Eigencones and the PRV conjecture

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November 3, 2009

Abstract

Let G be a complex semisimple simply connected algebraic group. Given two irreducible representations V_1 and V_2 of G, we are interested in some components of $V_1 \otimes V_2$. Consider two geometric realizations of V_1 and V_2 using the Borel-Weil-Bott theorem. Namely, for $i = 1, 2$, let $\mathcal{L}_i$ be a G-linearized line bundle on G/B such that $\mathrm{H}^{q_i}(G/B, \mathcal{L}_i)$ is isomorphic to V_i. Assume that the cup product

$$\mathrm{H}^{q_1}(G/B, \mathcal{L}_1) \otimes \mathrm{H}^{q_2}(G/B, \mathcal{L}_2) \longrightarrow \mathrm{H}^{q_1+q_2}(G/B, \mathcal{L}_1 \otimes \mathcal{L}_2)$$

is non zero. Then, $\mathrm{H}^{q_1+q_2}(G/B, \mathcal{L}_1 \otimes \mathcal{L}_2)$ is an irreducible component of $V_1 \otimes V_2$; such a component is said to be *cohomological*. Solving a Dimitrov-Roth conjecture, we prove here that the cohomological components of $V_1 \otimes V_2$ are exactly the PRV components of stable multiplicity one. Note that Dimitrov-Roth already obtained some particular cases. We also characterize these components in terms of the geometry of the Eigencone of G. Along the way, we prove that the structure coefficients of the Belkale-Kumar product on $\mathrm{H}^*(G/B, \mathbb{Z})$ in the Schubert basis are zero or one.

1 Introduction

Let G be a complex semisimple simply connected algebraic group with a fixed Borel subgroup B and maximal torus $T \subset B$. Let $X(T)$ denote the character group of T. For any dominant $\lambda \in X(T)$, V_λ denotes the irreducible G-module of highest weight λ. We will denote by $\mathrm{LR}(G)$ the set of triples (λ, μ, ν) of dominant weights such that $V_\lambda \otimes V_\mu \otimes V_\nu$ contains non zero G-invariant vectors. Note that, (λ, μ, ν) belongs to $\mathrm{LR}(G)$ if and only if V_ν^* is a submodule of $V_\lambda \otimes V_\mu$.

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Let W denote the Weyl group of T and w_0 denote the longest element of W. The most obvious component of $V_\lambda \otimes V_\mu$ is $V_{\lambda+\mu}$ corresponding to the point $(\lambda,\, \mu,\, -w_0(\lambda+\mu))$ in $\mathrm{LR}(G)$. Following Dimitrov-Roth, we present three natural generalizations of these elements of $\mathrm{LR}(G)$. Our main result which was conjectured and partially proved by Dimitrov-Roth in [DR09b, DR09a] asserts that these three generalizations actually coincide.

The PRV conjecture. Let $(\lambda,\, \mu,\, \nu)$ be a triple of dominant weights. In 1966, Parthasarathy, Ranga-Rao and Varadarajan proved in [PRRV67] that if there exists $w \in W$ such that $w\lambda + ww_0\mu + w_0\nu = 0$ then $(\lambda,\, \mu,\, \nu) \in \mathrm{LR}(G)$; and more precisely that $(V_{k\lambda} \otimes V_{k\mu} \otimes V_{k\nu})^G$ has dimension one for any positive integer k (here, V^G denotes the subspace of G-invariant vectors in the G-module V). Kumar [Kum89] and Mathieu [Mat89] independently proved the PRV conjecture which asserts that $(\lambda,\, \mu,\, \nu) \in \mathrm{LR}(G)$ if there exist $u, v, w \in W$ such that $u\lambda + v\mu + w\nu = 0$. Unlike the original PRV situation, $(V_\lambda \otimes V_\mu \otimes V_\nu)^G$ may have dimension greater than one. Here, we are interested in the set of triple of dominant weights $(\lambda,\, \mu,\, \nu)$ such that:

(i) $\exists u, v, w \in W$ s.t. $u\lambda + v\mu + w\nu = 0$; and,

(ii) $\dim(V_{k\lambda} \otimes V_{k\mu} \otimes V_{k\nu})^G = 1$ for any $k \geq 1$.

Such a point in $\mathrm{LR}(G)$ is said to have the PRV property (Property (i)) and to have stable multiplicity one (Property (ii)).

Cohomological component of $V_\lambda \otimes V_\nu$. Consider the complete flag variety $X = G/B$. For $\lambda \in X(T)$, we denote by $\mathcal{L}_\lambda$ the G-linearized line bundle on X such that B acts on the fiber over B/B by the character $-\lambda$. If λ is dominant the Borel-Weil theorem asserts that $\mathrm{H}^0(X, \mathcal{L}_\lambda)$ is isomorphic to V_λ^*. We also set $\lambda^* = -w_0\lambda$. The points $(\lambda,\, \mu,\, -w_0(\lambda+\mu))$ of $LR(G)$ have the following geometric interpretation: the morphism

$$\mathrm{H}^0(X, \mathcal{L}_\lambda) \otimes \mathrm{H}^0(X, \mathcal{L}_\mu) \longrightarrow \mathrm{H}^0(X, \mathcal{L}_{\lambda+\mu}), \tag{1}$$

given by the product of sections is non zero.

Following Dimitrov-Roth (see [DR09b, DR09a]), we are now going to introduce a natural generalization of these points of $\mathrm{LR}(G)$ coming from the Borel-Weil-Bott theorem. Let $l(w)$ denote the length of $w \in W$ and ρ denote the half sum of the positive roots. For $w \in W$ and $\lambda \in X(T)$, we set:

$$w \cdot \lambda = w(\lambda + \rho) - \rho. \tag{2}$$

The Borel-Weil-Bott theorem asserts that for any dominant weight λ and any $w \in W$, $\mathrm{H}^{l(w)}(X, \mathcal{L}_{w\cdot\lambda})$ is isomorphic to V_λ^*. Let $(\lambda,\, \mu,\, \nu)$ be a triple of dominant weights. We will say that $(\lambda,\, \mu,\, \nu^*)$ is a *cohomological point* of $\mathrm{LR}(G)$ if the cup product:

$$\mathrm{H}^{l(u)}(X, \mathcal{L}_{u\cdot\lambda}) \otimes \mathrm{H}^{l(v)}(X, \mathcal{L}_{v\cdot\mu}) \longrightarrow \mathrm{H}^{l(w)}(X, \mathcal{L}_{w\cdot\nu}), \tag{3}$$

is non zero for some $u, v, w \in W$ such that $l(w) = l(u)+l(v)$ and $u\cdot\lambda + v\cdot\mu = w\cdot\nu$.

Regularly extremal points. Let $\mathcal{L}R(G)$ denote the cone generated by the semigroup $\mathrm{LR}(G)$ in the rational vector space $X(T)^3_{\mathbb{Q}} = (X(T)\otimes\mathbb{Q})^3$. Let $X(T)^+_{\mathbb{Q}}$ (resp. $X(T)^{++}_{\mathbb{Q}}$) denote the cone generated by dominant (resp. strictly dominant) weights of T. Since the semigroup $\mathrm{LR}(G)$ is finitely generated, $\mathcal{L}R(G)$ is a closed convex polyhedral cone contained in $(X(T)^+_{\mathbb{Q}})^3$. A face of $\mathcal{L}R(G)$ which intersects $(X(T)^{++}_{\mathbb{Q}})^3$ is said to be *regular*. In [Res07] and [Res08a], the regular faces are parameterized bijectively. In particular, it is proved that the dimension of any regular face is greater or equal to $2r$ (where r is the rank of G). A point in $\mathrm{LR}(G)$ is said to be *regularly extremal* if it belongs to a regular face of $\mathcal{L}R(G)$ of dimension $2r$. Note that a regularly extremal point is not necessarily regular but it is only a limit of regular points in $\mathcal{L}R(G)$ which belongs to a minimal regular face.

The main result. We can now state

Theorem 1 *Let $(\lambda,\, \mu,\, \nu)$ be a triple of dominant weights. The following are equivalent:*

(i) $(\lambda,\, \mu,\, \nu)$ satisfies the PRV property and has stable multiplicity one;

(ii) $(\lambda,\, \mu,\, \nu)$ is a cohomological point in $\mathrm{LR}(G)$;

(iii) $(\lambda,\, \mu,\, \nu)$ is regularly extremal.

Theorem 1 was conjectured in [DR09b]. In [DR09a], Dimitrov-Roth prove it when λ, μ or ν is strictly dominant. Note that this case also follows easily from [Res07, Theorem G]. In [DR09a], Dimitrov-Roth also prove the case when G is a simple classical group. Here, we present a proof independent of the type of G semisimple.

We now introduce some notation to characterize in a more concrete way the points satisfying Theorem 1. Let Φ^+ denote the set of positive roots.

For $w \in W$, we consider the following *set of inversions of* w:

$$\Phi_w := \{\alpha \in \Phi^+ \, : \, -w\alpha \in \Phi^+\}. \tag{4}$$

We also set $\Phi_w^c = \Phi^+ - \Phi_w$.

We denote by $\sqcup$ the disjoint union. For example, Condition (5) below means that Φ^+ is the disjoint union of Φ_u, Φ_v and Φ_w. By [DR09a, Theorem I], we have:

Theorem 2 *Let $(\lambda,\, \mu,\, \nu)$ be a triple of dominant weights. Then, $(\lambda,\, \mu,\, \nu)$ satisfies (Assertion ((ii)) of) Theorem 1 if and only if there exist u, v and w in W such that*

$$\Phi^+ = \Phi_u \sqcup \Phi_v \sqcup \Phi_w, \tag{5}$$

and

$$u^{-1}\lambda + v^{-1}\mu + w^{-1}\nu = 0. \tag{6}$$

The Belkale-Kumar product for complete flag manifolds. Consider the cohomology ring $\mathrm{H}^*(X, \mathbb{Z})$. For $w \in W$, we will denote by σ_w the cycle class in cohomology of $\overline{BwB/B}$. The Poincaré dual $\sigma^\vee$ of σ_w is $\sigma_{w_0 w}$. It is well known that $\mathrm{H}^*(X, \mathbb{Z}) = \bigoplus_{w \in W} \sigma_w$. Along the way, we prove the following:

Theorem 3 *Let u, v and w in W such that $\Phi^+ = \Phi_u \sqcup \Phi_v \sqcup \Phi_w$. Then, we have:*

$$\sigma_u^\vee \cdot \sigma_v^\vee \cdot \sigma_w^\vee = \sigma_e.$$

In [BK06], Belkale-Kumar defined a new product on $\mathrm{H}^*(X, \mathbb{Z})$. Theorem 3 actually asserts that the structure coefficients of this product in the Schubert basis are zero or one. It allows to compute very easily in this ring. Note that particular cases of Theorem 3 were obtained in [Ric09, Ric08, Res08b]. The question to know if Theorem 3 holds was explicitly asked in [DR09b] and [Res08b].

In this paper, we are interested in the question of the existence of non-zero G-invariant vectors in the tensor product of three irreducible G-modules. All the results can be easily generalized to the case of the tensor product of s such G-modules, for any $s \geq 3$.

2 GIT cones

2.1 Definitions

Let X be a smooth irreducible projective variety endowed with an algebraic action of G. We assume that the group $\mathrm{Pic}^{\mathrm{G}}(\mathrm{X})$ of G-linearized line bundles on X has finite rank. In this work, X will always be a product of flag manifolds of G. We consider the following semigroup:

$$\mathrm{TC}^G(X) = \{\mathcal{L} \in \mathrm{Pic}^{\mathrm{G}}(\mathrm{X}) \ :\ \mathrm{H}^0(\mathrm{X},\mathcal{L})^{\mathrm{G}} \neq \{0\}\}. \tag{7}$$

The Borel-Weil theorem allows to identify $\mathrm{TC}^G((G/B)^3)$ with $\mathrm{LR}(G)$. We will denote by $\mathcal{TC}^G(X)$ the cone generated by $\mathrm{TC}^G(X)$ in $\mathrm{Pic}^{\mathrm{G}}(\mathrm{X})_{\mathbb{Q}} = \mathrm{Pic}^{\mathrm{G}}(\mathrm{X}) \otimes \mathbb{Q}$. The set of ample G-linearized line bundles on X generates an open convex cone $\mathrm{Pic}^{\mathrm{G}}(\mathrm{X})^{++}_{\mathbb{Q}}$ in $\mathrm{Pic}^{\mathrm{G}}(\mathrm{X})_{\mathbb{Q}}$. We set:

$$\mathcal{AC}^G(X) = \mathrm{Pic}^{\mathrm{G}}(\mathrm{X})^{++}_{\mathbb{Q}} \cap \mathcal{TC}^{\mathrm{G}}(\mathrm{X}). \tag{8}$$

For example, $\mathcal{TC}^G((G/B)^3)$ is $\mathcal{L}R(G)$ and $\mathcal{AC}^G((G/B)^3)$ is the intersection of $\mathcal{L}R(G)$ with the interior of the dominant chamber of $X(T^3)_{\mathbb{Q}}$.

For any $\mathcal{L} \in \mathrm{Pic}^{\mathrm{G}}(\mathrm{X})$, we set

$$X^{\mathrm{ss}}(\mathcal{L}) = \{x \in X \ :\ \exists n > 0 \text{ and } \sigma \in \mathrm{H}^0(X, \mathcal{L}^{\otimes n})^G \text{ s. t. } \sigma(x) \neq 0\}.$$

Note that this definition of $X^{\mathrm{ss}}(\mathcal{L})$ is like in [MFK94] if $\mathcal{L}$ is ample but not in general. We consider the following projective variety:

$$X^{\mathrm{ss}}(\mathcal{L})/\!/G := \mathrm{Proj} \bigoplus_{n \geq 0} \mathrm{H}^0(X, \mathcal{L}^{\otimes n})^G,$$

and the natural G-invariant morphism

$$\pi \ :\ X^{\mathrm{ss}}(\mathcal{L}) \longrightarrow X^{\mathrm{ss}}(\mathcal{L})/\!/G.$$

If $\mathcal{L}$ is ample π is a good quotient.

2.2 Covering pairs

2.2.1 — Let P be a parabolic subgroup of G containing B. Le W_P denote the Weyl group of P and W^P denote the set of minimal length representatives of elements in W/W_P. For $u \in W^P$, we will denote by σ_u^P the cycle class in $\mathrm{H}^{2\dim(G/P)-2l(u)}(G/P, \mathbb{Z})$ of $\overline{BuP/P}$. Let us consider the tangent space T_u of $u^{-1}BuP/P$ at the point P.

Using Kleiman's transversality theorem, Belkale-Kumar showed in [BK06, Proposition 2] the following important lemma:

Lemma 1 *Let now u, v and w in W^P such that $l(u) + l(v) + l(w) = \dim G/P$. The product $\sigma_u^P \cdot \sigma_v^P \cdot \sigma_w^P$ is non zero if and only if there exist $p_1, p_2, p_3 \in P$ such that the natural map*

$$T_P(G/P) \longrightarrow \frac{T_P(G/P)}{p_1 T_u} \oplus \frac{T_P(G/P)}{p_2 T_v} \oplus \frac{T_P(G/P)}{p_3 T_w},$$

is an isomorphism.

Then, Belkale-Kumar defined Levi-movability:

Definition. The triple $(u,\, v,\, w)$ is said to be *Levi-movable* if there exist $l_1, l_2, l_3 \in L$ such that the natural map

$$T_P(G/P) \longrightarrow \frac{T_P(G/P)}{l_1 T_u} \oplus \frac{T_P(G/P)}{l_2 T_v} \oplus \frac{T_P(G/P)}{l_3 T_w},$$

is an isomorphism.

We define $c_{uvw} \in \mathbb{Z}_{\geq 0}$ by

$$\sigma_u^P . \sigma_v^P = \sum_{w \in W^P} c_{uvw} (\sigma_w^P)^\vee,$$

where $(\sigma_w^P)^\vee$ denotes the Poincaré dual class of σ_w^P. Belkale-Kumar set:

$$c_{uvw}^{\odot_0} = \begin{cases} c_{uvw} & \text{if } (u,\, v,\, w) \text{ is Levi} - \text{movable;} \\ 0 & \text{otherwise.} \end{cases}$$

They define on the group $\mathrm{H}^*(G/P, \mathbb{Z})$ a bilinear product $\odot_0$ by the formula:

$$\sigma_u^P \odot_0 \sigma_v^P = \sum_{w \in W^P} c_{uvw}^{\odot_0} (\sigma_w^P)^\vee.$$

By [BK06, Definition 18], we have:

Theorem 4 *The product $\odot_0$ is commutative, associative and satisfies Poincaré duality.*

Note that T_u is stable by T. This implies that for $P = B$, $(u,\, v\, w)$ is Levi-movable if and only if

$$l(u) + l(v) + l(w) = 2l(w_0), \text{ and} \tag{9}$$
$$T_u \cap T_v \cap T_w = \{0\}. \tag{10}$$

Since the weights of T in T_u are precisely $-\Phi_u^c$, one can easily checks that $(u,\, v\; w)$ is Levi-movable if and only if

$$\Phi_u^c \sqcup \Phi_v^c \sqcup \Phi_w^c = \Phi^+. \tag{11}$$

2.2.2 — Let H be a subtorus of T and C be an irreducible subvariety of the H-fixed point set X^H in X. Let $\mathcal{L} \in \mathrm{Pic^G(X)}$. There exists a unique character $\mu^{\mathcal{L}}(C, H)$ of H such that

$$h.\tilde{x} = \mu^{\mathcal{L}}(C, H)(h^{-1})\tilde{x}, \tag{12}$$

for any $h \in H$ and $\tilde{x} \in \mathcal{L}$ over C. Analogously, if λ is a one parameter subgroup of G and C is an irreducible subvariety of $X^\lambda = X^{\mathrm{Im}\lambda}$, we will denote by $\mu^{\mathcal{L}}(C, \lambda)$ the integer such that:

$$\lambda(t)\tilde{x} = t^{-\mu^{\mathcal{L}}(C,\lambda)}\tilde{x}, \tag{13}$$

for all $t \in \mathbb{C}^*$ and $\tilde{x}$ as above.

We will consider the parabolic subgroup $P(\lambda)$ (see [MFK94]) defined by

$$P(\lambda) = \{g \in G \;:\; \lim_{t\to 0} \lambda(t)g\lambda(t^{-1}) \text{ exists in } G\}. \tag{14}$$

We also denote by G^λ the centralizer of λ in G; it is a Levi subgroup of $P(\lambda)$. Now, C is an irreducible component of X^λ. We denote by C^+ the corresponding Białynicki-Birula cell:

$$C^+ = \{x \in X \;:\; \lim_{t\to 0} \lambda(t)x \in C\}. \tag{15}$$

One can easily check that C^+ is $P(\lambda)$-stable. We consider the fiber product $G \times_{P(\lambda)} C^+$ and the morphism

$$\begin{array}{cccc} \eta \;: & G \times_{P(\lambda)} C^+ & \longrightarrow & X \\ & [g : x] & \longmapsto & g.x. \end{array} \tag{16}$$

Definition. The pair (C, λ) is said to be *generically finite* if η is dominant with finite general fibers. It is said to be *well generically finite* if it is generically finite and there exists a point $x \in C$ such that the tangent map of η at $[e : x]$ is invertible. It is said to be *well covering* if it is well generically finite and η is birational.

2.2.3 — Set $X = (G/B)^3$. Let λ be a dominant regular one parameter subgroup; $P(\lambda) = B$. The group λ has only isolated fixed points in X parameterized by W^3. Let $(u,\, v,\, w) \in W$ and $z = (u^{-1}B,\, v^{-1}B,\, w^{-1}B) \in X$. Set $C = \{z\}$. It is well known that $C^+ = Bu^{-1}B \times Bv^{-1}B \times Bw^{-1}B \subset X$. Consider now

$$\eta\ :\ G \times_B C^+ \longrightarrow X.$$

Let $x = (g_1B,\, g_2B,\, g_3B) \in X$. The projection $G \times_B C^+ \longrightarrow G/B$ induces an isomorphism between $\eta^{-1}(x)$ and $g_1BuB \cap g_2BvB \cap g_3BwB$. With the Kleiman theorem, this implies that

$$\begin{array}{rcl}
(C, \lambda) \text{ is generically finite} & \iff & \sigma_u.\sigma_v.\sigma_w = d[\mathrm{pt}] \text{ with } d > 0; \\
(C, \lambda) \text{ is well generically finite} & \iff & \sigma_u.\sigma_v.\sigma_w = d[\mathrm{pt}] \text{ with } d > 0, \text{ and} \\
& & \eta^{-1}(z) \text{ is finite;} \\
(C, \lambda) \text{ is well covering} & \iff & \sigma_u.\sigma_v.\sigma_w = [\mathrm{pt}], \text{ and} \\
& & \eta^{-1}(z) = \{[e : z]\}.
\end{array}$$

2.3 PRV points in $\mathrm{LR}(G)$

In this subsection, $X = (G/B)^3$. We have the following very easy lemma:

Lemma 2 *Let $(\lambda,\, \mu,\, \nu)$ be a triple of dominant weights. Then, $(\lambda,\, \mu,\, \nu)$ has the PRV property if and only if there exists an irreducible component C of X^T such that $\mu^{\mathcal{L}_{(\lambda,\,\mu,\,\nu)}}(C, T)$ is trivial.*

Proof. The irreducible components of X^T are the singletons $\{(uB,\, vB,\, wB)\}$ for $u,\, v,\, w \in W$. Moreover, a direct computation shows that

$$\mu^{\mathcal{L}_{(\lambda,\,\mu,\,\nu)}}(\{(uB,\, vB,\, wB)\}, T) = -(u\lambda + v\mu + w\nu).$$

The lemma follows. $\square$

We also make the following obvious observation:

Lemma 3 *Let $(\lambda,\, \mu,\, \nu)$ be a point in $\mathrm{LR}(G)$ with the PRV property. Then, $(\lambda,\, \mu,\, \nu)$ has stable multiplicity one if and only if $X^{\mathrm{ss}}(\mathcal{L}_{(\lambda,\,\mu,\,\nu)})/\!/G$ is a point.*

2.4 Cohomological points in $\mathrm{LR}(G)$

We now recall [DR09a, Theorem 1]:

Theorem 5 *Let $(\lambda,\, \mu,\, \nu)$ be a triple of dominant weights. Then, $(\lambda,\, \mu,\, \nu)$ is a cohomological point of $\mathrm{LR}(G)$ if and only if there exist $u,\, v,\, w \in W$ such that* *(i)* $u^{-1}\lambda + v^{-1}\mu + w^{-1}\nu = 0$, *and*

(ii) $\Phi^+ = \Phi_u \sqcup \Phi_v \sqcup \Phi_w$.

2.5 Regularly extremal points in $\mathrm{LR}(G)$

We now recall a result from [Res07, Res08a] which describes the regularly extremal points in $\mathrm{LR}(G)$. Indeed, in [Res07, Res08a], we describe the minimal regular faces of $\mathcal{L}R(G)$, and the Kumar-Mathieu version of the PRV conjecture proves that $\mathrm{LR}(G)$ is saturated along these faces (that is, any triple of dominant weights which belongs to $\mathcal{L}R(G)$ belongs to $LR(G)$).

Theorem 6 *Let $(\lambda,\, \mu,\, \nu)$ be a triple of dominant weights. Then, $(\lambda,\, \mu,\, \nu)$ is a regularly extremal point of* $\mathrm{LR}(G)$ *if and only if there exist $u,\, v,\, w \in W$ such that*

(i) $u^{-1}\lambda + v^{-1}\mu + w^{-1}\nu = 0$,

(ii) $\Phi^+ = \Phi^c_u \sqcup \Phi^c_v \sqcup \Phi^c_w$, *and*

(iii) $\sigma_u \cdot \sigma_v \cdot \sigma_w = \sigma_e$.

3 The Belkale-Kumar product for complete flag manifolds

Theorem 7 *The non-zero structure coefficients of the ring* $(\mathrm{H}^*(G/B, \mathbb{Z}), \odot_0)$ *in the Schubert basis are equal to* 1.

Proof. Let $(u,\, v,\, w) \in W^3$ such that

$$\sigma_u \odot_0 \sigma_v \odot_0 \sigma_w = d[\mathrm{pt}].$$

Note that d is the coefficient of $\sigma_w^\vee$ in the expression of $\sigma_u \odot_0 \sigma_v$ as a linear combination of Schubert classes. So, we have to prove that if $d \neq 0$ then $d = 1$.

Set $X = (G/B)^3$, $z = (u^{-1}B,\, v^{-1}B,\, w^{-1}B)$ and $C^+ = Bu^{-1}B \times Bv^{-1}B \times Bw^{-1}B)$. Consider the following morphism

$$\begin{array}{cccc} \eta: & G \times_B C^+ & \longrightarrow & X \\ & [g:x] & \longmapsto & gx. \end{array}$$

Since $(u,\, v,\, w)$ is Levi-movable, the tangent map of η is invertible at $[e:z]$ and so at any point of C^+. It follows that η is a covering of degree d. In particular d is the cardinality of the fiber $\eta^{-1}(z)$.

Consider the natural projection $\pi : G\times_B C^+ \longrightarrow G/B$. Choose a one parameter subgroup λ of T such that $P(\lambda)=B$; that is, λ is dominant and regular. The map π identifies $\eta^{-1}(z)$ with the set of $gB\in G/B$ such that $g^{-1}z\in C^+$. Since $\lim_{t\to 0}\lambda(t)(g^{-1}z)=z$, [Res07, Lemma 12] implies that $g^{-1}z\in Bz$. So, $g\in G_zB$. Finally, $\eta^{-1}(z)=G_z.B\subset G/B$.

It remains to prove that G_z is connected. Let $g\in G_z$. Since T and gTg^{-1} are maximal tori of G_z°, there exists $h\in G_z^\circ$ such that $gTg^{-1}=hTh^{-1}$. Then, $h^{-1}g$ normalizes T. But, $h^{-1}g$ fixes $u^{-1}B$. We deduce that $h^{-1}g$ belongs to T and so to G_z°. It follows that g belongs to G_z°. $\square$

4 The main theorem

Lemmas 2 and 3, Theorems 5 and 6 show that Theorem 1 is equivalent to the following

Theorem 8 *Let $(\lambda,\, \mu,\, \nu)$ be a triple of dominant weights. Then, the following are equivalent*

(i) there exist $u,\, v,\, w\in W$ such that

  - *(a) $u^{-1}\lambda+v^{-1}\mu+w^{-1}\nu=0$, and*
  - *(b) $X^{\rm ss}(\mathcal{L}_{(\lambda,\,\mu,\,\nu)})/\!/G$ is a point.*

(ii) there exist $u,\, v,\, w\in W$ such that

  - *(a) $u^{-1}\lambda+v^{-1}\mu+w^{-1}\nu=0$, and*
  - *(b) $\Phi^+=\Phi_u\sqcup\Phi_v\sqcup\Phi_w$,*

(iii) there exist $u,\, v,\, w\in W$ such that

  - *(a) $u^{-1}\lambda+v^{-1}\mu+w^{-1}\nu=0$,*
  - *(b) $\Phi^+=\Phi^c_u\sqcup\Phi^c_v\sqcup\Phi^c_w$, and*
  - *(c) $\sigma_u\cdot\sigma_v\cdot\sigma_w=\sigma_e$.*

We first prove

Lemma 4 *Let G be a reductive group and Y be a product of flag varieties of G. We assume that $\mathcal{AC}^G(Y)=\mathrm{Pic}^{\mathrm{G}}(\mathrm{Y})^{++}_{\mathbb{Q}}$.*

Then, Y is a point.

Proof. We are going to prove that if Y is not a point, then $\mathcal{AC}^G(Y)$ is not equal to $\mathrm{Pic^G(Y)}^{++}_{\mathbb{Q}}$. If $Y = G/P_1$ with P_1 a strict parabolic subgroup of G, $\mathcal{AC}^G(Y)$ is empty. If $Y = G/P_1 \times G/P_2$ with P_1 and P_2 two strict parabolic subgroups of G, a weight (λ, μ) belongs to $\mathcal{AC}^G(Y)$ if and only if $\mu = -w_0\lambda$. In particular, $\mathcal{AC}^G(Y)$ has empty interior.

Let us now assume that, $Y = G/P_1 \times G/P_2 \times G/P_3$ with P_1, P_2 and P_3 three strict parabolic subgroups of G. Let $(\lambda,\, \mu,\, \nu)$ be three weights such that $\mathcal{L}_{(\lambda,\, \mu,\, \nu)}$ is an ample line bundle on Y. The set of $\gamma \in X(T) \otimes \mathbb{Q}$ such that there exists a positive integer k such that $V^*_{k\gamma}$ is contained in $V_{k\lambda} \otimes V_{k\mu}$ is a compact polytope (namely, a moment polytope). In particular, there exists n such that for any positive integer k, $V^*_{kn\nu}$ is not a submodule of $V_{k\lambda} \otimes V_{k\mu}$. So, the ample element $\mathcal{L}_{(\lambda,\, \mu,\, n\nu)}$ does not belong to $\mathcal{AC}^G(Y)$.

The case when Y is a product of more than three flag varieties works similarly. $\square$

Proof.[of Theorem 8] Note that for any $u \in W$, we have

$$\begin{aligned}\Phi_{w_0u} &= \{\alpha \in \Phi^+ \,|\, -w_0u\alpha \in \Phi^+\}\\ &= \{\alpha \in \Phi^+ \,|\, u\alpha \in \Phi^+\}\\ &= \Phi^c_u;\end{aligned}$$

and

$$\begin{aligned}\Phi_{uw_0} &= \{\alpha \in \Phi^+ \,|\, u(-w_0\alpha) \in \Phi^+\}\\ &= -w_0\{\alpha \in \Phi^+ \,|\, u\alpha \in \Phi^+\}\\ &= -w_0\Phi^c_u.\end{aligned}$$

Assume that Assertion (iii) is satisfied for u, v and w in W. Then, we have:

$$\begin{aligned}\Phi^+ = -w_0\Phi^+ &= (-w_0\Phi^c_u) \sqcup (-w_0\Phi^c_v) \sqcup (-w_0\Phi^c_w)\\ &= \Phi_{uw_0} \sqcup \Phi_{vw_0} \sqcup \Phi_{ww_0}.\end{aligned}$$

So, uw_0, vw_0 and ww_0 satisfy Assertion (ii).

Conversely, assume that Assertion (ii) is satisfied for u', v' and w' in W. Set $u = u'w_0$, $v = v'w_0$ and $w = w'w_0$. The above proof shows that $\Phi^+ = \Phi^c_u \sqcup \Phi^c_v \sqcup \Phi^c_w$. Theorem 7 shows that $\sigma_u.\sigma_v.\sigma_w = \sigma_e$. The identity $u^{-1}\lambda + v^{-1}\mu + w^{-1}\nu = 0$ follows from $u'^{-1}\lambda + v'^{-1}\mu + w'^{-1}\nu = 0$. Finally, Assertion (iii) holds.

Let us assume that Assertion (iii) is satisfied and set $C = \{(u^{-1}B,\, v^{-1}B,\, w^{-1}B)\}$. By [Res07, Proposition 9], there exists a dominant morphism from C to

$X^{\mathrm{ss}}(\mathcal{L}_{(\lambda,\,\mu,\,\nu)})/\!/G$; it follows that $X^{\mathrm{ss}}(\mathcal{L}_{(\lambda,\,\mu,\,\nu)})/\!/G$ is a point.

Let us assume that $(\lambda,\, \mu,\, \nu)$ satisfies Assertion (i). If $\overline{X}$ is a product of three flag manifolds for G, there exists a unique G^3-equivariant map $p\,:\, X \longrightarrow \overline{X}$. There exists a unique such variety $\overline{X}$, such that $\mathcal{L}_{(\lambda,\,\mu,\,\nu)}$ is the pullback by p of an ample G-linearized line bundle $\overline{\mathcal{L}}$ on $\overline{X}$. Consider the image $\overline{z}$ of $(u^{-1}B,\, v^{-1}B,\, w^{-1}B)$ by p.

The condition $u^{-1}\lambda + v^{-1}\mu + w^{-1}\nu = 0$ implies that T acts trivially on the fiber in $\overline{\mathcal{L}}$ over $\overline{z}$. Since T has finite index in its normalizer $N(T)$ in G, $\overline{z}$ is semitable for $\overline{\mathcal{L}}$ and the action of $N(T)$. A Luna theorem (see [Res07, Proposition 8] for an adapted version) shows that $\overline{z}$ is semistable for $\overline{\mathcal{L}}$ and the action of G. In particular, $\overline{\mathcal{L}}$ belongs to $\mathcal{AC}^G(\overline{X})$.

Let $\overline{\mathcal{F}}$ be the face of $\mathcal{TC}^G(\overline{X})$ containing $\overline{\mathcal{L}}$ in its relative interior. By [Res07, Theorem H], there exists a well covering pair $(\overline{C}, \lambda)$ of $\overline{X}$ such that $\overline{\mathcal{F}}$ is the set of $\mathcal{L} \in \mathcal{TC}^G(\overline{X})$ such that $\mu^{\mathcal{L}}(\overline{C}, \lambda) = 0$. The first step of this proof is to show that there exists such a pair where $\overline{C}$ is a singleton.

By [Res09], there exists a well covering pair $(\overline{C}, \lambda)$ of $\overline{X}$ such that

(i) λ is a dominant one parameter subgroup of T;

(ii) $\overline{\mathcal{F}}$ is the set of $\mathcal{L} \in \mathcal{TC}^G(\overline{X})_{\mathbb{Q}}$ such that $\mu^{\mathcal{L}}(\overline{C}, \lambda) = 0$;

(iii) $\overline{\mathcal{L}}_{|\overline{C}}$ belongs to the relative interior of $\mathcal{AC}^{G^\lambda}(\overline{C})$;

(iv) if K is the kernel of the action of G^λ on $\overline{C}$, $\mathcal{AC}^{G^\lambda}(\overline{C})$ spans the subspace $\mathrm{Pic^G(\overline{X})^K_{\mathbb{Q}}}$.

We claim that $\overline{C}$ is a singleton. We mention that the proof of the claim will use Lemma 4.

We first prove that $G.\overline{z}$ is the unique closed G-orbit in $\overline{X}^{\mathrm{ss}}(\overline{\mathcal{L}})$. Since $\overline{X}^{\mathrm{ss}}(\overline{\mathcal{L}})/\!/G = X^{\mathrm{ss}}(\mathcal{L}_{(\lambda,\,\mu,\,\nu)})/\!/G$ is a point, $\overline{X}^{\mathrm{ss}}(\overline{\mathcal{L}})$ is affine and contains a unique closed G-orbit. Since $\overline{z}$ is fixed by T and B/T is unipotent, $B.\overline{z}$ is closed in the affine variety $\overline{X}^{\mathrm{ss}}(\overline{\mathcal{L}})$. Since G/B is complete, we deduce that $G.\overline{z}$ is closed in $\overline{X}^{\mathrm{ss}}(\overline{\mathcal{L}})$.

By [Res07, Proposition 10], $\overline{C}$ intersects $G.\overline{z}$. Up to changing $\overline{z}$ by another point in $W.\overline{z}$, one may assume that $\overline{z} \in \overline{C}$.

We claim that $\mathcal{AC}^{G^\lambda}(\overline{C})$ is the set of points in $\mathrm{Pic}^{G^\lambda}(\overline{C})^{++}_{\mathbb{Q}}$ with trivial action of K°. By Condition (iv), it is sufficient to prove that $\mathcal{AC}^{G^\lambda}(\overline{C})$ is the intersection of $\mathrm{Pic}^{G^\lambda}(\overline{C})^{++}_{\mathbb{Q}}$ and a linear subspace. The kernel of $\mu^\bullet(\overline{z},T)$ will be this subspace. By [Lun75, Corollary 1] (see also, [Res07, Proposition 8]), if $\mathcal{M}\in \mathrm{Pic}^{G^\lambda}(\overline{C})^{++}_{\mathbb{Q}}$ satisfy $\mu^{\mathcal{M}}(\overline{z},T)=0$ then $\overline{z}$ is semistable for $\mathcal{M}$ and $\mathcal{M}$ belongs to $\mathcal{AC}^{G^\lambda}(\overline{C})$. Since $\overline{C}^{\rm ss}(\overline{\mathcal{L}}_{|\overline{C}})/\!/G^\lambda$ is a point, $G^\lambda\overline{z}$ is the unique closed G^λ-orbit in $\overline{C}^{\rm ss}(\overline{\mathcal{L}}_{|\overline{C}})$. But, $\overline{\mathcal{L}}_{|\overline{C}}$ belongs to the relative interior of $\mathcal{AC}^G(\overline{C})$. It follows that $G^\lambda.\overline{z}$ is the only closed G^λ-orbit in $\overline{C}^{\rm ss}(\mathcal{M})$ for any $\mathcal{M}$ in the relative interior of $\mathcal{AC}^G(\overline{C})$. In particular, $\mu^{\mathcal{M}}(\overline{z},T)=0$. This implies that $\mathcal{AC}^{G^\lambda}(\overline{C})$ is contained in the kernel of $\mu^\bullet(\overline{z},T)$.

The claim and Lemma 4 below imply that $\overline{C}$ is one point; so, $\overline{C}=\{\overline{z}\}$. This ends the first step.

The second step consists in proving that $G_{\overline{z}}=G^\lambda$. Consider $\overline{\eta}\,:\,G\times_{P(\lambda)}\overline{C}^+\longrightarrow\overline{X}$. Since $(\overline{C},\lambda)$ is well covering, $\overline{\eta}^{-1}(z)$ is only one point. This implies that $G_{\overline{z}}$ is contained in $P(\lambda)$. On the other hand, G^λ is connected and acts on each irreducible component of $\overline{X}^\lambda$. We deduce that G^λ fixes $\overline{z}$. Moreover, $G.\overline{z}$ is affine, and $G_{\overline{z}}$ is reductive. This implies that $G_{\overline{z}}=G^\lambda$.

The third step consists in raising $(\overline{C},\lambda)$ to a well covering pair (C,λ) of X. Let P, Q and R be the parabolic subgroups of G containing B such that $\overline{X}=G/P\times G/Q\times G/R$. Up to multiplying u by an element of W_P on the left, we may assume that $\overline{BuP(\lambda)}=\overline{PuP(\lambda)}$. Similarly, we choose v and w without changing $\overline{z}=(u^{-1}P,\,v^{-1}Q,\,w^{-1}R)$. Since $(\overline{C},\lambda)$ is well covering, [Res07, Proposition 11] shows that:

$$[\overline{BuP(\lambda)}]\odot_0[\overline{BvP(\lambda)}]\odot_0[\overline{BwP(\lambda)}]=[\mathrm{pt}]\in \mathrm{H}^*(G/P(\lambda),\mathbb{Z}). \qquad (17)$$

Set $C=G^\lambda u^{-1}B\times G^\lambda v^{-1}B\times G^\lambda w^{-1}B\subset G/B^3$. Then, [Res07, Proposition 11] shows that (C,λ) is a well covering pair of X. The corresponding face $\mathcal{F}$ of $\mathcal{LR}(G)$ contains $\overline{\mathcal{F}}$.

The forth step consists in perturbing (C,λ) to obtain a well covering pair (C',λ') with a **regular** one parameter subgroup λ' such that the corresponding face $\mathcal{F}'$ of $\mathcal{LR}(G)$ still contains $\overline{\mathcal{F}}$. Let us recall that the map $W^{P(\lambda)}\times W_{P(\lambda)}\longrightarrow W$, $(u,v)\mapsto uv$ is a bijection. For $w\in W$, we will denote by $\bar{w}$ the unique element of $W_{P(\lambda)}$ such that $w\in W^{P(\lambda)}w$. Since $G_{\overline{z}}=G^\lambda$, one can multiply u, v and w on the right by elements of W_{G^λ} to obtain:

(i) $\overline{z} = (u^{-1}P,\, v^{-1}Q,\, w^{-1}R)$,

(ii) $[\overline{BuP(\lambda)}]\odot_0[\overline{BvP(\lambda)}]\odot_0[\overline{BwP(\lambda)}] = [\mathrm{pt}] \in \mathrm{H}^0(G/P(\lambda), \mathbb{Z})$,

(iii) $[\overline{B^\lambda \bar{u} B^\lambda}]\odot_0[\overline{B^\lambda \bar{v} B^\lambda}]\odot_0[\overline{B^\lambda \bar{w} B^\lambda}] = [\mathrm{pt}] \in \mathrm{H}^0(G^\lambda/B^\lambda, \mathbb{Z})$.

We claim that

$$[\overline{BuB}]\odot_0[\overline{BvB}]\odot_0[\overline{BwB}] = [\mathrm{pt}] \in \mathrm{H}^0(G/B, \mathbb{Z}). \tag{18}$$

Set $z = (u^{-1}B,\, v^{-1}B,\, w^{-1}B)$ and $C^+ = B^3.z$. Consider the morphism $\eta : G\times_B C^+ \longrightarrow X$. To prove the claim, we have to prove that η is birational and that $\eta^{-1}(z) = \{[e : z]\}$. By [Res08b] or [Ric08], $[\overline{BuB}]\cdot[\overline{BvB}]\cdot[\overline{BwB}] = [\mathrm{pt}]$ and η is birational. Let now $g \in G$ such that $g^{-1}z \in C^+$. It remains to prove that $g \in B$. Since $C^+ = B^3.z$ and $\overline{C}^+ = P(\lambda)^3 p(z)$, $g^{-1}\overline{z} \in \overline{C}^+$. But, $(\overline{C}^+, \lambda)$ is well covering, and so, $g \in P(\lambda)$. Since $P(\lambda) = G^\lambda B$, we may assume that $g \in G^\lambda$.

Consider now, the subvariety $F = G^\lambda u^{-1}B \times G^\lambda v^{-1}B \times G^\lambda w^{-1}B$ of X. There is a unique G^λ-equivariant isomorphism from F onto $(G^\lambda/B^\lambda)^3$ and $C^+\cap F$ maps onto $B^\lambda \bar{u}^{-1}B^\lambda \times B^\lambda \bar{v}^{-1}B^\lambda \times B^\lambda \bar{w}^{-1}B^\lambda$ by this isomorphism. Now, since $g^{-1}z \in C^+\cap F$ and $g \in G^\lambda$, Condition (iii) implies that $g \in B^\lambda$.

Finally, Condition 18 means that (u, v, w) satisfies Assertion (iii). $\square$

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