



HAL
open science

Ball throwing on spheres

Anne Estrade, Jacques Istas

► **To cite this version:**

Anne Estrade, Jacques Istas. Ball throwing on spheres. Bernoulli, 2010, 16 (4), pp.953-970.
10.3150/09-BEJ241 . hal-00332812v2

HAL Id: hal-00332812

<https://hal.science/hal-00332812v2>

Submitted on 5 Jun 2009

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

BALL THROWING ON SPHERES

ANNE ESTRADE AND JACQUES ISTAS

ABSTRACT. Ball throwing on Euclidean spaces has been considered for a while. A suitable renormalization leads to a fractional Brownian motion as limit object. In this paper we investigate ball throwing on spheres. A different behavior is exhibited: we still get a Gaussian limit but which is no longer a fractional Brownian motion. However the limit is locally self-similar when the self-similarity index H is less than $1/2$.

KEYWORDS. fractional Brownian motion, overlapping balls, scaling, self-similarity, spheres

1. INTRODUCTION

Random balls models have been studied for a long time and are known as germ-grain models, shot-noise, or micropulses. The common point of those models consists in throwing balls that eventually overlap at random in an n -dimensional space. Many random phenomena can be modeled through this procedure and many application fields are concerned: Internet traffic in dimension one, communication network or imaging in dimension two, biology or materials sciences in dimension three. A pioneer work is due to Wicksell [24] with the study of corpuscles. The literature on germ-grain models deals with two main axes. Either the research focuses on the geometrical or morphological aspect of the union of random balls (see [20] or [21] and references therein), or it is interested in the number of balls covering each point. This second point of view is currently known as shot-noise or spot-noise (see [6] for existence). In dimension three, the shot-noise process is a natural candidate for modeling porous or granular media, and more generally heterogeneous media with irregularities at any scale. The idea is to build a microscopic model which yields a macroscopic field with self-similar properties. The same idea is expected in dimension one for Internet traffic for instance [25]. A usual way for catching self-similarity is to deal with scaling limits. Roughly speaking, the balls are dilated with a scaling parameter λ and one lets λ go either to 0 or to infinity. We quote for instance [4] and [12] for the case $\lambda \rightarrow 0^+$, [11] and [2] for the case $\lambda \rightarrow +\infty$, [5] and [3] where both cases are considered.

In the present paper, we follow a procedure which is similar to [2] and [3]. Let us describe it precisely. A collection of random balls in \mathbb{R}^n whose centers and radii are chosen according to a random Poisson measure on $\mathbb{R}^n \times \mathbb{R}^+$ is considered. The Poisson intensity is prescribed as follows

$$\nu(dx, dr) = r^{-n-1+2H} dx dr ,$$

Date: June 5, 2009.

for some real parameter H . Since the Lebesgue measure dx is invariant with respect to isometry, so is the random balls model, and so will be any (eventual) limit. As the distribution of the radii follows a homogeneous distribution, a self-similar scaling limit may be expected. Indeed, with additional technical conditions, the scaling limits of such random balls models are isometry invariant self-similar Gaussian fields. The self-similarity index depends on the parameter H . When $0 < H < 1/2$, the Gaussian field is nothing but the well-known fractional Brownian motion [16, 18, 19].

Manifold indexed fields that share properties with Euclidean self-similar fields have been and still are extensively studied (e.g. [8, 9, 13, 14, 15, 17, 22, 23]). In this paper, we wonder what happens when balls are thrown onto a sphere, and no longer onto a Euclidean space. More precisely, is there a scaling limit of random balls models, and, when it exists, is this scaling limit a fractional Brownian field indexed by the surface for $0 < H < 1/2$?

The random field is still obtained by throwing overlapping balls in a Poissonian way. The Poisson intensity is chosen as follows

$$\nu(dx, dr) = f(r) \sigma(dx) dr.$$

The Lebesgue measure dx has been replaced by the surface measure $\sigma(dx)$. The function f , that manages the distribution of the radii, is still equivalent to $r^{-n-1+2H}$, at least for small r , where n stands for the surface dimension. It turns out that the results are completely different in the two cases (Euclidean, spherical). In the spherical case, there is a Gaussian scaling limit for any H . But it is no longer a fractional Brownian field, as defined by [13]. We then investigate the local behavior, in the tangent bundle, of this scaling limit in the spirit of local self-similarity [1, 7, 15]. It is locally asymptotically self-similar, with a Euclidean fractional Brownian field as tangent field. Our microscopic model has led to a local self-similar macroscopic model.

The paper is organized as follows. In Section 2, the spherical model is introduced and we prove the existence of a scaling limit. In Section 3, we study the locally self-similar property of the asymptotic field. Section 4 is devoted to a comparative analysis between Euclidean and spherical cases. Eventually, some technical computations are presented in the Appendix.

2. SCALING LIMIT

We work on \mathbb{S}_n the n -dimensional unit sphere, $n \geq 1$:

$$\mathbb{S}_n = \{(x_i)_{1 \leq i \leq n+1} \in \mathbb{R}^{n+1}; \sum_{1 \leq i \leq n+1} x_i^2 = 1\}.$$

2.1. Spherical caps. For $x, y \in \mathbb{S}_n$, let $d(x, y)$ denote the distance between x and y on \mathbb{S}_n , i.e. the non-oriented angle between Ox and Oy where O denotes the origin of \mathbb{R}^{n+1} . For $r \geq 0$, $B(x, r)$ denotes the closed ball on S centered at x with radius r :

$$B(x, r) = \{y \in \mathbb{S}_n; d(x, y) \leq r\}.$$

Let us notice that for $r < \pi$, $B(x, r)$ is a spherical cap on the unit sphere \mathbb{S}_n , centered at x with opening angle r and that for $r \geq \pi$, $B(x, r) = \mathbb{S}_n$.

Denoting $\sigma(dx)$ the surface measure on \mathbb{S}_n , we prescribe $\phi(r)$ as the surface of any ball on \mathbb{S}_n with radius r ,

$$\phi(r) := \sigma(B(x, r)) , \quad x \in \mathbb{S}_n, r \geq 0 .$$

We also introduce the following function defined for z and z' , two points in \mathbb{S}_n and $r \in \mathbb{R}^+$,

$$(1) \quad \Psi(z, z', r) := \int_{\mathbb{S}_n} \mathbf{1}_{d(x, z) < r} \mathbf{1}_{d(x, z') < r} \sigma(dx) .$$

Actually $\Psi(z, z', r)$ denotes the surface measure of the set of all points in \mathbb{S}_n that belong to both balls $B(z, r)$ and $B(z', r)$. Clearly $\Psi(z, z', r)$ only depends on the distance $d(z, z')$ between z and z' . We write

$$(2) \quad \psi(d(z, z'), r) = \Psi(z, z', r)$$

and note that it satisfies the following: $\forall (u, r) \in [0, \pi] \times \mathbb{R}^+$

- $0 \leq \psi(u, r) \leq \sigma(\mathbb{S}_n) \wedge \phi(r)$
- if $r < u/2$ then $\psi(u, r) = 0$ and if $r > \pi$ then $\psi(u, r) = \sigma(\mathbb{S}_n)$
- $\psi(0, r) = \phi(r) \sim cr^n$ as $r \rightarrow 0^+$.

In what follows we consider a family of balls $B(X_j, R_j)$ generated at random, following a strategy described in the next section.

2.2. Poisson point process. We consider a Poisson point process $(X_j, R_j)_j$ in $\mathbb{S}_n \times \mathbb{R}^+$, or equivalently $N(dx, dr)$ a Poisson random measure on $\mathbb{S}_n \times \mathbb{R}^+$, with intensity

$$\nu(dx, dr) = f(r) \sigma(dx) dr$$

where f satisfies the following assumption **A**(H) for some $H > 0$:

- $\text{supp}(f) \subset [0, \pi)$
- f is bounded on any compact subset of $(0, \pi)$
- $f(r) \sim r^{-n-1+2H}$ as $r \rightarrow 0^+$.

Remarks :

- 1) The first condition ensures that no balls of radius R_j on the sphere self-intersect.
- 2) Since $\phi(r) \sim cr^n$, $r \rightarrow 0^+$, the last condition implies that $\int_{\mathbb{R}^+} \phi(r) f(r) dr < +\infty$, which means that the mean surface, with respect to f , of the balls $B(X_j, R_j)$ is finite.

2.3. Random field. Let \mathcal{M} denote the space of signed measures μ on \mathbb{S}_n with finite total variation $|\mu|(\mathbb{S}_n)$, with $|\mu|$ the total variation measure of μ . For any $\mu \in \mathcal{M}$, we define

$$(3) \quad X(\mu) = \int_{\mathbb{S}_n \times \mathbb{R}^+} \mu(B(x, r)) N(dx, dr) .$$

Note that the stochastic integral in (3) is well defined since

$$\begin{aligned} \int_{\mathbb{S}_n \times \mathbb{R}^+} |\mu(B(x, r))| f(r) \sigma(dx) dr &\leq \int_{\mathbb{S}_n} \int_{\mathbb{S}_n} \int_{\mathbb{R}^+} \mathbf{1}_{d(x, y) < r} f(r) \sigma(dx) |\mu|(dy) dr \\ &= |\mu|(\mathbb{S}_n) \left(\int_{\mathbb{R}^+} \phi(r) f(r) dr \right) < +\infty . \end{aligned}$$

In the particular case where μ is a Dirac measure δ_z for some point $z \in \mathbb{S}_n$ we simply denote

$$(4) \quad X(z) = X(\delta_z) = \int_{\mathbb{S}_n \times \mathbb{R}^+} \mathbf{1}_{B(x,r)}(z) N(dx, dr) .$$

The pointwise field $\{X(z); z \in \mathbb{S}_n\}$ corresponds to the number of random balls (X_j, R_j) covering each point of \mathbb{S}_n . Each random variable $X(z)$ has a Poisson distribution with mean $\int_{\mathbb{R}^+} \phi(r) f(r) dr$.

Furthermore for any $\mu \in \mathcal{M}$,

$$\mathbb{E}(X(\mu)) = \mu(\mathbb{S}_n) \left(\int_{\mathbb{R}^+} \phi(r) f(r) dr \right)$$

and

$$\text{Var}(X(\mu)) = \int_{\mathbb{S}_n \times \mathbb{R}^+} \mu(B(x,r))^2 f(r) \sigma(dx) dr \in (0, +\infty] .$$

2.4. Key lemma. For $H > 0$, we would like to compute the integral

$$\int_{\mathbb{S}_n \times \mathbb{R}^+} \mu(B(x,r))^2 r^{-n-1+2H} \sigma(dx) dr$$

which is a candidate for the variance of an eventually scaling limit. We first introduce \mathcal{M}^H the set of measures for which the above integral does converge:

$$\mathcal{M}^H = \mathcal{M} \text{ if } 2H < n ; \mathcal{M}^H = \{\mu \in \mathcal{M} ; \mu(\mathbb{S}_n) = 0\} \text{ if } 2H > n .$$

The following lemma deals with the function ψ prescribed by (2).

Lemma 2.1. *Let $H > 0$ with $2H \neq n$. We introduce*

$$\psi^{(H)} = \psi \text{ if } 0 < 2H < n ; \psi^{(H)} = \psi - \sigma(\mathbb{S}_n) \text{ if } 2H > n .$$

Then for all $u \in [0, \pi]$,

$$\int_{\mathbb{R}^+} |\psi^{(H)}(u, r)| r^{-n-1+2H} dr < +\infty .$$

Furthermore, denoting

$$(5) \quad K_H(u) = \int_{\mathbb{R}^+} \psi^{(H)}(u, r) r^{-n-1+2H} dr$$

for any u in $[0, \pi]$, we have for all $\mu \in \mathcal{M}^H$,

$$0 \leq \int_{\mathbb{S}_n \times \mathbb{R}^+} \mu(B(x,r))^2 r^{-n-1+2H} \sigma(dx) dr = \int_{\mathbb{S}_n \times \mathbb{S}_n} K_H(d(z, z')) \mu(dz) \mu(dz') < +\infty .$$

Remark 2.2.

- 1) For x, y in \mathbb{S}_n , the difference of Dirac measures $\delta_x - \delta_y$ belongs to \mathcal{M}^H for any H .
- 2) In the case $2H > n$, since any $\mu \in \mathcal{M}^H$ is centered, the rhs integral is not changed when a constant is added to the kernel K_H .
- 3) This lemma proves that the kernel K_H defines a covariance function on \mathcal{M}^H .

Proof. Using the properties of ψ , we get in the case $0 < 2H < n$,

$$\begin{aligned} 0 \leq \int_{\mathbb{R}^+} \psi(u, r) r^{-n-1+2H} dr &\leq \int_{(0, \pi)} \phi(r) r^{-n-1+2H} dr + \sigma(\mathbb{S}_n) \int_{(\pi, \infty)} r^{-n-1+2H} dr \\ &< +\infty . \end{aligned}$$

In the same vein, in the case $2H > n$, we get

$$\begin{aligned} 0 \leq \int_{\mathbb{R}^+} (\sigma(\mathbb{S}_n) - \psi(u, r)) r^{-n-1+2H} dr &\leq \sigma(\mathbb{S}_n) \int_{(0, \pi)} r^{-n-1+2H} dr \\ &< +\infty . \end{aligned}$$

We have just established that there exists a finite constant C_H such that

$$(6) \quad \forall u \in [0, \pi] , \quad \int_{\mathbb{R}^+} |\psi^{(H)}(u, r)| r^{-n-1+2H} dr \leq C_H .$$

The first statement is proved.

Let us denote for $\mu \in \mathcal{M}^H$

$$I_H(\mu) = \int_{\mathbb{S}_n \times \mathbb{R}^+} \mu(B(x, r))^2 r^{-n-1+2H} \sigma(dx) dr$$

and start with proving that $I_H(\mu)$ is finite. We will essentially use Fubini's Theorem in the following lines.

$$\begin{aligned} I_H(\mu) &= \int_{\mathbb{R}^+} \left(\int_{\mathbb{S}_n} \mu(B(x, r))^2 \sigma(dx) \right) r^{-n-1+2H} dr \\ &= \int_{\mathbb{R}^+} \left(\int_{\mathbb{S}_n \times \mathbb{S}_n} \Psi(z, z', r) \mu(dz) \mu(dz') \right) r^{-n-1+2H} dr . \end{aligned}$$

Since $\mu \in \mathcal{M}^H$ is centered in the case $2H > n$, one can change ψ into $\psi^{(H)}$ within the previous integral. Then

$$\begin{aligned} I_H(\mu) &\leq \int_{\mathbb{R}^+} \left(\int_{\mathbb{S}_n \times \mathbb{S}_n} |\psi^{(H)}(d(z, z'), r)| |\mu|(dz) |\mu|(dz') \right) r^{-n-1+2H} dr \\ &\leq \int_{\mathbb{S}_n \times \mathbb{S}_n} \left(\int_{\mathbb{R}^+} |\psi^{(H)}(d(z, z'), r)| r^{-n-1+2H} dr \right) |\mu|(dz) |\mu|(dz') \\ &\leq C_H |\mu|(\mathbb{S}_n)^2 < +\infty . \end{aligned}$$

Following the same lines (except for the last one) without the “|” allows the computation of $I_H(\mu)$ and concludes the proof. \square

An explicit value for the kernel K_H is available starting from its definition. The point is to compute $\psi^{(H)}$. We give in the Appendix a recurrence formula for $\psi^{(H)}$, based on the dimension n of the unit sphere \mathbb{S}_n (see Lemma 4.1).

2.5. Scaling. Let $\rho > 0$ and λ be any positive function on $(0, +\infty)$. We consider the scaled Poisson measure N_ρ obtained from the original Poisson measure N by taking the image under the map $(x, r) \in S \times \mathbb{R}^+ \mapsto (x, \rho r)$ and multiplying the intensity by $\lambda(\rho)$. Hence N_ρ is still a Poisson random measure with intensity

$$\nu_\rho(dx, dr) = \lambda(\rho)\rho^{-1}f(\rho^{-1}r)\sigma(dx)dr.$$

We also introduce the scaled random field X_ρ defined on \mathcal{M} by

$$(7) \quad X_\rho(\mu) = \int_{\mathbb{S}_n \times \mathbb{R}^+} \mu(B(x, r)) N_\rho(dx, dr).$$

Theorem 2.3. *Let $H > 0$ with $2H \neq n$ and let f satisfy $\mathbf{A}(H)$. For all positive functions λ such that $\lambda(\rho)\rho^{n-2H} \xrightarrow{\rho \rightarrow +\infty} +\infty$, the limit*

$$\left\{ \frac{X_\rho(\mu) - \mathbb{E}(X_\rho(\mu))}{\sqrt{\lambda(\rho)\rho^{n-2H}}}; \mu \in \mathcal{M}^H \right\} \xrightarrow[\rho \rightarrow +\infty]{fdd} \{W_H(\mu); \mu \in \mathcal{M}^H\}$$

holds in the sense of finite dimensional distributions of the random functionals. Here W_H is the centered Gaussian random linear functional on \mathcal{M}^H with

$$(8) \quad \text{Cov}(W_H(\mu), W_H(\nu)) = \int_{\mathbb{S}_n \times \mathbb{S}_n} K_H(d(z, z')) \mu(dz)\nu(dz'),$$

where K_H is the kernel introduced in Lemma 2.1.

The theorem can be rephrased in term of the pointwise field $\{X(z); z \in \mathbb{S}_n\}$ defined in (4).

Corollary 2.4. *Let $H > 0$ with $2H \neq n$ and let f satisfy $\mathbf{A}(H)$. For all positive functions λ such that $\lambda(\rho)\rho^{n-2H} \xrightarrow{\rho \rightarrow +\infty} +\infty$,*

- if $0 < 2H < n$ then

$$\left\{ \frac{X_\rho(z) - \mathbb{E}(X_\rho(z))}{\sqrt{\lambda(\rho)\rho^{n-2H}}}; z \in \mathbb{S}_n \right\} \xrightarrow[\rho \rightarrow +\infty]{fdd} \{W_H(z); z \in \mathbb{S}_n\}$$

where W_H is the centered Gaussian random field on \mathbb{S}_n with

$$\text{Cov}(W_H(z), W_H(z')) = K_H(d(z, z')).$$

- if $2H > n$ then for any fixed point $z_0 \in \mathbb{S}_n$,

$$\left\{ \frac{X_\rho(z) - X_\rho(z_0)}{\sqrt{\lambda(\rho)\rho^{n-2H}}}; z \in \mathbb{S}_n \right\} \xrightarrow[\rho \rightarrow +\infty]{fdd} \{W_{H, z_0}(z); z \in \mathbb{S}_n\}$$

where W_{H, z_0} is the centered Gaussian random field on \mathbb{S}_n with

$$\text{Cov}(W_{H, z_0}(z), W_{H, z_0}(z')) = K_H(d(z, z')) - K_H(d(z, z_0)) - K_H(d(z', z_0)) + K_H(0).$$

Proof. of Theorem 2.3.

Let us denote $n(\rho) := \sqrt{\lambda(\rho)\rho^{n-2H}}$. The characteristic function of the normalized field $(X_\rho(\cdot) - \mathbb{E}(X_\rho(\cdot)))/n(\rho)$ is then given by

$$(9) \quad \mathbb{E} \left(\exp \left(i \frac{X_\rho(\mu) - \mathbb{E}(X_\rho(\mu))}{n(\rho)} \right) \right) = \exp \left(\int_{\mathbb{S}_n \times \mathbb{R}^+} G_\rho(x, r) \, dr \sigma(dx) \right)$$

where

$$(10) \quad G_\rho(x, r) = \left(e^{i \frac{\mu(B(x, r))}{n(\rho)}} - 1 - i \frac{\mu(B(x, r))}{n(\rho)} \right) \lambda(\rho) \rho^{-1} f(\rho^{-1}r) .$$

We will make use of Lebesgue's Theorem in order to get the limit of $\int_{\mathbb{S}_n \times \mathbb{R}^+} G_\rho(x, r) \, dr \sigma(dx)$ as $\rho \rightarrow +\infty$.

On the one hand, $n(\rho)$ tends to $+\infty$ so that $\left(e^{i \frac{\mu(B(x, r))}{n(\rho)}} - 1 - i \frac{\mu(B(x, r))}{n(\rho)} \right)$ behaves like $-\frac{1}{2} \left(\frac{\mu(B(x, r))}{n(\rho)} \right)^2$. Together with the assumption $\mathbf{A}(H)$, it yields the following asymptotic. For all $(x, r) \in \mathbb{S}_n \times \mathbb{R}^+$,

$$(11) \quad G_\rho(x, r) \xrightarrow{\rho \rightarrow +\infty} -\frac{1}{2} \mu(B(x, r))^2 r^{-n-1+2H} .$$

On the other hand, since $\frac{|\mu(B(x, r))|}{n(\rho)} \leq |\mu|(\mathbb{S}_n)$ for ρ large enough, we note that there exists some positive constant K such that for all x, r, ρ ,

$$\left| e^{i \frac{\mu(B(x, r))}{n(\rho)}} - 1 - i \frac{\mu(B(x, r))}{n(\rho)} \right| \leq K \left(\frac{\mu(B(x, r))}{n(\rho)} \right)^2 .$$

Therefore

$$|G_\rho(x, r)| \leq K \mu(B(x, r))^2 \rho^{-n-1+2H} f(\rho^{-1}r) .$$

There exists $C > 0$ such that for all $r \in \mathbb{R}^+$, $f(r) \leq Cr^{-n-1+2H}$. Then we get

$$(12) \quad |G_\rho(x, r)| \leq KC \mu(B(x, r))^2 r^{-n-1+2H}$$

where the right hand side is integrable on $\mathbb{S}_n \times \mathbb{R}^+$ by Lemma 2.1.

Applying Lebesgue's Theorem yields

$$\begin{aligned} \int_{\mathbb{S}_n \times \mathbb{R}^+} G_\rho(x, r) \sigma(dx) \, dr &\xrightarrow{\rho \rightarrow +\infty} -\frac{1}{2} \int_{\mathbb{S}_n \times \mathbb{R}^+} \mu(B(x, r))^2 r^{-n-1+2H} \sigma(dx) \, dr \\ &= -\frac{1}{2} \int_{\mathbb{S}_n \times \mathbb{S}_n} K_H(d(z, z')) \mu(dz) \mu(dz') . \end{aligned}$$

Hence $(X_\rho(\mu) - \mathbb{E}(X_\rho(\mu)))/n(\rho)$ converges in distribution to the centered Gaussian random variable $W(\mu)$ whose variance is equal to

$$\mathbb{E}(W(\mu)^2) = C \int_{\mathbb{S}_n \times \mathbb{S}_n} K_H(d(z, z')) \mu(dz) \mu(dz') .$$

By linearity, the covariance of W satisfies (8). □

Remark 2.5.

- 1) The pointwise limit field $\{W_H(z); z \in \mathbb{S}_n\}$ in Corollary 2.4 is stationary, i.e. its distribution is invariant under the isometry group of \mathbb{S}_n , whereas the increments of $\{W_{H,z_0}(z); z \in \mathbb{S}_n\}$ are distribution invariant under the group of all isometries of \mathbb{S}_n which keep the point z_0 invariant.
- 2) When $0 < H < 1/2$ the Gaussian field W_H does not coincide with the field introduced in [13] as the spherical fractional Brownian motion on \mathbb{S}_n .

Indeed, let us have a look at the case $n = 1$, it is easy to obtain the following piecewise expression for $\psi = \psi_1: \forall (u, r) \in [0, \pi] \times \mathbb{R}^+$,

$$\begin{aligned} \psi_1(u, r) &= 0 \text{ for } 0 \leq r < u/2 \\ &= 2r - u \text{ for } u/2 \leq r \leq \pi - u/2 \\ &= 4r - 2\pi \text{ for } \pi - u/2 \leq r \leq \pi \\ &= 2\pi \text{ for } r > \pi \end{aligned}$$

and to compute

$$K_H(u) = \frac{1}{H(1-2H)2^{2H}} (2(2H)^{2H} - u^{2H} - (2\pi - u)^{2H}) ,$$

Actually, we compute the variance of the increments of W_H

$$\begin{aligned} \mathbb{E}(W_H(z) - W_H(z'))^2 &= 2K_H(0) - 2K_H(d(z, z')) \\ &= \frac{2}{H(1-2H)2^{2H}} [d^{2H}(z, z') + (2\pi - d(z, z'))^{2H} - (2\pi)^{2H}]. \end{aligned}$$

The spherical fractional Brownian motion B_H , introduced in [13], satisfies

$$\mathbb{E}(B_H(z) - B_H(z'))^2 = d^{2H}(z, z').$$

Even up to a constant, processes W_H and B_H are clearly different. The Euclidean situation is therefore different. Indeed, [3], the variance of the increments of the corresponding field W_H is proportional to $|z - z'|^{2H}$.

3. LOCAL SELF-SIMILAR BEHAVIOR

We wonder whether the limit field W_H obtained in the previous section satisfies a local asymptotic self-similar (*lass*) property. More precisely we will let a “dilation” of order ε act on W_H near a fixed point A in \mathbb{S}_n and as in [15], up to a renormalization factor, we look for an asymptotic behavior as ε goes to 0. An H -self-similar tangent field T_H is expected. Recall that W_H is defined on a subspace \mathcal{M}^H of measures on \mathbb{S}_n , so that T_H will be defined on a subspace of measures on the tangent space $\mathcal{T}_A\mathbb{S}_n$ of \mathbb{S}_n .

3.1. Dilation. Let us fix a point A in \mathbb{S}_n and consider $\mathcal{T}_A\mathbb{S}_n$ the tangent space of \mathbb{S}_n at A . It can be identified with \mathbb{R}^n and A with the null vector of \mathbb{R}^n .

Let $1 < \delta < \pi$. The exponential map at point A , denoted by \exp , is a diffeomorphism between the Euclidean ball $\{y \in \mathbb{R}^n, \|y\| < \delta\}$ and $\overset{\circ}{B}(A, \delta) \subset \mathbb{S}_n$, where $\|\cdot\|$ denotes the

Euclidean norm in \mathbb{R}^n and $\overset{\circ}{B}(A, \delta)$ the open ball with center A and radius δ in \mathbb{S}_n . Furthermore for all $y, y' \in \mathbb{R}^n$ such that $\|y\|, \|y'\| < \delta$,

$$d(A, \exp y) = \|y\| \quad \text{and} \quad d(\exp y, \exp y') \leq \|y - y'\| .$$

We refer to [10] for precisions on the exponential map.

Let τ be a signed measure on \mathbb{R}^n . We define the dilated measure τ_ε by

$$\forall B \in \mathcal{B}(\mathbb{R}^n) \quad \tau_\varepsilon(B) = \tau(B/\varepsilon)$$

and then map it by the application \exp , defining the measure $\mu_\varepsilon = \exp^* \tau_\varepsilon$ on $\overset{\circ}{B}(A, \delta)$ by

$$(13) \quad \forall C \in \mathcal{B}(\overset{\circ}{B}(A, \delta)) \quad \mu_\varepsilon(C) = \exp^* \tau_\varepsilon(C) = \tau_\varepsilon(\exp^{-1}(C)).$$

We then consider the measure μ_ε as a measure on the whole sphere \mathbb{S}_n with support included in $\overset{\circ}{B}(A, \delta)$.

At last, we define the dilation of W_H within a “neighborhood of A ” by the following procedure. For any finite measure τ on \mathbb{R}^n , we consider $\mu_\varepsilon = \exp^* \tau_\varepsilon$ as defined by (13) and compute $W_H(\mu_\varepsilon)$. We will establish the convergence in distribution of $\varepsilon^{-H} W_H(\exp^* \tau_\varepsilon)$ for any τ in an appropriate space of measures on \mathbb{R}^n . Since $W_H(\mu_\varepsilon)$ is Gaussian, we will focus on its variance.

3.2. Asymptotic of the kernel K_H . For $0 < H < 1/2$, we already mentioned that the kernel $K_H(0) - K_H(u)$ is not proportional to u^{2H} . As a consequence, one cannot expect W_H to be self-similar. Nevertheless, as we are looking for an asymptotic local self-similarity, only the behavior of K_H near zero is relevant. Actually we will establish that, up to a constant, $K_H(0) - K_H(u)$ behaves like u^{2H} when $u \rightarrow 0^+$.

Lemma 3.1. *Let $0 < H < 1/2$. The kernel K_H defined by (5) satisfies*

$$K_H(u) = K_1 - K_2 u^{2H} + o(u^{2H}), \quad u \rightarrow 0^+$$

where $K_1 = K_H(0)$ and K_2 are nonnegative constants.

Proof. Let us state that the assumption $H < 1/2$ implies $H < n/2$ so that in that case K_H is prescribed by

$$K_H(u) = \int_{\mathbb{R}^+} \psi(u, r) r^{-n-1+2H} dr, \quad u \in [0, \pi].$$

We note that $K_H(0) < +\infty$ since $\psi(0, r) \sim cr^n$ as $r \rightarrow 0^+$ and $\psi(0, r) = \sigma(\mathbb{S}_n)$ for $r > \pi$. Then, subtracting $K_H(0)$ and remarking that $\psi(0, r) = \psi(u, r) = \sigma(\mathbb{S}_n)$ for $r > \pi$, we

write

$$\begin{aligned} K_H(0) - K_H(u) &= \int_0^\pi (\psi(0, r) - \psi(u, r)) r^{-n-1+2H} dr \\ &= \int_0^\delta (\psi(0, r) - \psi(u, r)) r^{-n-1+2H} dr \\ &\quad + \int_\delta^\pi (\psi(0, r) - \psi(u, r)) r^{-n-1+2H} dr \end{aligned}$$

where we recall that $\delta \in (1, \pi)$ is such that the exponential map is a diffeomorphism between $\{\|y\| < \delta\} \subset \mathbb{R}^n$ and $\mathring{B}(A, \delta) \subset \mathbb{S}_n$.

The second term is of order u , and therefore is negligible with respect to u^{2H} , since ψ is clearly Lipschitz on the compact interval $[\delta, \pi]$.

We now focus on the first term. Performing the change of variable $r \mapsto r/u$, we write it as

$$\int_0^\delta (\psi(0, r) - \psi(u, r)) r^{-n-1+2H} dr = u^{2H} \int_{\mathbb{R}^+} \Delta(u, r) r^{-n-1+2H} dr ,$$

where

$$\Delta(u, r) := \mathbf{1}_{ur < \delta} u^{-n} (\psi(u, ur) - \psi(0, ur)) .$$

Their only remains to prove that $\int_{\mathbb{R}^+} \Delta(u, r) r^{-n-1+2H} dr$ admits a finite limit K_2 as $u \rightarrow 0^+$. We will use Lebesgue's Theorem and start with establishing the simple convergence of $\Delta(u, r)$ for any given $r \in \mathbb{R}^+$.

We fix a unit vector \mathbf{v} in \mathbb{R}^n and a point $A' = \exp \mathbf{v}$ in \mathbb{S}_n . We then consider for any $u \in (0, \delta)$, the point $A'_u := \exp(u\mathbf{v}) \in \mathbb{S}_n$ located on the geodesic between A and A' such that $d(A, A'_u) = \|u\mathbf{v}\| = u$. We can then use (1) and (2) to write

$$\psi(u, \cdot) = \Psi(A, A'_u, \cdot) = \int_{\mathbb{S}_n} \mathbf{1}_{d(A, z) < \cdot} \mathbf{1}_{d(A'_u, z) < \cdot} d\sigma(z)$$

and

$$\psi(0, \cdot) = \Psi(A, A, \cdot) = \int_{\mathbb{S}_n} \mathbf{1}_{d(A, z) < \cdot} d\sigma(z)$$

in order to express $\Delta(u, r)$ as

$$\Delta(u, r) = \mathbf{1}_{ur < \delta} u^{-n} \int_{\mathbb{S}_n} \mathbf{1}_{d(A, z) < ur} \mathbf{1}_{d(A'_u, z) > ur} d\sigma(z) .$$

Since $ur < \delta$ the above integral runs on $\mathring{B}(A, ur) \subset \mathring{B}(A, \delta)$ and we can perform the exponential change of variable to get

$$\begin{aligned} \Delta(u, r) &= \mathbf{1}_{ur < \delta} u^{-n} \int_{\mathbb{R}^n} \mathbf{1}_{\|y\| < ur} \mathbf{1}_{d(\exp(u\mathbf{v}), \exp(y)) > ur} d\sigma(\exp(y)) \\ &= \mathbf{1}_{ur < \delta} \int_{\mathbb{R}^n} \mathbf{1}_{\|y\| < r} \mathbf{1}_{d(\exp(u\mathbf{v}), \exp(uy)) > ur} \tilde{\sigma}(uy) dy . \end{aligned}$$

In the last integral, the image by \exp of the surface measure $d\sigma(\exp(y))$ is written as $\tilde{\sigma}(y)dy$ where dy denotes the Lebesgue measure on \mathbb{R}^n .

We use the fact that $d(\exp(ux), \exp(ux')) \sim u\|x - x'\|$ as $u \rightarrow 0^+$ to get the following limit for the integrand

$$\mathbf{1}_{d(\exp(uv), \exp(uy)) < ur} \tilde{\sigma}(uy) \longrightarrow \mathbf{1}_{\|v-y\| > r} \tilde{\sigma}(0) .$$

Since the integrand is clearly dominated by

$$\|\sigma\|_\infty := \sup\{\tilde{\sigma}(y), \|y\| \leq \delta\} ,$$

Lebesgue's Theorem yields for all $r \in \mathbb{R}^+$,

$$\Delta(u, r) \longrightarrow \tilde{\sigma}(0) \int_{\mathbb{R}^n} \mathbf{1}_{\|y\| < r} \mathbf{1}_{\|v-y\| > r} dy .$$

We recall that $d(\exp x, \exp x') \leq \|x - x'\|$ for all $x, x' \in \mathbb{R}^n$ with norm less than δ . Therefore for all u ,

$$\Delta(u, r) \leq \|\sigma\|_\infty \int_{\mathbb{R}^n} \mathbf{1}_{\|y\| < r} \mathbf{1}_{\|v-y\| > r} dy$$

where the right hand side belongs to $L^1(\mathbb{R}^+, r^{-n-1+2H} dr)$ (see [2] Lemma A.2).

Using Lebesgue's Theorem for the last time, we obtain

$$\int_{\mathbb{R}^+} \Delta(u, r) r^{-n-1+2H} dr \xrightarrow{u \rightarrow 0^+} K_2$$

where

$$K_2 = \tilde{\sigma}(0) \int_{\mathbb{R}^n \times \mathbb{R}^+} \mathbf{1}_{\|y\| < r} \mathbf{1}_{\|v-y\| > r} r^{-n-1+2H} dy dr \in (0, +\infty) .$$

□

Let us remark that the proof makes it clear that the case $H > 1/2$ is dramatically different. The kernel $K_H(0) - K - H(u)$ behaves like u near zero and loses its $2H$ power.

3.3. Main result.

Let $0 < H < 1/2$. We consider the following space of measures on $\mathcal{T}_A \mathbb{S}_n \cong \mathbb{R}^n$

$$\begin{aligned} \mathfrak{M}^H &= \{\text{measures } \tau \text{ on } \mathbb{R}^n \text{ with finite total variation such that} \\ &\tau(\mathbb{R}^n) = 0 \text{ and } \int_{\mathbb{R}^n \times \mathbb{R}^n} \|x - x'\|^{2H} |\tau|(dx) |\tau|(dx') < +\infty\} . \end{aligned}$$

For any measure $\tau \in \mathfrak{M}^H$, we compute the variance of $W_H(\mu_\varepsilon)$ where $\mu_\varepsilon = \exp^* \tau_\varepsilon$ is defined by (13).

By Lemma 2.1, since μ_ε belongs to $\mathcal{M} = \mathcal{M}^H$ in the case $H < 1/2$,

$$\text{var}(W_H(\mu_\varepsilon)) = \int_{B(A, \delta) \times B(A, \delta)} K_H(d(z, z')) \mu_\varepsilon(dz) \mu_\varepsilon(dz') .$$

Performing an exponential change of variable followed by a dilation in \mathbb{R}^n , we get

$$\begin{aligned} \text{var}(W_H(\mu_\varepsilon)) &= \int_{\mathbb{R}^n \times \mathbb{R}^n} \mathbf{1}_{\|y\| < \delta} \mathbf{1}_{\|y'\| < \delta} K_H(d(\exp(y), \exp(y'))) \tau_\varepsilon(dy) \tau_\varepsilon(dy') \\ &= \int_{\mathbb{R}^n \times \mathbb{R}^n} \mathbf{1}_{\|x\| < \delta/\varepsilon} \mathbf{1}_{\|x'\| < \delta/\varepsilon} K_H(d(\exp(\varepsilon x), \exp(\varepsilon x'))) \tau(dx) \tau(dx'). \end{aligned}$$

Denoting $\widetilde{K}_H(u) = K_H(u) - K_H(0)$,

$$\begin{aligned} \text{var}(W_H(\mu_\varepsilon)) &= \int_{\mathbb{R}^n \times \mathbb{R}^n} \mathbf{1}_{\|x\| < \delta/\varepsilon} \mathbf{1}_{\|x'\| < \delta/\varepsilon} \widetilde{K}_H(d(\exp(\varepsilon x), \exp(\varepsilon x'))) \tau(dx) \tau(dx') \\ &\quad + K_H(0) \tau(\{\|x\| < \delta/\varepsilon\})^2. \end{aligned}$$

Let us admit for a while that

$$(14) \quad \frac{\tau(\{\|x\| < \delta/\varepsilon\})^2}{\varepsilon^{2H}} \xrightarrow{\varepsilon \rightarrow 0^+} 0.$$

Then, applying Lebesgue's Theorem with the convergence argument on \widetilde{K}_H obtained in Lemma 3.1, yields

$$(15) \quad \frac{\text{var}(W_H(\mu_\varepsilon))}{\varepsilon^{2H}} \xrightarrow{\varepsilon \rightarrow 0^+} -K_2 \int_{\mathbb{R}^n \times \mathbb{R}^n} \|x - x'\|^{2H} \tau(dx) \tau(dx').$$

Let us now establish (14) where we recall that τ is any measure in \mathfrak{M}^H . In particular, the total mass of τ is zero so that

$$\begin{aligned} \frac{\tau(\{\|x\| < \delta/\varepsilon\})}{\varepsilon^H} &= - \frac{\tau(\{\|x\| > \delta/\varepsilon\})}{\varepsilon^H} \\ &= - \int_{\mathbb{R}^n} \varepsilon^{-H} \mathbf{1}_{\|x\| > \delta/\varepsilon} \tau(dx). \end{aligned}$$

For any fixed $x \in \mathbb{R}^n$, $\varepsilon^{-H} \mathbf{1}_{\|x\| > \delta/\varepsilon}$ is zero when ε is small enough. Moreover $\varepsilon^{-H} \mathbf{1}_{\|x\| > \delta/\varepsilon}$ is dominated by $\delta^{-H} \|x\|^H$ which belongs to $L^1(\mathbb{R}^n, |\tau|(dx))$ since τ belongs to \mathfrak{M}^H . Lebesgue's Theorem applies once more.

We deduce from asymptotic (15) the following theorem.

Theorem 3.2. *Let $0 < H < 1/2$. The limit*

$$\frac{W_H(\exp^* \tau_\varepsilon)}{\varepsilon^H} \xrightarrow[\varepsilon \rightarrow 0^+]{fdd} T_H(\tau)$$

holds for all $\tau \in \mathfrak{M}^H$, in the sense of finite dimensional distributions of the random functionals. Here T_H is the centered Gaussian random linear functional on \mathfrak{M}^H with

$$(16) \quad \text{Cov}(T_H(\tau), T_H(\tau')) = -K_2 \int_{\mathbb{R}^n \times \mathbb{R}^n} \|x - x'\|^{2H} \tau(dx) \tau'(dx'),$$

As for Theorem 2.3, Theorem 3.2 can be rephrased in terms of pointwise fields. Indeed, $\delta_x - \delta_O$ belongs to \mathfrak{M}^H for all x in \mathbb{R}^n . Let us apply Theorem 3.2 with $\tau = \delta_x - \delta_O$. Then $T_H(\delta_x - \delta_O)$ has the covariance

$$\text{Cov}(T_H(\delta_x - \delta_O), T_H(\delta_{x'} - \delta_O)) = K_2(\|x\|^{2H} + \|x'\|^{2H} - \|x - x'\|^{2H}),$$

and the field $\{T_H(\delta_x - \delta_O); x \in \mathbb{R}^n\}$ is a Euclidean fractional Brownian field.

4. COMPARATIVE ANALYSIS

In this section, we aim to discuss the differences and the analogies between the Euclidean and the spherical case.

Let us first be concerned with the existence of a scaling limit random field. The variance of this limit field should be

$$\mathbf{V} = \int_{\mathbb{M}_n} \int_{\mathbb{R}^+} \mu(B(x, r))^2 \sigma(dx) r^{-n-1+2H} dr,$$

where \mathbb{M}_n is the n -dimensional corresponding surface with its surface measure σ . When speaking of the Euclidean case $\mathbb{M}_n = \mathbb{R}^n$ we refer to [3]. In the present paper, we studied the case $\mathbb{M}_n = \mathbb{S}_n$. Moreover, in this discussion, the hyperbolic case $\mathbb{M}_n = \mathbb{H}_n = \{(x_i)_{1 \leq i \leq n+1} \in \mathbb{R}^{n+1}; x_{n+1}^2 - \sum_{1 \leq i \leq n} x_i^2 = 1, x_{n+1} \geq 1\}$ is evoked.

In the Euclidean case, the random fields are defined on the space of measures with vanishing total mass. So let us first consider measures μ such that $\mu(\mathbb{M}_n) = 0$. Hence, whatever the surface \mathbb{M}_n , the integral \mathbf{V} involves the integral of the surface of the symmetric difference between two balls of same radius r . As r goes to infinity, three different behaviors emerge.

- $\mathbb{M}_n = \mathbb{S}_n$: this surface vanishes
- $\mathbb{M}_n = \mathbb{R}^n$: the order of magnitude of this surface is r^{n-1}
- $\mathbb{M}_n = \mathbb{H}_n$: the surface grows exponentially

The consequences are the following.

- $\mathbb{M}_n = \mathbb{S}_n$: any positive H is admissible
- $\mathbb{M}_n = \mathbb{R}^n$: the range of admissible H is $(0, 1/2)$
- $\mathbb{M}_n = \mathbb{H}_n$: no H is admissible

In the Euclidean case, the restriction $\mu(\mathbb{R}^n) = 0$ is mandatory whereas it is unnecessary in the spherical case for $H < n/2$. Indeed the integral \mathbf{V} is clearly convergent.

Let us now discuss the (local) self-similarity of the limit field. Of course, we no longer consider the hyperbolic case.

- $\mathbb{M}_n = \mathbb{R}^n$: dilating a ball is a homogeneous operation. Therefore, the limit field is self-similar.
- $\mathbb{M}_n = \mathbb{S}_n$: dilation is no longer homogeneous. Only local self-similarity can be expected. The natural framework of this local self-similarity is the tangent bundle, where the situation is Euclidean. Hence we have to come back to the restricting condition $H < 1/2$.

APPENDIX

Recurrence formula for the ψ_n 's.

Recall that the functions ψ_n 's are defined by (1) and (2)

$$\psi_n(u, r) = \Psi_n(M, M', r) = \int_{\mathbb{S}_n} \mathbf{1}_{d(M, N) < r} \mathbf{1}_{d(M', N) < r} d\sigma_n(N), \quad (u, r) \in [0, \pi] \times \mathbb{R}^+,$$

for any pair (M, M') in \mathbb{S}_n such that $d(M, M') = u$. Here σ_n stands for the surface measure on \mathbb{S}_n .

Lemma 4.1. *The family of functions $\psi_n, n \geq 2$ satisfies the following recursion:*
 $\forall (u, r) \in [0, \pi] \times \mathbb{R}^+,$

$$\psi_n(u, r) = \int_{-\sin r}^{\sin r} (1 - a^2)^{n/2} \psi_{n-1} \left(u, \arccos \left(\frac{\cos r}{\sqrt{1 - a^2}} \right) \right) da .$$

Proof. An arbitrary point of \mathbb{S}_n is parameterized either in Cartesian coordinates, $(x_i)_{1 \leq i \leq n+1}$, or in spherical ones

$$(\phi_i)_{1 \leq i \leq n} \in [0, \pi]^{n-1} \times [0, 2\pi)$$

with

$$\begin{aligned} x_1 &= \cos \phi_1 \\ x_2 &= \sin \phi_1 \cos \phi_2 \\ x_3 &= \sin \phi_1 \sin \phi_2 \cos \phi_3 \\ &\dots \\ x_n &= \sin \phi_1 \sin \phi_2 \dots \sin \phi_{n-1} \cos \phi_n \\ x_{n+1} &= \sin \phi_1 \sin \phi_2 \dots \sin \phi_{n-1} \sin \phi_n \end{aligned}$$

Let M be the point $(\phi_i)_{1 \leq i \leq n} = (0, \dots, 0)$. One can write the ball $B_n(M, r)$ of radius r , which is a spherical cap on \mathbb{S}_n with opening angle r as follows,

$$B_n(M, r) = \{(\phi_i)_{1 \leq i \leq n} \in \mathbb{S}_n ; \phi_1 \leq r\}$$

or in Cartesian coordinates

$$B_n(M, r) = \{(x_i)_{1 \leq i \leq n+1} \in \mathbb{S}_n ; x_1 \geq \cos r\} .$$

Let $a \in (-1, 1)$ and let P_a be the hyperplane of \mathbb{R}^{n+1} defined by $x_{n+1} = a$. Let us consider the intersection $P_a \cap B_n(M, r)$.

- If $1 - a^2 < \cos^2 r$ then $P_a \cap B_n(M, r) = \emptyset$.
- If $1 - a^2 \geq \cos^2 r$ then

$$\begin{aligned} P_a \cap B_n(M, r) &= \{(x_i)_{1 \leq i \leq n+1} \in \mathbb{S}_n ; x_1 \geq \cos r \text{ and } x_{n+1} = a\} \\ &= \{(x_i)_{1 \leq i \leq n} \in \mathbb{R}^n ; x_1 \geq \cos r \text{ and } \sum_{1 \leq i \leq n} x_i^2 = 1 - a^2\} \times \{a\}. \end{aligned}$$

In other words, denoting $\mathbb{S}_{n-1}(R)$ the $(n-1)$ -dimensional sphere of radius R ,

$$P_a \cap B_n(M, r) = B_{n-1, \sqrt{1-a^2}}(M(a), r(a)) \times \{a\}$$

where $B_{n-1, \sqrt{1-a^2}}(M(a), r(a))$ is the spherical cap on $\mathbb{S}_{n-1}(\sqrt{1-a^2})$, centered at $M(a) = (\sqrt{1-a^2}, 0, \dots, 0)$ and with opening angle $r(a) = \arccos \left(\frac{\cos r}{\sqrt{1-a^2}} \right)$.

Let now M' be defined in spherical coordinates by $(\phi_i)_{1 \leq i \leq n} = (u, 0, \dots, 0)$, so that $d(M, M') = u$. The intersection $P_a \cap B_n(M', r)$ is the map of $P_a \cap B_n(M, r)$ by the rotation of angle u and center C in the plane $x_3 = \dots = x_{n+1} = 0$. So

- if $1 - a^2 < \cos^2 r$ then $P_a \cap B_n(M', r) = \emptyset$.
- if $1 - a^2 \geq \cos^2 r$ then

$$P_a \cap B_n(M', r) = B_{n-1, \sqrt{1-a^2}}(M'(a), r(a)) \times \{a\}$$

where the $(n-1)$ -dimensional spherical cap $B_{n-1, \sqrt{1-a^2}}(M'(a), r(a))$ is now centered at $M'(a) = (\sqrt{1-a^2} \cos u, \sqrt{1-a^2} \sin u, 0, \dots, 0)$.

We define $\psi_{n-1, R}(u, r)$ as the intersection surface of two spherical caps on $\mathbb{S}_{n-1}(R)$, whose centers are at a distance Ru and with the same opening angle r .

By homogeneity, this leads to

$$\psi_{n-1, R}(u, r) = R^{n-1} \psi_{n-1, 1}(u, r) = R^n \psi_{n-1}(u, r).$$

The surface measure σ_n of \mathbb{S}_n can be written as

$$d\sigma_n(x_1, \dots, x_n, a) = \sqrt{1-a^2} d\sigma_{n-1, \sqrt{1-a^2}}(x_1, \dots, x_n) \times da$$

where $\sigma_{n-1, R}$ is the surface measure of $\mathbb{S}_{n-1}(R)$.

We then obtain

$$\begin{aligned} \psi_n(u, r) &= \int_{-1}^1 \mathbf{1}_{1-a^2 \geq \cos^2 r} \psi_{n-1, \sqrt{1-a^2}}(u, \arccos\left(\frac{\cos r}{\sqrt{1-a^2}}\right)) \sqrt{1-a^2} da \\ &= \int_{-\sin r}^{\sin r} (\sqrt{1-a^2})^n \psi_{n-1}(u, \arccos\left(\frac{\cos r}{\sqrt{1-a^2}}\right)) da, \end{aligned}$$

and Lemma 4.1 is proved. \square

ACKNOWLEDGEMENTS

The authors are very grateful to the referee for his/her careful reading and useful comments.

REFERENCES

- [1] Benassi, A., Jaffard, S. and Roux, D. *Gaussian processes and Pseudodifferential Elliptic operators*, Rev. Mat. Iberoamericana, **13** (1), 19–90 (1997)
- [2] Biermé, H., and Estrade, A. *Poisson random balls: self-similarity and X-Ray images*, Adv. Appl. Prob. **38**, 853–872 (2006)
- [3] Biermé, H., Estrade, A. and Kaj I. *Self-similar random fields and rescaled random balls model*, preprint (2008) available on <http://hal.archives-ouvertes.fr/hal-00161614-v2/>
- [4] Cioczek-Georges, R. and Mandelbrot, B.B. *A class of micropulses and antipersistent fractional Brownian motion*, Stoch. Proc. Appl. **60**(1), 1–18 (1995)
- [5] Cohen, S. and Taqqu, M. *Small and large scale behavior of the Poissonized Telecom Process*, Methodology and Computing in Applied Probability, **6**(4), 363–379 (2004)
- [6] Dalay, D.J. *The definition of multi-dimensional generalization of shot-noise*, J. Appl. Prob. **8**, 128–135 (1971)
- [7] Dobrushin, R.L. *Automodel generalized random fields and their renorm group* In: Dobrushin, R.L. and Sinai, Ya.G. (eds.) Multicomponent Random Systems, pp. 153–198, Dekker, New York (1980)
- [8] Faraut, J. *Fonction brownienne sur une variété riemannienne*, Séminaire de probabilités de Strasbourg **7**, 61–76 (1973)
- [9] Gangolli, R. *Positive definite kernels on homogeneous spaces and certain stochastic processes related to Lévy's Brownian motion of several parameters*, Ann. Inst. Poincaré **3**, 121–226 (1967)
- [10] Helgason, S. *Differential Geometry and Symmetric Spaces*, Academic Press, (1962)

- [11] Heinrich, L. and Schmidt, V. *Normal convergence of multidimensional shot noise and rate of this convergence*, Adv. Appl. Prob. **17**, 709–730 (1985)
- [12] Kaj, I., Leskelä, L., Norros, I. and Schmidt, V. *Scaling limits for random fields with long-range dependence*, Ann. Probab. **35**(2), 528–550 (2007)
- [13] Istas, J. *Spherical and hyperbolic fractional Brownian motion*, Elec. Comm. Prob. **10**, 254–262 (2005)
- [14] Istas, J. *On fractional stable fields indexed by metric spaces*, Elec. Comm. Prob. **11**, 242–251 (2006)
- [15] Istas, J., Lacaux, C. *On locally self-similar fractional random fields indexed by a manifold*, preprint (2009)
- [16] Kolmogorov, A. *Wienersche Spiralen und einige andere interessante Kurven im Hilbertsche Raum. (German)*, C. R. (Dokl.) Acad. Sci. URSS **26**, 115–118 (1940)
- [17] Lévy, P. *Processus stochastiques et mouvement brownien*, Gauthier-Villars (1965)
- [18] Mandelbrot, B.B. and Van Ness, J.W. *Fractional Brownian Motions, Fractional Noises and Applications*, SIAM Review **10**, 422–437 (1968)
- [19] Samorodnitsky, G. and Taqqu, M. *Stable non-Gaussian random processes: stochastic models with infinite variance*, Chapman & Hall, New York (1994)
- [20] Serra, J. *Image Analysis and Mathematical Morphology*, Academic Press, (1983)
- [21] Stoyan, D., Kendall, W.S. and Mecke, J. *Stochastic geometry and its applications*, Wiley (1995)
- [22] Takenaka, S. *Integral-geometric construction of self-similar stable processes*, Nagoya Math. J. **123** 1–12, (1991).
- [23] Takenaka, S., Kubo, I. and Urakawa, H. *Brownian motion parametrized with metric space of constant curvature*, Nagoya Math. J. **82** 131–140 (1981)
- [24] Wicksell, S.D. *The corpuscle problem: A Mathematical Study of a Biometric Problem*, Biometrika **17**, 84–99 (1925)
- [25] Willingsler, W., Taqqu, M.S., Sherman, R., and Wilson, D.V. *Self-similarity through high variability: statistical analysis of Ethernet LAN traffic at source level*, IEE/ACM Transactions in Networking **5**, 71–86 (1995)

ANNE ESTRADE, MAP5 UMR CNRS 8145, UNIVERSITÉ PARIS DESCARTES, 45 RUE DES SAINTS-PÈRES, F-75270 PARIS CEDEX 06 & ANR-05-BLAN-0017

E-mail address: `anne.estrade@parisdescartes.fr`

JACQUES ISTAS, LABORATOIRE JEAN KUNTZMANN, UNIVERSITÉ DE GRENOBLE ET CNRS, F-38041 GRENOBLE CEDEX 9

E-mail address: `Jacques.Istas@imag.fr`