

GREQAM

Groupement de Recherche en Economie
Quantitative d'Aix-Marseille - UMR-CNRS 6579
Ecole des Hautes Etudes en Sciences Sociales
Universités d'Aix-Marseille II et III

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DOES DYNAMIC EFFICIENCY RULE OUT SUNSPOT FLUCTUATIONS ?

Jean-Pierre DRUGEON
Carine NOURRY
Alain VENDITTI

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Abstract : The aim of this paper is to discuss the role of the elasticity of capital-labor substitution on the local determinacy properties of the steady state in a two-sector economy with CES technologies and sector-specific externalities.

Keywords : Sector-specific externalities, constant returns, capital-labor substitution, indeterminacy.

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Does dynamic efficiency rule out sunspot fluctuations ?*

By Jean-Pierre DRUGEON, Carine NOURRY and Alain
VENDITTI

We consider a two-sector two-periods overlapping generations model with inelastic labor, consumption in both periods of life and homothetic preferences. We first show that the steady state capital per capita is lower than its Golden-Rule counterpart and any converging competitive equilibrium is dynamically efficient if and only if the share of first period consumption over the wage income is large enough. Second we prove that a slightly stronger restriction on this share ensures the local determinacy of equilibria and thus rules out sunspot fluctuations. Third we show that saddle-point stability holds under mild additional conditions on the technology. Using calibrations based upon well-documented empirical evidences for the US economy, we finally provide new channels for the assessment of the plausibility of dynamic efficiency and local determinacy of competitive equilibria.

Keywords: Two-sector OLG model, dynamic efficiency, local determinacy.

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1 Introduction

Contrary to models with a finite number of infinitely-lived agents in which the competitive equilibrium is dynamically efficient and generically locally determinate, dynamic inefficiency and local indeterminacy arise within models with an infinite number of finitely-lived agents.¹ These properties are central issues in growth and stabilization policies analysis. Indeed, if sunspot fluctuations appear to occur along dynamically inefficient competitive equilibria, policy interventions could be used at the same time to stabilize the economy and to improve welfare.

In two-sector intertemporal equilibrium models, the existence of endogenous fluctuations requires a capital intensive consumption good sector. The input allocations across sectors, which are driven by Rybczinsky and Stolper-Samuelson effects, then generate oscillations of the capital accumulation path.² Within an overlapping generations framework, macroeconomic instability associated with sunspot fluctuations based upon self-fulfilling expectations may also arise depending on the properties of preferences, i.e. the saving rate and the elasticity of intertemporal substitution in consumption.

Dynamic inefficiency is related to the over-accumulation of the capital stock with respect to the Golden-Rule.³ In other words, it is associated

¹See Kehoe *et al.* [18], Shell [26] and Woodford [30].

²See Benhabib and Nishimura [4] for optimal growth models, and Galor [15], Venditti [29] for OLG models.

³The feasibility of dynamic inefficiency in the aggregate OLG model with production has been demonstrated by Diamond [12]. Galor and Ryder [16] have shown that dynamic efficiency occurs if the stationary wage rate is lower than the stationary capital-labor ratio.

with a too large saving rate, and a Pareto-improvement can be achieved by allowing the current generation to devour a portion of the capital stock while leaving the consumption of all future generations intact.

The central question raised in this paper is to know whether a decrease of the saving rate to reach dynamic efficiency may have some consequences on the intertemporal allocations of capital and thus on the occurrence of endogenous fluctuations. We will show that the answer is positive since some conditions related to dynamic efficiency of equilibria allow to exclude the aggregate volatility based upon sunspot equilibria.⁴

We consider a formulation of the two-sector OLG model based upon generic sectoral technologies that are summarized by a social production function which characterizes the factor-price frontier associated with interior temporary equilibria.⁵ We also assume a life-cycle utility function which is linearly homogeneous with respect to young and old consumptions so that the propensity to consume, or equivalently the share of first period consumption over the wage income, only depends on the gross rate of return on financial assets. Building on this property, we follow the same procedure as in Lloyd-Braga *et al.* [21] and use a scaling parameter to give simple conditions on the propensity to consume for the existence of a normalized steady state which remains invariant while the elasticity of intertemporal substitution in consumption is varied.

⁴In an aggregate Diamond [13] model, Tirole [28] shows that macroeconomic instability based upon asset bubbles is ruled out if the competitive equilibrium is dynamically efficient.

⁵Such a formulation is the standard way to analyze multisector optimal growth models (see Burmeister *et al.* [7]).

In a first step we focus on the dynamic efficiency properties of equilibria. We show that the normalized steady state is lower than the Golden-Rule capital stock if and only if the share of first period consumption over the wage income is large enough.⁶ We finally prove that under this condition, any competitive equilibrium converging to the normalized steady state is dynamically efficient. Dynamic efficiency in the two-sector OLG model has not been studied until the recent contribution of Cremers [10]. She shows, as in the current paper, that dynamic efficiency is associated with a stationary capital stock lower than the Golden-Rule level. However, her results are based upon unnecessary and restrictive assumptions implying a globally unique monotonically converging perfect foresight equilibrium. Moreover, she does not provide any conditions on the fundamentals that ensure the existence of a steady state which is exceeded by the Golden-Rule. In opposition to her, we provide a complete and general analysis. In particular our dynamic efficiency condition applies even if the convergence process of the competitive equilibrium is non-monotonic.

In a second step we focus on the local determinacy properties of the normalized steady state. The analysis is performed under the standard assumption of gross substitution in consumption. We also assume that the consumption good is capital intensive. We show indeed that, when the stationary capital-labor ratio is lower than the Golden-Rule level, this restriction is a necessary condition for the occurrence of local indeterminacy. On this

⁶In opposition to this result, it is worth noticing that within infinite-horizon models, dynamic efficiency of the equilibrium is obtained for any value of the propensity to consume.

basis, we first prove that the normalized steady state is locally determinate if the share of first period consumption over the wage income is larger than 50%. Such a condition appears to be a consequence of dynamic efficiency when the share of capital in total income is less than $1/3$ while it implies dynamic efficiency when the share of capital is larger than $1/3$. We thus provide a sufficient condition for dynamic efficiency that also rules out sunspot fluctuations. Second, we prove that saddle-point stability of the normalized steady state holds under mild additional conditions on the elasticity of the rental rate of capital, *i.e.*, slight restrictions on the sectoral elasticities of capital-labor substitution.

We finally consider calibrations of the main parameters based upon well-documented empirical evidences for the US. Using national accounting data over the last three decades, we refer to Baxter [2] and Takahashi *et al.* [27] to show that a two-sector representation of the U.S. economy is characterized by a capital-intensive aggregate consumption good sector over the whole period. Using the Penn World Data set, we derive the average over the last fifty years of the annual ratio of consumption expenditures over GDP. We also refer to Duffy and Papageorgiou [14] to show that capital and labor have an elasticity of substitution significantly greater than unity. On this basis, we present numerical simulations to show that dynamic efficiency holds and the NSS is saddle-point stable for any value of the elasticity of intertemporal substitution in consumption. We thus provide new channels for the assessment of the plausibility of dynamic efficiency and local determinacy of competitive equilibria.⁷

⁷Abel *et al.* [1] use a stochastic version of the aggregate Diamond [12] model to assess

This paper is organized as follows: The next section sets up the basic model. In Section 3 we prove the existence of a normalized steady state and we give conditions for dynamic efficiency of the intertemporal competitive equilibrium. Section 4 provides the characteristic polynomial and presents the geometrical method used for the local stability analysis. In Section 5 we present our main results on the joint occurrence of dynamic efficiency and local determinacy. Section 6 discusses the empirical plausibility of our conditions while Section 7 contains some concluding comments. All the proofs are given in a final Appendix.

2 The model

2.1 Production

The technological side is formalized as in standard optimal growth models. There are two produced goods, one consumption good y_0 and one capital good y . The consumption good cannot be used as capital so it is entirely consumed, and the capital good cannot be consumed. There are two inputs, capital and labor. We assume complete depreciation of capital within one period and that labor is inelastically supplied. Each good is produced with a standard constant returns to scale technology:

$$y_0 = f^0(k^0, l^0), \quad y = f^1(k^1, l^1)$$

with $k^0 + k^1 \leq k$, k being the total stock of capital, and $l^0 + l^1 \leq \ell$, ℓ being the total amount of labor.

the plausibility of dynamic efficiency in the US economy.

Assumption 1. *Each production function $f^i : \mathbb{R}_+^2 \rightarrow \mathbb{R}_+$, $i = 0, 1$, is C^2 , increasing in each argument, concave, homogeneous of degree one and such that for any $x > 0$, $f_1^i(0, x) = f_2^i(x, 0) = +\infty$, $f_1^i(+\infty, x) = f_2^i(x, +\infty) = 0$.*

Notice that by definition we have $y \leq f^1(k, \ell)$. Assumption 1 then implies that there exists $\bar{k} > 0$ solution of $k - f^1(k, \ell) = 0$ such that $f^1(k, \ell) > k$ when $k < \bar{k}$, while $f^1(k, \ell) < k$ when $k > \bar{k}$. It follows that it is not possible to maintain stocks over \bar{k} . The set of admissible 3-uples (k, y, ℓ) is thus defined as follows

$$\tilde{\mathcal{K}} = \{(k, y, \ell) \in \mathbb{R}_+^3 \mid 0 < \ell, 0 \leq k \leq \bar{k}, 0 \leq y \leq f^1(k, \ell)\} \quad (1)$$

There are two representative firms, one for each sector. For any given (k, y, ℓ) , profit maximization in each representative firm is equivalent to solving the following problem of optimal allocation of productive factors between the two sectors:

$$\begin{aligned} T(k, y, \ell) &= \max_{k^0, k^1, l^0, l^1} f^0(k^0, l^0) \\ \text{s.t. } & y \leq f^1(k^1, l^1) \\ & k^0 + k^1 \leq k \\ & l^0 + l^1 \leq \ell \\ & k^0, k^1, l^0, l^1 \geq 0 \end{aligned} \quad (2)$$

The social production function $T(k, y, \ell)$ describes the frontier of the production possibility set associated with interior temporary equilibria such that $(k, y, \ell) \in \tilde{\mathcal{K}}$, and gives the maximal output of the consumption good. Under Assumption 1, for any $(k, y, \ell) \in \tilde{\mathcal{K}}$, $T(k, y, \ell)$ is homogeneous of degree one, concave and we assume in the following that it is C^2 . Denoting w the wage rate, r the gross rental rate of capital and p the price of investment good, all

in terms of the price of the consumption good, we derive from the envelope theorem that the first derivatives of the social production function give

$$r = T_1(k, y, \ell), \quad p = -T_2(k, y, \ell), \quad w = T_3(k, y, \ell) \quad (3)$$

Building on the homogeneity of $T(k, y, \ell)$, the share of capital in total income is then given by

$$s(k, y, \ell) = \frac{rk}{T(k, y, \ell) + py} \in (0, 1) \quad (4)$$

2.2 Consumption and savings

The economy is populated by finitely-lived agents. In each period t , N_t persons are born, and they live for two periods. In their first period of life (when *young*), the agents are endowed with one unit of labor that they supply inelastically to firms. Their income directly results from the real wage. They allocate this income between current consumption and savings which are invested in the firms. In their second period of life (when *old*), they are retired. Their income is given by the return on the savings made at time t . As they do not care about events occurring after their death, they consume their income entirely. The preferences of a representative agent born at time t are thus defined over his consumption bundle (c_t , when he is young, and d_{t+1} , when he is old) and are summarized by the utility function $u(c_t, d_{t+1}/B)$, with $B > 0$ a scaling parameter.

Assumption 2. $u(c, d/B)$ is C^r over \mathbb{R}_+^2 for r large enough, increasing with respect to each argument ($u_1(c, d/B) > 0$, $u_2(c, d/B) > 0$), concave over \mathbb{R}_{++}^2 and homogeneous of degree one. Moreover, for all $c, d > 0$,

$\lim_{d/cB \rightarrow 0} u_1/u_2 = 0$ and $\lim_{d/cB \rightarrow +\infty} u_1/u_2 = +\infty$, where u_1/u_2 stands for $u_1(1, d/cB)/u_2(1, d/cB)$.⁸

Homogeneity is introduced to write the capital accumulation equation as a function of the share of young agents' consumption over the wage income.⁹

Each agent is assumed to have $1 + n > 0$ children so that population is increasing at constant rate n , *i.e.*, $N_{t+1} = (1 + n)N_t$. Under perfect foresight, and considering w_t and R_{t+1} as given, a young agent maximizes his utility function over his life-cycle as follows:

$$\begin{aligned} \max_{c_t, d_{t+1}, \phi_t} \quad & u(c_t, d_{t+1}/B) \\ \text{s.t.} \quad & w_t = c_t + \phi_t \\ & R_{t+1}\phi_t = d_{t+1} \\ & c_t, d_{t+1}, \phi_t \geq 0 \end{aligned} \tag{5}$$

Assumption 2 implies the existence and uniqueness of interior solutions for optimal saving ϕ_t . Using the homogeneity of $u(c_t, d_{t+1}/B)$, the first order conditions state as:

$$\frac{u_1(1, d_{t+1}/c_t B)}{u_2(1, d_{t+1}/c_t B)} \equiv g(d_{t+1}/c_t B) = R_{t+1}/B \tag{6}$$

$$c_t + \frac{d_{t+1}}{R_{t+1}} = w_t \tag{7}$$

$$\phi_t = w_t - c_t \tag{8}$$

The normality of c_t implies $g'(d/cB) > 0$. It follows that:

$$d_{t+1}/c_t B = g^{-1}(R_{t+1}/B) \equiv h(R_{t+1}/B) \tag{9}$$

⁸These boundary conditions imply the standard Inada conditions.

⁹Since preferences are characterized by some utility function up to a monotone increasing transformation, all our results are compatible with a homothetic utility function.

Combining (6)-(7) with the Euler's identity $cu(1, d/cB) \equiv cu_1(1, d/cB) + (d/B)u_2(1, d/cB)$, while taking into account (9), we then derive:

$$c_t = \frac{u_1(1, h(R_{t+1}/B))}{u(1, h(R_{t+1}/B))} w_t \equiv \alpha(R_{t+1}/B) w_t \quad (10)$$

with $\alpha(R/B) \in (0, 1)$ the propensity to consume of the young, or equivalently the share of first period consumption over the wage income. We also conclude that the first order condition (8) becomes:

$$\phi_t = \phi(w_t, R_{t+1}/B) \equiv (1 - \alpha(R_{t+1}/B)) w_t \quad (11)$$

In the rest of the paper we introduce the following standard Assumption:

Assumption 3. *The consumption levels c_t and d_{t+1} are gross substitute.*

Such a restriction, which is satisfied if and only if the elasticity of intertemporal substitution in consumption $\gamma(R/B)$, as given by:¹⁰

$$\gamma(R/B) = \frac{u_1(1, h(R/B))u_2(1, h(R/B))}{u(1, h(R/B))u_{12}(1, h(R/B))} \quad (12)$$

is larger than 1, implies that the saving function (11) is increasing with respect to the gross rate of return R .

2.3 Perfect-foresight competitive equilibrium

Total labor is given by the number N_t of young households, *i.e.*, $\ell_t = N_t$, and is thus increasing at rate n , *i.e.*, $\ell_{t+1} = (1 + n)\ell_t$. We then define a perfect-foresight competitive equilibrium:

¹⁰Taking the elasticity of (6), and considering the Euler's identity applied to u_1 , *i.e.*, $u_{11}(1, h(R/B)) + u_{12}(1, h(R/B))h(R/B) \equiv 0$, generate $\gamma(R/B)$ as the elasticity of $h(R/B)$.

Definition 1. A sequence $\{k_t^0, k_t^1, l_t^0, l_t^1, k_t, y_t, \ell_t, c_t, d_t, r_t, w_t, p_t\}_{t=0}^\infty$, with $(k_0, \ell_0) = (\hat{k}_0, \hat{\ell}_0)$ given, is a perfect-foresight competitive equilibrium if:

- i) $\{k_t^0, k_t^1, l_t^0, l_t^1\}$ solves (2) given $(k_t, y_t, \ell_t) \in \tilde{\mathcal{K}}$;
- ii) $c_t = \alpha(R_{t+1}/B)w_t$;
- iii) $\ell_t(1 - \alpha(R_{t+1}/B))w_t = p_t y_t$;
- iv) $y_t = k_{t+1}$;
- v) $\ell_{t+1} = (1 + n)\ell_t$;
- vi) $\ell_t[c_t + d_t/(1 + n)] = T(k_t, y_t, \ell_t)$;
- vii) (r_t, w_t, p_t) is given by (3);
- viii) $R_{t+1} = r_{t+1}/p_t$.¹¹

Let us denote $\kappa_t = k_t/\ell_t$ the capital-labor ratio at time $t \geq 0$ and $\bar{\kappa}$, solution of $\kappa - f^1(\kappa, 1) = 0$, the maximal admissible value of κ . The set of admissible paths given by (1) can then be redefined as follows

$$\mathcal{K} = \{(\kappa_t, \kappa_{t+1}) \in \mathbb{R}_+^2 \mid 0 \leq \kappa_t \leq \bar{\kappa}, 0 \leq \kappa_{t+1} \leq f^1(\kappa_t, 1)/(1 + n)\} \quad (13)$$

Since $T(k, y, \ell)$ is linearly homogeneous, we derive from Definition 1 that a perfect-foresight competitive equilibrium satisfies the following difference equation:

$$(1 + n)\kappa_{t+1} + \frac{T_3(\kappa_t, (1+n)\kappa_{t+1}, 1)}{T_2(\kappa_t, (1+n)\kappa_{t+1}, 1)} \left[1 - \alpha \left(-\frac{T_1(\kappa_{t+1}, (1+n)\kappa_{t+2}, 1)}{T_2(\kappa_t, (1+n)\kappa_{t+1}, 1)B} \right) \right] = 0 \quad (14)$$

with $(\kappa_t, \kappa_{t+1}) \in \mathcal{K}$ and $\kappa_0 = \hat{\kappa}_0 = \hat{k}_0/\hat{\ell}_0$ given.

¹¹Starting from the equality *vi*) in Definition 1 and using the budget constraints of the representative agent with the homogeneity of $T(k, y, \ell)$ we get $\ell_{t+1}(w_{t+1} - \phi_{t+1} + R_{t+1}\phi_t) = r_{t+1}k_{t+1} - p_{t+1}y_{t+1} + w_{t+1}\ell_{t+1}$. The result is obtained after obvious simplifications.

3 Steady state and dynamic efficiency

3.1 A normalized steady state

A steady state is defined as $\kappa_t = \kappa^*$, $p_t = p^* = -T_2(\kappa^*, (1+n)\kappa^*, 1)$, $r_t = r^* = T_1(\kappa^*, (1+n)\kappa^*, 1)$, $w_t = w^* = T_3(\kappa^*, (1+n)\kappa^*, 1)$ and $R^* = r^*/p^*$ for all t with κ^* solution of the following equation

$$(1+n)\kappa + \frac{T_3(\kappa, (1+n)\kappa, 1)}{T_2(\kappa, (1+n)\kappa, 1)} \left[1 - \alpha \left(-\frac{T_1(\kappa, (1+n)\kappa, 1)}{T_2(\kappa, (1+n)\kappa, 1)B} \right) \right] = 0 \quad (15)$$

We will consider in the following a family of economies parameterized by the elasticity of intertemporal substitution in consumption $\gamma(R/B)$. The steady state κ^* clearly depends on the characteristics of technologies and preferences. As a result, varying $\gamma(R/B)$ generates modifications of the value of the stationary capital-labor ratio and thus implies variations of all the other shares and elasticities characterizing the technologies and preferences. This property significantly complexifies the local stability and bifurcation analysis. To simplify, we follow the same procedure as in Lloyd-Braga *et al.* [21]: building on the homogeneity property of the utility function, we use the scaling parameter B in order to give conditions for the existence of a normalized steady state $\kappa^* = \theta \in (0, \bar{\kappa})$ which will remain invariant as the elasticity of intertemporal substitution in consumption is varied. Therefore, for a given set of parameters characterizing the technologies and preferences, we will be able to isolate the role of $\gamma(R/B)$ on the local stability properties of competitive equilibria.

Let us denote $z = R/B$. Under Assumption 3, the share of first period consumption $\alpha(z)$ is a monotone decreasing function with $\lim_{z \rightarrow 0} \alpha(z) = \alpha_{sup}$

and $\lim_{z \rightarrow +\infty} \alpha(z) = \alpha_{inf}$. By definition, we have $(\alpha_{inf}, \alpha_{sup}) \subseteq (0, 1)$. Now let us define¹²

$$\Phi_\theta = 1 + (1+n) \frac{\theta T_2(\theta, (1+n)\theta, 1)}{T_3(\theta, (1+n)\theta, 1)} \in (0, 1)$$

By choosing appropriately the sectoral technologies and the value of $\theta \in (0, \bar{\kappa})$, we may derive a corresponding $\Phi_\theta \in (\alpha_{inf}, \alpha_{sup})$. We then get:

Proposition 1. *Under Assumptions 1-3, let $\theta \in (0, \bar{\kappa})$ be such that $\Phi_\theta \in (\alpha_{inf}, \alpha_{sup})$. Then there exists a unique value $B^* > 0$ for the scaling parameter solution of*

$$\alpha \left(-\frac{T_1(\theta, (1+n)\theta, 1)}{T_2(\theta, (1+n)\theta, 1)B} \right) = \Phi_\theta \quad (16)$$

such that $\kappa^* = \theta$ is a steady state if and only if $B = B^*$.

Proof: See Appendix 8.1.

In the rest of the paper we will assume that $B = B^*$ in order to guarantee the existence of one normalized steady state (NSS in the sequel).

Remark 1: When the utility function is CES, *i.e.*, $\gamma(R/B) = \gamma$ is constant, we get $\alpha(z) = 1/[1 + \delta^\gamma z^{\gamma-1}]$ with $\delta \in (0, 1)$ the rate of preference for present consumption. When $\gamma > 1$, we have $(\alpha_{inf}, \alpha_{sup}) = (0, 1)$ and the existence of a NSS is ensured.¹³

¹²From Definition 1, we derive indeed $1 + (1+n)\theta T_2(\theta, (1+n)\theta, 1)/T_3(\theta, (1+n)\theta, 1) = 1 - py/wl = 1 - \phi/w \in (0, 1)$.

¹³See Section 6 and Appendix 8.7 for a complete CES illustration.

3.2 Dynamic efficiency

Our aim is to analyze the dynamic efficiency properties of the equilibrium path around the NSS. Let us evaluate all the shares and elasticities previously defined at the NSS. From (4), (10), (12), and considering $B = B^*$ as given by Proposition 1, let $s = s(\theta, (1+n)\theta, 1)$, $\alpha = \alpha(-T_1(\theta, (1+n)\theta, 1)/T_2(\theta, (1+n)\theta, 1)B^*)$ and $\gamma = \gamma(-T_1(\theta, (1+n)\theta, 1)/T_2(\theta, (1+n)\theta, 1)B^*)$. From Definition 1, (16) and the homogeneity of $T(k, y, \ell)$, considering that $\theta T_2/T_3 = (T_2/T_1)(\theta T_1/T_3) = -s/R(1-s)$, we derive the stationary gross rate of return along the NSS:

$$R^* = \frac{(1+n)s}{(1-\alpha)(1-s)} \quad (17)$$

It is well-known since Diamond [12] that if too much capital is accumulated in the long run, the economy is dynamically inefficient. Such a situation occurs if the population growth factor $1+n$ exceeds the steady state marginal product of capital. Following Phelps [22], it is then said that the capital-labor ratio exceeds the Golden-Rule level. In a two-sector OLG model, the Golden-Rule level of capital-labor ratio, denoted $\hat{\kappa}$, is characterized from the total stationary consumption which is given by the social production function:

$$c + d/(1+n) = T(\kappa, (1+n)\kappa, 1) \quad (18)$$

Along a stationary path of capital stock, the highest utility is defined as the maximum of $u(c, d)$ subject to (18). There is no other restriction than the non-negativity of capital and consumptions, and the maximum of utility implies the maximum of the production of the consumption good $T(\kappa, (1+n)\kappa, 1)$. As in the aggregate Diamond [12] formulation, the Golden-Rule capital-labor ratio $\hat{\kappa}$ is independent of the allocation of total consumption

between young and old:

Proposition 2. *Under Assumptions 1-2, the stationary gross rate of return on financial assets $R(\kappa) = -T_1(\kappa, (1+n)\kappa, 1)/T_2(\kappa, (1+n)\kappa, 1)$ is a decreasing function of κ , i.e., $R'(\kappa) < 0$, and there exists a unique optimal stationary path $(\hat{\kappa}, \hat{c}, \hat{d})$ which is characterized by the following conditions:*

$$R(\hat{\kappa}) \equiv \hat{R} = 1 + n, \quad \hat{c} + \frac{\hat{d}}{1+n} = T(\hat{\kappa}, (1+n)\hat{\kappa}, 1), \quad u_1(\hat{c}, \hat{d}) = (1+n)u_2(\hat{c}, \hat{d})$$

with $\hat{\kappa}$ the Golden-Rule capital-labor ratio.

Proof: See Appendix 8.2.

We now characterize the dynamic efficiency properties of equilibrium paths. Building on Proposition 2, they are appraised through the comparison of the NSS with respect to the Golden-Rule. We introduce the following definitions adapted from Cass [8]:

Definition 2. *A path of capital stocks per capita $\{\kappa_t\}_{t \geq 0}$ is feasible if, for all $t \geq 0$, $(\kappa_t, \kappa_{t+1}) \in \mathcal{K}$ and $T(\kappa_t, (1+n)\kappa_{t+1}, 1) \geq 0$.*

Definition 3. *A feasible path of capital stocks per capita $\{\kappa_t\}_{t \geq 0}$ is inefficient if there exists another feasible path $\{\kappa'_t\}_{t \geq 0}$ such that:*

- i) $\kappa'_0 = \kappa_0$ and $\forall t \geq 0$, $T(\kappa'_t, (1+n)\kappa'_{t+1}, 1) \geq T(\kappa_t, (1+n)\kappa_{t+1}, 1)$;*
- ii) there exists $t \geq 0$ such that $T(\kappa'_t, (1+n)\kappa'_{t+1}, 1) > T(\kappa_t, (1+n)\kappa_{t+1}, 1)$.*

A feasible path is efficient if it is not inefficient.

Considering the stationary gross rate of return as defined by (17), we then derive a condition on the share of first period consumption over the wage income α to get a NSS lower than the Golden-Rule level and to ensure the dynamic efficiency of equilibria:

Proposition 3. *Under Assumptions 1-3, let $\underline{\alpha} = 1 - s/(1 - s)$. Then:*

i) the NSS is characterized by an under-accumulation of capital if and only if $\alpha \geq \underline{\alpha}$, i.e., if and only if $R^ \geq \hat{R} = 1 + n$;*

ii) an intertemporal competitive equilibrium converging towards the NSS is dynamically efficient if $\alpha \in (\underline{\alpha}, 1)$ and dynamically inefficient if $\alpha \in (0, \underline{\alpha})$.¹⁴

Proof: See Appendix 8.3.

Notice from the definition of the bound $\underline{\alpha}$ that if the labor income is relatively lower than the capital income, i.e., $s \geq 1/2$, then a young agent does not have enough wage resources to provide a large amount of savings so that an under-accumulation of capital is obtained without additional restriction. On the contrary, if the labor income is relatively larger than the capital income, i.e., $s < 1/2$, then a young agent receives enough wage resources to be able to provide a large amount of savings. In this case, over-accumulation of capital can be avoided provided his share of first period consumption over the wage income is large enough. Notice also that our condition for an under-accumulation of capital $\alpha \geq \underline{\alpha}$ can be equivalently reformulated as in Phelps [22], namely the aggregate saving rate $\varsigma \equiv (1 - \alpha)w/(w + rk) = (1 - \alpha)(1 - s)$ needs to be lower than the share of capital in total income s , i.e., $\varsigma \leq s$.

The criterion for dynamic efficiency provided in Proposition 3 is first based upon a NSS which is characterized by an under-accumulation of capital, *i.e.*, a large enough amount of first period consumption. Second, it is

¹⁴Dynamic efficiency in the rather exceptional case when the stationary capital-labor ratio just corresponds to the Golden Rule is a difficult problem that has been analyzed in details by Cass [8].

based upon a NSS which is stable. Our model consists in one predetermined variable, the current capital stock, and one forward variable, the next period capital stock. Therefore, stability of the NSS can be understood in two different ways: if the dimension of the stable manifold is equal to one, then the NSS is saddle-point stable. For a given initial capital stock, there exists a unique converging equilibrium path. In such a case, the NSS is locally determinate.¹⁵ If on the contrary the dimension of the stable manifold is equal to two, there exists a continuum of equilibrium paths starting from the same initial capital stock and converging to the NSS. In this case, the NSS is locally indeterminate. The dynamic efficiency property of the NSS will be appraised through these two types of stability. In particular, we will show that a slightly stronger restriction than the dynamic efficiency property allows to rule out the existence of local indeterminacy.

4 Local properties of the normalized steady state

4.1 Characteristic polynomial

In order to derive a tractable formulation for the characteristic polynomial, we introduce the relative capital intensity difference across sectors

$$b \equiv \frac{l^1}{y} \left(\frac{k^1}{l^1} - \frac{k^0}{l^0} \right) \quad (19)$$

¹⁵If the dimension of the stable manifold is equal to zero, the NSS is totally unstable but we still call it locally determinate.

and the elasticity of the rental rate of capital

$$\varepsilon_{rk} = -T_{11}(\theta, (1+n)\theta, 1)\theta/T_1(\theta, (1+n)\theta, 1) \quad (20)$$

evaluated at the NSS. Assuming that $b \neq 0$, let us linearize the difference equation (14) around the NSS:

Lemma 1. *Under Assumptions 1-3, the characteristic polynomial is*

$$\mathcal{P}(\lambda) = \lambda^2 - \lambda\mathcal{T} + \mathcal{D} \quad (21)$$

with

$$\mathcal{D} = \frac{s[(1+n)b\alpha(\gamma-1)+1-\alpha+\alpha(1+n)b]}{(1+n)b(1-\alpha)(1-s)\alpha(\gamma-1)}, \quad \mathcal{T} = \frac{1}{(1+n)b\varepsilon_{rk}\alpha(\gamma-1)} + \frac{1+\mathcal{D}(1+n)^2b^2}{(1+n)b}$$

Proof: See Appendix 8.4.

4.2 Geometrical method

Under Assumptions 1-3, Proposition 1 shows that when the scaling parameter satisfies $B = B^*$, the NSS remains constant as the elasticity of intertemporal substitution γ is made to vary. As in Grandmont *et al.* [17], we will then study the variations of the trace $\mathcal{T}(\gamma)$ and the determinant $\mathcal{D}(\gamma)$ in the $(\mathcal{T}, \mathcal{D})$ plane as γ varies continuously within $(1, +\infty)$. This methodology allows to characterize the local stability of the NSS, as well as the occurrence of local bifurcations. Indeed, from Proposition 1, solving \mathcal{T} and \mathcal{D} with respect to $\alpha(\gamma - 1)$ yields the following linear relationship $\Delta(\mathcal{T})$:

$$\mathcal{D} = \Delta(\mathcal{T}) = \mathcal{S}\mathcal{T} - \frac{\varepsilon_{rk}s[1-\alpha+\alpha(1+n)b]-s(1+n)b}{(1+n)b[(1-\alpha)(1-s)+\varepsilon_{rk}s(1+n)b(1-\alpha+\alpha(1+n)b)]} \quad (22)$$

where the slope \mathcal{S} of $\Delta(\mathcal{T})$ is

$$\mathcal{S} = \frac{\varepsilon_{rk}s[1-\alpha+\alpha(1+n)b]}{(1-\alpha)(1-s)+\varepsilon_{rk}s(1+n)b[1-\alpha+\alpha(1+n)b]}. \quad (23)$$

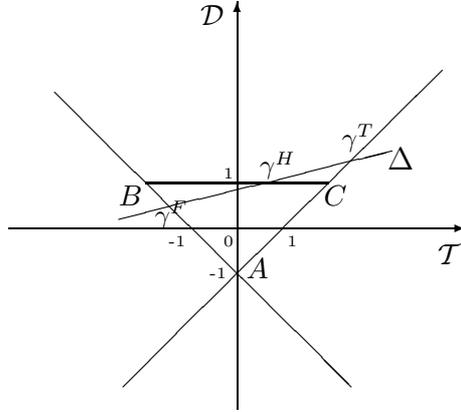


Figure 1: Stability triangle and $\Delta(\mathcal{T})$ line.

As γ spans the interval $(1, +\infty)$, $\mathcal{T}(\gamma)$ and $\mathcal{D}(\gamma)$ vary linearly along the line $\Delta(\mathcal{T})$. Figure 1 provides an illustration of $\Delta(\mathcal{T})$.

We also introduce three other relevant lines: line AC ($\mathcal{D} = \mathcal{T} - 1$) along which one characteristic root is equal to 1, line AB ($\mathcal{D} = -\mathcal{T} - 1$) along which one characteristic root is equal to -1 and segment BC ($\mathcal{D} = 1$, $|\mathcal{T}| < 2$) along which the characteristic roots are complex conjugate with modulus equal to 1. These lines divide the space $(\mathcal{T}, \mathcal{D})$ into three different types of regions according to the number of characteristic roots with modulus less than 1. When $(\mathcal{T}, \mathcal{D})$ belongs to the interior of triangle ABC , the NSS is locally indeterminate. Let γ^F , γ^T and γ^H in $(1, +\infty)$ be the values of γ at which $\Delta(\mathcal{T})$ respectively crosses the lines AB , AC and the segment BC . As γ respectively goes through γ^F , γ^T or γ^H , a flip, transcritical or Hopf bifurcation generically occurs.¹⁶

¹⁶When γ goes through γ^T , one characteristic root crosses 1. If $B = B^*$, the existence of the NSS is always ensured and a saddle-node bifurcation cannot occur. Depending on the number of steady states, the critical value γ^T will be associated with an exchange of

We will show later on that when different values for the shares s , α , b and the elasticity ε_{rk} are considered, the location of the line $\Delta(\mathcal{T})$ on the $(\mathcal{T}, \mathcal{D})$ plane is modified so that all the possible configurations for the stability properties of the NSS will be derived.

Notice from Lemma 1 that the determinant can be formulated as

$$\mathcal{D} = 1 + \frac{s[1-\alpha+\alpha(1+n)b]+(1+n)b\alpha(\gamma-1)(1-s)(\alpha-\underline{\alpha})}{(1+n)b(1-\alpha)(1-s)\alpha(\gamma-1)}$$

Under gross substitutability, *i.e.*, $\gamma > 1$, it follows that when $\alpha > \underline{\alpha}$, the NSS is locally determinate as soon as $b \geq 0$ since $\mathcal{D} > 1$. Therefore, local indeterminacy of the NSS, which is necessarily based on the occurrence of $\mathcal{D} < 1$, requires a capital intensive consumption good, *i.e.*, $b < 0$. In the rest of the paper we will then focus on this configuration and we will also restrict the share of capital in total income in order to get a positive value for the bound $\underline{\alpha} = 1 - s/(1 - s)$:

Assumption 4. $b < 0$ and $s \in (0, 1/2)$.

As $\gamma \in (1, +\infty)$, the fundamental properties of $\Delta(\mathcal{T})$ are characterized from the consideration of its extremities. The starting point of the pair $(\mathcal{T}(\gamma), \mathcal{D}(\gamma))$ is indeed obtained when $\gamma = +\infty$:

$$\begin{aligned} \lim_{\gamma \rightarrow +\infty} \mathcal{D}(\gamma) &= \mathcal{D}_\infty = \frac{s}{(1-\alpha)(1-s)} \\ \lim_{\gamma \rightarrow +\infty} \mathcal{T}(\gamma) &= \mathcal{T}_\infty = \frac{(1-\alpha)(1-s)+(1+n)^2 b^2 s}{(1+n)b(1-\alpha)(1-s)} \end{aligned} \tag{24}$$

stability between the NSS and another (resp. two others) steady state through a transcritical (resp. pitchfork) bifurcation. However, pitchfork bifurcations require some second derivative of the map which defines the dynamical system (14) to be equal to zero. As shown in Ruelle [25], this is a non-generic configuration. In order to simplify the exposition we then concentrate on the generic case and we associate in the rest of the paper the existence of one eigenvalue going through 1 to a transcritical bifurcation.

while the end point is obtained when $\gamma = 1$:

$$\begin{aligned}
\mathcal{D}(1) = \mathcal{D}_1 = \pm\infty &\Leftrightarrow b[1 - \alpha + \alpha(1 + n)b] \gtrless 0 \\
\mathcal{T}(1) = \mathcal{T}_1 = \pm\infty & \\
\Leftrightarrow b[(1 - \alpha)(1 - s) + \varepsilon_{rk}(1 + n)bs[1 - \alpha + \alpha(1 + n)b]] &\gtrless 0
\end{aligned} \tag{25}$$

Moreover, we get

$$\mathcal{D}'(\gamma) = -\frac{s[1 - \alpha + \alpha(1 + n)b]}{(1 + n)b(1 - \alpha)(1 - s)\alpha(\gamma - 1)^2} \tag{26}$$

It follows that $\mathcal{D}'(\gamma) \gtrless 0$ if and only if $\mathcal{D}_1 = \mp\infty$.

The next Lemmas provide a precise characterization of $\Delta(\mathcal{T})$. A first one gives informations on the starting point $(\mathcal{T}_\infty, \mathcal{D}_\infty)$ and $\mathcal{D}'(\gamma)$:

Lemma 2. *Under Assumptions 1-4, let $\alpha > \underline{\alpha}$. For given s , α , b and ε_{rk} , the following results hold:*

- i) $\mathcal{D}_\infty > 1$;
- ii) $\mathcal{D}'(\gamma) > 0$ if and only if $b \in (-(1 - \alpha)/(1 + n)\alpha, 0)$;
- iii) $\mathcal{T}_\infty < 0$;
- iv) $1 + \mathcal{T}_\infty + \mathcal{D}_\infty < 0$ if and only if $b \in (-\infty, -1/(1 + n)) \cup (-(1 - \alpha)(1 - s)/(1 + n)s, 0)$.

Proof: See Appendix 8.5.

Lemma 2 exhibits three critical bounds on b which appear to be crucial for the stability properties of the NSS: $b_0 = -1/(1 + n)$, $b_1 = -(1 - \alpha)/(1 + n)\alpha$ and $b_2 = -(1 - \alpha)(1 - s)/(1 + n)s$. We obtain the following comparisons:

$$\begin{aligned}
b_1 > b_0 &\Leftrightarrow \alpha > 1/2, & b_2 > b_1 &\Leftrightarrow \alpha < s/(1 - s), \\
\underline{\alpha} < 1/2 &\Leftrightarrow s > 1/3, & s/(1 - s) > 1/2 &\Leftrightarrow s > 1/3
\end{aligned} \tag{27}$$

A second Lemma then provides additional informations on the slope \mathcal{S} and on the intersections of $\Delta(\mathcal{T})$ with the lines AB and AC :

Lemma 3. *Under Assumptions 1-4, let $\alpha > \underline{\alpha}$. There exists $\bar{\varepsilon}_{rk} > 0$ such that for given s , α , b and ε_{rk} , the following results hold:*

1 - When $b \in (b_1, 0)$:

a: $\Delta(\mathcal{T}) = 1$ implies $\mathcal{T} < -2$, and if $\varepsilon_{rk} \in (0, \bar{\varepsilon}_{rk})$, $\mathcal{S} \in (0, 1)$;

b: $\Delta(\mathcal{T}) = -1$ implies $\mathcal{T} < 0$ i) when $\alpha > \max\{\underline{\alpha}, 1/2\}$, or ii) when $s \in (1/3, 1/2)$, $\alpha \in (\underline{\alpha}, 1/2)$ and $\varepsilon_{rk} \in (0, \bar{\varepsilon}_{rk})$;

2 - When $b \in (-\infty, b_1)$, $\mathcal{S} \in (-1, 0)$ i) when $\alpha > \max\{\underline{\alpha}, 1/2\}$ and $\varepsilon_{rk} \in (0, \bar{\varepsilon}_{rk})$, or ii) when $s \in (1/3, 1/2)$ and $\alpha \in (\underline{\alpha}, 1/2)$.

Proof: See Appendix 8.6.

To sum up, in graphical terms, the relevant part of $\Delta(\mathcal{T})$ is thus a half-line starting in $(\mathcal{T}_\infty, \mathcal{D}_\infty)$, with $\mathcal{T}_\infty < 0$, $\mathcal{D}_\infty > 1$, and pointing upwards or downwards, to the right or to the left depending on the sign of $\mathcal{D}'(\gamma)$.

5 Dynamic efficiency and local determinacy

If $\alpha > \underline{\alpha}$, $\Delta(\mathcal{T})$ starts within an area in which local determinacy necessarily holds since $\mathcal{D}_\infty > 1$. The possible occurrence of local indeterminacy requires therefore that $\mathcal{D}(\gamma)$ is an increasing function. Let us first introduce as suggested by Lemma 3 a slightly stronger condition on the share α by assuming that $\alpha > \max\{\underline{\alpha}, 1/2\}$. This inequality implies $b_1 > b_0$. While Lemma 2 suggests that local indeterminacy might occur when $b > b_1 = -(1 - \alpha)/(1 + n)\alpha$, we derive from Lemma 3 that $\mathcal{D} = 1$ implies $\mathcal{T} < -2$ (see 1-a) and $\mathcal{D} = -1$

implies $\mathcal{T} < 0$ (see 1-b-i)). As a result, when $\alpha > \max\{\underline{\alpha}, 1/2\}$, $\Delta(\mathcal{T})$ can only cross the line AB when $\mathcal{D}(\gamma) > 1$ or the line AC when $\mathcal{D}(\gamma) < -1$ and local indeterminacy is ruled out.

Proposition 4. *Under Assumptions 1-4, let the NSS be characterized by an under-accumulation of capital with $\alpha > \max\{\underline{\alpha}, 1/2\}$. Then the NSS is locally determinate.*

As already mentioned before, we know that in two-sector models with a gross substitutability assumption, the existence of endogenous fluctuations is fundamentally based upon a capital intensive consumption good sector. The input allocations across sectors, which are driven by Rybczinsky and Stolper-Samuelson effects, indeed generate oscillations of the capital accumulation path which may be large enough to be propagated in the economy through the saving behavior of the agents. However, when $\alpha > \max\{\underline{\alpha}, 1/2\}$, the propensity to save is too low so that the amount of capital accumulated is not sufficient to lead to amplified fluctuations based upon self-fulfilling expectations. It is worth noticing that, as shown by (27), such a mechanism is also based upon a trade-off between the value of the share of capital in total income s and the value of the share of first period consumption in wage income α . Indeed, when $s \leq 1/3$, we get $\underline{\alpha} \geq 1/2$ and the condition of Proposition 4 simply becomes $\alpha > \underline{\alpha}$. On the contrary, when $s \in (1/3, 1/2)$, local indeterminacy is ruled out if $\alpha > 1/2 (> \underline{\alpha})$.

Proposition 4 provides a link between under-accumulation of capital and local determinacy. Moreover, using the results of Proposition 3, dynamic

efficiency also requires stability of the equilibrium path. But the local determinacy property is compatible with a saddle-point stable or a totally unstable NSS. We need therefore to find conditions for the saddle-point stability of the NSS. From Lemmas 2 and 3 we may derive the following geometrical representations when $\alpha > \max\{\underline{\alpha}, 1/2\}$. In Figure 2 we consider first extreme values for b with $\bar{b} = \max\{b_1, b_2\}$ and $b \in (-\infty, b_0) \cup (\bar{b}, 0)$.

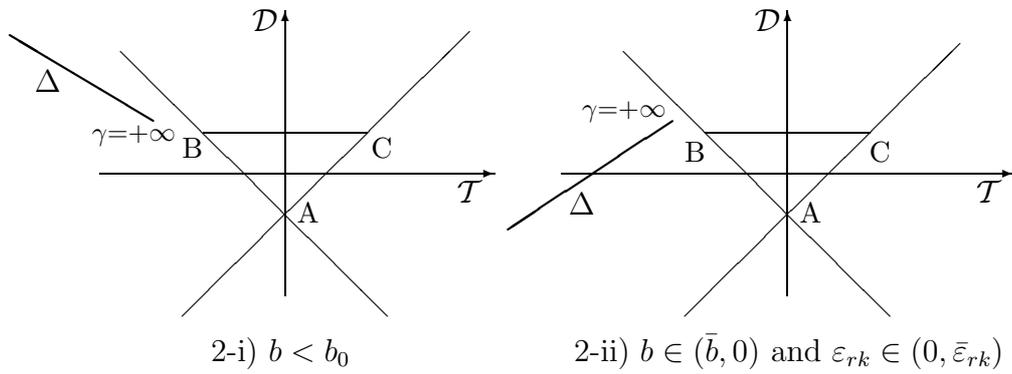


Figure 2: Δ -half-line with $\alpha > \max\{\underline{\alpha}, 1/2\}$.

In both cases, saddle-point stability is obtained for any value of the elasticity of intertemporal substitution in consumption $\gamma > 1$. Notice that for values of b close to zero, a low elasticity of the rental rate of capital allows to rule out the existence of a transcritical bifurcation since $\mathcal{S} \in (0, 1)$.¹⁷

Consider now the case of intermediary values for b . Two different types of configurations may be derived depending on whether b_1 is lower or larger than b_2 . Let us start in Figure 3 with the case $b_1 < b_2$ which, as shown by (27), is obtained when $\alpha < s/(1 - s)$ and $s > 1/3$.

¹⁷It can be shown indeed that when $b < b_0$, $\mathcal{S} \in (-1, 0)$ for any $\varepsilon_{rk} > 0$, but when $b \in (\bar{b}, 0)$, $\mathcal{S} \in (0, 1)$ requires $\varepsilon_{rk} \in (0, \bar{\varepsilon}_{rk})$.

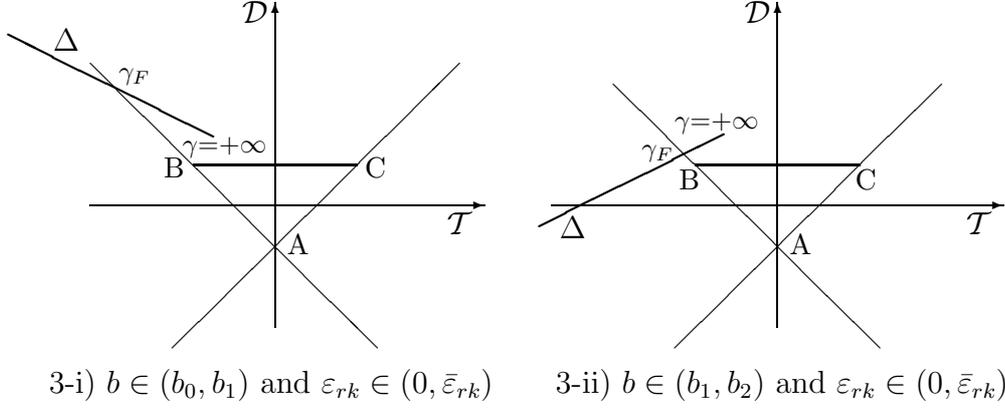


Figure 3: Δ -half-line with $\alpha \in (\max\{\underline{\alpha}, 1/2\}, s/(1-s))$.

In both cases, saddle-point stability is obtained for low enough values of the elasticity of intertemporal substitution in consumption, *i.e.*, $\gamma \in (1, \gamma_F)$, with γ_F a flip bifurcation value. Notice that if $b \in (b_0, b_1)$, a low elasticity of the rental rate of capital is introduced to get $\mathcal{S} \in (-1, 0)$ and thus the possible existence of saddle-point stability, while if $b \in (b_1, b_2)$ it is introduced to rule out the existence of a transcritical bifurcation since $\mathcal{S} \in (0, 1)$.

Let us finally consider the case $b_1 > b_2$ which is obtained when $\alpha > s/(1-s)$. If $b \in (b_0, b_2)$, we get the same picture as in Figure 3-i): a low elasticity of the rental rate of capital is introduced to get $\mathcal{S} \in (-1, 0)$ and thus the existence of saddle-point stability when $\gamma \in (1, \gamma_F)$, with γ_F a flip bifurcation value. On the contrary, if $b \in (b_2, b_1)$, we get the same picture as in Figure 2-i): the restriction $\varepsilon_{rk} \in (0, \bar{\varepsilon}_{rk})$ allows to rule out the existence of a flip bifurcation and to ensure saddle-point stability for any $\gamma > 1$.

All these results are summarized in the following Proposition:

Proposition 5. *Under Assumptions 1-4, let the NSS be characterized by an under-accumulation of capital with $\alpha > \max\{\underline{\alpha}, 1/2\}$ and consider the bound $\bar{b} = \max\{b_1, b_2\}$. The following results hold:*

- 1- *When $b < b_0$, the NSS is saddle-point stable for any $\gamma > 1$;*
- 2- *When $b \in (b_0, \bar{b})$, there exist $\bar{\varepsilon}_{rk} > 0$ and $\gamma_F \in (1, +\infty)$ such that if $\varepsilon_{rk} \in (0, \bar{\varepsilon}_{rk})$, the NSS is saddle-point stable for $\gamma \in (1, \gamma_F)$, undergoes a flip bifurcation at $\gamma = \gamma_F$ and becomes locally unstable for $\gamma > \gamma_F$. Notice that $\gamma_F = +\infty$ and the NSS is saddle-point stable for any $\gamma > 1$ if $b > b_2$.*
- 3- *When $b > \bar{b}$, there exists $\bar{\varepsilon}_{rk} > 0$ such that if $\varepsilon_{rk} \in (0, \bar{\varepsilon}_{rk})$, the NSS is saddle-point stable for any $\gamma > 1$.*

When $\alpha > \max\{\underline{\alpha}, 1/2\}$, mild additional conditions on the elasticity of intertemporal substitution in consumption γ and the elasticity of the rental rate of capital ε_{rk} ensure that the NSS is saddle-point stable and thus that the equilibrium path is locally determinate and dynamically efficient.

Up to now, we have proved that local indeterminacy is ruled out under a slightly stronger condition than dynamic efficiency, *i.e.*, $\alpha > \max\{\underline{\alpha}, 1/2\}$. We will show in Section 6 below that such a condition is empirically plausible when applied to the US economy. However, we have already noticed that if $s \in (1/3, 1/2)$, then $\underline{\alpha} < 1/2$ and an under-accumulation of capital becomes compatible with $\alpha < 1/2$. Such a configuration cannot be a priori rejected by the data.

Consider Lemma 3 with $s \in (1/3, 1/2)$ and $\alpha \in (\underline{\alpha}, 1/2)$. We derive from (27) that $b_1 < b_0 < b_2 < 0$. Using also Lemma 2, notice that Figure 2-i) illustrates the case $b < b_1$ while Figure 2-ii) illustrates the case $b \in$

$(b_1, b_0) \cup (b_2, 0)$, and finally Figure 3-ii) illustrates the case $b \in (b_0, b_2)$. All these results are summarized in the following Proposition:

Proposition 6. *Under Assumptions 1-4, let $s \in (1/3, 1/2)$ and the NSS be characterized by an under-accumulation of capital with $\alpha \in (\underline{\alpha}, 1/2)$. The following results hold:*

- 1- *When $b < b_1$, the NSS is saddle-point stable for any $\gamma > 1$;*
- 2- *When $b \in (b_1, b_0) \cup (b_2, 0)$, there exists $\bar{\varepsilon}_{rk} > 0$ such that if $\varepsilon_{rk} \in (0, \bar{\varepsilon}_{rk})$, the NSS is saddle-point stable for any $\gamma > 1$;*
- 3- *When $b \in (b_0, b_2)$, there exist $\bar{\varepsilon}_{rk} > 0$ and $\gamma_F \in (1, +\infty)$ such that if $\varepsilon_{rk} \in (0, \bar{\varepsilon}_{rk})$, the NSS is saddle-point stable for $\gamma \in (1, \gamma_F)$, undergoes a flip bifurcation at $\gamma = \gamma_F$ and becomes locally unstable for $\gamma > \gamma_F$.*

We thus prove with Proposition 6 that even in the case $\alpha \in (\underline{\alpha}, 1/2)$, local indeterminacy is easily ruled out. We will show in the following Section that once we calibrate the main parameters using empirically plausible values, dynamic efficiency holds and the NSS is saddle-point stable for any $\gamma > 1$.

6 Plausibility of dynamic efficiency and local determinacy

Our aim is now to find whether or not the conditions under which dynamic efficiency and local determinacy hold simultaneously are empirically plausible. As shown in Section 5, the values of different parameters have to be discussed: the share of first period consumption over the wage income α , the capital intensity difference across sectors b , the share of capital in total

income s , the aggregate elasticity of capital-labor substitution Σ through ε_{rk} and the elasticity of intertemporal substitution in consumption γ . We consider an economy with CES preferences and technologies to discuss the plausibility of our results.¹⁸ The calibrations will be based upon the following empirical evidences for the US economy:

- Considering that one period within the overlapping generations model corresponds to 30 years, the rate of growth of young households is computed from the increase of the labor force between 1970 and 2000. Using the data given by the US Department of Labor, we get $n = 0.723$.¹⁹

- Over the period 1950–2000 for the U.S., the annual ratio of consumption expenditures over GDP averages at 67% with a minimum of 63.8% in 1981 and a maximum of 69.5% in 1969. Notice also that over the last decade 1990 – 2000 the average is 68.1%.²⁰

- The share of capital in total income is often calibrated at the standard value $s = 1/3$. But, as shown recently, one may suspect that the US economy is characterized by a larger share of capital.²¹ We will also consider a share

¹⁸All the technical details are provided in Appendix 8.7.

¹⁹As shown at <http://stats.bls.gov/cps/cpsaat1.pdf>, the civilian labor force is equal to 82771 thousands in 1970 and 142583 thousands in 2000.

²⁰These numbers have been computed using the Penn World Data set available at <http://www.bized.ac.uk/dataserv/penndata/pennhome.htm>.

²¹Checchi and Garcia-Penalosa [9] show that over the period 1960-2003, the main OECD countries (US, France, UK and Germany) are characterized by a labour share which belongs to the interval (0.45,0.35). For the US we have more precisely $1 - s \in (0.55, 0.59)$ or equivalently $s \in (0.41, 0.45)$ with a mean $\bar{s} = 0.42$. However, using data that do not take into account the self-employed, they under-estimate the labour share and thus

109), for the richest group of countries. Since we consider a two-sector model, direct comparisons with these estimates are not possible. Denoting σ_i the elasticity of capital-labor substitution in sector $i = 0, 1$, we need therefore to define an aggregate elasticity of substitution between capital and labor. Using Drugeon [13], such an elasticity, denoted Σ , is obtained as a weighted sum of the sectoral elasticities σ_i ,²³ and is closely linked with the elasticity of the interest rate ε_{rk} , namely:

$$\Sigma = \frac{y_0 + py}{pyk_0y_0} (pyk_0l_0\sigma_0 + y_0k_1l_1\sigma_1), \quad \varepsilon_{rk} = (l_0/y_0)^2 \frac{w(y_0+py)}{\Sigma} \quad (28)$$

In the following numerical illustration we will then consider Σ to provide comparisons with the estimates of Duffy and Papageorgiou [14].²⁴

Let us now calibrate the model focussing mainly on the plausible values for s and α . In our model, total consumption over GDP is given at the steady state by $\ell[c + d/(1 + n)]/(T + py) = (1 - s)\alpha + s$.

i) Consider first the standard value $s = 1/3$. We compute $\underline{\alpha} = 1/2$ and the condition to rule out local indeterminacy is the same as the condition for dynamic efficiency. Using sectoral elasticities of capital-labor substitution

²³As shown in Drugeon [13], the weighted sum is defined with the aggregate shares of capital and labor income in national income, the aggregate shares of the consumption good and the investment good in the national product, and the sectoral shares of capital and labor income in total production cost. The expression (28) is obtained after straightforward simplifications.

²⁴Notice that Duffy and Papageorgiou [14] provide a structural estimation of the aggregate production function. Since we consider a two-sector model, their specification then corresponds to a biased estimation of our aggregate elasticity Σ . However, up to our knowledge, there is no available paper in the literature which deals with estimations of production functions in a two-sector framework.

such as $\sigma_0 \approx 1.25$ and $\sigma_1 \approx 1.053$, we get $\alpha = 0.5157$ and a ratio of consumption expenditures over GDP of 67.7% compatible with the previous estimates. We also find $b \approx -0.258$, an aggregate elasticity of capital-labor substitution $\Sigma \approx 1.136$ which belongs to the plausible interval, and $\varepsilon_{rk} \approx 0.496$. As shown by case 3 of Proposition 5 the NSS is saddle-point stable and dynamic efficiency holds for any $\gamma > 1$.

ii) Consider now a lower, but still greater than $1/2$, share of first period consumption $\alpha = 0.501$, with sectoral elasticities $\sigma_0 \approx 1.11$ and $\sigma_1 \approx 1.47$. The corresponding share of capital becomes $s = 0.356$ and $\underline{\alpha} \approx 0.447$. We then find a ratio of consumption expenditures over GDP of 67.8% still compatible with the previous estimates. We also get $b \approx -0.338$, an aggregate elasticity of capital-labor substitution $\Sigma \approx 1.06$ still within the plausible interval, and $\varepsilon_{rk} \approx 0.474$. Again, case 3 of Proposition 5 shows that the NSS is saddle-point stable and dynamic efficiency holds for any $\gamma > 1$.

These two illustrations show that the restriction $\alpha > \max\{\underline{\alpha}, 1/2\}$ appears to be empirically plausible.

iii) Consider finally the case with $\alpha < 1/2$. Using sectoral elasticities of capital-labor substitution such as $\sigma_0 \approx 1.37$ and $\sigma_1 \approx 3.45$, we get $s = 0.38$, $\underline{\alpha} \approx 0.388$ and $\alpha \approx 0.47 \in (\underline{\alpha}, 1/2)$. The corresponding annual ratio of consumption expenditures over GDP is equal to 67.14% and is again fully compatible with the estimates. We then find $b \approx -0.585$, an aggregate elasticity of capital-labor substitution $\Sigma \approx 1.145$ which belongs to the plausible interval, and $\varepsilon_{rk} \approx 0.337$. It follows from case 2 of Proposition 6 that even in this case, the NSS is saddle-point stable and dynamic efficiency holds for any $\gamma > 1$.

We then prove with these three simulations that as soon as the main parameters are calibrated from empirical evidences, dynamic efficiency holds and the NSS is saddle-point stable for any $\gamma > 1$. Notice also from (28) that saddle-point stability is associated with a low elasticity of the rental rate of capital, *i.e.*, large enough sectoral elasticities of capital-labor substitution.²⁵

7 Concluding comments

We have considered a two-sector two-periods overlapping generations model with inelastic labor, consumption in both periods of life and linearly homogeneous preferences. We have first proved the existence of a normalized steady state. We have then shown that the NSS is lower than the Golden-Rule level and any converging competitive equilibrium is dynamically efficient if and only if the share of first period consumption over the wage income is large enough. Finally we have proved that a slightly stronger restriction on this share ensures the local determinacy of equilibria and thus rules out sunspot fluctuations. Using calibrations based upon well-documented empirical evidences for the US economy, we have then provided new channels for the assessment of the plausibility of dynamic efficiency and local determinacy of competitive equilibria.

The main limitation of our approach results from the local dimension of our dynamic efficiency analysis, *i.e.*, in a neighborhood of the NSS. Using the Cass [8] criterion, a global analysis could be considered. However, within a

²⁵Such a conclusion is in line with the contribution of Reichlin [24] (see also Ralf [23]) in which local indeterminacy is obtained with Leontief technologies in both sector.

two-sector framework, the possibility of non-monotonous equilibrium paths and of factor intensity reversal introduce strong difficulties that go beyond the scope of the present paper. These points are then left for future researches.

Panthéon-Sorbonne-Economie, Centre National de la Recherche Scientifique, Paris,

Université de la Méditerranée, GREQAM, Marseille,

and

GREQAM, Centre National de la Recherche Scientifique, Marseille.

8 Appendix

8.1 Proof of Proposition 1

Consider the set \mathcal{K} as defined by (13). Then $\kappa^* = \theta \in (0, \bar{\kappa})$ is a solution of (15) if

$$\alpha \left(-\frac{T_1(\theta, (1+n)\theta, 1)}{T_2(\theta, (1+n)\theta, 1)B} \right) = 1 + (1+n) \frac{\theta T_2(\theta, (1+n)\theta, 1)}{T_3(\theta, (1+n)\theta, 1)} \equiv \Phi_\theta \in (0, 1) \quad (29)$$

From (7), (9), (10), one gets the identity $R/(h(R/B)B) \equiv \alpha(R/B)/(1 - \alpha(R/B))$. Taking elasticities of both sides yields

$$\alpha'(R/B) \frac{R}{\alpha(R/B)B} = (1 - \gamma(R/B))(1 - \alpha(R/B)) \quad (30)$$

Therefore, Assumption 3, which is equivalent to $\gamma(R/B) > 1$, implies that $\alpha(z)$ is a monotone decreasing function with $\lim_{z \rightarrow 0} \alpha(z) = \alpha_{sup}$, $\lim_{z \rightarrow +\infty} \alpha(z) = \alpha_{inf}$ and $(\alpha_{inf}, \alpha_{sup}) \subseteq (0, 1)$. The following Figure provides an illustration of $\alpha(z)$.

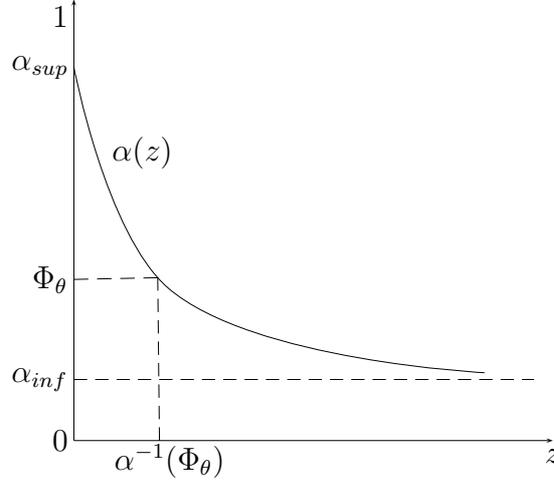


Figure 4: Share of first period consumption $\alpha(z)$.

It follows that $\alpha(z)$ admits an inverse function defined over $(\alpha_{inf}, \alpha_{sup})$. Let $\theta \in (0, \bar{\kappa})$ be such that $\Phi_\theta \in (\alpha_{inf}, \alpha_{sup})$. We then derive

$$B^* = -\frac{T_1(\theta, (1+n)\theta, 1)}{T_2(\theta, (1+n)\theta, 1)\alpha^{-1}(\Phi_\theta)} \quad (31)$$

and $\kappa^* = \theta$ is a steady state if and only if $B = B^*$. *Q.E.D.*

8.2 Proof of Proposition 2

It is shown in Benhabib and Nishimura [4, 5] and Bosi *et al.* [6] that

$$T_{12} = -T_{11}b, \quad T_{22} = T_{11}b^2 < 0, \quad T_{31} = -T_{11}a > 0, \quad T_{32} = T_{11}ab \quad (32)$$

with $a \equiv k^0/l^0 > 0$ the capital-labor ratio in the consumption good sector, b the relative capital intensity difference across sectors as defined by (19) and $T_{11} < 0$. Considering that $R(\kappa) = -T_1(\kappa, (1+n)\kappa, 1)/T_2(\kappa, (1+n)\kappa, 1)$, a straightforward computation then gives:

$$R'(\kappa) = -\frac{T_{11}}{T_2}[1 - (1+n)b](1 - Rb)$$

Linear homogeneity of $T(k, y, \ell)$ implies:

$$a\ell/k = 1 - (1 + n)b > 0 \quad (33)$$

Consider the input coefficients in each sector as defined by

$$a_{00} = l^0/y_0, \quad a_{10} = k^0/y_0, \quad a_{01} = l^1/y, \quad a_{11} = k^1/y$$

Linear homogeneity of $f^1(k_1, l_1)$ gives $wa_{01} + ra_{11} = p$ or equivalently $p(1 - Ra_{11}) = wa_{01} > 0$. From the definition of b given by (19) we finally obtain

$$1 - Rb = \frac{a_{00}(1 - Ra_{11}) + Ra_{10}a_{01}}{a_{00}} > 0$$

and $R'(\kappa) < 0$. Consider now the first order condition for a maximum of the total stationary consumption (18) with respect to κ :

$$-\frac{T_1(\kappa, (1+n)\kappa, 1)}{T_2(\kappa, (1+n)\kappa, 1)} = 1 + n \quad (34)$$

This is equivalent to the equation defining the stationary capital-labor ratio of a two-sector optimal growth model with inelastic labor supply, no discounting and full depreciation of capital. Since $R'(\kappa) < 0$, the proof of Theorem 3.1 in Becker and Tsyganov [3] applies and there exists a unique solution $\hat{\kappa}$ of (34). Along a stationary path of capital stocks, the highest utility is finally defined as the maximum of $u(c, d)$ subject to (18). *Q.E.D.*

8.3 Proof of Proposition 3

From (17), we get $R^* > 1 + n$ if and only if $\alpha > \underline{\alpha}$. The rest of the proof is based upon arguments similar to the ones introduced in Chapter 2 (Proposition 2.4, p. 83) of de la Croix and Michel [11].²⁶

²⁶We provide an extension to the two-sector framework of the dynamic efficiency property derived within an aggregate Diamond model. The whole argument is based upon

Let us start with the case in which the NSS is characterized by an under-accumulation of capital, *i.e.*, $-T_1(\theta, (1+n)\theta, 1)/T_2(\theta, (1+n)\theta, 1) > 1+n$. We will show that increasing consumption for one period t_1 without reducing it at any other period leads to a contradiction. Recall that the ratio of total consumption over labor is given by $\mathcal{C}_t = c_t + d_t/(1+n) = T(\kappa_t, (1+n)\kappa_{t+1}, 1)$. Under-accumulation implies that $-T_1(\theta, (1+n)\theta, 1)/T_2(\theta, (1+n)\theta, 1) > z(1+n)$ for some $z > 1$. Along a converging equilibrium path we have for t large enough, say $t \geq t_0$, $-T_1(\kappa_t, (1+n)\kappa_{t+1}, 1)/T_2(\kappa_t, (1+n)\kappa_{t+1}, 1) > z(1+n)$. At any date t , the difference from another feasible path $\tilde{\kappa}_t$ satisfies

$$\Delta\mathcal{C}_t = \tilde{\mathcal{C}}_t - \mathcal{C}_t = T(\tilde{\kappa}_t, (1+n)\tilde{\kappa}_{t+1}, 1) - T(\kappa_t, (1+n)\kappa_{t+1}, 1)$$

Concavity of T then implies

$$\begin{aligned} \Delta\mathcal{C}_t &\leq T_1(\kappa_t, (1+n)\kappa_{t+1}, 1)(\tilde{\kappa}_t - \kappa_t) \\ &\quad + (1+n)T_2(\kappa_t, (1+n)\kappa_{t+1}, 1)(\tilde{\kappa}_{t+1} - \kappa_{t+1}) \\ \Leftrightarrow \tilde{\kappa}_{t+1} - \kappa_{t+1} &\leq -\frac{T_1(\kappa_t, (1+n)\kappa_{t+1}, 1)}{(1+n)T_2(\kappa_t, (1+n)\kappa_{t+1}, 1)}(\tilde{\kappa}_t - \kappa_t) + \frac{\Delta\mathcal{C}_t}{(1+n)T_2(\kappa_t, (1+n)\kappa_{t+1}, 1)} \end{aligned}$$

Assume therefore that consumption never decreases. It follows that capital never increases. Indeed, by induction, if $\tilde{\kappa}_t - \kappa_t \leq 0$, which obviously holds at $t = 0$, and if $\Delta\mathcal{C}_t \geq 0$ then the previous inequality implies $\tilde{\kappa}_{t+1} - \kappa_{t+1} \leq 0$. Assume moreover that there is some date $t_1 > 0$ such that $\Delta\mathcal{C}_{t_1} > 0$. Then the previous argument implies $\tilde{\kappa}_t - \kappa_t < 0$ for any $t > t_1$ and, for $t > t_2 = \max\{t_0, t_1\}$, we get

$$\tilde{\kappa}_{t+1} - \kappa_{t+1} < z(\tilde{\kappa}_t - \kappa_t)$$

the concavity of the social production function T and does not depend on the sign of the capital intensity difference across sectors.

since $-T_1(\kappa_t, (1+n)\kappa_{t+1}, 1)/T_2(\kappa_t, (1+n)\kappa_{t+1}, 1) > z(1+n)$. It follows that

$$\tilde{\kappa}_{t+1} - \kappa_{t+1} < z^{t-t_2}(\tilde{\kappa}_{t_2} - \kappa_{t_2}) < 0$$

As $z > 1$ and κ_{t+1} converges to the NSS, we have that $\tilde{\kappa}_{t+1} - \kappa_{t+1}$ converges to $-\infty$ and $\tilde{\kappa}_{t+1}$ becomes negative, which is not possible.

Consider now the case in which the NSS is characterized by an over-accumulation of capital, *i.e.*, $-T_1(\theta, (1+n)\theta, 1)/T_2(\theta, (1+n)\theta, 1) < 1+n$. We will show that we can lower the stock of capital and increase consumption at one date without reducing consumption at another date. Consider an equilibrium path converging to the NSS. In a neighborhood $(\theta - 2\epsilon, \theta + 2\epsilon)$ of the NSS we have $-T_1(\kappa, (1+n)\kappa, 1)/T_2(\kappa, (1+n)\kappa, 1) < 1+n$. After some date t_0 , we then have $\kappa_t \in (\theta - \epsilon, \theta + \epsilon)$, $-T_1(\kappa_t, (1+n)\kappa_{t+1}, 1)/T_2(\kappa_t, (1+n)\kappa_{t+1}, 1) < 1+n$ and $-T_1(\kappa_t - \epsilon, (1+n)(\kappa_{t+1} - \epsilon), 1)/T_2(\kappa_t - \epsilon, (1+n)(\kappa_{t+1} - \epsilon), 1) < 1+n$. Let us decrease the stock of capital by ϵ after date t_0 and forever. Concavity of $T(k, y, \ell)$ with respect to y implies:

$$\begin{aligned} T(\kappa_{t_0}, (1+n)(\kappa_{t_0+1} - \epsilon), 1) & - T(\kappa_{t_0}, (1+n)\kappa_{t_0+1}, 1) \\ & \geq -T_2(\kappa_{t_0}, (1+n)(\kappa_{t_0+1} - \epsilon), 1)(1+n)\epsilon \end{aligned}$$

Therefore, investment κ_{t_0+1} is reduced by ϵ and consumption $T(\kappa_{t_0}, (1+n)(\kappa_{t_0+1} - \epsilon), 1)$ is increased by at least $-T_2(\kappa_{t_0}, (1+n)\kappa_{t_0+1}, 1)(1+n)\epsilon$. At $t > t_0$ the new consumption is $T(\kappa_t - \epsilon, (1+n)(\kappa_{t+1} - \epsilon), 1) \equiv \phi(\epsilon)$ with

$$\begin{aligned} \phi'(\epsilon) & = -T_1(\kappa_t - \epsilon, (1+n)(\kappa_{t+1} - \epsilon), 1) \\ & - (1+n)T_2(\kappa_t - \epsilon, (1+n)(\kappa_{t+1} - \epsilon), 1) \end{aligned}$$

Since over-accumulation implies $\phi'(\epsilon) > 0$ we conclude that consumption can be increased for all periods and the path is dynamically inefficient. *Q.E.D.*

8.4 Proof of Lemma 1

Let us define the elasticity of the rental rate of capital

$$\varepsilon_{rk} = -T_{11}(\theta, (1+n)\theta, 1)\theta/T_1(\theta, (1+n)\theta, 1)$$

the elasticity of the price of investment good

$$\varepsilon_{py} = T_{22}(\theta, (1+n)\theta, 1)(1+n)\theta/T_2(\theta, (1+n)\theta, 1)$$

and the elasticity of the wage rate

$$\varepsilon_{wk} = T_{31}(\theta, (1+n)\theta, 1)\theta/T_3(\theta, (1+n)\theta, 1)$$

all evaluated at the NSS. Total differentiation of (14) using these expressions with (3), (4) and (30) evaluated at the NSS gives

$$\mathcal{D} = \frac{(1+n)b\varepsilon_{wk} + \varepsilon_{py}[1 + \alpha(\gamma-1)]}{(1+n)^2 b^2 \varepsilon_{rk} \alpha(\gamma-1)}, \quad \mathcal{T} = \frac{1 + (1+n)b\varepsilon_{wk} + \varepsilon_{py} + \alpha(\gamma-1)(\varepsilon_{rk} + \varepsilon_{py})}{(1+n)b\varepsilon_{rk} \alpha(\gamma-1)} \quad (35)$$

Considering (32) with $T_1\theta/T_3 = s/(1-s)$, $-T_1\theta/T_2 = R^* = s/(1-\alpha)(1-s)$ and the fact that (33) implies $a = [1 - (1+n)b]\theta$ we derive

$$\varepsilon_{py} = \frac{\varepsilon_{rk}(1+n)^2 b^2 s}{(1-\alpha)(1-s)} \quad \text{and} \quad \varepsilon_{wk} = \frac{\varepsilon_{rk}[1 - (1+n)b]s}{(1-s)}$$

Substituting these expressions into (35) gives the result. *Q.E.D.*

8.5 Proof of Lemma 2

i) We get from (17), (24) and Proposition 3 that $\mathcal{D}_\infty = R^*/(1+n) > 1$ iff $\alpha > \underline{\alpha}$.

ii) The result follows from (25) and (26).

iii) The result immediately follows from (24) and Assumption 4.

iv) Obvious computations from (24) give

$$1 + \mathcal{T}_\infty + \mathcal{D}_\infty = [1 + (1 + n)b] \frac{(1-\alpha)(1-s) + (1+n)bs}{(1+n)b(1-\alpha)(1-s)} \quad (36)$$

The result follows from the fact that $-(1 - \alpha)(1 - s)/s > -1$ when $\alpha > \underline{\alpha}$.

Q.E.D.

8.6 Proof of Lemma 3

1 - a: Solving $\mathcal{D} = 1$ in Lemma 1 gives

$$\alpha(1 - \gamma) = \frac{s[1-\alpha+\alpha(1+n)b]}{(1+n)b[\alpha(1-s)-(1-2s)]} \quad (37)$$

Under $\alpha > \underline{\alpha}$, since $\gamma > 1$, (37) can be satisfied only if $b \in (-(1 - \alpha)/(1 + n)\alpha, 0)$. Substituting $\mathcal{D} = 1$ into the expression of \mathcal{T} allows to get

$$\mathcal{T} + 2 = \frac{1}{(1+n)b\varepsilon_{rk}\alpha(\gamma-1)} + \frac{[1+(1+n)b]^2}{b} < 0$$

Moreover, when $b \in (-(1 - \alpha)/(1 + n)\alpha, 0)$, we derive from (23) that $\mathcal{S} \in (0, 1)$ if

$$\varepsilon_{rk} < \frac{(1-\alpha)(1-s)}{[1-(1+n)b]s[1-\alpha+\alpha(1+n)b]} \equiv \varepsilon_{rk}^1$$

b: Solving $\mathcal{D} = -1$ in Lemma 1 gives

$$\alpha(1 - \gamma) = \frac{s[1-\alpha+\alpha(1+n)b]}{(1+n)b[1-\alpha(1-s)]} \quad (38)$$

Since $\gamma > 1$, (38) can be satisfied only if $b \in (-(1 - \alpha)/(1 + n)\alpha, 0)$.

i) If $\alpha > 1/2$ then $-(1 - \alpha)/(1 + n)\alpha > -1/(1 + n)$ and substituting $\mathcal{D} = -1$ into the expression of \mathcal{T} allows to get under $b \in (-(1 - \alpha)/(1 + n)\alpha, 0)$

$$\mathcal{T} = \frac{1}{(1+n)b\varepsilon_{rk}\alpha(\gamma-1)} + \frac{1-(1+n)^2b^2}{(1+n)b} < 0 \quad (39)$$

ii) Now let $s \in (1/3, 1/2)$, so that $\underline{\alpha} < 1/2$, and $\alpha \in (\underline{\alpha}, 1/2)$. It follows that $-(1 - \alpha)/(1 + n)\alpha < -1/(1 + n)$. Substituting (38) into (39) gives

$$\mathcal{T} = \frac{\varepsilon_{rk}s[1-\alpha+\alpha(1+n)b][1-(1+n)^2b^2]-(1+n)b[1-\alpha(1-s)]}{(1+n)b\varepsilon_{rk}s[1-\alpha+\alpha(1+n)b]} \quad (40)$$

Therefore when $b \in (-1/(1 + n), 0)$, $\mathcal{T} < 0$ but when $b \in (-(1 - \alpha)/(1 + n)\alpha, -1/(1 + n))$, $\mathcal{T} < 0$ iff

$$\varepsilon_{rk} < \frac{(1+n)b(1-\alpha+\alpha s)}{[1-(1+n)^2b^2]s[1-\alpha+\alpha(1+n)b]} \equiv \varepsilon_{rk}^2 \quad (41)$$

2 - Let $b < -(1 - \alpha)/(1 + n)\alpha$. (23) implies that $\mathcal{S} \in (-1, 0)$ iff

$$1 + [1 + (1 + n)b]\varepsilon_{rk}\frac{s}{(1-\alpha)(1-s)}[1 - \alpha + \alpha(1 + n)b] > 0$$

i) Recall that under $\alpha > 1/2$ we have $-(1 - \alpha)/(1 + n)\alpha > -1/(1 + n)$. Therefore, when $b \in (-1/(1 + n), -(1 - \alpha)/(1 + n)\alpha)$ the previous inequality is satisfied if

$$\varepsilon_{rk} < -\frac{(1-\alpha)(1-s)}{[1+(1+n)b]s[1-\alpha+\alpha(1+n)b]} \equiv \varepsilon_{rk}^3$$

and when $b < -1/(1 + n)$ it holds for any $\varepsilon_{rk} > 0$.

ii) Assume now that $s \in (1/3, 1/2)$ and $\alpha \in (\underline{\alpha}, 1/2)$. Since $-(1 - \alpha)/(1 + n)\alpha < -1/(1 + n)$ we derive that $\mathcal{S} \in (-1, 0)$ for any $\varepsilon_{rk} > 0$.

The proof is completed by taking $\bar{\varepsilon}_{rk} = \min\{\varepsilon_{rk}^1, \varepsilon_{rk}^2, \varepsilon_{rk}^3\}$. *Q.E.D.*

8.7 A CES example

Let

$$u(c, d) = [c^{1-1/\gamma} + \delta(d/B)^{1-1/\gamma}]^{\gamma/(\gamma-1)}$$

$$f^0(k^0, l^0) = [\alpha_1(k^0)^{-\rho_0} + \alpha_2(l^0)^{-\rho_0}]^{-1/\rho_0}$$

$$f^1(k^1, l^1) = [\beta_1(k^1)^{-\rho_1} + \beta_2(l^1)^{-\rho_1}]^{-1/\rho_1}$$

with $\delta > 0$, $\gamma > 1$, $\alpha_i, \beta_i > 0$, $i = 1, 2$, such that $\alpha_1 + \alpha_2 = \beta_1 + \beta_2 = 1$ and $\rho_i > -1$, $i = 0, 1$. The sectoral elasticities of capital-labor substitution are given by $\sigma_0 = 1/(1 + \rho_0)$ and $\sigma_1 = 1/(1 + \rho_1)$. We get from (9) and (10):

$$d/cB = (\delta R/B)^\gamma, \quad \alpha(R/B) = \frac{1}{1 + \delta^\gamma (R/B)^{\gamma-1}} \quad (42)$$

The Lagrangian associated with the optimization program (2) is

$$\begin{aligned} \mathcal{L} = & [\alpha_1 (k^0)^{-\rho_0} + \alpha_2 (l^0)^{-\rho_0}]^{-1/\rho_0} + p \left[[\beta_1 (k^1)^{-\rho_1} + \beta_2 (l^1)^{-\rho_1}]^{-1/\rho_1} - y \right] \\ & + r[k - k^0 - k^1] + w[\ell - l^0 - l^1] \end{aligned}$$

We derive the first order conditions:

$$\alpha_1 (y_0/k^0)^{1+\rho_0} - r = 0, \quad \alpha_2 (y_0/l^0)^{1+\rho_0} - w = 0 \quad (43)$$

$$p\beta_1 (y/k^1)^{1+\rho_1} - r = 0, \quad p\beta_2 (y/l^1)^{1+\rho_1} - w = 0 \quad (44)$$

Using $k^0 = k - k^1$, $l^0 = \ell - l^1$, $\kappa = k/\ell$, and manipulating (43)-(44) give

$$l^1/\ell = (k^1/\ell) \left(\frac{(k^1/y)^{\rho_1} - \beta_1}{\beta_2} \right)^{-1/\rho_1} \quad (45)$$

$$\frac{\alpha_1 \beta_2}{\alpha_2 \beta_1} = \left(\frac{\kappa - k^1/\ell}{1 - l^1/\ell} \right)^{1+\rho_0} \left(\frac{l^1}{k^1} \right)^{1+\rho_1} \quad (46)$$

$$r = \alpha_1 \left[\alpha_1 + \alpha_2 \left(\frac{\alpha_1 \beta_2}{\alpha_2 \beta_1} \right)^{\frac{\rho_0}{1+\rho_0}} \left(\frac{(k^1/y)^{\rho_1} - \beta_1}{\beta_2} \right)^{\frac{\rho_0(1+\rho_1)}{\rho_1(1+\rho_0)}} \right]^{-\frac{1+\rho_0}{\rho_0}} \quad (47)$$

By solving (45)-(46) with respect to k^1/ℓ and substituting $y/\ell = (1+n)\kappa$, we get the optimal demand function for capital in the investment good sector evaluated at the steady state $k^1/\ell = g(\kappa, (1+n)\kappa, 1)$. Denoting $g = g(\kappa, (1+n)\kappa, 1)$ and $x = \kappa/g$, (45) becomes

$$l^1/\ell = (1+n)\kappa \left(\frac{1 - \beta_1[(1+n)x]^{\rho_1}}{\beta_2} \right)^{-1/\rho_1} \quad (48)$$

Notice that in order to be correctly defined, this labor demand function needs to be such that $1 - \beta_1[(1+n)x]^{\rho_1} > 0$. We immediately derive

$$b = \frac{1 - \left(\frac{\alpha_1 \beta_2}{\alpha_2 \beta_1}\right)^{\frac{1}{1+\rho_0}} \left(\frac{[(1+n)x]^{-\rho_1 - \beta_1}}{\beta_2}\right)^{\frac{\rho_1 - \rho_0}{\rho_1(1+\rho_0)}}}{(1+n)x} \quad (49)$$

Moreover we have from (44)

$$\frac{r}{p} = \beta_1 [(1+n)x]^{1+\rho_1} \quad (50)$$

Combining (43) and (46) we finally get

$$\frac{w}{r} = \frac{\beta_2}{\beta_1} [(1+n)x]^{-(1+\rho_1)} \left(\frac{1 - \beta_1 [(1+n)x]^{\rho_1}}{\beta_2}\right)^{\frac{1+\rho_1}{\rho_1}} \quad (51)$$

Substituting all this into (42) with $R = r/p$ and solving equation (15) with respect to B gives

$$B^* = \left(\frac{(1+n)\kappa \delta^{-\gamma} [\beta_1 [(1+n)x]^{1+\rho_1}]^{1-\gamma}}{\beta_2 \left(\frac{1 - \beta_1 [(1+n)x]^{\rho_1}}{\beta_2}\right)^{\frac{1+\rho_1}{\rho_1}} - (1+n)\kappa} \right)^{\frac{1}{1-\gamma}} \quad (52)$$

Using (49), equation (46) can be written as follows

$$\frac{\alpha_1 \beta_2}{\alpha_2 \beta_1} = \frac{[\kappa(x-1)]^{1+\rho_0} (1+n)^{1+\rho_1} x^{\rho_1 - \rho_0} \left(\frac{1 - \beta_1 [(1+n)x]^{\rho_1}}{\beta_2}\right)^{\frac{\rho_0 - \rho_1}{\rho_1}}}{\left[\left(\frac{1 - \beta_1 [(1+n)x]^{\rho_1}}{\beta_2}\right)^{\frac{1}{\rho_1}} - (1+n)\kappa\right]^{1+\rho_0}} \quad (53)$$

The set \mathcal{K} as defined by (13) requires $\kappa_t \in (0, \bar{\kappa})$ with $\bar{\kappa} = 1$. The numerical simulations are computed according to the following procedure: We choose values for the parameters $\delta, \alpha_i, \beta_i, \rho_i, i = 0, 1$, and the NSS $\kappa = \theta \in (0, 1)$. Then we solve (53) for the corresponding value of x and we derive the normalization constant B from (52) and $g = k^1/\ell = x/\kappa$. Substituting all this into (47)-(51), we compute the corresponding values for $\alpha, b, \Sigma, \varepsilon_{rk}$ and the bounds $\underline{\alpha}, b_0, b_1, b_2, \bar{\varepsilon}_{rk}$.

We calibrate the model on the basis of quarterly data. Since one period in the OLG model corresponds to 30 years, *i.e.*, 120 quarters, the corresponding

rate of time preference is $\delta = 0.99^{120} \approx 0.3$. The three numerical illustrations are obtained with the following set of parameters' values:

i) If $\alpha_1 = 0.4339$, $\beta_1 = 0.15$, $\rho_0 = -0.2$, $\rho_1 = -0.05$, $\theta = 0.165$, we also get $\bar{b} \approx -0.545$ and $\bar{\varepsilon}_{rk} \approx 2.63$.

ii) If $\alpha_1 = 0.444$, $\beta_1 = 0.25$, $\rho_0 = -0.1$, $\rho_1 = -0.32$, $\theta = 0.15$, we also get $\bar{b} \approx -0.524$ and $\bar{\varepsilon}_{rk} \approx 2.75$.

iii) If $\alpha_1 = 0.58$, $\beta_1 = 0.47$, $\rho_0 = -0.27$, $\rho_1 = -0.71$, $\theta = 0.127$, we also get $b_0 \approx -0.58$, $b_1 \approx -0.653$ and $\bar{\varepsilon}_{rk} \approx 7.8$.

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