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Abstract. *In this paper, I prove the existence of a regular eigenform for suitable weights, on fractals with the only assumptions that the boundary cells are separated and the union of the interior cells is connected. This result improves previous results, and works for many of the usually considered finitely ramified fractals.*

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1. Introduction

The subpject of this paper is that of analysis on finitely ramified fractals. The Sierpinski Gasket, the Vicsek Set and the Lindstrøm Snowflake are finitely ramified fractals, while the Sierpinski Carpet is not. The essential reason is that, in the Sierpinski Carpet, some two cells intesect at a segment line and not only at finitely many points. J. Kigami introduced in [3] a general class of finitely ramified fractals, called P.C.F. self-similar sets. The general theory of P.C.F. self-similar sets and many examples can be found in [4]. In this paper, I essentially consider P.C.F. self-similar sets with a mild additional hypothesis − as e.g. considered in [10] − that is, I require that every point in the initial set is a fixed point of one of the contractions defining the fractal. I will only consider connected fractals.

One of the main problems in analysis on finitely ramified fractals is the construction of self-similar Dirichlet forms, i.e. energies, on them, and the basic tool to do this is the construction of a self-similar *discrete* Dirichlet form defined on a finite subset $V^{(0)}$ of the fractal, which is a sort of *boundary* of the fractal, but not in a topological sense. Such Dirichlet forms on $V^{(0)}$ are self-similar in the sense that they are eigenforms, that is, eigenfunctions of a special nonlinear operator Λ_r, depending on a set of positive numbers r_i (called weights) put on the cells, often called renormalization operator. More precisely, I will call r-eigenform an eigenfunction of Λ_r and G-eigenform (short for generalized eigenform) an r-eigenform fo some r. In [5], [9] and [6] criteria for the existence of an eigenforms with prescribed weights are discussed. In particular, in [5], T. Linstrøm proved that there exists an eigenforms on the nested fractals with all weights equal to 1, C. Sabot in [9] proved a rather general criterion, and V. Metz in [6] improved the results in [9].

In this paper I consider, instead, the problem whether on a given fractal there exists a set of weights r such that the operator Λ_r has an eigenform, in other words whether there exist a G-eigenform. In fact, an open problem is whether a G-eigenform exists on every P.C.F.self-similar set. Results about this problem, so far, have been given in [1], in [7] and in [8]. In [1], a method is described that permits to prove the existence of a G-eigenform on fractals with three vertices with some additional, relatively mild, conditions, and on fractals with more than three vertices, but with stronger symmetry assumptions. In [7], the existence of a G-eigenform is proved in the general case of fractals with three vertices. In [8] the existence of a G-eigenform is proved on a relatively general class of fractals, called nicely separated fractals, with an arbitrary number of vertices, and with no symmetries. However, the eigenform turns out to be regular only on a subclass of those fractals, which in any case contains new and nontrivial examples of fractals having a G-eigenform. In [1], the method consists of approximating a collapsed simpler structure in which there is existence and uniqueness of the eigenform, by putting weights tending to infinity on the interior (i.e., containing no vertices) cells. In [7] the existence result follows from a connectedness argument. In [8], the existence of a G-eigenform is proved using an approximation method combined with a fixed point argument. Namely, if we say that a map is stably fixed if it is continuous and maps a suitable nonempty compact and convex set into its interior, we note that a sufficiently close approximation of a stably fixed map has a fixed point. Now, in [8], roughly speaking, Λ_r, or more precisely, a sort of normalization of it, on the given fractal, tends, when some weights tend to infinity (and possibly other weights to 0), to a stably fixed map.

In the present paper, I merely require that the boundary (i.e., containing a vertex) cells are mutually disjoint and that the set of interior cells is connected, and prove that on such fractals there exists a regular G-eigenform. Such a result improves both the result of [1], in that it does not require any sort of symmetry, and that of [8], in that it does not require any technical condition on the intersection of the boundary cells with the interior cells. The method of proof is similar to that in [8], in the sense that here I use the idea (introduced in [1]) of putting weights tending to infinity on the interior cells, and the corresponding Λ_r approximate a stably fixed map on a collapsed structure.

The difference is that in [8] the map is a rather technical form of normalization of Λ_r, and this as a natural normalization of Λ_r (e.g., Λ_r divided by its norm) fails to be a stably fixed map. In the present paper, instead of using the map $E \mapsto \Lambda_r(E)$ with fixed r, I use a map of the form $E \mapsto \Lambda_{r(E)}(E)$, where r is a suitable continuous function. A natural normalization of the map obtained by this simple device is in fact a stably fixed map. The proof in the present paper definitely simplifies that in [8].

2. Notation

In this section, I introduce the notation and the setting. I will use the obvious notation $\mathbb{R}^A$ to denote the linear space of the functions from a set A to $\mathbb{R}$ and for every $t \in \mathbb{R}$, t_A will denote the function in $\mathbb{R}^A$ taking the value t at every point of A. In case $A = \{1, ..., n\}$, I will write $\mathbb{R}^n$ for $\mathbb{R}^A$ as usual, and t_n for t_A. I will use the restriction of an element of $\mathbb{R}^A$ to a subset B of A in its obvious sense, which can also be interpreted as the projection on $\mathbb{R}^B$. Given a set $V = \{P_1, ..., P_N\}$, $N \geq 2$, let $J = \{\{j_1, j_2\} : j_1, j_2 = 1, ..., N, j_1 \neq j_2\}$. Note that $\#J = M$ where $M = \frac{n(n-1)}{2}$. I will denote by $\widetilde{\mathcal{D}}(V)$ or simply $\widetilde{\mathcal{D}}$ the set of the irreducible Dirichlet forms on V, i.e., the set of the functionals E from $\mathbb{R}^V$ into $\mathbb{R}$ of the form

$$E(u) = \sum_{\{j_1,j_2\}\in J} c_{\{j_1,j_2\}}(E)\big(u(P_{j_1}) - u(P_{j_2})\big)^2$$

with $c_{\{j_1,j_2\}}(E) \geq 0$, such that moreover $E(u) = 0$ if and only if u is constant. Every $E \in \widetilde{\mathcal{D}}$ is uniquely determined by its coefficients, namely

$$c_{\{j_1,j_2\}}(E) = \frac{1}{4}\Big(E(\chi_{\{P_{j_1}\}} - \chi_{\{P_{j_2}\}}) - E(\chi_{\{P_{j_1}\}} + \chi_{\{P_{j_2}\}})\Big)$$

χ_A denoting the characteristic function of a set A. Thus, we could identify $E \in \widetilde{\mathcal{D}}$ with the set of its coefficients in $[0,+\infty[^J$. However, as I will also define the set of effective resistances, in order to avoid possible confusion in the interpretation of an element of $[0,+\infty[^J$, I will explicitly define this identification as a map. Let $\tilde{\mathcal{Q}}$ be the set of $q \in [0,+\infty[^J$ such that the graph $\big\{\{j_1,j_2\} : q_{\{j_1,j_2\}} > 0\big\}$ on $\{1,...,N\}$ is connected. Note that $\tilde{\mathcal{Q}} \supseteq]0,+\infty[^J$. Now, we define the bijection $\overline{I}$ from $\tilde{\mathcal{Q}}$ to $\widetilde{\mathcal{D}}$ in the following way:

Given $q \in \tilde{\mathcal{Q}}$ we put $\overline{I}(q)$ to be the element of $\widetilde{\mathcal{D}}$ such that $c_d\big(\overline{I}(q)\big) = q_d$ for every $d \in J$. I now recall the notion of the restriction of an element of $\widetilde{\mathcal{D}}$. Given $E \in \widetilde{\mathcal{D}}$, a nonempty subset V' of V and $u \in \mathbb{R}^{V'}$, I denote by $E_{V'}(u) = \inf\big\{E(v) : v : V \to \mathbb{R}, v = u \text{ on } V'\big\}$. The infimum is in fact a minimum and is attained at a unique function v satisfying $\min u \leq v \leq \max u$. I denote such a function by $H_{V,V',E}(u)$. I recall the following standard facts:

Lemma 2.1 *The function $v := H_{V,V',E}(u)$ is the unique function from V to $\mathbb{R}$ having the following properties:*
i) $v = u$ on V'
ii) $\sum_{j'\neq j} c_{\{j,j'\}}(v(P_j) - v(P_{j'})) = 0$ for every $P_j \in V \setminus V'$.
Also, $H_{V,V',E}$ is linear on $\mathbb{R}^{V'}$, and $H_{V,V',E}(c) = c$ for every constant c. ■

Corollary 2.2. *The maps $(q,u) \mapsto H_{V,V',\overline{I}(q)}(u)$ and $(q,u) \mapsto \overline{I}(q)_{V'}(u)$ are continuous.* ■

Given $q \in \tilde{\mathcal{Q}}$, we recall that the *effective resistance* $\hat{R}(q)_{\{A,B\}}$ for $A, B \subseteq V$, $A \cap B = \emptyset$, $A \neq \emptyset$, $B \neq \emptyset$, is defined as $\frac{1}{\overline{I}(q)_{A\cup B}(\chi_B)}$. Put also

$$\overline{R}(q)_{\{j_1,j_2\}} = \hat{R}(q)_{\{\{P_{j_1}\},\{P_{j_2}\}\}} = \frac{1}{\inf\big\{\overline{I}(q)(v) |\ v : V \to \mathbb{R}, v(P_{j_1}) = 0, v(P_{j_2}) = 1\ \big\}}$$

when $\{j_1,j_2\} \in J$. This is the most usual case (effective resistance between two points). It is known that the map

$$\overline{R} : \tilde{\mathcal{Q}} \to]0,+\infty[^J$$

is one-to-one (by [4] , Theorem 2.1.12) and continuous, thus its restriction to $]0,+\infty[^J$ is open, and the inverse map from the image $\widehat{U} := \overline{R}\big(]0,+\infty[^J\big)$ onto $]0,+\infty[^J$ is continuous.

Moreover, since $\overline{R}(1_J)$ is clearly constant, using a homogeneity argument we see that $1_J \in \widehat{U}$. Let $\delta_1 \in]0,1[$ be such that

$$[1-\delta_1, 1+\delta_1]^J \subseteq \widehat{U}\,.$$

Now, given $R \in \widehat{U}$ we can naturally extend R to $\{A,B\}$ such that $A, B \subseteq V$, $A \cap B = \emptyset$, $A \neq \emptyset$, $B \neq \emptyset$, putting $R_{A,B} = \hat{R}\big(\overline{R}^{-1}(R)\big)_{\{A,B\}}$. By the previous considerations we have

Corollary 2.3. *the map $R \mapsto R_{A,B}$ from $\widehat{U}$ to $\mathbb{R}$ is continuous.* ∎

I will now define the fractal setting, which is based on that in [7]. This kind of approach was firstly given in [2]. We define a fractal by giving a *fractal triple*, i.e., a triple $(V^{(0)}, V^{(1)}, \Psi)$ where $V^{(0)}$ and $V^{(1)}$ are finite sets with $\#V^{(0)} \geq 2$, and Ψ is a finite set of one-to-one maps from $V^{(0)}$ into $V^{(1)}$ satisfying

$$V^{(1)} = \bigcup_{\psi \in \Psi} \psi(V^{(0)})\,. \tag{2.1}$$

We put $V^{(0)} = \{P_1, ..., P_N\}$, and of course $N \geq 2$. A set of the form $\psi(V^{(0)})$ with $\psi \in \Psi$ will be called a cell or a 1-cell. We require that

a) For each $j = 1, ..., N$ there exists a (unique) map $\psi_j \in \Psi$ such that $\psi_j(P_j) = P_j$, and $\Psi = \{\psi_1, ..., \psi_k\}$, with $k \geq N$.

b) $P_j \notin \psi_i(V^{(0)})$ when $i \neq j$.

c) Any two points in $V^{(1)}$ can be connected by a path whose any edge belongs to a 1-cell, depending of the edge.

Of course, it immediately follows $V^{(0)} \subseteq V^{(1)}$. Let $W =]0, +\infty[^k$. We put $\widetilde{\mathcal{D}} = \widetilde{\mathcal{D}}(V^{(0)})$, and in general we will consider the previous setting with $V = V^{(0)}$. Put $V_i = \psi_i(V^{(0)})$ for each $i = 1, ..., k$.

Next, I recall the definition of the renormalization operator Λ_r. For every $u \in \mathbb{R}^{V^{(0)}}$, every $E \in \widetilde{\mathcal{D}}$ and every $r \in W$, let

$$\Lambda_r(E)(u) = \inf\Big\{S_r(E,v) : v \in \mathbb{R}^{V^{(1)}}, v = u \text{ on } V^{(0)}\Big\},$$

$$S_r(E,v) := \sum_{i=1}^{k} r_i E(v \circ \psi_i)\,.$$

It is well known that the infimum is attained at a unique function. In the next section, I will introduce the class of fractals considered in this paper, where I will prove the existence of a G-eigenform in $\widetilde{\mathcal{D}}$, in other words, the existence of $E \in \widetilde{\mathcal{D}}$ such that $\Lambda_r(E) = \rho E$ for some $\rho > 0$ and $r \in W$. Define $\Lambda^* : \widehat{U} \times W \to \widehat{U}$ by

$$\Lambda^*(R,r)(= \Lambda^*_r(R)) = \tilde{R}\big(\Lambda_r \tilde{R}^{-1}(R)\big)\,,$$

where, $\tilde{R} = \overline{R} \circ \overline{I}^{-1}$. Since, by a standard argument, the map $(q, r) \mapsto \overline{I}^{-1}\Big(\Lambda_r(\overline{I}(q))\Big)$ from $\tilde{\mathcal{Q}} \times W \to \tilde{\mathcal{Q}}$ is continuous, it follows that Λ^* is continuous. Note that in the previous considerations, in particular in order to state that in fact Λ^* takes values in $\widehat{U}$, I implicitly use the well-known fact that Λ_r maps $\overline{I}(]0, +\infty[^J)$ into itself. In the following, as done in [1], I will sometimes permit that the weights can also assume the value $+\infty$. More precisely, suppose $\{1, ..., N\} \subseteq A \subseteq \{1, ..., k\}$, $s \in]0, +\infty[^A$ and $t \in]0, +\infty]$. Then, we define $\iota_t(s) \in W$ by

$$\iota_t(s)_i = \begin{cases} s_i & \text{if } i \in A \\ t & \text{if } i \notin A \,, \end{cases}$$

and put $\iota(s) := \iota_{+\infty}(s)$. Define

$$\Lambda_{s,A}(E)(u) := \Lambda_{\iota(s)}(E)(u) =$$

$$\inf\Big\{\sum_{i\in A} s_i E(v \circ \psi_i) : v \in \mathbb{R}^{V^{(1)}}, v = u \text{ on } V^{(0)},\ v \text{ constant on } V_i\ \forall\, i \notin A\Big\}$$

where we use the convention $0 \times (+\infty) = 0$. Note that, if the set $\{V_i : i \notin A\}$ is connected, then the condition v constant on $V_i\ \forall\, i \notin A$ amounts to v constant on $\bigcup\limits_{i\notin A} V_i$. Note also that the infimum is in fact a minimum, attained at a function v satisfying $\min u \le v \le \max u$. It is easily seen that $\Lambda_{s,A}(E) \in \widetilde{\mathcal{D}}$. Note that, putting

$$K_{j_1,j_2} := \big\{v \in \mathbb{R}^{V^{(1)}} : 0 \le v \le 1, v(P_{j_1}) = 0, v(P_{j_2}) = 1\big\}\,,$$

and $E = \overline{I}\big(\overline{R}^{-1}(R)\big)$, we have

$$\Lambda^*(R, \iota_t(s))_{\{j_1,j_2\}} = \Big(\inf\Big\{S_{\iota_t(s)}(E, v) : v \in K_{j_1,j_2}\Big\}\Big)^{-1}.$$

Put now

$$\Lambda^*(R, \iota(s))_{\{j_1,j_2\}} := \Big(\inf\Big\{S_{\iota(s)}(E, v) : v \in K_{j_1,j_2}\Big\}\Big)^{-1},$$

and note that

$$\Lambda^*(R, \iota(s))_{\{j_1,j_2\}} = \Big(\inf\Big\{\sum_{i\in A} s_i E(v \circ \psi_i) : v \in K_{j_1,j_2},\ v \text{ constant on } V_i\ \forall\, i \notin A\Big\}\Big)^{-1}. \tag{2.2}$$

Lemma 2.4. *For every $\{j_1, j_2\} \in J$ we have*

$$\Lambda^*(R, \iota_t(s))_{\{j_1,j_2\}} \underset{t\to+\infty}{\longrightarrow} \Lambda^*(R, \iota(s))_{\{j_1,j_2\}}$$

uniformly for $(R, s) \in K$ for every K compact subset of $\widehat{U} \times]0, +\infty[^A$.

Proof. A similar statement is proved in [1], Section 5. So, I will only sketch the proof. Using the previous formulas as well as the compactness of K_{j_1,j_2}, it is simple to prove that $\Lambda^*(R,\iota_t(s))_{\{j_1,j_2\}}$ is continuous with respect to $(R,s)\in\widehat{U}\times]0,+\infty[^A$ for $t\in]0,+\infty]$. Moreover

$$\Lambda^*(R,\iota_t(s))_{\{j_1,j_2\}} \underset{t\to+\infty}{\longrightarrow} \Lambda^*(R,\iota(s))_{\{j_1,j_2\}}$$

and the function on the left-hand side is monotonically decreasing in t, thus using the Dini Theorem, the convergence is uniform on K. ∎

3. Fractals with connected interior.

We will say that the fractal triple is *a connected interior fractal triple* if
i) $V_{j_1}\cap V_{j_2}=\emptyset$ if $j_1,j_2=1,...,N$, $j_1\neq j_2$,
ii) the set $\{V_i : i=N+1,...,k\}$ is connected.

We will assume that i) and ii) hold in all this section. As a consequence, defining

$$\widetilde{V} := \bigcup_{i=N+1,\ldots,k} V_i,$$

and, for $i=1,..,N$,

$$\phi(i) := \big\{P_j : j=1,...,N, \psi_i(P_j)\in\widetilde{V}\big\},$$

we have that $\tilde{V}\neq\emptyset$, and $P_i\notin\phi(i)\neq\emptyset$, for every $i=1,...,N$.

Note that this class of fractals generalizes both that considered in [1], and that considered in [8] when the eigenform is regular. I will prove that on these fractals there exists a G-eigenform. To this aim, I will evaluate $\Lambda^*(R,\iota(s))_{\{j_1,j_2\}}$ when $A=\{1,...,N\}$. Namely,

Lemma 3.1. *Let $(R,s)\in\widehat{U}\times]0,+\infty[^N$. Then, for every $\{j_1,j_2\}\in J$ we have*

$$\Lambda^*(R,\iota(s))_{\{j_1,j_2\}} = \frac{R_{\{P_{j_1},\phi(j_1)\}}}{s_{j_1}} + \frac{R_{\{P_{j_2},\phi(j_2)\}}}{s_{j_2}}\,.$$

Proof (Sketch). Let R and s be as above, and let $E=\overline{I}(\overline{R}^{-1}(R))$. Then,

$$\frac{1}{\Lambda^*(R,\iota(s))_{\{j_1,j_2\}}} = \inf\big\{s_{j_1}E(v\circ\psi_{j_1})+s_{j_2}E(v\circ\psi_{j_2}) : v\in K_{j_1,j_2}, v \text{ constant on } \widetilde{V}\big\}$$

$$= \inf\left\{\frac{s_{j_1}z^2}{R_{\{P_{j_1},\phi(j_1)\}}}+\frac{s_{j_2}(1-z)^2}{R_{\{P_{j_2},\phi(j_2)\}}} : z\in[0,1]\right\},$$

for every $\{j_1,j_2\}\in J$, where we use formula (2.2) and ii) in the definition of connected interior fractal triple in the first equality, and i) in the second equality. We conclude using a simple calculation (resistances in series). ∎

Theorem 3.2 *There exist $R\in\widehat{U}$, $s\in]0,+\infty[^N$ and $t>0$ such that $\Lambda^*(R,\iota_t(s))=R$.*

Proof. For $R \in \widehat{U}$, define $\sigma(R)_j := 2R_{\{\{P_j\},\phi(j)\}}$ for every $j = 1, ..., N$. Then,

$$\Lambda^*_\infty(R, \sigma(R))_{\{j_1,j_2\}} = 1\,.$$

Moreover, $\sigma : \widehat{U} \to]0, +\infty[^N$ is continuous by Corollary 2.3. Using Lemma 2.4 and Lemma 3.1, we thus see that, for sufficiently large t, the map $R \mapsto \Lambda^*\Big(R, \iota_t\big(\sigma(R)\big)\Big)$ maps $[1-\delta_1, 1+\delta_1]^J$ into itself. As it is continuous, it has a fixed point. ∎

Corollary 3.3 There exist $E \in \widetilde{\mathcal{D}}$ and $r \in W$ such that $\Lambda_r(E) = E$.

Proof. It suffices to take $E = \overline{R}^{-1}(R)$, where R is as in Theorem 3.2 and $r = \iota_t(s)$. ∎

Remark 3.4 It follows that the eigenform obtained as above is regular (this means that $r_i > \rho$ for every $i = 1, ..., k$) as, by a known and standard argument, this is always true when $r_i \le r_{i'}$ when $i \le N$, $i' > N$. See for example [4] for the notion of regular eigenform. ∎

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