A LASALLE'S INVARIANCE THEOREM FOR NONSMOOTH LAGRANGIAN DYNAMICAL SYSTEMS

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Abstract

A key condition for the statement of Lasalle's invariance theorem is the continuity of the trajectories of the dynamical systems with respect to initial conditions. Systems with discontinuous flows generally don't present such a continuity, but in the particular case of nonsmooth Lagrangian dynamical systems we will see that in fact this continuity property holds in most cases. We will then be able to propose a a LaSalle's invariance theorem for nonsmooth dynamical systems satisfying this property through the use of general time-invariant flows.

Key words

LaSalle's invariance theorem, nonsmooth dynamical systems, continuity of trajectories with respect to initial conditions, hybrid systems

1 Introduction

Originating in the analysis of the non-permanent contact between perfectly rigid bodies, the mathematical analysis of nonsmooth Lagrangian dynamical systems is very recent. It concerns Lagrangian dynamical systems with coordinates constrained to stay inside some closed sets, what leads to introduce mathematical tools which are unusual in control theory, velocities with locally bounded variations, measure accelerations, measure differential inclusions to name a few.

Control theory, and in particular stability theory is usually presented for dynamical systems with states that vary continuously with time (Bacciotti and Ceragioli, 1999) (Branicky, 1998) (Khalil, 1996) (Orlov, 2003) (Zubov, 1964), but it is not the case of nonsmooth lagrangian dynamical systems. Stability theory is hopefully not strictly bound to continuity properties, and some stability results have already been proposed for discontinuous dynamical systems both in the usual framework of hybrid systems (Ye *et al.*, 1998) and in the framework of nonsmooth dynamical systems (Chareyron and Wieber, 2004).

But a theorem equivalent to LaSalle's invariance theorem for nonsmooth dynamical systems still needs to be stated, so we propose here to extend this theorem through the framework of nonsmooth dynamical systems.

First, we are going to spend some time in section 2 to present this framework. Then, assuming that the trajectories of the system dynamics are continuous with respect to initial conditions, we will propose in section 3 a LaSalle's invariance theorem through the use of general time-invariant flows. Section 4 is then devoted to the application of LaSalle's invariance theorem to the case of nonsmooth Lagrangian dynamical systems. Building on this theorem, we will be able to conclude on the attractivity of equilibrium points of nonsmooth Lagrangian dynamical systems. Continuous dependance of trajectories with respect to initial conditions appears to be a key property for being able to apply LaSalle's invariance theorem. Systems with discontinuous flows generally don't present such a continuity, but in the particular case of nonsmooth Lagrangian dynamical systems we will see in section 5 that in fact this continuity property holds in most cases.

2 Nonsmooth Lagrangian dynamical systems

We're interested with Lagrangian dynamical systems which may experience non-permanent contacts with their environments, such as for example dynamical systems of non-penetrating perfectly rigid bodies. We will see that this assumption of non-permanent contacts is at the origin of a *nonsmooth* behavior of the dynamics. This section aims therefore at presenting such nonsmooth Lagrangian dynamical systems. Note that it is mostly identical to what already appears in section 2 of (Chareyron and Wieber, 2005).

2.1 Systems with non-permanent contacts

With n the number of degrees of freedom of the dynamical system, let us consider a time-variation of generalized coordinates $\boldsymbol{q} : \mathbb{R} \to \mathbb{R}^n$ and the related velocity $\dot{\boldsymbol{q}} : \mathbb{R} \to \mathbb{R}^n$:

$$\forall\, t, t_0 \in \mathbb{R},\ \boldsymbol{q}(t) = \boldsymbol{q}(t_0) + \int_{t_0}^{t} \dot{\boldsymbol{q}}(\tau)\, d\tau.$$

We're interested here with Lagrangian dynamical systems which may experience non-permanent contacts of perfectly rigid bodies. Geometrically speaking, the non-overlapping of rigid bodies can be expressed as a constraint on the position of the corresponding dynamical system, a constraint that will take the form here of a closed set $\Phi \subset \mathbb{R}^n$ in which the generalized coordinates are bound to stay (Moreau, 1988*b*):

$$\forall\, t \in \mathbb{R},\ \boldsymbol{q}(t) \in \Phi.$$

This way, contact phases between two or more rigid bodies correspond to phases when $\boldsymbol{q}(t)$ lies on the boundary of Φ, and non-contact phases to phases when $\boldsymbol{q}(t)$ lies in the interior of Φ. We will suppose that this closed set is time-invariant.

We can define then for all $\boldsymbol{q} \in \Phi$ the tangent cone (Hiriart-Urruty and Lemaréchal, 1996)

$$\begin{aligned}\mathcal{T}(\boldsymbol{q}) = \big\{\boldsymbol{v} \in \mathbb{R}^n :\ & \exists\, \tau_k \to 0,\ \tau_k > 0,\\ & \exists\, \boldsymbol{q}_k \to \boldsymbol{q},\ \boldsymbol{q}_k \in \Phi\\ & \text{with } \tfrac{\boldsymbol{q}_k - \boldsymbol{q}}{\tau_k} \to \boldsymbol{v}\big\},\end{aligned}$$

and we can readily observe that if the velocity $\dot{\boldsymbol{q}}(t)$ has a left and right limit at an instant t, then obviously $-\dot{\boldsymbol{q}}^-(t) \in \mathcal{T}(\boldsymbol{q}(t))$ and $\dot{\boldsymbol{q}}^+(t) \in \mathcal{T}(\boldsymbol{q}(t))$.

Now, note that $\mathcal{T}(\boldsymbol{q}) = \mathbb{R}^n$ in the interior of the domain Φ, but it reduces to a half-space or even less on its boundary (Fig. 1): if the system reaches this boundary with a velocity $\dot{\boldsymbol{q}}^- \notin \mathcal{T}(\boldsymbol{q})$, it won't be able to continue its movement with a velocity $\dot{\boldsymbol{q}}^+ = \dot{\boldsymbol{q}}^-$ and still stay in Φ (Fig. 1). A discontinuity of the velocity will have to occur then, corresponding to an impact between contacting rigid bodies, the landmark of *nonsmooth* dynamical systems.

We can also define for all $\boldsymbol{q} \in \Phi$ the normal cone (Hiriart-Urruty and Lemaréchal, 1996)

$$\mathcal{N}(\boldsymbol{q}) = \big\{\boldsymbol{v} \in \mathbb{R}^n :\ \forall\, \boldsymbol{q}' \in \Phi,\ \boldsymbol{v}^T(\boldsymbol{q}' - \boldsymbol{q}) \le 0\big\},$$

and we will see in the inclusion (4) of section 2.3 that it is directly related to the reaction forces arising from the contacts between rigid bodies.

Now, note that $\mathcal{N}(\boldsymbol{q}) = \{0\}$ in the interior of the domain Φ, and it contains at least a half-line of $\mathbb{R}^n$ on its boundary (Fig. 1): this will imply the obvious observation that non-zero contact forces may be experienced only when there is a contact, that is precisely when $\boldsymbol{q}(t)$ lies on the boundary of the domain Φ.

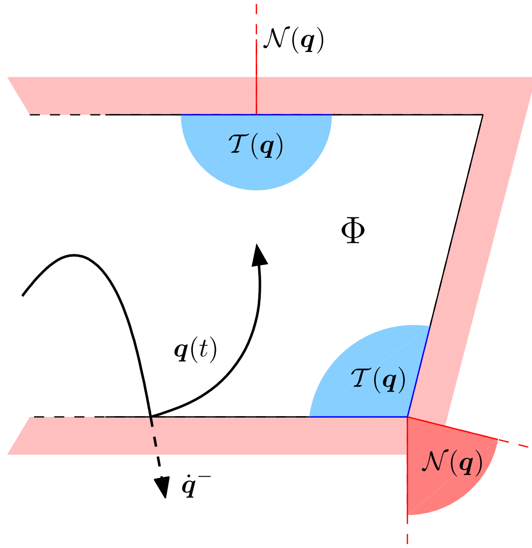

Figure 1. Examples of tangent cones $\mathcal{T}(\boldsymbol{q})$ and normal cones $\mathcal{N}(\boldsymbol{q})$ on the boundary of the domain Φ, and example of a trajectory $\boldsymbol{q}(t) \in \Phi$ that reaches this boundary with a velocity $\dot{\boldsymbol{q}}^- \notin \mathcal{T}(\boldsymbol{q})$.

In the end, note that with these definitions, the state $(\boldsymbol{q}(t), \dot{\boldsymbol{q}}(t))$ appears now to stay inside the set

$$\Omega = \big\{(\boldsymbol{q}, \dot{\boldsymbol{q}}) :\ \boldsymbol{q} \in \Phi,\ \dot{\boldsymbol{q}} \in \mathcal{T}(\boldsymbol{q})\big\}.$$

2.2 Nonsmooth Lagrangian dynamics

The dynamics of Lagrangian systems subject to Lebesgues-integrable forces are usually expressed as differential equations

$$\boldsymbol{M}(\boldsymbol{q})\, \frac{d\dot{\boldsymbol{q}}}{dt} + \boldsymbol{N}(\boldsymbol{q}, \dot{\boldsymbol{q}})\, \dot{\boldsymbol{q}} = \boldsymbol{f},$$

with $\boldsymbol{M}(\boldsymbol{q})$ the symmetric positive definite inertia matrix that we will suppose to be a C^1 function of $\boldsymbol{q}$, $\boldsymbol{N}(\boldsymbol{q}, \dot{\boldsymbol{q}})\, \dot{\boldsymbol{q}}$ the corresponding nonlinear effects and $\boldsymbol{f}$ the Lebesgues-integrable forces, dt being the Lebesgues measure. Classically, solutions to these differential equations lead to smooth motions with locally absolutely continuous velocities $\dot{\boldsymbol{q}}(t)$.

But we have seen that discontinuities of the velocities may occur when the coordinates of such systems are constrained to stay inside closed sets. These classical differential equations must therefore be turned into measure differential equations (Moreau, 1988*b*; Moreau, 2001)

$$\boldsymbol{M}(\boldsymbol{q})\, d\dot{\boldsymbol{q}} + \boldsymbol{N}(\boldsymbol{q}, \dot{\boldsymbol{q}})\, \dot{\boldsymbol{q}}\, dt = \boldsymbol{f}\, dt + d\boldsymbol{r}, \qquad (1)$$

with $d\boldsymbol{r}$ the reaction forces arising from the contacts between rigid bodies, an abstract measure which may not be Lebesgues-integrable. This way, the measure acceleration $d\dot{\boldsymbol{q}}$ may not be Lebesgues-integrable either so that the velocity may not be locally absolutely continuous anymore but only with locally bounded variation, $\dot{\boldsymbol{q}} \in \mathrm{lbv}(\mathbb{R}, \mathbb{R}^n)$ (Moreau, 1988*b*) (Moreau, 2001) (see the remark below for more details). Functions with locally bounded variation have left and right limits at every instant, and we have for every compact subinterval $[\sigma, \tau] \subset \mathbb{R}$

$$\int_{[\sigma,\tau]} d\dot{\boldsymbol{q}} = \dot{\boldsymbol{q}}^+(\tau) - \dot{\boldsymbol{q}}^-(\sigma).$$

Considering then the integral of the measure differential equations (1) over a singleton $\{\tau\}$, we have

$$\int_{\{\tau\}} \boldsymbol{M}(\boldsymbol{q})\, d\dot{\boldsymbol{q}} = \boldsymbol{M}(\boldsymbol{q}) \int_{\{\tau\}} d\dot{\boldsymbol{q}} = \boldsymbol{M}(\boldsymbol{q})\big(\dot{\boldsymbol{q}}^+(\tau)-\dot{\boldsymbol{q}}^-(\tau)\big),$$

$$\int_{\{\tau\}} \big(\boldsymbol{N}(\boldsymbol{q},\dot{\boldsymbol{q}})\,\dot{\boldsymbol{q}}-\boldsymbol{f}\big)dt = \big(\boldsymbol{N}(\boldsymbol{q},\dot{\boldsymbol{q}})\,\dot{\boldsymbol{q}}-\boldsymbol{f}\big)\int_{\{\tau\}} dt = 0,$$

leading to the following relationship between possible discontinuities of the velocities and possible atoms of the contact forces,

$$\boldsymbol{M}(\boldsymbol{q})\,\big(\dot{\boldsymbol{q}}^+(\tau) - \dot{\boldsymbol{q}}^-(\tau)\big) = \int_{\{\tau\}} d\boldsymbol{r},$$

or, $\boldsymbol{M}(\boldsymbol{q})$ being invertible,

$$\dot{\boldsymbol{q}}^+(\tau) = \dot{\boldsymbol{q}}^-(\tau) + \boldsymbol{M}(\boldsymbol{q})^{-1}\int_{\{\tau\}} d\boldsymbol{r}. \qquad (2)$$

Remark 1. *A function f has a locally bounded variation on $\mathbb{R}$ if its variation on any compact interval $[t_0,\, t_n]$ is finite:*

$$\mathrm{var}(f;[t_0,t_n]) = \sup_{t_0\leq\cdots\leq t_n} \sum_{i=1}^{n} ||f(t_i)-f(t_{i-1})|| < +\infty.$$

Rather than for this definition, it is for their properties that functions with bounded variations are of any use to us. First of all, as we have already seen, functions with locally bounded variation have left and right limits at every instant, and if df denotes the differential measure of the function f, we have for every compact subinterval $[\sigma,\tau] \subset \mathbb{R}$

$$\int_{[\sigma,\tau]} df = f^+(\tau) - f^-(\sigma).$$

*Functions with locally bounded variation can also be decomposed into the sum of a continuous function and a countable set of discontinuous step functions (Moreau, 1988*a*). In specific cases, as when the definition of the dynamics (1) is piecewise analytic, its solutions can be shown to be piecewise continuous with possibly infinitely (countably) many discontinuities (Ballard, 2000). In this case, it is possible to focus distinctly on each continuous piece and each discontinuity as in the framework of hybrid systems (Lygeros* et al.*, 2003),(Branicky, 1998). But this is usually done through an ordering of the discontinuities strictly increasing with time, what is problematic when having to go through accumulations of impacts. The framework of nonsmooth analysis appears more appropriate for the analysis of impacting systems, even though the calculus rules for functions with bounded variation require some care as shown in the following proposition that will be used in section 4.2,*

Proposition 1. *If $x \in lbv(I, \mathbb{R}^n)$, $y \in lbv(I, \mathbb{R}^n)$, and $\Phi(x,y) : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is a continuous bilinear mapping function, then $\Phi(x(t),y(t)) \in lbv(I, \mathbb{R})$ and*

$$d\Phi(x,y) = \Phi(dx, \frac{y^+ + y^-}{2}) + \Phi(\frac{x^+ + x^-}{2}, dy).$$

2.3 Frictionless unilateral interactions

Following (Moreau, 1988*b*), we will consider that the interactions with the constraints are perfectly rigid unilateral and frictionless. Expressing the $\mathbb{R}^n$ valued measure $d\boldsymbol{r}$ as the product of a non-negative real measure $d\mu$ and an $\mathbb{R}^n$ valued function $\boldsymbol{r}'_\mu \in L^1_{loc}([0,T], d\mu; \mathbb{R}^n)$,

$$d\boldsymbol{r} = \boldsymbol{r}'_\mu\, d\mu, \qquad (3)$$

this corresponds to the inclusion

$$\forall\, t \in \mathbb{R},\ -\boldsymbol{r}'_\mu(t) \in \mathcal{N}(\boldsymbol{q}(t)) \qquad (4)$$

which implies especially a complementarity between the interaction forces $\boldsymbol{r}'_\mu(t)$ and the coordinates $\boldsymbol{q}(t)$ of the system as has been pointed out in section 2.1: non-zero contact forces may be experienced only when there is a contact, when $\boldsymbol{q}(t)$ lies on the boundary of the domain Φ.

We will consider moreover that the impulsive behavior of these interactions is ruled by a coefficient of restitution $e \in [0,1]$, perfectly elastic when $e = 1$, perfectly inelastic when $e = 0$. With

$$\dot{\boldsymbol{q}}_e = \dot{\boldsymbol{q}}^+ - \frac{e}{2}(\dot{\boldsymbol{q}}^+ - \dot{\boldsymbol{q}}^-), \qquad (5)$$

this corresponds to a complementarity condition between the interaction forces $\boldsymbol{r}'_\mu(t)$ and the velocity $\dot{\boldsymbol{q}}_e(t)$,

$$\forall\, t \in \mathbb{R},\ \dot{\boldsymbol{q}}_e(t)^T \boldsymbol{r}'_\mu(t) = 0. \qquad (6)$$

For a more in-depth presentation of these concepts and equations which may have subtle implications, the interested reader should definitely refer to (Moreau, 1988*b*).

3 LaSalle's invariance theorem for dynamical systems with state discontinuities

Control theory, and in particular stability theory is usually presented for dynamical systems with states that vary continuously with time (Bacciotti and Ceragioli, 1999) (Branicky, 1998) (Khalil, 1996) (Orlov, 2003) (Zubov, 1964), but we have seen that in the case of nonsmooth lagrangian dynamical systems, the velocity and thus the state $x(t) = (\boldsymbol{q}(t), \dot{\boldsymbol{q}}(t))$ may present discontinuities. Stability theory is hopefully not strictly bound to continuity properties, and some stability results have already been proposed for discontinuous dynamical systems both in the usual framework of hybrid systems (Ye *et al.*, 1998) and in the framework of nonsmooth dynamical systems (Chareyron and Wieber, 2004). As presented in the remark 1, this second framework seems more appropriate. Now, we have been able in (Chareyron and Wieber, 2004) to derive some stability theory in this framework through the use of general time-invariant flows. We are going therefore to propose a version of LaSalle's invariance theorem in the same setting.

3.1 General time-invariant flows

Let us consider therefore a time-invariant flow on a metric space $\mathcal{X}$, an application $X : \mathbb{R}\times\mathcal{X} \to \mathcal{X}$ which may not be differentiable nor even continuou such that

$$\forall\, x \in \mathcal{X},\; X(0,x) = x,$$
$$\forall\, x \in \mathcal{X},\; \forall\, t,s \in \mathbb{R}^+,\; X(t, X(s,x)) = X(t+s, x).$$

Given a position x_0, the function $t \to X(t, x_0)$ is called a motion of the dynamical system with initial position x_0, and the set of positions $\{X(t,x_0),\;\; t \in \mathbb{R}^+\}$ will be called the corresponding trajectory.

LaSalle's invariance theorem needs that the trajectories of the flow be continuous with respect to initial conditions,

$$\forall\, x_0 \in \mathcal{X},\; \forall\, t \in \mathbb{R},\; x \to x_0 \Longrightarrow X(t,x) \to X(t,x_0). \tag{7}$$

Systems with discontinuous flows generally don't present such a continuity, but in the particular case of nonsmooth Lagrangian dynamical systems, we will see in section 5 that in fact this continuity property holds in most cases.

3.2 Invariance of limit sets

LaSalle's invariance theorem is based on the invariance property of positive limit sets, as defined by

Definition 1. *A point p is said to be a positive limit point of the motion $X(t,x_0)$ if there exists a sequence $\{t_n\}$ such that $t_n \to +\infty$ and $X(t_n,x_0) \to p$ when $n \to \infty$. The set of all positive limits points is called the positive limit set $\mathcal{L}^+$.*

In the smooth case, theorem 3 of (Zubov, 1964) ensures that the positive limit set is a closed and invariant set. When considering a system with states discontinuities, the closedness of the positive limit set can always be proved, but not its invariance.

The invariance of the positive limit set can be obtained with the additionnal assumption of continuous dependence of trajectories of the dynamical system with respect to initial conditions. This can be shown identically to what is done in lemma 3.1 of (Khalil, 1996),

Lemma 1. *If a flow $X(t,x)$ is continuous with respect to initial conditions, then the limit set $\mathcal{L}^+$ of any motion $X(t,x_0)$ is invariant.*

Proof. Given a motion $X(t,x_0)$ and a point p in the corresponding limit set $\mathcal{L}^+$, there exists a sequence $\{t_n\}$ such that $t_n \to +\infty$ and $X(t_n,x_0) \to p$ when $n \to \infty$. Thanks to the continuity with respect to initial conditions and following the definition of the limit set $\mathcal{L}^+$, we have that

$$\begin{aligned}\forall\, t > 0,\; X(t,p) &= \lim_{n\to\infty} X(t, X(t_n,x_0)) \\ &= \lim_{n\to\infty} X(t+t_n, x_0) \;\in\; \mathcal{L}^+\end{aligned}$$

what shows the invariance of the limit set.

3.3 LaSalle's invariance theorem

Let's propose now a version of LaSalle's invariance theorem in this framework of general time-invariant flows. This theorem is built on the analysis of the variation with time of a function $V(x)$, what is usually done with the help of its time derivative $\dot{V}(x)$, but such a derivative may not exist in this framework. We need therefore to state this theorem without the use of time derivatives what leads to the following variation of theorem 3.4 in (Khalil, 1996):

Theorem 1. *Let $\Omega \subset \mathcal{X}$ be a compact set and $V : \mathcal{X} \to \mathbb{R}$ a continuous function such that*

(i) the set Ω is positively invariant,

$$\forall\, x \in \Omega,\; \forall\, t \geq 0,\; X(t,x) \in \Omega,$$

(ii) the function V is a non-increasing function of time when starting in Ω,

$$\forall\, x \in \Omega,\; \forall\, t \geq 0,\; V(X(t,x)) \leq V(x),$$

(iii) the subset $\mathcal{E} \subset \Omega$ gathers all the states where the function V is stationnary with time,

$$\forall\, x \in \Omega,\ \forall\, t > 0,\ V(X(t,x)) = V(x) \Longrightarrow x \in \mathcal{E}.$$

If the trajectories of the dynamical system are continuous with respect to initial conditions, then every trajectory starting in Ω converges asymptotically as $t \to \infty$ to the largest invariant subset of $\mathcal{E}$.

Proof. The proof is exactly the same as the one of theorem 3.4 in (Khalil, 1996). Note that it relies strongly on the application of lemma 1.

4 Application to nonsmooth Lagrangian dynamical systems

This section is devoted to the application of LaSalle's invariance theorem as stated in the previous section to the case of nonsmooth Lagrangian dynamical systems. We need to suppose then that the forces $\boldsymbol{f}$ acting on the dynamics (1) derive from a potential plus a strictly dissipative term. Note that most of the derivations of this section are very similar to what appeared in (Chareyron and Wieber, 2005) for the Lyapunov stability analysis of nonsmooth Lagrangian dynamical systems, but applied now to prove the attractivity of equilibrium points through the use of LaSalle's invariance theorem.

4.1 A preliminary on Stieljes measures

The theorem 1 is based on an analysis of the time variation of a function $V(x)$. In the specific case of nonsmooth Lagrangian dynamical systems, we have seen that the states $x(t) = (\boldsymbol{q}(t), \dot{\boldsymbol{q}}(t))$ have locally bounded variations: $x \in \text{lbv}([t_0,T],\, \mathbb{R}^n \times \mathbb{R}^n)$. If a function $V : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ is locally Lipschitz continuous, for example if it is convex or $\mathcal{C}^1$, then $V(x(t))$ will also have locally bounded variations: $(V \circ x) \in \text{lbv}([t_0,T], \mathbb{R})$. In this case, the variations of $V \circ x$ will be directly related to the sign of the associated Stieljes measure $d(V \circ x)$. Gathering these results, the following trivial lemma is going to be a cornerstone of this section:

Lemma 2. *Let $x \in lbv([t_0,T],\, \mathbb{R}^n \times \mathbb{R}^n)$ be a function with locally bounded variations and $V : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R}$ be locally Lipschitz continuous, then $V \circ x$ is nonincreasing if and only if $d(V \circ x) \leq 0$ and constant if and only if $d(V \circ x) = 0$.*

4.2 Attractivity of the equilibria

Let us consider now that the Lebesgues-integrable forces $\boldsymbol{f}$ acting on the dynamics (1) derive from a C^1 potential function $P(\boldsymbol{q})$, with an additionnal strictly dissipative term $\boldsymbol{h}$:

$$\boldsymbol{f} = -\frac{dP}{d\boldsymbol{q}}(\boldsymbol{q}) + \boldsymbol{h}, \text{ with } \dot{\boldsymbol{q}}^T \boldsymbol{h}\, dt \leq 0 \text{ and } \tag{8}$$

$$\dot{\boldsymbol{q}}^T \boldsymbol{h}\, dt = 0 \Longrightarrow \dot{\boldsymbol{q}}\, dt = 0.$$

With

$$K(\boldsymbol{q}, \dot{\boldsymbol{q}}) = \frac{1}{2}\, \dot{\boldsymbol{q}}^T \boldsymbol{M}(\boldsymbol{q})\, \dot{\boldsymbol{q}}$$

the kinetic energy of the Lagrangian dynamics (1), the total energy of the system $K(\boldsymbol{q}, \dot{\boldsymbol{q}}) + P(\boldsymbol{q})$ can be shown to be a non-increasing function of time. We briefly go through the main steps, refer to (Chareyron and Wieber, 2004) for the full computation steps. Applying calculus rules specific to the differentiation of lbv functions (Moreau, 1988*a*; Moreau, 1988*b*) and the fact that $\frac{1}{2}(\dot{\boldsymbol{q}}^+ + \dot{\boldsymbol{q}}^-)\, dt = \dot{\boldsymbol{q}}\, dt$, we have

$$dK = \frac{1}{2} \dot{\boldsymbol{q}}^T \dot{\boldsymbol{M}}(\boldsymbol{q}, \dot{\boldsymbol{q}})\, \dot{\boldsymbol{q}}\, dt + \frac{1}{2} (\dot{\boldsymbol{q}}^+ + \dot{\boldsymbol{q}}^-)^T \boldsymbol{M}(\boldsymbol{q})\, d\dot{\boldsymbol{q}}.$$

With the dynamics (1), this leads to

$$dK = \frac{1}{2} \dot{\boldsymbol{q}}^T \left(\dot{\boldsymbol{M}}(\boldsymbol{q}, \dot{\boldsymbol{q}}) - 2\, \boldsymbol{N}(\boldsymbol{q}, \dot{\boldsymbol{q}}) \right) \dot{\boldsymbol{q}}\, dt + \frac{1}{2} (\dot{\boldsymbol{q}}^+ + \dot{\boldsymbol{q}}^-)^T d\boldsymbol{r} + \dot{\boldsymbol{q}}^T \boldsymbol{f}\, dt,$$

but since $\dot{\boldsymbol{M}}(\boldsymbol{q}, \dot{\boldsymbol{q}}) - 2\, \boldsymbol{N}(\boldsymbol{q}, \dot{\boldsymbol{q}})$ is an antisymmetric matrix, the first term here is void:

$$dK = \frac{1}{2} (\dot{\boldsymbol{q}}^+ + \dot{\boldsymbol{q}}^-)^T d\boldsymbol{r} + \dot{\boldsymbol{q}}^T \boldsymbol{f}\, dt, \tag{9}$$

what relates the variation of the kinetic energy dK to the power exerted by the forces $d\boldsymbol{r}$ and $\boldsymbol{f}\, dt$. Now, with the help of relations (5), (3) and (2), the first term can be shown to be non-positive, equation (9) can be reduced to

$$dK \leq \dot{\boldsymbol{q}}^T \boldsymbol{f}\, dt.$$

Now, with forces $\boldsymbol{f}$ defined as in (8), this results in

$$dK \leq -\dot{\boldsymbol{q}}^T \frac{dP}{d\boldsymbol{q}}(\boldsymbol{q})\, dt + \dot{\boldsymbol{q}}^T \boldsymbol{h}\, dt = -dP + \dot{\boldsymbol{q}}^T \boldsymbol{h}\, dt,$$

and we end up with

$$dK + dP \leq \dot{\boldsymbol{q}}^T \boldsymbol{h}\, dt \leq 0. \tag{10}$$

Lemma 4.1 tells us then that this energy of the system is a nonincreasing function of time, so that it naturally satisfies condition (ii) of theorem 1 whatever the set Ω. We can observe also through the same lemma and condition (10) that if it is constant over a time interval then on this interval

$$dK + dP = 0 \Longrightarrow \dot{\boldsymbol{q}}^T \boldsymbol{h} dt = 0 \Longrightarrow \dot{\boldsymbol{q}} dt = 0.$$

If we consider then without loss of generality that the velocity $\dot{\boldsymbol{q}}$ is right-continuous (Moreau, 1988*a*), so that

$$\dot{\boldsymbol{q}}dt = 0 \implies \dot{\boldsymbol{q}} = 0.$$

This implies that condition (iii) of theorem 1 is satisfied by the set of states with zero velocity,

$$\mathcal{E} = \Omega \cap (\Phi \times \{0\}),$$

of which the largest invariant subset is by construction the set of equilibrium points that lie inside Ω, what leads to the following application of theorem 1, showing the attractivity of the equilibrium point of nonsmooth Lagrangian dynamical systems,

Theorem 2. *If the forces f acting on a nonsmooth Lagrangian dynamical system derive as in (8) from a C^1 potential function $P(\boldsymbol{q})$ with a strictly dissipative term and if the trajectories of this system are continuous with respect to initial conditions, then if there is a compact set $\Omega \subset \Phi \times \mathbb{R}^n$ that is positively invariant, every trajectory starting in this set converges asymptotically as $t \to \infty$ to the equilibrium points of the system that lie inside this set.*

5 Continuous dependence with respect to initial conditions

The continuous dependence of trajectories with respect to initial conditions appears to be a key property for being able to apply LaSalle's invariance theorem. Section 5.1 illustrates the influence that a discontinuity of the trajectories with respect to initial conditions may have on the invariance property of the limit sets. Section 5.2 presents then when and where such a discontinuity can occur in the case of nonsmooth Lagrangian dynamical systems and section 5.3 gives additionnal details on the physical meaning of such discontinuities.

5.1 Discontinuity with respect to initial conditions

The results presented in section 3.2 hold under the assumption that solutions depend continuously on initial conditions, we're now interested to see what can happen if they do not. Based on Remark 3.2 in (Chellaboina and Bhat, 2003), let us consider the following dynamical system, with $\mathcal{X} = \mathbb{R}$,

$$\forall\ t > 0,\quad X(t, x_0) = \begin{cases} x_0\, e^{-t} & \text{if}\ \ x_0 \neq 0, \\ e^{-t} & \text{if}\ \ x_0 = 0. \end{cases}$$

We can observe that the trajectories of this flow are not continuous with respect to initial conditions for the initial position $x_0 = 0$. Now, for every $x_0 \in \mathbb{R}$, the trajectory $X(t, x_0)$ approaches the set $\mathcal{L}^+ = \{0\}$ but this set is obviously not invariant. The conclusions of lemma 1 appear therefore to be contradicted just because of this discontinuity. Since lemma 1 is crucial to derive LaSalle's invariance theorem, this discontinuity impairs the applicability of this theorem.

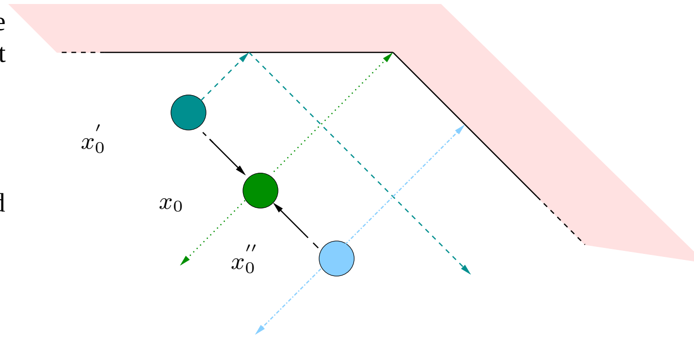

Figure 2. An example of nonsmooth Lagrangian dynamical system where trajectories are not continous with respect to initial conditions

Note however that the limit set $\mathcal{L}^+$ appears to be non-invariant here because the point where the discontinuity occurs for an initial position $x_0 = 0$ which belongs to this limit set. The question now is therefore where and when do such discontinuity appear?

5.2 The case of nonsmooth Lagrangian dynamical systems

Let us consider the classical example of a ball striking a corner. For any initial conditions x_0, the dynamics of the ball together with the impact law (4)-(6) determine completely the motion $X(t, x_0)$. The figure 2 shows the example of three trajectories for three initial conditions, x_0, $x_0^{'}$ and $x_0^{''}$.

When the initial conditions $x_0^{''} \to x_0$ then $X(t, x_0^{''}) \to X(t, x_0)$ but when $x_0^{'} \to x_0$ we obviously don't have $X(t, x_0^{'}) \to X(t, x_0)$: a discontinuity appears, and we can observe that it appears because an impact takes place at a corner. Now, when the corner is orthogonal (figure 3) or when it is acute and the impacts are perfectly inelastic $e = 0$, trajectories can be proved to be always continuous with respect to initial conditions (Ballard, 2000) (Paoli, 2004). It is also the case for one-degree of freedom systems (Schatzman, 1978) or when Φ is convex and its boundary is $\mathcal{C}^1$ at impact points (Ballard, 2000).

Depending on the shape of the set Φ, the trajectories of nonsmooth Lagrangian systems appear therefore to be continuous with respect to initial conditions in most cases, and discontinuous only in specific cases. It should be possible therefore to apply widely LaSalle's invariance theorem.

5.3 Impacts at corners, multiple impacts

Impact at corners are widely referred to as multiple impacts since they involve several contacts simultaneously. This corresponds to a very complex physical behavior, and it can be interesting to give more details about its model as induced by the impact law presented in section 2.3. Newton's law was first written for a single contact, and its generalization to the multiconstrained cases appears to be valid only within a cer-

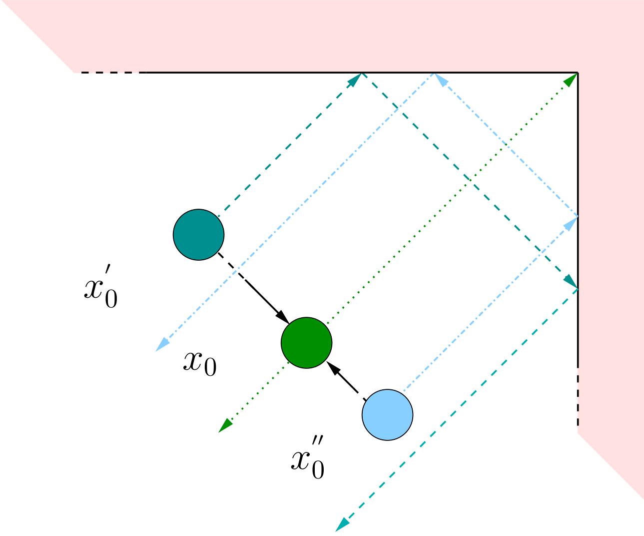

Figure 3. An example of nonsmooth Lagrangian dynamical system where trajectories are continous with respect to initial conditions

tain restricted framework. One should be aware that the physical impact process is an extremely complicated and highly dynamic event extending over the entire system and taking place within a very short time, so an impact law never can contain the full information about this physical impact process (Glocker, 2001).

An accurate model would require at least the elasticity of each of the participating bodies in order to take into account the wave propagation process initiated by the collision. When this process is condensed to an event taking place at a single instant in time, such as in the rigid body approach, one must not expect the equations of the contact law (4)-(6) to be already general enough to carry the whole physical information needed for computing reasonable approximates of post impact-velocities (Glocker, 2001).

In such situations, one has to abandon any hope of predicting the motion of the system: this is a consequence of the over-idealization made in the indeformability assumption. This fact is often illustrated by the example of Newton's cradle (Ballard, 2000). In addition to probably obtaining a trajectory that does not correspond to the "real" motion, this situation does not allow the computation of approximate solutions since round up errors may lead to a totally different trajectory.

In the real world, trajectories with impacts at corners are very sensitive to initial conditions, what is synthesized in the model (1)-(6) as a pure discontinuity.

6 Conclusion

In the usual control theory, stated for dynamical systems having continuous flows, the LaSalle's invariance theorem appears to be a powerful tool, and its extension to the case of dynamical systems with discontinuous flows would be of great interest. This paper was meant at analysing the possible extension of the LaSalle's invariance theorem for nonsmooth dynamical systems through the framework of nonsmooth dynamical systems. We saw that the extention of the LaSalle's invariance theorem for systems with discontinuous flows was possible under mild conditions : continuous dependance of the system trajectories with respect to initial conditions. Systems with discontinuous flows generally don't present such a continuity, but in the particular case of nonsmooth Lagrangian dynamical systems we saw that in fact this continuity property holds in most cases. Building on the LaSalle's invariance theorem for nonsmooth dynamical systems, and assuming that the forces acting on a nonsmooth Lagrangian dynamical system derive from a potential function with a strictly dissipative term, we have then been able to conclude on the attractivity of equilibrium points of this system.

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