q}}^T \frac{dP}{d\mathbf{q}}(\mathbf{q}) dt,$$

we are led to

$$dV = dK + dP \leq 0.$$

The function  $V(x(t))$  is therefore non-increasing with time, condition (ii) of the theorem of section 2.4 is also satisfied, and the proof that the state  $(\mathbf{q}_d, 0)$  is globally stable with the closed loop dynamics (8) is completed.

Note that what we have proved here is the stability of the state  $(\mathbf{q}_d, 0)$  only, and not of the contact forces  $\mathbf{r}_d dt$ : on the contrary to what appears in [3], non-zero contact forces can't be stable in our case since, as we have seen in section 2.1, non-zero contact forces may be experienced only on the boundary of the domain  $\Phi$ , when there is a contact. These forces may therefore jump to zero in every neighbourhood of any equilibrium position, what is not compatible with Lyapunov stability.

### 3.3 An example

Following [3], we can see for example that with a strictly convex quadratic potential function

$$P(\mathbf{q}) = \frac{1}{2}(\mathbf{q} - \mathbf{q}_d)^T \mathbf{W}(\mathbf{q} - \mathbf{q}_d) + \mathbf{r}_d^T(\mathbf{q} - \mathbf{q}_d)$$

with a symmetric positive definite matrix  $\mathbf{W}$ , the control law (7) becomes a strictly linear feedback

$$\mathbf{u} = -\mathbf{W}(\mathbf{q} - \mathbf{q}_d) - \mathbf{r}_d - \mathbf{C}\dot{\mathbf{q}} - \mathbf{e}_f$$

for which we know now that the equilibrium state  $(\mathbf{q}_d, 0)$ , where the contact forces are  $\mathbf{r}_d dt$ , is globally stable.

## 4 Conclusion

The stability of nonsmooth mechanical systems, a typical example of hybrid dynamical systems, has been studied here in the framework of convex and nonsmooth analysis.

The position and force control law proposed in [3] can be proved to be stable in this framework with no need for any assumptions concerning the state of the contacts experienced by the systems. This result is obtained with the help of differentiation rules for functions with locally bounded variations which are somehow different from the more usual ones for locally absolutely continuous functions, but which can be practiced in a very similar way, allowing to derive a Lyapunov stability analysis for nonsmooth dynamical systems very similar to what appears in the smooth case.

Extreme care must be taken though about the particularities of nonsmooth dynamical systems: if we can propose a stability theorem such as the one of section 2.4, which is very similar to usual theorems for smooth dynamical systems, it doesn't mean that the whole stability theory for smooth dynamics can be translated to the nonsmooth case without specific and sometimes subtle adaptations. The example of Lasalle's theorem discussed in section 2.4 or the fact that the contact forces can't be stable for physical reasons speak for themselves.

## References

1. Yabuta, T.: Nonlinear basic stability concept of the hybrid position/force control scheme for robot manipulators. *IEEE Trans. Robot. Automat.* **8** (1992) 663–670
2. Bourgeot, J.M., Brogliato, B.: Tracking control of Lagrangian complementarity systems. *Int. J. of Bifurcation and Chaos, special issue on Nonsmooth Dynamical Systems* **15** (2005)
3. Wang, D., McClamroch, H.: Position and force control for constrained manipulator motion: Lyapunov's direct method. *IEEE Trans. Robot. Automat.* **9** (1993) 308–313
4. Khalil, H.: *Nonlinear systems*. Prentice-Hall (1996)
5. Brogliato, B.: Some perspectives on the analysis and control of complementarity systems. *IEEE Transactions on Automatic Control* **48** (2003) 918–935
6. Brogliato, B., Heemels, W.: Hybrid systems modeling and control. *European Journal of Control. Special issue on "fundamental issues in control"* **9** (2003) 177–189
7. De Carlo, R., Branicky, S., Pettersson, S., Lennartson, B.: Perspectives and results on the stability and stabilizability of hybrid systems. *Proc. IEEE* **88** (2000) 1069–1082
8. Lygeros, J., Johansson, K., Simic, S., Zhang, J., Sastry, S.: Dynamical properties of hybrid automata. *IEEE Transactions on Automatic Control* **48** (2003) 2–17
9. Schatzman, M.: A class of nonlinear differential equations of second order in time. *Nonlinear Analysis, Theory, Methods & Applications* **2** (1978) 355–373
10. Moreau, J.J.: Liaisons unilatérales sans frottements et chocs inélastiques. *C.R. Acad.Sc.Paris* **296** (1983) 1473–1476
11. Moreau, J.J.: Unilateral contact and dry friction in finite freedom dynamics. In Moreau, J.J., Panagiotopoulos, P., eds.: *Nonsmooth mechanics and Applications*. Volume 302. Springer Verlag (1988) 1–82
12. Ballard, P.: The dynamics of discrete mechanical systems with perfect unilateral constraints. *Archive for Rational Mechanics and Analysis* (2000) 199–274
13. Ballard, P.: Formulation and well-posedness of the dynamics of rigid-body systems with perfect unilateral constraints. *Philosophical Transactions: Mathematical, Physical & Engineering Sciences* **359** (2001) 2327–2346
14. Hiriart-Urruty, J.B., Lemaréchal, C.: *Convex Analysis and Minimization Algorithms*. Springer Verlag, Heidelberg (1996) Two volumes - 2nd printing.
15. Moreau, J.J.: An introduction to unilateral dynamics. In Fremond, M., Maceri, F., eds.: *Novel approaches in civil engineering*. Springer Verlag (2001)
16. Moreau, J.J.: Bounded variation in time. In Moreau, J.J., Panagiotopoulos, P., Strang, G., eds.: *Topics in Nonsmooth Mechanics*. Birkhäuser, Basel Boston Berlin (1988) 1–74

17. Branicky, S.: Multiple lyapunov functions and other analysis tools for switched and hybrid systems. *IEEE Transactions on Automatic Control* **43** (1998) 475–482
18. Zubov, V.: *Methods of A.M. Lyapunov and their application*. Noordhoff (1964)
19. Orlov, Y.: Extended invariance principle for nonautonomous switched systems. *IEEE Transactions on Automatic Control* **48** (2003) 1448–1452
20. Bacciotti, A., Ceragioli, F.: Stability and stabilization of discontinuous systems and nonsmooth lyapunov functions. *ESAIM: COCV* **4** (1999) 361–376
21. Ye, H. Michel, A., Hou, L.: Stability theory for hybrid dynamical systems. *IEEE Transactions on Automatic Control* **43** (1998) 461–474
22. Chareyron, S., Wieber, P.: Stabilization and regulation of nonsmooth lagrangain systems. Technical report, INRIA Rhône-Alpes (2004) submitted.
23. Paoli, L.: Problemes de vibro-impact: etude de la dépendance par rapport aux donnees. *C.R. Acad. Sci. Paris* **339** (2004) 27–32
24. Tornambè, A.: Modeling and control of impact in mechanical systems: theory and experimental results. *IEEE Transactions on Automatic Control* **44** (1999) 294–309