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Self-interacting diffusions IV: Rate of convergence*

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Abstract

Self-interacting diffusions are processes living on a compact Riemannian manifold defined by a stochastic differential equation with a drift term depending on the past empirical measure μ_t of the process. The asymptotics of μ_t is governed by a deterministic dynamical system and under certain conditions (μ_t) converges almost surely towards a deterministic measure μ^* (see Benaïm, Ledoux, Raimond (2002) and Benaïm, Raimond (2005)). We are interested here in the rate of convergence of μ_t towards μ^*. A central limit theorem is proved. In particular, this shows that greater is the interaction repelling faster is the convergence.

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1 Introduction

Self-interacting diffusions

Let M be a smooth compact Riemannian manifold and $V : M \times M \to \mathbb{R}$ a sufficiently smooth mapping[1]. For all finite Borel measure μ, let $V\mu : M \to \mathbb{R}$ be the smooth function defined by

$$V\mu(x) = \int_M V(x,y)\mu(dy).$$

Let (e_α) be a finite family of vector fields on M such that

$$\sum_\alpha e_\alpha(e_\alpha f)(x) = \Delta f(x),$$

where Δ is the Laplace operator on M and $e_\alpha(f)$ stands for the Lie derivative of f along e_α. Let (B^α) be a family of independent Brownian motions.

A *self-interacting diffusion* on M associated to V can be defined as the solution to the stochastic differential equation (SDE)

$$dX_t = \sum_\alpha e_\alpha(X_t) \circ dB_t^\alpha - \nabla(V\mu_t)(X_t)dt.$$

where

$$\mu_t = \frac{1}{t}\int_0^t \delta_{X_s} ds$$

is the empirical occupation measure of (X_t).

In absence of drift (i.e $V = 0$), (X_t) is just a Brownian motion on M but in general it defines a non Markovian process whose behavior at time t depends on its past trajectories through μ_t. This type of process was introduced in Benaim, Ledoux and Raimond (2002) (hence after referred as [3]) and further analyzed in a series of papers by Benaim and Raimond (2003, 2005, 2007) (hence after referred as [4], [5] and [6]). We refer the reader to these papers for more details and especially to [3] for a detailed construction of the process and its elementary properties. For a general overview of processes with reinforcement we refer the reader to the recent survey paper by Pemantle (2007) ([15]).

[1]The mapping $V_x : M \to \mathbb{R}$ defined by $V_x(y) = V(x,y)$ is C^2 and its derivatives are continuous in (x,y)

Notation and Background

Standing Notation We let $\mathcal{M}(M)$ denote the space of finite Borel measures on M, $\mathcal{P}(M) \subset \mathcal{M}(M)$ the space of probability measures. If I is a metric space (typically, $I = M, \mathbb{R}^+ \times M$ or $[0,T] \times M$) we let $C(I)$ denote the space of real valued continuous functions on I equipped with the topology of uniform convergence on compact sets. When I is compact and $f \in C(I)$ we let $\|f\| = \sup_{x \in I} |f(x)|$. The normalized Riemann measure on M will be denoted by λ.

Let $\mu \in \mathcal{P}(M)$ and $f : M \to \mathbb{R}$ a nonnegative or $\mu-$integrable Borel function. We write μf for $\int f d\mu$, and $f\mu$ for the measure defined as $f\mu(A) = \int_A f d\mu$. We let $L^2(\mu)$ denote the space of such functions for which $\mu|f|^2 < \infty$, equipped with the inner product

$$\langle f, g \rangle_\mu = \mu(fg)$$

and the norm

$$\|f\|_\mu = \sqrt{\mu f^2}.$$

We simply write L^2 for $L^2(\lambda)$.

Of fundamental importance in the analysis of the asymptotics of (μ_t) is the mapping $\Pi : \mathcal{M}(M) \to \mathcal{P}(M)$ defined by

$$\Pi(\mu) = \xi(V\mu)\lambda \tag{1}$$

where $\xi : C(M) \to C(M)$ is the function defined by

$$\xi(f)(x) = \frac{e^{-f(x)}}{\int_M e^{-f(y)}\lambda(dy)}. \tag{2}$$

In [3], it is shown that the asymptotics of μ_t can be precisely related to the long term behavior of a certain semiflow on $\mathcal{P}(M)$ induced by the ordinary differential equation (ODE) on $\mathcal{M}(M)$:

$$\dot{\mu} = -\mu + \Pi(\mu). \tag{3}$$

Depending on the nature of V, the dynamics of (3) can either be convergent or nonconvergent leading to similar behaviors for $\{\mu_t\}$(see [3]). When V is symmetric, (3) happens to be a *quasigradient* and the following convergence result hold.

Theorem 1.1 ([5]) *Assume that V is symmetric. i.e. $V(x,y) = V(y,x)$. Then the limit set of $\{\mu_t\}$ (for the topology of weak* convergence) is almost surely a compact connected subset of*

$$\mathsf{Fix}(\Pi) = \{\mu \in \mathcal{P}(M) : \mu = \Pi(\mu)\}.$$

In particular, if $\mathsf{Fix}(\Pi)$ is finite then (μ_t) converges almost surely toward a fixed point of Π. This holds for a generic function V (see [5]).

Sufficient conditions ensuring that $\mathsf{Fix}(\Pi)$ has cardinal one are as follows:

Theorem 1.2 ([5], [6]) *Assume that V is symmetric and that one of the two following conditions hold*

(i) *Up to an additive constant V is a Mercer kernel, That is*

$$V(x,y) = K(x,y) + C$$

and

$$\int K(x,y)f(x)f(y)\lambda(dx)\lambda(dy) \geq 0$$

for all $f \in L^2$.

(ii) *For all $x \in M, y \in M, u \in T_xM, v \in T_yM$*

$$\mathsf{Ric}_x(u,u) + \mathsf{Ric}_y(v,v) + \mathsf{Hess}_{x,y}V((u,v),(u,v)) \geq K(\|u\|^2 + \|v\|^2)$$

where K is some positive constant. Here Ric_x stands for the Ricci tensor at x and $\mathsf{Hess}_{x,y}$ is the Hessian of V at (x,y).

Then $\mathsf{Fix}(\Pi)$ reduces to a singleton $\{\mu^*\}$ and $\mu_t \to \mu^*$ with probability one.

As observed in [6] the condition (i) in Theorem 1.2 seems well suited to describe *self-repelling diffusions*. On the other hand, it is not clearly related to the geometry of M. Condition (ii) has a more geometrical flavor and is robust to smooth perturbations (of M and V). It can be seen as a Bakry-Emery type condition for self interacting diffusions.

In [5], it is also proved that every stable (for the ODE (3)) fixed point of Π has a positive probability to be a limit point for μ_t; and any unstable fixed point cannot be a limit point for μ_t.

Organisation of the paper

Let $\mu^* \in \mathsf{Fix}(\Pi)$. We will assume that

Hypothesis 1.3 *μ_t converges a.s. towards μ^*.*

Sufficient conditions are given by Theorem 1.2

In this paper we intend to study the rate of this convergence. Let

$$\Delta_t = e^{t/2}(\mu_{e^t} - \mu^*).$$

It will be shown that, under some conditions to be specified later, for all $g = (g_1, \ldots, g_n) \in C(M)^n$ the process

$$[\Delta_s g_1, \ldots, \Delta_s g_n, V\Delta_s]_{s \geq t}$$

converges in law, as $t \to \infty$, toward a certain stationary Ornstein-Uhlenbeck process (Z^g, Z) on $\mathbb{R}^n \times C(M)$. This process is defined in Section 2. The main result is stated in section 3 and some examples are developed. It is in particular observed that a strong repelling interaction gives a faster convergence. The section 4 is a proof section. The appendix, section 5, contains general material on random variables and Ornstein-Uhlenbeck processes on $C(M)$.

In the following K (respectively C) denotes a positive constant (respectively a positive random constant). These constants may change from line to line.

2 The Ornstein-Uhlenbeck process (Z^g, Z).

Throughout all this section we let $\mu \in \mathcal{P}(M)$. For $x \in M$ we set $V_x : M \to \mathbb{R}$ defined by $V_x(y) = V(x, y)$.

2.1 The operator G_μ

Let $g \in C(M)$ and let $G_{\mu,g} : \mathbb{R} \times C(M) \to \mathbb{R}$ be the linear operator defined by

$$G_{\mu,g}(u, f) = u/2 + \mathrm{Cov}_\mu(g, f), \tag{4}$$

where Cov_μ is the covariance on $L^2(\mu)$, that is the bilinear form acting on $L^2 \times L^2$ defined by

$$\mathrm{Cov}_\mu(f,g) = \mu(fg) - (\mu f)(\mu g).$$

We define the linear operator $G_\mu : C(M) \to C(M)$ by

$$\begin{aligned} G_\mu f(x) &= G_{\mu,V_x}(f(x), f) \\ &= f(x)/2 + \mathrm{Cov}_\mu(V_x, f). \end{aligned} \tag{5}$$

It is easily seen that $\|G_\mu f\| \leq (2\|V\| + 1/2)\|f\|$. In particular, G_μ is a bounded operator. Let $\{e^{-tG_\mu}\}$ denotes the semigroup acting on $C(M)$ with generator $-G_\mu$. From now on we will assume the following:

Hypothesis 2.1 *There exists $\kappa > 0$ and $\widehat{\lambda} \in \mathcal{P}(M)$ such that $\mu << \widehat{\lambda}$ with $\|\frac{d\mu}{d\widehat{\lambda}}\|_\infty < \infty$, λ and $\widehat{\lambda}$ are equivalent measures with $\|\frac{d\lambda}{d\widehat{\lambda}}\|_\infty < \infty$ and $\|\frac{d\widehat{\lambda}}{d\lambda}\|_\infty < \infty$, and such that for all $f \in L^2(\widehat{\lambda})$,*

$$\langle G_\mu f, f\rangle_{\widehat{\lambda}} \geq \kappa \|f\|^2_{\widehat{\lambda}}.$$

Let

$$\lambda(-G_\mu) = \lim_{t\to\infty} \frac{\log(\|e^{-tG_\mu}\|)}{t}.$$

This limit exists by subadditivity. Then

Lemma 2.2 *Hypothesis 2.1 implies that $\lambda(-G_\mu) \leq -\kappa < 0$.*

Proof : For all $f \in L^2(\widehat{\lambda})$,

$$\begin{aligned} \frac{d}{dt}\|e^{-tG_\mu} f\|^2_{\widehat{\lambda}} &= -2\langle G_\mu e^{-tG_\mu} f, e^{-tG_\mu} f\rangle_{\widehat{\lambda}} \\ &\leq -2\kappa \|e^{-tG_\mu} f\|_{\widehat{\lambda}}. \end{aligned}$$

This implies that $\|e^{-tG_\mu} f\|_{\widehat{\lambda}} \leq e^{-\kappa t}\|f\|_{\widehat{\lambda}}$.

Denote by g_t the solution of the differential equation

$$\frac{dg_t}{dt} = \mathrm{Cov}_\mu(V_x, g_t)$$

with $g_0 = f$, where $f \in C(M)$. Note that $e^{-tG_\mu} f = e^{-t/2} g_t$. It is straightforward to check that (using the fact that $\|\frac{d\mu}{d\widehat{\lambda}}\|_\infty < \infty$)

$$\frac{d}{dt}\|g_t\|_{\widehat{\lambda}} \leq K\|g_t\|_{\widehat{\lambda}}$$

with K a constant depending only on V and μ. Thus

$$\sup_{t\in[0,1]} \|g_t\|_{\widehat{\lambda}} \leq K\|f\|_{\widehat{\lambda}}.$$

Now, since for all $x \in M$ and $t \in [0,1]$

$$\left|\frac{d}{dt}g_t(x)\right| \leq K\|g_t\|_{\widehat{\lambda}} \leq K\|f\|_{\widehat{\lambda}},$$

we have $\|g_1\| \leq K\|f\|_{\widehat{\lambda}}$. This implies that

$$\|e^{-G_\mu}f\| \leq K\|f\|_{\widehat{\lambda}}.$$

Now for all $t > 1$, and $f \in C(M)$,

$$\begin{aligned}
\|e^{-tG_\mu}f\| &= \|e^{-G_\mu}e^{-(t-1)G_\mu}f\| \\
&\leq K\|e^{-(t-1)G_\mu}f\|_{\widehat{\lambda}} \\
&\leq Ke^{-\kappa(t-1)}\|f\|_{\widehat{\lambda}} \\
&\leq Ke^{-\kappa t}\|f\|_{\infty}.
\end{aligned}$$

This implies that $\|e^{-tG_\mu}\| \leq Ke^{-\kappa t}$, which proves the lemma. **QED**

The *adjoint* of G_μ is the operator on $\mathcal{M}(M)$ defined by the relation

$$m(G_\mu f) = (G_\mu^* m)f$$

for all $m \in \mathcal{M}(M)$ and $f \in C(M)$. It is not hard to verify that

$$G_\mu^* m = \frac{1}{2}m + (Vm)\mu - (\mu(Vm))\mu. \tag{6}$$

2.2 The generator A_μ and its inverse Q_μ

Let H^2 be the Sobolev space of real valued functions on M, associated with the norm $\|f\|_H^2 = \|f\|_\lambda^2 + \|\nabla f\|_\lambda^2$. Since $\Pi(\mu)$ and λ are equivalent measures with continuous Radon-Nykodim derivative, $L^2(\Pi(\mu)) = L^2(\lambda) := L^2$. We denote by K_μ the projection operator, acting on $L^2(\Pi(\mu))$, defined by

$$K_\mu f = f - \Pi(\mu)f.$$

We denote by A_μ the operator acting on H^2 defined by

$$A_\mu f = \frac{1}{2}\Delta f - \langle \nabla V\mu, \nabla f\rangle.$$

Note that for f and g in L^2,

$$\langle A_\mu f, g\rangle_{\Pi(\mu)} = -\frac{1}{2}\int \langle \nabla f, \nabla g\rangle(x)\Pi(\mu)(dx)$$

where $\langle \cdot, \cdot\rangle$ denotes the Riemannian inner product on M.

For all $f \in C(M)$ there exists $\mathsf{Q}_\mu f \in H^2$ such that $\Pi(\mu)(\mathsf{Q}_\mu f) = 0$ and

$$f - \Pi(\mu)f = K_\mu f = -A_\mu \mathsf{Q}_\mu f. \tag{7}$$

Note that if P_t^μ denotes the semigroup with generator A_μ, then

$$\mathsf{Q}_\mu f = \int_0^\infty P_t^\mu K_\mu f dt.$$

Since there exists $p_t^\mu(\cdot, \cdot)$ such that

$$P_t^\mu f(x) = \int_M p_t^\mu(x, y) f(y)\Pi(\mu)(dy),$$

we have

$$\mathsf{Q}_\mu f(x) = \int_M q_\mu(x, y) f(y)\Pi(\mu)(dy)$$

where

$$q_\mu(x, y) = \int_0^\infty (p_t^\mu(x, y) - 1)dt.$$

Then, as shown in [3], $\mathsf{Q}_\mu f$ is C^1 and there exists a constant K such that for all $f \in C(M)$ and $\mu \in \mathcal{P}(M)$,

$$\|\mathsf{Q}_\mu f\|_\infty \leq K\|f\|_\infty \tag{8}$$

$$\|\nabla \mathsf{Q}_\mu f\|_\infty \leq K\|f\|_\infty. \tag{9}$$

Finally, note that for f and g in L^2,

$$\begin{aligned} \int \langle \nabla \mathsf{Q}_\mu f, \nabla \mathsf{Q}_\mu g\rangle(x)\Pi(\mu)(dx) &= -2\langle A_\mu \mathsf{Q}_\mu f, \mathsf{Q}_\mu g\rangle_{\Pi(\mu)} \\ &= 2\langle f, \mathsf{Q}_\mu g\rangle_{\Pi(\mu)}. \end{aligned} \tag{10}$$

2.3 The covariance C_μ

We let $\widehat{C}_\mu$ denote the bilinear continuous form $\widehat{C}_\mu : C(M) \times C(M) \to \mathbb{R}$ defined by

$$\widehat{C}_\mu(f,g) = 2\langle f, \mathsf{Q}_\mu g\rangle_{\Pi(\mu)}.$$

This form is symmetric (see its expression given by (10)). Note also that for some constant depending on μ,

$$|\widehat{C}_\mu(f,g)| \leq K\|f\| \times \|g\|.$$

We let C_μ denote the mapping $C_\mu : M \times M \to \mathbb{R}$ defined by

$$C_\mu(x,y) = \widehat{C}_\mu(V_x, V_y).$$

Then C_μ is a covariance function (or a Mercer kernel), i.e. it is continuous, symmetric and $\sum_{i,j} \lambda_i \lambda_j C_\mu(x_i, x_j) \geq 0$.

2.4 The process Z

We now define an Ornstein-Uhlenbeck process on $C(M)$ of covariance C_μ and drift $-G_\mu$. This heavily relies on the general construction given in the appendix.

A *Brownian motion on $C(M)$ with covariance C_μ* is a $C(M)$-valued stochastic process $W = \{W_t\}_{t\geq 0}$ such that

(i) $W_0 = 0$;

(ii) $t \mapsto W_t$ is continuous;

(iii) For every finite subset $S \subset \mathbb{R} \times M$, $\{W_t(x)\}_{(t,x)\in S}$ is a centered Gaussian random vector;

(iv) $\mathsf{E}[W_s(x)W_t(y)] = (s \wedge t)C_\mu(x,y)$.

Lemma 2.3 *There exists a Brownian motion on $C(M)$ with covariance C_μ.*

Proof : Let

$$\begin{aligned} d_{C_\mu}(x,y) &:= \sqrt{C_\mu(x,x) - 2C_\mu(x,y) + C_\mu(y,y)} \\ &= \|\nabla \mathsf{Q}_\mu(V_x - V_y)\|_{\Pi(\mu)} \\ &\leq K\|V_x - V_y\| \end{aligned}$$

where the last inequality follows from (9). Then

$$d_{C_\mu}(x,y) \leq Kd(x,y)$$

and the result follows from Proposition 5.8 and Remark 5.7 in the appendix. **QED**

We say that a $C(M)$-valued process Z is an *Ornstein-Uhlenbeck process of covariance C_μ and drift $-G_\mu$* if

$$Z_t = Z_0 - \int_0^t G_\mu Z_s ds + W_t \tag{11}$$

where

(i) W is a $C(M)$-valued Brownian motion of covariance C_μ;

(ii) Z_0 is a $C(M)$-valued random variable;

(iii) W and Z_0 are independent.

Note that we can think of Z as a solution to the linear SDE

$$dZ_t = dW_t - G_\mu Z_t dt.$$

It follows from section 5.3 in the appendix that such a process exists and defines a Markov process. Furthermore

Proposition 2.4 *Under hypothesis 2.1,*

(i) *(Z_t) converges in law toward a $C(M)$-valued random variable Z_∞;*

(ii) *Z_∞ is* Gaussian*, in the sense that for every finite set $S \subset M$, $\{Z_\infty(x)\}_{x\in S}$ is a centered Gaussian random vector;*

(iii) *Let π^μ denotes the law of Z_∞. Then π^g is characterized by its variance*

$$\mathsf{Var}(\pi^\mu) : \mathcal{M}(M) \to \mathbb{R},$$

$$m \mapsto \mathsf{E}((mZ_\infty)^2),$$

and for all $m \in \mathcal{M}$,

$$\begin{aligned}\mathsf{Var}(\pi^\mu)(m) &= \int_0^\infty \int_{M\times M} C_\mu(x,y) m_t(dx) m_t(dy) dt \\ &= \int_0^\infty \widehat{C}_\mu(Vm_t, Vm_t) dt\end{aligned}$$

where

$$m_t = e^{-tG_\mu^*} m.$$

Proof : This follows from Proposition 5.16 in the appendix. Example 5.18 shows that assertion (iii) of this proposition is satisfied. **QED**

2.5 The process Z^g.

For $g = (g_1, \ldots, g_n) \in C(M)^n$, let $\tilde{M} = \{1, \ldots, n\} \cup M$ be the disjoint union of $\{1, \ldots, n\}$ and M, and $C_\mu^g : \tilde{M} \times \tilde{M} \to \mathbb{R}$ be the function defined by

$$C_\mu^g(x,y) = \begin{cases} \widehat{C}_\mu(g_x, g_y) \text{ for } x, y \in \{1, \ldots, n\}, \\ C_\mu(x,y) \text{ for } x, y \in M, \\ \widehat{C}_\mu(V_x, g_y) \text{ for } x \in M, y \in \{1, \ldots, n\}. \end{cases}$$

Then C_μ^g is a Mercer kernel (see section 5.2).

A *Brownian motion on* $\mathbb{R}^n \times C(M)$ *with covariance* C_μ^g is a $\mathbb{R}^n \times C(M)$-valued stochastic process $(W^g, W) = \{(W_t^{g_1}, \ldots, W_t^{g_n}, W_t)\}_{t\geq 0}$ such that:

(i) $W = \{W_t\}_{t\geq 0}$ is a $C(M)$-valued Brownian motion with covariance C_μ;

(ii) For every finite subset $S \subset \mathbb{R} \times M, \{W_t^g, W_t(x)\}_{(t,x)\in S}$ is a centered Gaussian random vector;

(iii) $\mathsf{E}(W_s^{g_i} W_t^{g_j}) = (s \wedge t)\widehat{C}_\mu(g_i, g_j)$ and
$\mathsf{E}(W_s(x) W_t^{g_i}) = (s \wedge t)\widehat{C}_\mu(V_x, g_i)$.

Lemma 2.5 *There exists a Brownian motion on* $\mathbb{R}^n \times C(M)$ *with covariance* C_μ^g.

Proof : Let $\tilde{d}$ be the distance on $\tilde{M}$ defined by

$$\tilde{d}(x,y) = \begin{cases} \mathbf{1}_{x\neq y} \text{ for } x, y \in \{1, \ldots, n\}, \\ d(x,y) \text{ for } x, y \in M, \\ d(x, x_0) + 1 \text{ for } x \in M, y \in \{1, \ldots, n\} \end{cases}$$

where x_0 is some arbitrary point in M. This makes $\tilde{M}$ a compact metric space, and it is easy to show that the function C_μ^g verifies hypothesis 5.6 (use the proof of Lemma 2.3). The result follows by application of Proposition 5.8. **QED**

Let now be $Z_t^g = (Z_t^{g_1}, \ldots, Z_t^{g_n}) \in \mathbb{R}^n$ denote the solution to the SDE

$$dZ_t^{g_i} = dW_t^{g_i} - (Z_t^{g_i}/2 + \mathrm{Cov}_\mu(Z_t, g_i))\, dt,\ i = 1, \ldots, n \tag{12}$$

where (W^g, W) is as above and $Z = (Z_t)$ is given by (11).

The following result generalizes Proposition 2.4.

Proposition 2.6 *Under hypothesis 2.1,*

(i) *The process (Z_t^g, Z_t) converges in law toward a centered $\mathbb{R}^n \times C(M)$ valued Gaussian random variable (Z_∞^g, Z_∞).*

(ii) *Let $\pi^{g,\mu}$ denotes the law of (Z_∞^g, Z_∞). Then $\pi^{g,\mu}$ is characterized by its variance*

$$\mathsf{Var}(\pi^{g,\mu}) : \mathbb{R}^n \times \mathcal{M}(M) \to \mathbb{R},$$
$$(u, m) \mapsto \mathsf{E}\left((mZ_\infty + \langle u, Z_\infty^g\rangle)^2\right);$$

and for all $u \in \mathbb{R}^n, m \in \mathcal{M}(M)$,

$$\mathsf{Var}(\pi^{g,\mu})(u, m) = \int_0^\infty \widehat{C}_\mu(f_t, f_t)dt$$

with

$$f_t = e^{-t/2} \sum_i u_i g_i + V m_t,$$

and where m_t is defined by

$$m_t f = m_0(e^{-tG_\mu} f) + \sum_{i=1}^n u_i \int_0^t e^{-s/2} Cov_\mu(g_i, e^{-(t-s)G_\mu} f)ds. \tag{13}$$

Proof : Let $G_\mu^g : \mathbb{R}^n \times C(M) \to \mathbb{R}^n \times C(M)$ be the operator defined by

$$G_\mu^g = \begin{pmatrix} I/2 & A_\mu^g \\ 0 & G_\mu \end{pmatrix} \tag{14}$$

where $A^g_\mu : C(M) \to \mathbb{R}^n$ is the linear map defined by

$$A^g_\mu(f) = \Big(\mathrm{Cov}_\mu(f, g_1), \ldots, \mathrm{Cov}_\mu(f, g_n)\Big).$$

Then (Z^g, Z) is a $C(\tilde{M})$-valued Ornstein-Uhlenbeck process of covariance C^g_μ and drift $-G^g_\mu$. It is not hard to verify that under hypothesis 2.1, the assumptions of Proposition 5.16 hold, so that (Z^g_t, Z_t) converges in law toward a centered $\mathbb{R}^n \times C(M)$ valued Gaussian random variable (Z^g_∞, Z_∞) with variance

$$\mathsf{Var}(\pi^{g,\mu})(u, m) = \int_0^\infty \widehat{C}_\mu(f_t, f_t)dt$$

with $f_t = \sum_i u_t(i)g_i + Vm_t$ and where $(u_t, m_t) = e^{-t(G^g_\mu)^*}(u, m)$. Now

$$(G^g_\mu)^* = \begin{pmatrix} I/2 & 0 \\ (A^g_\mu)^* & (G_\mu)^* \end{pmatrix}$$

and $(A^g_\mu)^* u = \sum_i u_i(g_i - \mu g_i)\mu$. Thus $u_t = e^{-t/2}u$ and

$$\frac{dm_t}{dt} = -(A^g_\mu)^* u_t - (G_\mu)^* m_t$$

Thus m_t is the solution with $m_0 = m$ of

$$\frac{dm_t}{dt} = -e^{-t/2}\left(\sum_i u_i(g_i - \mu g_i)\right)\mu - G^*_\mu m_t \tag{15}$$

Note that (15) is equivalent to

$$\frac{d}{dt}(m_t f) = -e^{-t/2}\mathrm{Cov}_\mu\left(\sum_i u_i g_i, f\right) - m_t(G_\mu f)$$

for all $f \in C(M)$, and $m_0 = m$. From which we deduce that

$$m_t = e^{-tG^*_\mu}m_0 - \int_0^t e^{-s/2}e^{-(t-s)G^*_\mu}\left(\sum_i u_i(g_i - \mu g_i)\mu\right)ds$$

which implies the formula for m_t given by (13). **QED**

For further reference we call (Z^g, Z) an *Ornstein-Uhlenbeck process of covariance C^g_μ and drift $-G^g_\mu$*. It is called *stationary* when its initial distribution is $\pi^{g,\mu}$.

3 A central limit theorem for μ_t

We state here the main results of this article. We assume $\mu^* \in \mathsf{Fix}(\Pi)$ satisfies hypotheses 1.3 and 2.1. Set $\Delta_t = e^{t/2}(\mu_{e^t} - \mu^*)$, $D_t = V\Delta_t$ and $D_{t+\cdot} = \{D_{t+s} : s \geq 0\}$. Then

Theorem 3.1 *$D_{t+\cdot}$ converges in law, as $t \to \infty$, towards a stationary Ornstein-Uhlenbeck process of covariance C_{μ^*} and drift $-G_{\mu^*}$.*

For $g = (g_1, \ldots, g_n) \in C(M)^n$, we set $D_t^g = (\Delta_t g, D_t)$ and $D_{t+\cdot}^g = \{D_{t+s}^g : s \geq 0\}$. Then

Theorem 3.2 *$(D_{t+s}^g)_{s\geq 0})$ converges in law towards a stationary Ornstein-Uhlenbeck process of covariance $C_{\mu^*}^g$ and drift $-G_{\mu^*}^g$.*

Define $\widehat{C} : C(M) \times C(M) \to \mathbb{R}$ the symmetric bilinear form defined by

$$\widehat{C}(f, g) = \int_0^\infty \widehat{C}_{\mu^*}(f_t, g_t)dt, \tag{16}$$

with (g_t is defined by the same formula, with g in place of f)

$$f_t(x) = e^{-t/2} f(x) - \int_0^t e^{-s/2} \mathrm{Cov}_{\mu^*}(f, e^{-(t-s)G_{\mu^*}} V_x)ds. \tag{17}$$

Corollary 3.3 *$\Delta_t g$ converges in law towards a centered Gaussian variable Z_∞^g of covariance*

$$\mathsf{E}[Z_\infty^{g_i} Z_\infty^{g_j}] = \widehat{C}(f, g).$$

Proof : Follows from theorem 3.2 and the calculus of $\mathsf{Var}(\pi^{g,\mu})(u, 0)$. **QED**

3.1 Examples

3.1.1 Diffusions

Suppose $V(x, y) = V(x)$, so that (X_t) is just a standard diffusion on M with invariant measure $\mu^* = \frac{exp(-V)\lambda}{\lambda \exp(-V)}$.

Let $f \in C(M)$. Then f_t defined by (17) is equal to (using $e^{-tG_{\mu^*}}1 = e^{-t/2}1) = e^{-t/2}f$. Thus

$$\widehat{C}(f, g) = 2\mu^*(f\mathsf{Q}_{\mu^*}g). \tag{18}$$

Corollary 3.3 says that

Theorem 3.4 *For all $g \in C(M)^n$, Δ_t^g converges in law toward a centered Gaussian variable $(Z_\infty^{g_1}, \ldots, Z_\infty^{g_n})$, with covariance given by*

$$\mathsf{E}(Z_\infty^{g_i} Z_\infty^{g_j}) = 2\mu^*(g_i \mathsf{Q}_{\mu^*} g_j).$$

Remark 3.5 *This central limit theorem for Brownian motions on compact manifolds has already been considered by Baxter and Brosamler in [1] and [2]; and by Bhattacharya in [7] for ergodic diffusions.*

3.1.2 The case $\mu^* = \lambda$ and V symmetric.

Suppose here that $\mu^* = \lambda$ and that V is symmetric. We assume (without loss of generality since $\Pi(\lambda) = \lambda$ implies that $V\lambda$ is a constant function) that $V\lambda = 0$.

Since V is compact and symmetric, there exists an orthonormal basis $(e_\alpha)_{i\geq 0}$ in $L^2(\lambda)$ and a sequence of reals $(\lambda_\alpha)_{\alpha\geq 0}$ such that e_0 is a constant function and

$$V = \sum_{\alpha\geq 1} \lambda_\alpha e_\alpha \otimes e_\alpha.$$

Assume that for all α, $1/2 + \lambda_\alpha > 0$. Then hypothesis 2.1 holds with $\widehat{\lambda} = \lambda$, and the convergence of μ_t towards λ holds with positive probability (see [6]).

Let $f \in C(M)$ and f_t defined by (17), denoting $f^\alpha = \langle f, e_\alpha\rangle_\lambda$ and $f_t^\alpha = \langle f_t, e_\alpha\rangle_\lambda$, we have $f_t^0 = e^{-t/2} f^0$ and for $\alpha \geq 1$,

$$\begin{aligned} f_t^\alpha &= e^{-t/2} f^\alpha - \lambda_\alpha e^{-(1/2+\lambda_\alpha)t} \left(\frac{e^{\lambda_\alpha t} - 1}{\lambda_\alpha} \right) f^\alpha \\ &= e^{-(1/2+\lambda_\alpha)t} f^\alpha. \end{aligned}$$

Using the fact that

$$\widehat{C}_\lambda(f, g) = 2\lambda(f \mathsf{Q}_\lambda g),$$

this implies that

$$\widehat{C}(f, g) = 2 \sum_{\alpha\geq 1} \sum_{\beta\geq 1} \frac{1}{1 + \lambda_\alpha + \lambda_\beta} \langle f, e_\alpha\rangle_\lambda \langle g, e_\beta\rangle_\lambda \lambda(e_\alpha \mathsf{Q}_\lambda e_\beta).$$

This, with corollary 3.3, proves

Theorem 3.6 *Assume hypothesis 1.3 and that* $1/2+\lambda_\alpha > 0$ *for all* α*. Then for all* $g \in C(M)^n$*,* Δ_t^g *converges in law toward a centered Gaussian variable* $(Z_\infty^{g_1}, \ldots, Z_\infty^{g_n})$*, with covariance given by*

$$\mathsf{E}(Z_\infty^{g_i} Z_\infty^{g_j}) = \widehat{C}(g_i, g_j).$$

In particular,

$$\mathsf{E}(Z_\infty^{e_\alpha} Z_\infty^{e_\beta}) = \frac{2}{1+\lambda_\alpha+\lambda_\beta}\lambda(e_\alpha \mathsf{Q}_\lambda e_\beta).$$

Note that when all λ_α are positive, which corresponds to what is named a self-repelling interaction in [6], the rate of convergence of μ_t towards λ is bigger than when there is no interaction, and the bigger is the interaction (that is larger λ_α's) faster is the convergence.

4 Proof of the main results

We assume hypothesis 1.3 and μ^* satisfies hypothesis 2.1. It is possible to choose κ in hypothesis 2.1 such that $\kappa < 1/2$. In the following κ will denote such constant. Note that we have $\lambda(-G_{\mu^*}) < -\kappa$. Such κ exists when hypothesis 2.1 holds.

4.1 A lemma satisfied by Q_μ

We denote by $\mathcal{X}(M)$ the space of continuous vector fields on M, and equip the spaces $\mathcal{P}(M)$ and $\mathcal{X}(M)$ respectively with the weak convergence topology and with the uniform convergence topology.

Lemma 4.1 *For all* $f \in C(M)$*, the mapping* $\mu \mapsto \nabla \mathsf{Q}_\mu f$ *is a continuous mapping from* $\mathcal{P}(M)$ *in* $\mathcal{X}(M)$*.*

Proof : Let μ and ν be in $\mathcal{M}(M)$, and $f \in C(M)$. Set $g = \mathsf{Q}_\mu f$. Then $f = -A_\mu g + \Pi(\mu) f$ and

$$\begin{aligned}
\|\nabla \mathsf{Q}_\mu f - \nabla \mathsf{Q}_\nu f\|_\infty &= \|-\nabla \mathsf{Q}_\mu A_\mu g + \nabla \mathsf{Q}_\nu A_\mu g\|_\infty \\
&= \|\nabla g + \nabla \mathsf{Q}_\nu A_\mu g\|_\infty \\
&\leq \|\nabla(g + \mathsf{Q}_\nu A_\nu g)\|_\infty + \|\nabla \mathsf{Q}_\nu (A_\mu - A_\nu) g\|_\infty
\end{aligned}$$

since $\nabla(g + \mathsf{Q}_\nu A_\nu g) = 0$ and $(A_\mu - A_\nu)g = \langle \nabla V_{\mu-\nu}, \nabla g\rangle$, we get

$$\|\nabla \mathsf{Q}_\mu f - \nabla \mathsf{Q}_\nu f\|_\infty \leq K \|\langle \nabla V_{\mu-\nu}, \nabla g\rangle\|_\infty. \tag{19}$$

Using the fact that $(x, y) \mapsto \nabla V_x(y)$ is uniformly continuous, the right hand term of (19) converges towards 0, when $d(\mu, \nu)$ converges towards 0, d being a distance compatible with the weak convergence. **QED**

4.2 The process Δ

Set $h_t = V\mu_t$ and $h^* = V\mu^*$. Recall $\Delta_t = e^{t/2}(\mu_{e^t} - \mu^*)$ and $D_t = V\Delta_t$. Note that $D_t(x) = \Delta_t V_x$.

To simplify the notation, we set $K_s = K_{\mu_s}$, $\mathsf{Q}_s = \mathsf{Q}_{\mu_s}$ and $A_s = A_{\mu_s}$. Let $(M_t^f)_{t\geq 1}$ be the martingale defined by

$$M_t^f = \sum_\alpha \int_1^t e_\alpha (\mathsf{Q}_s f)(X_s) dB_s^\alpha.$$

The quadratic covariation of M^f and M^g (with f and g in $C(M)$) is given by

$$\langle M^f, M^g\rangle_t = \int_1^t \langle \nabla \mathsf{Q}_s f, \nabla \mathsf{Q}_s g\rangle (X_s) ds.$$

Then for all $t \geq 1$ (with $\dot{\mathsf{Q}}_t = \frac{d}{dt}\mathsf{Q}_t$) ,

$$\mathsf{Q}_t f(X_t) - \mathsf{Q}_1 f(X_1) = M_t^f + \int_1^t \dot{\mathsf{Q}}_s f(X_s) ds - \int_1^t K_s f(X_s) ds.$$

Thus

$$\begin{aligned}
\mu_t f &= \frac{1}{t}\int_1^t K_s f(X_s) ds + \frac{1}{t}\int_1^t \Pi(\mu_s) f ds + \frac{1}{t}\int_0^1 f(X_s) ds \\
&= -\frac{1}{t}\left(\mathsf{Q}_t f(X_t) - \mathsf{Q}_1 f(X_1) - \int_1^t \dot{\mathsf{Q}}_s f(X_s) ds\right) \\
&\quad + \frac{M_t^f}{t} + \frac{1}{t}\int_1^t \langle \xi(h_s), f\rangle_\lambda ds + \frac{1}{t}\int_0^1 f(X_s) ds.
\end{aligned}$$

Note that (D_t) is a continuous process taking its values in $C(M)$ and that $D_t = e^{t/2}(h_{e^t} - h^*)$. For $f \in C(M)$ (using the fact that $\mu^* f = \langle \xi(h^*), f\rangle_\lambda$),

$$\Delta_t f = \sum_{i=1}^5 \Delta_t^i f \tag{20}$$

with

$$\begin{aligned}
\Delta_t^1 f &= e^{-t/2}\left(-\mathsf{Q}_{e^t} f(X_{e^t}) + \mathsf{Q}_1 f(X_1) + \int_1^{e^t} \dot{\mathsf{Q}}_s f(X_s) ds\right) \\
\Delta_t^2 f &= e^{-t/2} M_{e^t}^f \\
\Delta_t^3 f &= e^{-t/2} \int_1^{e^t} \langle \xi(h_s) - \xi(h^*) - D\xi(h^*)(h_s - h^*), f\rangle_\lambda ds \\
\Delta_t^4 f &= e^{-t/2} \int_1^{e^t} \langle D\xi(h^*)(h_s - h^*), f\rangle_\lambda ds \\
\Delta_t^5 f &= e^{-t/2}\left(\int_0^1 f(X_s) ds - \mu^* f\right).
\end{aligned}$$

Then $D_t = \sum_{i=1}^5 D_t^i$, where $D_t^i = V\Delta_t^i$. Finally, note that

$$\langle D\xi(h^*)(h - h^*), f\rangle_\lambda = -\mathrm{Cov}_{\mu^*}(h - h^*, f). \tag{21}$$

4.3 First estimates

We recall some estimates from [3]: There exists a constant K such that for all $f \in C(M)$ and $t > 0$,

$$\begin{aligned}
\|\mathsf{Q}_t f\|_\infty &\le K\|f\|_\infty \\
\|\nabla \mathsf{Q}_t f\|_\infty &\le K\|f\|_\infty \\
\|\dot{\mathsf{Q}}_t f\|_\infty &\le \frac{K}{t}\|f\|_\infty.
\end{aligned}$$

These estimates imply in particular that

$$\langle M^f - M^g\rangle_t \le K\|f - g\|_\infty \times t$$

and that

Lemma 4.2 *There exists a constant K depending on $\|V\|_\infty$ such that for all $t \ge 1$, and all $f \in C(M)$*

$$\|\Delta_t^1 f\|_\infty + \|\Delta_t^5 f\|_\infty \le K \times (1+t)e^{-t/2}\|f\|_\infty, \tag{22}$$

which implies that $((\Delta^1 + \Delta^5)_{t+s})_{s\ge 0}$ and $((D^1 + D^5)_{t+s})_{s\ge 0}$ both converge towards 0 (respectively in $\mathcal{M}(M)$ and in $C(\mathbb{R}^+ \times M)$).

We also have

Lemma 4.3 *There exists a constant K such that for all $t \geq 0$ and all $f \in C(M)$,*

$$\begin{aligned}
\mathsf{E}[(\Delta_t^2 f)^2] &\leq K\|f\|_\infty^2, \\
|\Delta_t^3 f| &\leq K\|f\|_\lambda \times e^{-t/2} \int_0^t \|D_s\|_\lambda^2 ds, \\
|\Delta_t^4 f| &\leq K\|f\|_\lambda \times e^{-t/2} \int_0^t e^{s/2} \|D_s\|_\lambda ds.
\end{aligned}$$

Proof : The first estimate follows from

$$\begin{aligned}
\mathsf{E}[(\Delta_t^2 f)^2] &= e^{-t}\mathsf{E}[(M_{e^t}^f)^2] = e^{-t}\mathsf{E}[\langle M^f \rangle_{e^t}] \\
&\leq e^{-t} \int_1^{e^t} \|\nabla \mathsf{Q}_s f\|_\infty^2 ds \\
&\leq K\|f\|_\infty^2.
\end{aligned}$$

The second estimate follows from the fact that

$$\|\xi(h) - \xi(h^*) - D\xi(h^*)(h - h^*)\|_\lambda = O(\|h - h^*\|_\lambda^2).$$

The last estimate follows easily after having remarked that

$$\begin{aligned}
|\langle D\xi(h^*)(h_s - h^*), f\rangle| &= |\mathrm{Cov}_{\mu^*}(h_s - h^*, f)| \\
&\leq K\|f\|_\lambda \times \|h_s - h^*\|_\lambda \\
&\leq K\|f\|_\lambda \times s^{-1/2}\|D_{\log(s)}\|_\lambda.
\end{aligned}$$

This proves this lemma. **QED**

4.4 The processes Δ' and D'

Set $\Delta' = \Delta^2 + \Delta^3 + \Delta^4$ and $D' = D^2 + D^3 + D^4$. For $g \in C(M)$, set

$$\epsilon_t^g = e^{t/2}\langle \xi(h_{e^t}) - \xi(h^*) - D\xi(h^*)(h_{e^t} - h^*), g\rangle_\lambda.$$

Then

$$d\Delta_t' g = -\frac{\Delta_t' g}{2} dt + dN_t^g + \epsilon_t^g dt + \langle D\xi(h^*)(D_t), g\rangle_\lambda dt$$

where for all $g \in C(M)$, N^g is a martingale. Moreover, for f and g in $C(M)$,

$$\langle N^f, N^g \rangle_t = \int_0^t \langle \nabla \mathsf{Q}_{e^s} f(X_{e^s}), \nabla \mathsf{Q}_{e^s} g(X_{e^s}) \rangle ds.$$

Then, for all x,

$$dD'_t(x) = -\frac{D'_t(x)}{2} dt + dM_t(x) + \epsilon_t(x) dt + \langle D\xi(h^*)(D_t), V_x \rangle_\lambda dt$$

where M is the martingale in $C(M)$ defined by $M(x) = N^{V_x}$ and $\epsilon_t(x) = \epsilon_t^{V_x}$. We also have

$$G_{\mu^*}(D')_t(x) = \frac{D'_t(x)}{2} - \langle D\xi(h^*)(D'_t), V_x \rangle_\lambda.$$

Denoting $L_{\mu^*} = L_{-G_{\mu^*}}$ (defined by equation (32) in the appendix), this implies that

$$\begin{aligned} dL_{\mu^*}(D')_t(x) &= dD'_t(x) + G_{\mu^*}(D')_t(x) dt \\ &= dM_t(x) + \langle D\xi(h^*)((D^1 + D^5)_t), V_x \rangle_\lambda dt + \epsilon_t(x) dt \end{aligned}$$

Thus

$$L_{\mu^*}(D')_t(x) = M_t(x) + \int_0^t \epsilon'_s(x) ds$$

with $\epsilon'_s(x) = \epsilon'_s V_x$ where for all $f \in C(M)$,

$$\epsilon'_s f = \epsilon^f_s + \langle D\xi(h^*)((D^1 + D^5)_s), f \rangle_\lambda.$$

Using lemma 5.10,

$$D'_t = L^{-1}_{\mu^*}(M)_t + \int_0^t e^{-(t-s)G_{\mu^*}} \epsilon'_s ds. \tag{23}$$

For $g = (g_1, \ldots, g_n) \in C(M)^n$, we denote $\Delta'_t g = (\Delta'_t g_1, \ldots, \Delta'_t g_n)$, $N^g = (N^{g_1}, \ldots, N^{g_n})$ and $\epsilon'_t g = (\epsilon'_t g_1, \ldots, \epsilon'_t g_n)$. Then, denoting $L^g_{\mu^*} = L_{-G^g_{\mu^*}}$ (with $G^g_{\mu^*}$ defined by (14)) we have

$$L^g_{\mu^*}(\Delta' g, D')_t = (N^g_t, M_t) + \int_0^t (\epsilon'_s g, \epsilon'_s) ds$$

so that (using lemma 5.10 and integrating by parts)

$$(\Delta'_t g, D'_t) = (L^g_{\mu^*})^{-1}(N^g, M)_t + \int_0^t e^{-(t-s)G^g_{\mu^*}}({\epsilon'}_s g, \epsilon'_s)ds. \tag{24}$$

Moreover

$$(L^g_{\mu^*})^{-1}(N^g, M)_t = \left(\widehat{N}^{g_1}_t, \ldots, \widehat{N}^{g_n}_t, L^{-1}_{\mu^*}(M)_t\right),$$

where

$$\widehat{N}^{g_i}_t = N^{g_i}_t - \int_0^t \left(\frac{N^{g_i}_s}{2} + \widehat{C}_{\mu^*}(L^{-1}_{\mu^*}(M)_s, g_i)\right) ds.$$

4.5 Estimation of ϵ'_t

4.5.1 Estimation of $\|L^{-1}_{\mu^*}(M)_t\|_\lambda$

Lemma 4.4 **(i)** *For all $\alpha \geq 2$, there exists a constant K_α such that for all $t \geq 0$,*

$$\mathsf{E}[\|L^{-1}_{\mu^*}(M)_t\|^\alpha_\lambda]^{1/\alpha} \leq K_\alpha.$$

(ii) *a.s. there exists C with $\mathsf{E}[C] < \infty$ such that for all $t \geq 0$,*

$$\|L^{-1}_{\mu^*}(M)_t\|_\lambda \leq C(1+t).$$

Proof : Since $\|L^{-1}_{\mu^*}(M)_t\|_\lambda \leq K\|L^{-1}_{\mu^*}(M)_t\|_{\widehat{\lambda}}$, we estimate $\|L^{-1}_{\mu^*}(M)_t\|_{\widehat{\lambda}}$. We have

$$dL^{-1}_{\mu^*}(M)_t = dM_t - G_{\mu^*}L^{-1}_{\mu^*}(M)_t dt.$$

Let N be the martingale defined by

$$N_t = \int_0^t \left\langle \frac{L^{-1}_{\mu^*}(M)_s}{\|L^{-1}_{\mu^*}(M)_s\|_{\widehat{\lambda}}}, dM_s \right\rangle_{\widehat{\lambda}}.$$

We have $\langle N\rangle_t \leq Kt$ for some constant K. Then

$$\begin{aligned} d\|L^{-1}_{\mu^*}(M)_t\|^2_{\widehat{\lambda}} &= 2\|L^{-1}_{\mu^*}(M)_t\|_{\widehat{\lambda}} dN_t - 2\langle L^{-1}_{\mu^*}(M)_t, G_{\mu^*}L^{-1}_{\mu^*}(M)_t\rangle_{\widehat{\lambda}} dt \\ &+ d\left(\int \langle M(x)\rangle_t \widehat{\lambda}(dx)\right). \end{aligned}$$

Note that there exists a constant K such that

$$\frac{d}{dt}\left(\int \langle M(x)\rangle_t \widehat{\lambda}(dx)\right) \leq K$$

and that (see hypothesis 2.1)

$$\langle L_{\mu^*}^{-1}(M)_t, G_{\mu^*} L_{\mu^*}^{-1}(M)_t \rangle_{\widehat{\lambda}} \geq \kappa \| L_{\mu^*}^{-1}(M)_t \|_{\widehat{\lambda}}^2.$$

This implies that

$$\frac{d}{dt} \mathsf{E}[\| L_{\mu^*}^{-1}(M)_t \|_{\widehat{\lambda}}^2] \leq -2\kappa \mathsf{E}[\| L_{\mu^*}^{-1}(M)_t \|_{\widehat{\lambda}}^2] + K$$

which implies (i) for $\alpha = 2$. For $\alpha > 2$, we find that

$$\begin{aligned} \frac{d}{dt} \mathsf{E}[\| L_{\mu^*}^{-1}(M)_t \|_{\widehat{\lambda}}^\alpha] &\leq -\alpha\kappa \mathsf{E}[\| L_{\mu^*}^{-1}(M)_t \|_{\widehat{\lambda}}^\alpha] + K \mathsf{E}[\| L_{\mu^*}^{-1}(M)_t \|_{\widehat{\lambda}}^{\alpha-2}] \\ &\leq -\alpha\kappa \mathsf{E}[\| L_{\mu^*}^{-1}(M)_t \|_{\widehat{\lambda}}^\alpha] + K \mathsf{E}[\| L_{\mu^*}^{-1}(M)_t \|_{\widehat{\lambda}}^\alpha]^{\frac{\alpha-2}{\alpha}} \end{aligned}$$

which implies that $\mathsf{E}[\| L_{\mu^*}^{-1}(M)_t \|_{\widehat{\lambda}}^\alpha]$ is bounded.

We now prove (ii). Fix $\alpha > 1$. Then there exists a constant K such that

$$\frac{\| L_{\mu^*}^{-1}(M)_t \|_{\widehat{\lambda}}^2}{(1+t)^\alpha} \leq \| L_{\mu^*}^{-1}(M)_0 \|_{\widehat{\lambda}}^2 + 2 \int_0^t \frac{\| L_{\mu^*}^{-1}(M)_s \|_{\widehat{\lambda}}}{(1+s)^\alpha} dN_s + K.$$

Then BDG inequality implies that

$$\mathsf{E}\left[\sup_{t \geq 0} \frac{\| L_{\mu^*}^{-1}(M)_t \|_{\widehat{\lambda}}^2}{(1+t)^\alpha} \right] \leq K + 2 \sup_{t \geq 0} \left(\int_0^t \frac{K ds}{(1+s)^{2\alpha}} \right)^{1/2}$$

which is finite. This implies the lemma by taking $\alpha = 2$. **QED**

4.5.2 Estimation of $\|D_t\|_\lambda$

Note that $|\epsilon_t^g| \leq K e^{-t/2} \|D_t\|_\lambda^2 \times \|g\|$. Thus

$$|\epsilon'_t g| \leq K e^{-t/2} (1 + t + \|D_t\|_\lambda^2) \times \|g\|.$$

This implies (using lemma 2.2 and the fact that $0 < \kappa < 1/2$)

Lemma 4.5 *There exists K such that*

$$\left\| \int_0^t e^{-(t-s)G_{\mu^*}} \epsilon'_s ds \right\| \leq K e^{-\kappa t} \left(1 + \int_0^t e^{-(1/2-\kappa)s} \|D_s\|_\lambda^2 ds \right). \qquad (25)$$

This lemma with lemma 4.4-(ii) imply the following

Lemma 4.6 *a.s. there exists C with $\mathsf{E}[C] < \infty$ such that*

$$\|D_t\|_\lambda \leq C \times \left[1 + t + \int_0^t e^{-s/2} \|D_s\|_\lambda^2 ds\right]. \tag{26}$$

Proof : First note that

$$\|D_t\|_\lambda \leq \|D'_t\|_\lambda + K(1+t)e^{-t/2}.$$

Using the expression of D'_t given by (23), we get

$$\begin{aligned} \|D'_t\|_\lambda &\leq \|L_{\mu^*}^{-1}(M)_t\|_\lambda + \left\| \int_0^t e^{-(t-s)G_{\mu^*}} \epsilon'_s ds \right\| \\ &\leq C(1+t) + Ke^{-\kappa t}\left(1 + \int_0^t e^{-(1/2-\kappa)s} \|D_s\|_\lambda^2 ds\right) \end{aligned}$$

which implies the lemma. **QED**

Lemma 4.7 *Let x and ϵ be real functions. If for all $t \geq 0$,*

$$x_t \leq \alpha + \int_0^t \epsilon_s x_s ds,$$

where α is a real constant, then

$$x_t \leq \alpha \exp\left(\int_0^t \epsilon_s ds\right).$$

Proof : Similarly to the proof of Gronwall's lemma, we set $y_t = \int_0^t \epsilon_s x_s ds$. Then,

$$\dot{y}_t \leq \alpha\epsilon_t + \epsilon_t y_t.$$

Take $\lambda_t = y_t \exp\left(-\int_0^t \epsilon_s ds\right)$, then

$$\dot{\lambda}_t \leq \alpha\epsilon_t \exp\left(-\int_0^t \epsilon_s ds\right)$$

and

$$\begin{aligned} y_t &\leq \alpha \int_0^t \epsilon_s \exp\left(\int_s^t \epsilon_u du\right) ds \\ &\leq \alpha \exp\left(\int_0^t \epsilon_u du\right) - \alpha. \end{aligned}$$

This implies the lemma. **QED**

This lemma implies that

$$\|D_t\|_\lambda \leq C(1+t) \times \exp\left(C \int_0^t e^{-s/2} \|D_s\|_\lambda ds\right).$$

Since hypothesis 1.3 implies that $\lim_{s\to\infty} e^{-s/2}\|D_s\|_\lambda = 0$, this proves that a.s. for all $\epsilon > 0$, there exists C_ϵ such that

$$\|D_t\|_\lambda \leq C_\epsilon e^{\epsilon t}.$$

Take $\epsilon < 1/4$. Then

$$\int_0^\infty e^{-s/2} \|D_s\|_\lambda^2 ds \leq C_\epsilon.$$

This implies

Lemma 4.8 *a.s., there exists C such that for all t,*

$$\|D_t\|_\lambda \leq C(1+t).$$

4.5.3 Estimation of ϵ'_t

Lemma 4.9 *a.s. there exists C such that for all $f \in C(M)$,*

$$|\epsilon'_t f| \leq C(1+t)^2 e^{-t/2} \|f\|$$

Proof : We have $|\epsilon'_t f| \leq |\epsilon_t^f| + K(1+t)e^{-t/2}\|f\|$ and

$$\begin{aligned} |\epsilon_t^f| &\leq K\|f\|_\lambda \times e^{-t/2} \|D_t\|_\lambda^2 \\ &\leq C\|f\| \times (1+t)^2 e^{-t/2} \end{aligned}$$

by lemma 4.8. **QED**

4.6 Estimation of $\|D_t - L_{\mu^*}^{-1}(M)_t\|$

Lemma 4.10 $\|D_t - L_{\mu^*}^{-1}(M)_t\| \leq Ce^{-\kappa t}$.

Proof : We have $\|D_t - D'_t\| \leq K(1+t)e^{-t/2}$. So to prove this lemma, it suffices to prove that (see the expression of D'_t given by (23))

$$\left\| \int_0^t e^{-(t-s)G_{\mu^*}} \epsilon'_s ds \right\| \leq Ce^{-\kappa t}.$$

This term is dominated by

$$K \int_0^t e^{-\kappa(t-s)} \|\epsilon'_s\| ds.$$

Using the previous lemma, it is also dominated by

$$Ce^{-\kappa t} \int_0^t e^{\kappa s}(1+s)^2 e^{-s/2} ds \leq Ce^{-\kappa t}$$

because $\kappa \in]0, 1/2[$. The lemma is proved. **QED**

In addition, for $g = (g_1, \ldots, g_n) \in C(M)^n$, setting

$$\Delta_t g = (\Delta_t g_1, \ldots, \Delta_t g_n),$$

Lemma 4.11 $\|(\Delta_t g, D_t) - (L_{\mu^*}^g)^{-1}(N^g, M)_t\| \leq C(1 + \|g\|)e^{-\kappa t}$.

Proof : We have $\|(\Delta_t g, D_t) - (\Delta'_t g, D'_t)\| \leq K(1 + \|g\|)(1+t)e^{-\kappa t}$. So to prove this lemma, using (24), it suffices to prove that

$$\left\| \int_0^t e^{-(t-s)G_{\mu^*}^g} ({\epsilon'}_s g, \epsilon'_s) ds \right\| \leq K(1 + \|g\|)e^{-\kappa t}. \tag{27}$$

Using hypothesis 2.1 and the definition of $G_{\mu^*}^g$, we have that for all positive t,

$$\|e^{-tG_{\mu^*}^g}\| \leq Ke^{-\kappa t}.$$

This implies

$$\|e^{-(t-s)G_{\mu^*}^g} ({\epsilon'}_s g, \epsilon'_s)\| \leq Ke^{-\kappa(t-s)} \|\epsilon'_s\| \times (1 + \|g\|).$$

Thus the term (27) is dominated by

$$K(1 + \|g\|) \int_0^t e^{-\kappa(t-s)} \|\epsilon'_s\| ds,$$

from which we prove (27) like in the previous lemma. **QED**

4.7 Tightness results

We refer the reader to section 5.1.2 in the appendix, where tightness criteria for families of $C(M)$-valued random variables are given. They will be used in this section.

4.7.1 Tightness of $(L_{\mu^*}^{-1}(M)_t)_{t\geq 0}$

In this section we prove the following lemma which in particular implies the tightness of $(D_t)_{t\geq 0}$ and of $(D'_t)_{t\geq 0}$.

Lemma 4.12 *$(L_{\mu^*}^{-1}(M)_t)_{t\geq 0}$ is tight.*

Proof : We have the relation (that defines $L_{\mu^*}^{-1}(M)$)

$$dL_{\mu^*}^{-1}(M)_t(x) = -G_{\mu^*}L_{\mu^*}^{-1}(M)_t(x)dt + dM_t(x).$$

Thus, using the expression of G_{μ^*}

$$dL_{\mu^*}^{-1}(M)_t(x) = -\frac{1}{2}L_{\mu^*}^{-1}(M)_t(x)dt + A_t(x)dt + dM_t(x),$$

with

$$A_t(x) = \widehat{C}_{\mu^*}(V_x, L_{\mu^*}^{-1}(M)_t).$$

Since μ^* is absolutely continuous with respect to λ, we have that

$$\|A_t\| \leq K\|L_{\mu^*}^{-1}(M)_t\|_\lambda$$

and therefore (using lemma 4.4 (i) for $\alpha = 2$)

$$\sup_t \mathsf{E}[\|A_t\|^2] < \infty.$$

We also have

$$\mathrm{Lip}(A_t) \leq K\|L_{\mu^*}^{-1}(M)_t\|_\lambda,$$

where $\mathrm{Lip}(A_t)$ is the Lipschitz constant of A_t (see (38))

In order to prove this tightness result, we first prove that for all x, $(L_{\mu^*}^{-1}(M)_t(x))_t$ is tight. Setting $Z_t^x = L_{\mu^*}^{-1}(M)_t(x)$ we have

$$\begin{aligned}\frac{d}{dt}\mathsf{E}[(Z_t^x)^2] &\leq -\mathsf{E}[(Z_t^x)^2] + 2\mathsf{E}[|Z_t^x| \times |A_t(x)|] + \frac{d}{dt}\mathsf{E}[\langle M(x)\rangle_t] \\ &\leq -\mathsf{E}[(Z_t^x)^2] + K\mathsf{E}[(Z_t^x))^2]^{1/2} + K\end{aligned}$$

which implies that $(L_{\mu^*}^{-1}(M)_t(x))_t$ is bounded in $L^2(\mathsf{P})$ and thus tight.

We now estimate $\mathsf{E}[|Z_t^x - Z_t^y|^\alpha]^{1/\alpha}$ for α greater than 2 and the dimension of M. Setting $Z_t^{x,y} = Z_t^x - Z_t^y$, we have

$$\begin{aligned}
\frac{d}{dt}\mathsf{E}[(Z_t^{x,y})^\alpha] &\leq -\frac{\alpha}{2}\mathsf{E}[(Z_t^{x,y})^\alpha] + \alpha\mathsf{E}[(Z_t^{x,y})^{\alpha-1}|A_t(x) - A_t(y)|] \\
&\quad + \frac{\alpha(\alpha-1)}{2}\mathsf{E}\left[(Z_t^{x,y})^{\alpha-2}\frac{d}{dt}\langle M(x) - M(y)\rangle_t\right] \\
&\leq -\frac{\alpha}{2}\mathsf{E}[(Z_t^{x,y})^\alpha] + \alpha d(x,y)\mathsf{E}[(Z_t^{x,y})^{\alpha-1}\mathrm{Lip}(A_t)] \\
&\quad + Kd(x,y)^2\mathsf{E}[(Z_t^{x,y})^{\alpha-2}] \\
&\leq -\frac{\alpha}{2}\mathsf{E}[(Z_t^{x,y})^\alpha] + Kd(x,y)\mathsf{E}[(Z_t^{x,y})^{\alpha-1}\|L^{-1}(M)_t\|_\lambda] \\
&\quad + Kd(x,y)^2\mathsf{E}[(Z_t^{x,y})^{\alpha-2}] \\
&\leq -\frac{\alpha}{2}\mathsf{E}[(Z_t^{x,y})^\alpha] + Kd(x,y)\mathsf{E}[(Z_t^{x,y})^\alpha]^{\frac{\alpha-1}{\alpha}}\mathsf{E}[\|L^{-1}(M)_t\|_\lambda^\alpha]^{1/\alpha} \\
&\quad + Kd(x,y)^2\mathsf{E}[(Z_t^{x,y})^\alpha]^{\frac{\alpha-2}{\alpha}} \\
&\leq -\frac{\alpha}{2}\mathsf{E}[(Z_t^{x,y})^\alpha] + Kd(x,y)\mathsf{E}[(Z_t^{x,y})^\alpha]^{\frac{\alpha-1}{\alpha}} \\
&\quad + Kd(x,y)^2\mathsf{E}[(Z_t^{x,y})^\alpha]^{\frac{\alpha-2}{\alpha}}.
\end{aligned}$$

Thus, if $x_t = \mathsf{E}[(Z_t^{x,y})^\alpha]/d(x,y)^\alpha$,

$$\frac{dx_t}{d}t \leq -\frac{\alpha}{2}x_t + Kx_t^{\frac{\alpha-1}{\alpha}} + Kx_t^{\frac{\alpha-2}{\alpha}}.$$

It is now an exercise to show that $x_t \leq K$ and so that

$$\mathsf{E}[(Z_t^{x,y})^\alpha]^{1/\alpha} \leq Kd(x,y).$$

Using corollary 5.3, this completes the proof for the tightness of $(L_{\mu^*}^{-1}(M)_t)_t$. **QED**

Remark 4.13 *Kolmogorov's theorem (see theorem 1.4.1 and its proof in Kunita (1990)), with the estimates given in the proof of this lemma, implies that*

$$\sup_t \mathsf{E}[\|L_{\mu^*}^{-1}(M)_t\|] < \infty.$$

4.7.2 Tightness of $((L^g_{\mu^*})^{-1}(N^g, M)_t)_{t\geq 0}$

Fix $g = (g_1, \ldots, g_n) \in C(M)^n$. Let $\widehat{\Delta} g$ be defined by the relation

$$(\widehat{\Delta} g, L^{-1}_{\mu^*}(M)) = (L^g_{\mu^*})^{-1}(N^g, M).$$

Set $A_t g = (A_t g_1, \ldots, A_t g_n)$ with $A_t g_i = \widehat{C}_{\mu^*}(g_i, L^{-1}_{\mu^*}(M)_t)$. Then

$$d\widehat{\Delta}_t g = dN^g_t - \frac{\widehat{\Delta}_t g}{2} dt + A_t g dt.$$

Thus,

$$\widehat{\Delta}_t g = e^{-t/2} \int_0^t e^{s/2} dN^g_s + e^{-t/2} \int_0^t e^{s/2} A_s g ds.$$

Using this expression it is easy to prove that $(\widehat{\Delta}_t g)_{t\geq 0}$ is bounded in $L^2(\mathsf{P})$. This implies, using also lemma 4.12

Lemma 4.14 *$((L^g_{\mu^*})^{-1}(N^g, M)_t)_{t\geq 0}$ is tight.*

4.8 Convergence in law of $(N^g, M)_{t+\cdot} - (N^g, M)_t$

In this section, we denote by E_t the conditional expectation with respect to $\mathcal{F}_{e^t}$. We also set $\mathsf{Q} = \mathsf{Q}_{\mu^*}$ and $C = \widehat{C}_{\mu^*}$.

4.8.1 Preliminary lemmas.

For $f \in C(M)$ and $t \geq 0$, set $N^{f,t}_s = N^f_{t+s} - N^f_t$.

Lemma 4.15 *For all f and g in $C(M)$,*

$$\lim_{t\to\infty} \langle N^{f,t}, N^{g,t} \rangle_s = s \times C(f, g).$$

Proof : Set

$$G(z) = \langle \nabla \mathsf{Q} f, \nabla \mathsf{Q} g \rangle(z) - C(f, g)$$

and

$$G_u(z) = \langle \nabla \mathsf{Q}_u f, \nabla \mathsf{Q}_u g \rangle(z) - C(f, g).$$

We have

$$\begin{aligned}\langle N^{f,t}, N^{g,t}\rangle_s - s \times C(f,g) &= \int_{e^t}^{e^{t+s}} G_u(X_u)\frac{du}{u} \\ &= \int_{e^t}^{e^{t+s}} (G_u - G)(X_u)\frac{du}{u} \\ &+ \int_{e^t}^{e^{t+s}} G(X_u)\frac{du}{u}.\end{aligned}$$

Integrating by parts, we get that

$$\int_{e^t}^{e^{t+s}} G(X_u)\frac{du}{u} = (\mu_{e^{t+s}}G - \mu_{e^t}G) + \int_0^s (\mu_{e^{t+u}}G)du.$$

Since $\mu^* G = 0$, this converges towards 0 on the event $\{\mu_t \to \mu^*\}$. The term $\int_{e^t}^{e^{t+s}} (G_u - G)(X_u)\frac{du}{u}$ converges towards 0 because $(\mu, z) \mapsto \nabla \mathsf{Q}_\mu f(z)$ is continuous. This proves the lemma. **QED**

Let $f_1, \ldots, f_n$ be in $C(M)$. Let (t_k) be an increasing sequence converging to ∞ such that the conditional law of $M^{n,k} = (N^{f_1,t_k}, \ldots, N^{f_n,t_k})$ given $\mathcal{F}_{e^{t_k}}$ converges in law towards a $\mathbb{R}^n$-valued process $W^n = (W_1, \ldots, W_n)$.

Lemma 4.16 *W^n is a centered Gaussian process such that for all i and j,*

$$\mathsf{E}[W_i^n(s)W_j^n(t)] = (s \wedge t)C(f_i, f_j).$$

Proof : We first prove that W^n is a martingale. For all k, $M^{n,k}$ is a martingale. For all $u \leq v$, Bürkholder-Davies-Gundy inequality (BDG inequality in the following) implies that $(M^{n,k}(v) - M^{n,k}(u))_k$ is bounded in L^2.

Let $l \geq 1$, $\varphi \in C(\mathbb{R}^l)$, $0 \leq s_1 \leq \cdots \leq s_l \leq u$ and $(i_1, \ldots, i_l) \in \{1, \ldots, n\}^l$. Then for all k and $i \in \{1, \ldots, n\}$, the martingale property implies that

$$\mathsf{E}_{t_k}[(M_i^{n,k}(v) - M_i^{n,k}(u))Z_k] = 0$$

where Z_k is of the form

$$Z_k = \varphi(M_{i_1}^{n,k}(s_1), \ldots, M_{i_l}^{n,k}(s_l)). \tag{28}$$

Using the convergence of the conditional law of $M^{n,k}$ given $\mathcal{F}_{e^{t_k}}$ towards the law of W^n and since $(M_i^{n,k}(v) - M_i^{n,k}(u))_k$ is uniformly integrable (because it is bounded in L^2), we prove that

$$\mathsf{E}[(W_i^n(v) - W_i^n(u))Z] = 0$$

where Z is of the form

$$Z = \varphi(W^n_{i_1}(s_1), \ldots, W^n_{i_l}(s_l)). \tag{29}$$

This implies that W^n is a martingale.

We now prove that for $(i,j) \in \{1,\ldots,n\}$ (with $C = C_{\mu^*}$),

$$\langle W^n_i, W^n_j \rangle_s = s \times C(f_i, f_j).$$

By definition of $\langle M^{n,k}_i, M^{n,k}_j \rangle$ (in the following $\langle \cdot,\cdot \rangle^v_u = \langle \cdot,\cdot \rangle_v - \langle \cdot,\cdot \rangle_u$)

$$\begin{aligned} \mathsf{E}_{t_k}\Big[\Big((M^{n,k}_i(v) - M^{n,k}_i(u))(M^{n,k}_j(v) - M^{n,k}_j(u)) \\ -\langle M^{n,k}_i, M^{n,k}_j \rangle^v_u\Big) Z_k\Big] = 0 \end{aligned} \tag{30}$$

where Z_k is of the form (28). Using the convergence in law and the fact that $(M^{n,k}(v) - M^{n,k}(u))^2_k$ is bounded in L^2 (still using BDG inequality), we prove that as $k \to \infty$,

$$\mathsf{E}_{t_k}[(M^{n,k}_i(v) - M^{n,k}_i(u))(M^{n,k}_j(v) - M^{n,k}_j(u))Z_k]$$

converges towards

$$\mathsf{E}[(W^n_i(v) - W^n_i(u))(W^n_j(v) - W^n_j(u))Z].$$

with Z of the form (29). Now,

$$\begin{aligned} \mathsf{E}_{t_k}[\langle M^{n,k}_i, M^{n,k}_j \rangle_v Z_k] - v \times \mathsf{E}[Z] \times C(x_i, x_j) \\ = \quad \mathsf{E}_{t_k}[(\langle M^{n,k}_i, M^{n,k}_j \rangle_v - v \times C(f_i, f_j))Z_k] \\ + \quad v \times (\mathsf{E}_{t_k}[Z_k] - \mathsf{E}[Z]) \times C(f_i, f_j) \end{aligned}$$

The convergence in L^2 of $\langle M^{n,k}_i, M^{n,k}_j \rangle_v$ towards $v \times C(f_i, f_j)$ shows that the first term converges towards 0. The convergence of the conditional law of $M^{n,k}$ with respect to $\mathcal{F}_{e^{t_k}}$ towards W^n shows that the second term converges towards 0. Thus

$$\mathsf{E}\left[\big((W^n_i(v) - W^n_i(u))(W^n_j(v) - W^n_j(u)) - (v-u)C(f_i, f_j)\big) Z\right] = 0.$$

This shows that $\langle W^n_i, W^n_j \rangle_s = s \times C(f_i, f_j)$. We conclude using Lévy's theorem. **QED**

4.8.2 Convergence in law of $M_{t+\cdot} - M_t$

In this section, we denote by $\mathcal{L}_t$ the conditional law of $M_{t+\cdot} - M_t$ knowing $\mathcal{F}_{e^t}$. Then $\mathcal{L}_t$ is a probability measure on $C(\mathbb{R}^+ \times M)$.

Proposition 4.17 *When $t \to \infty$, $\mathcal{L}_t$ converges weakly towards the law of a $C(M)$-valued Brownian motion of covariance C_{μ^*}.*

Proof : In the following, we will simply denote $M_{t+\cdot} - M_t$ by M^t. We first prove that

Lemma 4.18 *$\{\mathcal{L}_t : \ t \geq 0\}$ is tight.*

Proof : For all $x \in M$, t and u in $\mathbb{R}^+$,

$$\begin{aligned} \mathsf{E}_t[(M_u^t(x))^2] &= \mathsf{E}_t\left[\int_t^{t+u} d\langle M(x)\rangle_s\right] \\ &\leq Ku. \end{aligned}$$

This implies that for all $u \in \mathbb{R}^+$ and $x \in M$, $(M_u^t(x))_{t\geq 0}$ is tight.

Let $\alpha > 0$. We fix $T > 0$. Then for (u, x) and (v, y) in $[0, T] \times M$, using BDG inequality,

$$\begin{aligned} \mathsf{E}_t[|M_u^t(x) - M_v^t(y)|^\alpha]^{\frac{1}{\alpha}} &\leq \mathsf{E}_t[|M_u^t(x) - M_u^t(y)|^\alpha]^{\frac{1}{\alpha}} \\ &\quad + \mathsf{E}_t[|M_u^t(y) - M_v^t(y)|^\alpha]^{\frac{1}{\alpha}} \\ &\leq K_\alpha \times (\sqrt{T} d(x, y) + \sqrt{|v - u|}) \end{aligned}$$

where K_α is a positive constant depending only on α, $\|V\|$ and $\mathrm{Lip}(V)$ the Lipschitz constant of V.

We now let D_T be the distance on $[0, T] \times M$ defined by

$$D_T((u, x), (v, y)) = K_\alpha \times (\sqrt{T} d(x, y) + \sqrt{|v - u|}).$$

The covering number $N([0, T] \times M, D_T, \epsilon)$ is of order $\epsilon^{-d-1/2}$ as $\epsilon \to 0$. Taking $\alpha > d + 1/2$, we conclude using corollary 5.3. **QED**

Let (t_k) be an increasing sequence converging to ∞ and N a $C(M)$-valued random process (or a $C(\mathbb{R}^+ \times M)$ random variable) such that $\mathcal{L}_{t_k}$ converges in law towards N.

Lemma 4.19 *N is a $C(M)$-valued Brownian motion of covariance C_{μ^*}.*

Proof : Let W be a $C(M)$-valued Brownian motion of covariance C_{μ^*}. Using lemma 4.16, we prove that for all $(x_1, \ldots, x_n) \in M^n$, $(N(x_1), \ldots, N(x_n))$ has the same distribution as $(W(x_1), \ldots, X(x_n))$. This implies the lemma. **QED**

Since $\{\mathcal{L}_t\}$ is tight, this lemma implies that $\mathcal{L}_t$ converges weakly towards the law of a $C(M)$-valued Brownian motion of covariance C_{μ^*}. **QED**

4.8.3 Convergence in law of $(N^g, M)_{t+\cdot} - (N^g, M)_t$

In this section, we fix $g = (g_1, \ldots, g_n) \in C(M)^n$ and we denote by $\mathcal{L}_t^g$ the conditional law of $(N^g, M)_{t+\cdot} - (N^g, M)_t$ knowing $\mathcal{F}_{e^t}$. Then $\mathcal{L}_t^g$ is a probability measure on $C(\mathbb{R}^+ \times M \cup \{1, \ldots, n\})$. In the following we will denote $(N^{g,t}, M^t)$ the process $(N^g, M)_{t+\cdot} - (N^g, M)_t$.

Let $(W_t^f)_{(t,f)\in\mathbb{R}^+\times C(M)}$ be a $\mathcal{X}(M)$-valued Brownian motion of covariance $\widehat{C}_{\mu^*}$. Denoting $W_t(x) = W_t^{V_x}$, then $W = (W_t(x))_{(t,x)\in\mathbb{R}^+\times M}$ is a $C(M)$-valued Brownian motion of covariance C_{μ^*}. For $g = (g_1, \ldots, g_n) \in C(M)^n$, W^g will denote $(W^{g_1}, \ldots, W^{g_n})$. In the following we will simply denote (W^g, W) the process $(W_t^g, (W_t(x))_{x\in M})_{t\geq 0}$.

Proposition 4.20 *As t goes to ∞, $\mathcal{L}_t^g$ converges weakly towards the law of (W^g, W).*

Proof : We first prove that

Lemma 4.21 *$\{\mathcal{L}_t^g : \ t \geq 0\}$ is tight.*

Proof : This is a straightforward consequence of the tightness of $\{\mathcal{L}_t\}$ and of the fact that for all $\alpha > 0$, there exists K_α such that for all nonnegative u and v, $\mathsf{E}_t[|N_u^{g,t} - N_v^{g,t}|^\alpha]^{\frac{1}{\alpha}} \leq K_\alpha\sqrt{|v-u|}$. **QED**

Let (t_k) be an increasing sequence converging to ∞ and $(\tilde{N}^g, \tilde{M})$ a $\mathbb{R}^n \times C(M)$-valued random process (or a $C(\mathbb{R}^+ \times M \cup \{1, \ldots, n\})$ random variable) such that $\mathcal{L}_{t_k}^g$ converges in law towards $(\tilde{N}^g, \tilde{M})$. Then lemmas 4.15 and 4.16 imply that $(\tilde{N}^g, \tilde{M})$ has the same law as (W^g, W). Since $\{\mathcal{L}_t^g\}$ is tight, $\mathcal{L}_t^g$ convergences towards the law of (W^g, W). **QED**

4.9 Convergence in law of D

4.9.1 Convergence in law of $(D_{t+s} - e^{-sG_{\mu^*}} D_t)_{s\geq 0}$

We have

$$D'_{t+s} - e^{-sG_{\mu^*}} D'_t \;=\; L^{-1}_{\mu^*}(M^t)_s + \int_0^s e^{-(s-u)G_{\mu^*}} \epsilon'_{t+u} du.$$

Since (using lemma 4.9)

$$\left\| \int_0^s e^{-(s-u)G_{\mu^*}} \epsilon'_{t+u} du \right\| \leq K e^{-\kappa t}$$

and $\|D_t - D'_t\| \leq K(1+t)e^{-t/2}$, this proves that $(D_{t+s} - e^{-sG_{\mu^*}} D_t - L^{-1}_{\mu^*}(M_{t+\cdot} - M_t)_s)_{s\geq 0}$ converges towards 0. Since $L^{-1}_{\mu^*}$ is continuous, this proves that the law of $L^{-1}_{\mu^*}(M_{t+\cdot} - M_t)$ converges weakly towards $L^{-1}_{\mu^*}(W)$. Since $L^{-1}_{\mu^*}(W)$ is an Ornstein-Uhlenbeck process of covariance C_{μ^*} and drift $-G_{\mu^*}$ started from 0, we have

Theorem 4.22 *The conditional law of $(D_{t+s} - e^{-sG_{\mu^*}} D_t)_{s\geq 0}$ given $\mathcal{F}_{e^t}$ converges weakly towards an Ornstein-Uhlenbeck process of covariance C_{μ^*} and drift $-G_{\mu^*}$ started from 0.*

4.9.2 Convergence in law of $D_{t+\cdot}$

We can now prove theorem 3.1. We here denote by P_t the semigroup of an Ornstein-Uhlenbeck process of covariance C_{μ^*} and drift $-G_{\mu^*}$, and we denote by π its invariant probability measure.

We know that (as $t \to \infty$) $(D_{t+s} - e^{-sG_{\mu^*}} D_t)_{s\geq 0}$ converges in law towards $L^{-1}_{\mu^*}(W)$, where W is a $C(M)$-valued Brownian motion of covariance C_{μ^*}. Since $(D_t)_{t\geq 0}$ is tight, there exists $\nu \in \mathcal{P}(C(M))$ and an increasing sequence t_n converging towards ∞ such that D_{t_n} converges in law towards ν. Then $D_{t_n+\cdot}$ converges in law towards $(L^{-1}_{\mu^*}(W)_s + e^{-sG_{\mu^*}} Z_0)$, with Z_0 independent of W and distributed like ν. This proves that $D_{t_n+\cdot}$ converges in law towards an Ornstein-Uhlenbeck process of covariance C_{μ^*} and drift $-G_{\mu^*}$.

We now fix $t > 0$. Let s_n be a subsequence of t_n such that $D_{s_n-t+\cdot}$ converges in law. Then D_{s_n-t} converges towards a law we denote by ν_t and $D_{s_n-t+\cdot}$ converges in law towards an Ornstein-Uhlenbeck process of covariance C_{μ^*} and drift $-G_{\mu^*}$. Since $D_{s_n} = D_{s_n-t+t}$, D_{s_n} converges in law towards $\nu_t \mathsf{P}_t$. On the other hand D_{s_n} converges in law towards ν. Thus $\nu_t \mathsf{P}_t = \nu$.

Let φ be a Lipschitz bounded function on $C(M)$. Then

$$\begin{aligned} |\nu_t \mathsf{P}_t \varphi - \pi\varphi| &= \left| \int (\mathsf{P}_t\varphi(f) - \pi\varphi)\nu_t(df) \right| \\ &\leq \int |\mathsf{P}_t\varphi(f) - \mathsf{P}_t\varphi(0)|\nu_t(df) + |\mathsf{P}_t\varphi(0) - \pi\varphi| \end{aligned}$$

where the second term converges towards 0 (using (37)) and the first term is dominated by (using lemma 5.15)

$$Ke^{-\kappa t} \int \|f\|\nu_t(df).$$

It is easy to check that

$$\begin{aligned} \int \|f\|\nu_t(df) &= \lim_{k\to\infty} \int (\|f\| \wedge k)\nu_t(df) \\ &= \lim_{k\to\infty} \lim_{n\to\infty} \mathsf{E}[\|D_{s_n - t}\| \wedge k] \\ &\leq \lim_{n\to\infty} \mathsf{E}[\|D_{s_n - t}\|] \\ &\leq \sup_t \mathsf{E}[\|D_t\|]. \end{aligned}$$

Since

$$\begin{aligned} \|D_t\| &\leq \|D_t^1 + D_t^5\| + \|L_{\mu^*}^{-1}(M)_t\| \\ &\quad + \left\| \int_0^t e^{(t-s)G_{\mu^*}} \epsilon'_s ds \right\|, \end{aligned}$$

using the estimates (22), the proof of lemma 4.10 and remark 4.13, we get that

$$\sup_{t\geq 0} \mathsf{E}[\|D_t\|] < \infty.$$

Taking the limit, we prove $\nu\varphi = \pi\varphi$ for all Lipschitz bounded function φ on $C(M)$. This implies $\nu = \pi$, which proves the theorem. **QED**

4.9.3 Convergence in law of D^g

We can also prove theorem 3.2.

For $g = (g_1, \ldots, g_n) \in C(M)^n$, we set $D_t^g = (\Delta_t g, D_t)$, and $D_t'^g = (\Delta_t' g, D_t')$. Since $\|D_t^g - D_t'^g\| \leq K(1+t)e^{-t/2}$, instead of studying D^g, we can only study $D_t'^g$. Then

$$\begin{aligned} D'^g_{t+s} - e^{-sG^g_{\mu^*}} D'^g_t &= (L^g_{\mu^*})^{-1}(N^{g,t}, M^t)_s \\ &\quad + \int_0^s e^{-(s-u)G^g_{\mu^*}} (\epsilon'_{t+u} g, \epsilon'_{t+u}) du. \end{aligned}$$

The norm of the second term of the right hand side (using the proof of lemma 4.11) is dominated by

$$\begin{aligned} &\leq K(1+\|g\|) \int_0^s e^{-\kappa(s-u)} \|\epsilon'_{t+u}\| du \\ &\leq K \int_0^s e^{-\kappa(s-u)} (1+t+u)^2 e^{-(t+u)/2} du \\ &\leq k e^{-\kappa t} \end{aligned}$$

Like in section 4.9.1, since $(L^g_{\mu^*})^{-1}(W^g, W)$ is an Ornstein-Uhlenbeck process of covariance $C^g_{\mu^*}$ and drift $-G^g_{\mu^*}$ started from 0,

Theorem 4.23 *The conditional law of* $((\Delta^g, D)_{t+s} - e^{-sG^g_{\mu^*}} (\Delta^g, D)_t)_{s \geq 0}$ *given* $\mathcal{F}_{e^t}$ *converges weakly towards an Ornstein-Uhlenbeck process of covariance* $C^g_{\mu^*}$ *and drift* $-G^g_{\mu^*}$ *started from* 0.

From this theorem, like in section 4.9.2, we prove theorem 3.2. **QED**

5 Appendix : Random variables and Ornstein-Uhlenbeck processes on $C(M)$

5.1 $C(M)$-valued random variables

5.1.1 Generalities

Let (M, d) be a compact metric space (note that there is no assumption here that M is a manifold), $C(M)$ the space of real valued continuous functions on M equipped with the uniform norm $\|f\| = \sup_{x \in M} |f(x)|$. By classical results, $C(M)$ is a separable (see e.g [16]) Banach space (see e.g [11] or [9]) and its topological dual is the space $\mathcal{M}(M)$ of bounded signed measures on

M (see e.g [11] or [9]). For $\mu \in \mathcal{M}(M)$ and $f \in C(M)$ we use the notation $\mu f = \langle \mu, f \rangle = \int_M f d\mu$.

Let $(\Omega, \mathcal{F}, \mathsf{P})$ be a probability space. A $C(M)$*-valued random variable* is a Borel map $F : \Omega \to C(M)$.

For $x \in M$, let $\pi_x : C(M) \to \mathbb{R}$, denote the projection defined by

$$\pi_x(f) = f(x).$$

Lemma 5.1 *The Borel σ-field on $C(M)$ is the σ-field generated by the maps $\{\pi_x\}_{x \in M}$. In particular*

(i) *A map $F : \Omega \to C(M)$ is a $C(M)$-valued random variable if and only if $\{\pi_x(F)\}_{x \in M}$ is a family of real valued random variables.*

(ii) *The law of a $C(M)$-valued random variable is determined by its finite dimensional distributions (i.e the law of $\{\pi_x(F)\}_{x \in I}$ with $I \subset M$ finite).*

Proof : Let $\mathcal{A} = \sigma\{\pi_x, x \in M\}$ and $\mathcal{B}$ the Borel σ-field on $C(M)$. The maps π_x being continuous, $\mathcal{B}$ contains $\mathcal{A}$. Conversely, let $B_f(r) = \{g \in C(M) \ : \|g - f\| \leq r\}$ and let S be a countable dense subset of M. Then $B_f(r) = \cap_{x \in S}\{g \in C(M) : \ |\pi_x(f) - \pi_x(g)| \leq r\}$ Hence $B_f(r) \in \mathcal{A}$. Since $C(M)$ is separable, $\mathcal{B}$ is generated by the sets $\{B_f(r), f \in C(M), r \geq 0\}$. **QED**

5.1.2 Tightness criteria

Let $\mathcal{P}(C(M))$ be the space of Borel probability measures on $C(M)$. An element ν of $\mathcal{P}(C(M))$ is the law of a $C(M)$-valued random variable F, and $\nu = \mathsf{P}_F$. Recall that a sequence $\{\nu^n\}$ in $\mathcal{P}(C(M))$ is said *converging weakly* towards $\nu \in \mathcal{P}(C(M))$ if $\int \varphi d\nu_n \to \int \varphi d\nu$ for every bounded and continuous function $\varphi : C(M) \to \mathbb{R}$. A sequence $\{F_n\}$ of $C(M)$-valued random variable is said *converging in law* towards F a $C(M)$-valued random variable if $\{P_{F_n}\}$ converges in law towards P_F. A family $\mathcal{X} \subset \mathcal{P}(C(M))$ is said to be *tight* if for every $\epsilon > 0$ there exists some compact set $\mathcal{K} \subset C(M)$ such that $\mathsf{P}(\mathcal{K}) \geq 1 - \epsilon$ for all $\mathcal{P} \in \mathcal{X}$. A family of random variables is said to be tight if the family of their laws is tight.

Since $C(M)$ is a separable and complete, Prohorov theorem [8] asserts that $\mathcal{X} \subset \mathcal{P}(C(M))$ is tight if and only if it is relatively compact.

The next proposition gives a useful criterium for a class of random variables to be tight. It follows directly from [14] (Corollary 11.7 p. 307 and the remark following Theorem 11.2). A function $\psi : \mathbb{R}_+ \to \mathbb{R}_+$ is a Young function if it is convex, increasing and $\psi(0) = 0$. If Z is a real valued random variable, we let

$$\|Z\|_\psi = \inf\{c > 0 \ : \mathsf{E}(\psi(|Z|/c)) \leq 1\}.$$

For $\epsilon > 0$, we denote by $N(M, d; \epsilon)$ the covering number of E by balls of radius less than ϵ (i.e. the minimal number of balls of radius less than ϵ that cover E), and by D the diameter of M.

Proposition 5.2 *Let $(F_t)_{t\in I}$ be a family of $C(M)$-valued random variables and ψ a Young function. Assume that*

(i) *There exists $x \in E$ such that $(F_t(x))_{t\in I}$ is tight;*

(ii) $\|F_t(x) - F_t(y)\|_\psi \leq Kd(x, y)$;

(iii) $\int_0^D \psi^{-1}(N(M, d; \epsilon))d\epsilon < \infty$.

Then $(F_t)_{t\geq 0}$ is tight.

Corollary 5.3 *Suppose M is a compact finite dimensional manifold of dimension r, d the Riemannian distance, and*

$$[\mathsf{E}|F_t(x) - F_t(y)|^\alpha]^{1/\alpha} \leq Kd(x, y)$$

for some $\alpha > r$. Then conditions (ii) and (iii) of Proposition 5.2 hold true.

Proof : One has $N(E, d; \epsilon)$ is of order ϵ^{-r}; and for $\psi(x) = x^\alpha$, $\|\cdot\|_\psi$ is the L^α norm. Hence the result. **QED**

5.1.3 $C(M)$-valued Gaussian variable

Recall that a (centered) real-valued random variable Y with variance σ^2 is said to be *Gaussian* if it has distribution

$$P_Y(dx) = \frac{1}{\sqrt{2\pi}\sigma} \exp(-\frac{x^2}{2\sigma^2})dx.$$

Its characteristic function is then

$$\Phi_Y(t) = \mathsf{E}[\exp(itY)] = \exp(-\frac{t^2\sigma^2}{2}).$$

Here we adopt the convention that the zero function ($Y = 0$) is Gaussian with variance 0 and that all the Gaussian random variables are centered.

A family $\{Y_i\}_{i\in I}$ of real-valued random variables is said to be Gaussian if for all finite set $J \subset I$ and for all $\alpha \in \mathbb{R}^J$, $\sum_{j\in J} \alpha_j Y_j$ is Gaussian.

A $C(M)$-valued random variable F is said to be *Gaussian* if for all $\mu \in \mathcal{M}(M)$, $\langle \mu, F\rangle$ is Gaussian.

Lemma 5.4 *A $C(M)$-valued random variable F is Gaussian if and only if the family $\{\pi_x(F)\}$ is Gaussian.*

Proof : The direct implication is obvious. We prove the second. Assume that $\{\pi_x(F)\}$ is a Gaussian family. Let μ be a probability over M. By the strong law of large number and the separability of $C(M)$ there exists a nonempty set $\Lambda \subset M^{\mathbb{N}}$ (actually Λ has $\mu^{\mathbb{N}}$ measure 1) such that for all $(x_i) \in \Lambda$ and all $f \in C(M)$

$$\lim_{n\to\infty} \frac{1}{n} \sum_{i=1}^{n} f(x_i) = \langle \mu, f\rangle.$$

In particular $Y_n \to \langle \mu, F\rangle$ where

$$Y_n = \frac{1}{n} \sum_{i=1}^{n} F(x_i).$$

And, by Lebesgue theorem, $\Phi_{Y_n}(t) \to \Phi_{\langle\mu,F\rangle}(t)$. Since, by assumption Y_n is Gaussian, $\Phi_{Y_n}(t) = \exp(-t^2\sigma_n^2/2)$. Let $\sigma \in [0, \infty]$ be a limit point of (σ_n). Then $\Phi_{\langle\mu,F\rangle}(t) = \exp(-t^2\sigma^2/2)$. This proves that $\sigma < \infty$ (a characteristic function being continuous) and that $\langle \mu, F\rangle$ is Gaussian.

If now $\mu \in \mathcal{M}(M)$ by Jordan-Hann decomposition we may write $\mu = a\mu_1 - b\mu_2$ with $a, b \geq 0$ and μ_1, μ_2 probabilities. It follows from what precede that $\langle \mu, X \rangle$ is Gaussian. **QED**

Given a $C(M)$-valued Gaussian random variable F we let $\mathsf{Var}_F : \mathcal{M}(M) \to \mathbb{R}$ denote the *variance function* of F defined by

$$\mathsf{Var}_F(\mu) = \mathsf{E}(\langle \mu, F \rangle^2).$$

In view of lemma 5.1 (ii), the law of F is entirely determined by its variance function.

A useful property of Gaussian variables is the following.

Lemma 5.5 *Let M' be another compact metric space and $A : C(M) \to C(M')$ a bounded linear operator. Let F be a $C(M)$-valued Gaussian random variable. Then AF is a $C(M')$-valued Gaussian random variable with variance*

$$\mathsf{Var}_{AF} = \mathsf{Var}_F \circ A^*$$

where $A^* : \mathcal{M}(M') \to \mathcal{M}(M)$ is the adjoint of A.

Proof : follows from the duality $\langle \mu, AF \rangle = \langle A^*\mu, F \rangle$ and the definitions. **QED**

5.2 Brownian motions on $C(M)$.

Let $C : M \times M \to \mathbb{R}$ be a continuous symmetric (i.e $C(x, y) = C(y, x)$) function such that

$$\sum_{ij} a_i a_j C(x_i, x_j) \geq 0$$

for every finite sequence (a_i, x_i) with $a_i \in \mathbb{R}$ and $x_i \in M$. Such a function is sometimes called a *Mercer kernel*.

A *Brownian motion on $C(M)$ with covariance C* is a $C(M)$-valued stochastic process $W = \{W_t\}_{t \geq 0}$ such that $W_0 = 0$ and for each $T \geq 0$, $W^T = \{W_t(x) : t \leq T, x \in M\}$ is a $C([0, T] \times M)$-valued Gaussian random variable with variance

$$\mathsf{Var}_{W^T}(\nu) = \int_{([0,T] \times M)^2} (s \wedge t) C(x, y) \nu(dsdx) \nu(dtdy)$$

or equivalently

$$\mathsf{E}(W_t(x)W_s(y)) = (s \wedge t)C(x,y).$$

Let

$$d_C(x,y) = \sqrt{C(x,x) - 2C(x,y) + C(y,y)}.$$

The d_C is a pseudo-distance on M. For $\epsilon > 0$, let

$$\omega_C(\epsilon) = \sup\{\eta > 0 : d(x,y) \leq \eta \Rightarrow d_C(x,y) \leq \epsilon\}.$$

Then $N(M,d;\omega_C(\epsilon)) \geq N(M,d_C;\epsilon)$.

Hypothesis 5.6

$$\int_0^1 \log(N(d,M;\omega_C(\epsilon))\ d\epsilon < \infty$$

where $N(d,M;\eta)$ is the covering number of M by balls of radius less than η.

Remark 5.7 *Assume that M is a compact finite dimensional manifold and that $d_C(x,y) \leq Kd(x,y)^\alpha$ for some $\alpha > 0$. Then $\omega_C(\epsilon) \leq (\frac{\epsilon}{K})^{1/\alpha}$ and $N(d,M;\eta) = O(\eta^{-dim(M)})$; so that the preceding hypothesis holds.*

Proposition 5.8 *Under hypothesis 5.6 there exists a Brownian motion on $C(M)$ with covariance C.*

Proof : By Mercer Theorem (see e.g [10]) there exists a countable family of function $\Psi_i \in C(M), i \in \mathbb{N}$, such that

$$C(x,y) = \sum_i \Psi_i(x)\Psi_i(y)$$

and the convergence is uniform. Let $B^i, i \in \mathbb{N}$, be a family of independent standard Brownian motions. Set

$$W_t^n(x) = \sum_{i \leq n} B_t^i \Psi_i(x),\ n \geq 0.$$

Then, for each $(t,x) \in \mathbb{R}^+ \times M$, the sequence $(W_t^n(x))_{n\geq 1}$ is a Martingale. It is furthermore bounded in L^2 since

$$\mathsf{E}[(W_t^n(x))^2] = t\sum_{i\leq n} \Psi_i(x)^2 \leq tC(x,x).$$

Hence by Doob's convergence theorem one may define

$$W_t(x) = \sum_{i \geq 0} B_t^i \Psi_i(x).$$

Let now $S \subset \mathbb{R}^+ \times M$ be a countable and dense set. It is easily checked that the family $(W_t(x))_{(t,x)\in S}$ is a centered Gaussian family with covariance given by

$$\mathsf{E}[W_s(x)W_t(y)] = (s \wedge t)C(x,y),$$

In particular, for $t \geq s$

$$\begin{aligned} \mathsf{E}[(W_s(x) - W_t(y))^2] &= sC(x,x) - 2sC(x,y) + tC(y,y) \\ &= sd_C(x,y)^2 + (t-s)C(y,y) \\ &\leq K(t-s) + sd_C(x,y)^2 \end{aligned}$$

This later bound combined with classical results on Gaussian processes (see e.g Theorem 11.17 in [14]) implies that $(t,x) \mapsto W_t(x)$ admits a version uniformly continuous over $S_T = \{(t,x) \in S : t \leq T\}$. By density it can be extended to a continuous (in (t,x)) process

$$W = (W_t(x))_{\{(t,x)\in\mathbb{R}^+\times M\}}$$

The process W can be viewed as a $C(M)$-valued continuous random process with the desired covariance. **QED**

5.3 Ornstein-Ulhenbeck processes

Let $A : C(M) \to C(M)$ be a bounded operator and W a $C(M)$-valued Brownian motion with covariance C as defined in the preceding section.

An *Ornstein-Ulhenbeck process* with drift A, covariance C and initial condition $F_0 = f \in C(M)$ is defined to be a $C(M)$ valued stochastic process continuous in t, such that

$$F_t - f = \int_0^t AF_s ds + W_t. \tag{31}$$

Note that we may think of F as the solution to the "stochastic differential equation" on $C(M)$:

$$dF_t = AF_t dt + dW_t$$

with initial condition $F_0 = f \in C(M)$.

Our aim here is to construct such a solution and state some of its properties.

We let $(e^{tA})_{t\in\mathbb{R}}$ denote the linear flow induced by A. Recall that for each t, e^{tA} is the bounded operator on $C(M)$ defined by

$$e^{tA} = \sum_{k\in\mathbb{N}} \frac{(tA)^k}{k!}.$$

Given $T > 0$ we let $L_A : C(\mathbb{R}^+ \times M) \to C(\mathbb{R}^+ \times M)$ be defined by

$$L_A(f)_t = f_t - f_0 - \int_0^t Af_s ds, \qquad t \geq 0. \tag{32}$$

Given $T > 0$ we let $L_A^T : C([0,T] \times M) \to C([0,T] \times M)$ be defined by

$$L_A^T(f)_t = f_t - f_0 - \int_0^t Af_s ds, \qquad 0 \leq t \leq T. \tag{33}$$

Note that if for $f \in C(\mathbb{R}^+ \times M)$, we let $f^T \in C([0,T] \times M)$ be defined by $f_t^T = f_t$ for $t \in [0,T]$.

Lemma 5.9 *L_A^T is a bounded operator and its restriction to $C_0([0,T]\times M) = \{f \in C([0,T] \times M) \ : f_0 = 0\}$ is bijective with inverse $(L_A^T)^{-1}$ defined by*

$$(L_A^T)^{-1}(g)_t = g_t + \int_0^t e^{(t-s)A} Ag_s ds, \qquad t \in [0,T]. \tag{34}$$

Proof : Linearity of L_A^T is obvious. Also

$$\|L_A^T(f)\| \leq (2 + T\|A\|)\|f\|.$$

This proves that L_A^T is bounded.

Observe that $L_A^T(f) = 0$ implies that $f_t = e^{tA} f_0$. Hence L_A^T restricted to $C_0([0,T] \times M)$ is injective. Let $g \in C_0([0,T] \times M)$ and let f_t be given by the left hand side of (34). Then

$$h_t = L_A^T(f)_t - g_t = \int_0^t e^{(t-s)A} Ag_s ds - \int_0^t Af_s ds.$$

It is easily seen that h is differentiable and that $\frac{d}{dt}h_t = 0$. This proves that $h_t = h_0 = 0$. **QED**

We also have

Lemma 5.10 *The restriction of L_A to $C_0(\mathbb{R}^+ \times M) = \{f \in C(\mathbb{R}^+ \times M) : f_0 = 0\}$ is bijective with inverse $(L_A)^{-1}$ defined by*

$$L_A^{-1}(g)_t = g_t + \int_0^t e^{(t-s)A} A g_s ds. \tag{35}$$

The next lemma easily follows.

Lemma 5.11 *For all $f \in C(M)$ and $g \in C_0(\mathbb{R}^+ \times M)$ the solution to*

$$f_t = f + \int_0^t A f_s ds + g_t,$$

is given by

$$f_t = e^{tA} f + L_A^{-1}(g)_t.$$

If now W is a $C(M)$-valued Brownian motion as defined in the preceding section, one may define

$$F_t = e^{tA} f + L_A^{-1}(W)_t.$$

Such a process is the unique solution to (31). Note that, by Lemma 5.5 $(F_t - e^{tA} f)_{t \leq T}$ is a $C_0([0,T] \times M)$-valued Gaussian random variable. In particular

Proposition 5.12 *Let (F_t) be the solution to (31) with initial condition $F_0 = 0$. Then for each $t \geq 0$, F_t is a $C(M)$-valued Gaussian random variable with variance*

$$\mathsf{Var}_{F_t}(\mu) = \int_0^t \langle \mu, e^{sA} C e^{sA^*} \mu \rangle ds.$$

where $C : \mathcal{M}(M) \to C(M)$ is the operator defined by $C\mu(x) = \int_M C(x,y)\mu(dy)$.

Proof : Fix $T > 0$. To shorten notation let $G : C_0([0,T] \times M) \to C(M)$ be the operator defined by

$$G(g) = (L_A^T)^{-1}(g)_T.$$

Hence $F_T = G(W^T)$. By Lemma 5.5, F_T is Gaussian with variance

$$\mathsf{Var}_{F_T} = \mathsf{Var}_{W^T} \circ G^*.$$

Now, for all $\nu \in \mathcal{M}([0,T] \times M)$

$$\mathsf{Var}_{W^T}(\nu) = \langle \nu, \mathcal{C}\nu \rangle$$

where

$$\mathcal{C}\nu(s,x) = \int_{[0,T]\times M} (s \wedge u) C(x,y)\nu(dudy).$$

Thus

$$\mathsf{Var}_{F_T}(\mu) = \langle \mu, G\mathcal{C}G^*\mu \rangle.$$

Our next goal is to compute $G\mathcal{C}G^*\mu$. It easily follows from the definition of G that

$$G^*\mu = \delta_T \otimes \mu + dsA^* e^{(T-s)A^*}\mu.$$

Thus (integrating by parts)

$$\mathcal{C}G^*\mu = \mathcal{C}\nu_1 + \mathcal{C}\nu_2$$

with $\nu_1 = \delta_T \otimes \mu$ and $\nu_2 = dsA^* e^{(T-s)A*}\mu$. One has

$$\mathcal{C}\nu_1(s,x) = s(C\mu)(x);$$

$$\mathcal{C}\nu_2(s,x) = \int_0^T (s \wedge u)(C\dot{m}_u(\mu))(x)du$$

with $m_u(\mu) = -e^{(T-u)A^*}\mu$ and $\dot{m}_u(\mu)$ stands for the derivative of $u \mapsto m_u(\mu)$. Thus

$$\mathcal{C}\nu_2(s,x) = -sC\mu - \int_0^s Cm_u(\mu)(x)du$$

and

$$\mathcal{C}G^*\mu(s,x) = -\int_0^s Cm_u(\mu)(x)du.$$

Set

$$h_s(x) = \int_0^s Cm_u(\mu)(x)du.$$

Then

$$G\mathcal{C}G^*\mu = h_T + \int_0^T e^{(T-s)A} Ah_s ds = \int_0^T e^{(T-s)A}\dot{h}_s ds$$

$$= \int_0^T e^{(T-s)A} Ce^{(T-s)A^*}\mu ds = \int_0^T e^{sA} Ce^{sA^*}\mu ds.$$

QED

5.3.1 Asymptotic Behaviour

Let $\lambda(A) = \lim_{t\to\infty} \frac{\log(\|e^{tA}\|}{t}$ which exists by subadditivity. Then for some constant $K < \infty$, $\|e^{tA}\| \leq Ke^{\lambda(A)t}$ for all positive t. Let (F_t) denote the solution to (31), with $F_0 = f \in C(M)$.

Corollary 5.13 *Assume $\lambda(A) < 0$. Then for each $\mu \in \mathcal{M}(M)$ $(\langle \mu, F_t\rangle)$ converges in law toward a Gaussian random variable with variance*

$$\mathsf{V}(\mu) = \int_0^\infty \langle \mu, e^{sA} C e^{sA*} \mu\rangle ds.$$

Proof : follows from proposition 5.12 and Lemma 5.11 **QED**

Corollary 5.14 *Assume that $\lambda(A) < 0$. Set*

$$d_{\mathsf{V}}(x, y) = \sqrt{\mathsf{V}(\delta_x - \delta_y)}$$

and

$$\omega_{\mathsf{V}}(\epsilon) = \sup\{\eta > 0 : d(x, y) \leq \eta \Rightarrow d_{\mathsf{V}}(x, y) \leq \epsilon\}.$$

Assume furthermore that ω_{V} verifies the condition expressed by hypothesis 5.6. Then (F_t) converges in law toward a $C(M)$-valued Gaussian random variable with variance V.

Proof : Let ν_t denote the law of F_t. Corollary 5.13 and lemma 5.1 imply that every limit point of $\{\nu_t\}$ (for the weak* topology) is the law of a $C(M)$-valued Gaussian variable with variance V. The proof then reduces to show that (ν_t) is relatively compact or equivalently that $\{F_t\}$ is tight. We use Proposition 5.2. The first condition follows from Lemma 5.13. Let $\psi(x) = e^{x^2} - 1$. It is easily verified that for any real valued Gaussian random variable Z with variance σ^2, $\|Z\|_\Psi = \sigma\sqrt{8/3}$. Hence $\|F_t(x) - F_t(y)\|_\psi \leq 2d_{\mathsf{V}}(x, y)$ so that condition (ii) holds with the pseudo distance d_{V}. By definition of ω_{V}, $N(M, d; \omega_{\mathsf{V}}(\epsilon)) \geq N(M, d_{\mathsf{V}}; \epsilon)$ and since $\psi^{-1}(u) = \sqrt{\log(u-1)}$ condition (iii) is verified. **QED**

Denote by P_t the semigroup associated to an Ornstein-Uhlenbeck process of covariance C and drift A. Then for all bounded measurable $\varphi : C(M) \to \mathbb{R}$ and $f \in C(M)$,

$$\mathsf{P}_t\varphi(f) = \mathsf{E}[\varphi(F_t)]. \tag{36}$$

Denote by π the law of a $C(M)$-valued Gaussian random variable with variance V. Then π is the invariant probability measure of P_t, i.e. $\pi\mathsf{P}_t = \pi$. Corollary 5.14 implies that, when $\lambda(A) < 0$, for all $f \in C(M)$ and all bounded continuous $\varphi : C(M) \to \mathbb{R}$,

$$\lim_{t\to\infty} \mathsf{P}_t\varphi(f) = \pi\varphi. \tag{37}$$

Even thought we don't have the speed of convergence in the previous limit, we have

Lemma 5.15 *Assume that $\lambda(A) < 0$. For all bounded Lipschitz continuous $\varphi : C(M) \to \mathbb{R}$, all f and g in $C(M)$,*

$$|\mathsf{P}_t\varphi(f) - \mathsf{P}_t\varphi(g)| \leq Ke^{\lambda(A)t}\|f - g\|.$$

Proof : We have $\mathsf{P}_t\varphi(f) = \mathsf{E}[\varphi(L_A^{-1}(W)_t + e^{tA}f)]$. So, using the fact that φ is Lipschitz,

$$\begin{aligned} |\mathsf{P}_t\varphi(f) - \mathsf{P}_t\varphi(g)| &\leq K\|e^{tA}(f - g)\| \\ &\leq Ke^{\lambda(A)t}\|f - g\|. \end{aligned}$$

This proves the lemma. **QED**

To conclude this section we give a set of simple sufficient conditions ensuring that the hypotheses of corollary 5.14 are satisfied.

For $f \in C(M)$ we let

$$\mathrm{Lip}(f) = \sup_{x\neq y} \frac{|f(x) - f(y)|}{d(x, y)} \in \mathbb{R}^+ \cup \{\infty\}. \tag{38}$$

A map f is said to be Lipschitz provided $Lip(f) < \infty$.

Proposition 5.16 *Assume*

(i) *$N(d, M; \epsilon) = O(\epsilon^{-r})$ for some $r > 0$ (This holds in particular if M is a finite dimensional manifold).*

(ii) *$x \mapsto C(x, y)$ is Lipschitz uniformly in y. That is*

$$\sup_{z\in M} |C(x, z) - C(y, z)| \leq Kd(x, y)$$

for some $K \geq 0$.

(iii) *There exists $K > 0$ such that*

$$Lip(Af) \leq K(Lip(f) + \|f\|).$$

(iv) $\lambda(A) < 0$

Then the hypotheses, hence the conclusion, of corollary 5.14 are satisfied.

We begin with the following lemma.

Lemma 5.17 *Under hypotheses (iii) and (iv) of proposition 5.16*

$$Lip(e^{tA}f) \leq e^{Kt}(Lip(f) + K'\|f\|)$$

for some constants K, K'.

Proof : For all x, y

$$|e^{tA}f(x) - e^{tA}f(y)| = \left| \int_0^t [Ae^{sA}f(x) - Ae^{sA}f(y)]ds + f(x) - f(y) \right|$$

$$\leq K \left(\int_0^t \left[Lip(e^{sA}f) + \|e^{sA}f\| \right] ds + Lip(f) \right) d(x, y).$$

Since $\lambda = \lambda(A) < 0$, there exists $C > 0$ such that $\|e^{sA}\| \leq Ce^{-s\lambda}$. Thus

$$Lip(e^{tA}f) \leq K \int_0^t Lip(e^{sA}f)ds + \frac{KC}{\lambda}\|f\| + Lip(f)$$

and the result follows from Gronwal's lemma. **QED**

We now pass to the proof of the proposition. In what follows the constants may change from line to line.

Proof : Set $\mu = \delta_x - \delta_y$ and $f_s = Ce^{sA^*}\mu$ so that

$$\langle \mu, e^{sA}Ce^{sA^*}\mu \rangle = e^{sA}f_s(x) - e^{sA}f_s(y).$$

It follows from hypotheses (ii) and (iv) that

$$Lip(f_s) + \|f_s\| \leq Ke^{-s\lambda}$$

for some positive constants K and a. Therefore, by the preceding lemma,

$$Lip(e^{sA}f_s) \leq Ke^{s\alpha}$$

for some (other) positive constants K, α. Thus

$$\begin{aligned} d_{\mathsf{V}}(x,y)^2 &\leq d(x,y)\int_0^T Lip(e^{sA}f_s)ds + \int_T^\infty (e^{sA}f(x) - e^{sA}f(y))ds \\ &\leq d(x,y)\int_0^T Ke^{s\alpha}ds + 2\int_T^\infty \|e^{sA}f_s\|ds \\ &\leq K\left(d(x,y)e^{\alpha T} + \int_T^\infty e^{-s\lambda}ds\right) \\ &\leq K(d(x,y)e^{\alpha T} + e^{-\lambda T}). \end{aligned}$$

Let $\gamma = \frac{\alpha}{\lambda}$, $\epsilon > 0$, and $T = -\ln(\epsilon)/\lambda$. Then

$$d_{\mathsf{V}}^2(x,y) \leq K(\epsilon^{-\gamma}d(x,y) + \epsilon).$$

Therefore

$$d(x,y) \leq \epsilon^{\gamma+1} \Rightarrow d_{\mathsf{V}}^2(x,y) \leq K\epsilon,$$

so that $N(d, M; \omega_{\mathsf{V}}(\epsilon)) = O(\epsilon^{-2r(\gamma+1)})$ and hypothesis 5.6 holds true. **QED**

Example 5.18 Let

$$Af(x) = \int f(y)k(x,dy)$$

with

$$k(x,dy) = k_0(x,y)\mu(dy) + \sum_{i=1}^n a_i(x)\delta_{b_i(x)}$$

where

(i) μ is a bounded measure on M,

(ii) $k_0(x,y)$ is bounded and uniformly Lipschitz in x,

(iii) $a_i : M \to \mathbb{R}$ and $b_i : M \to M$ are Lipschitz.

Then hypothesis (iii) of proposition 5.16 is verified.

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