Iterative higher order sliding mode observer for nonlinear systems with unknown inputs

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Abstract

This paper deals with the problem of the design sliding mode observers for nonlinear systems subject to unknown inputs. In most approaches, sliding mode observers can be designed under the assumption that the system can be transformed into a specific canonical observable form. Then, the state and the unknown input of the system can be recovered in finite time. In this work, the class of systems for which unknown input sliding mode observers can be designed is enlarged by introducing an extended triangular observable form and a higher order sliding mode observers for which finite time convergence can be shown using Lyapunov stability arguments.

1 Introduction

In the eighties, P. Kokotovic, H Khalil and J. O'Reilly published a seminal book on singularly perturbed system [18], which was one of the first dealing with iterative methods applied in control theory. This book, as well as V. Utkin's one [23], have inspired many works on nonlinear control for constraining the system behaviour to evolve on a given submanifold of the state space. To the best of our knowledge, all control or observer methods able to constrain the system or observation error dynamics to a submanifold encounter some problems under sampling, because, in this case it is only guaranty to reach a neighbourhood of the submanifold. In a joint paper with Professor Khalil [5], we have also shown the necessity for systems under sampling to take care how the output measurement is realized with respect to the control setting for singularly perturbed systems or systems with fast actuators. Works of Professor Hassan Khalil also influenced strongly industrial applications and the way to teach nonlinear control. Actually, like the book of A. Isidori [13] in the eighties is one of the founding book of nonlinear system theory, the book of H. Khalil

[17] is the first book which allows engineers and students to deal rather easily with nonlinear control. For these and many other reasons as his kindness, we are very pleased to be involved in this special issue dedicated to the 60^{th} birthday of Professor Hassan Khalil.

The problem of designing observers for multivariable systems partially driven by unknown inputs has been widely studied. Such observers can be of important use for systems subject to disturbances or with inaccessible inputs, and in many applications such as fault detection and identification or parameter identification. Robust observers have to be designed. For linear systems, this can be achieved using a Luenberger type observers under some decoupling and detectability conditions [6, 12].

Robust observers for nonlinear systems subject to unknown inputs, such as high gain [11, 19] or sliding mode observers [2, 20], are usually designed under the assumptions that the system can be put into a set of triangular observable canonical forms (when the dynamics of the system is perfectly known, similar forms are also derived for the problem of nonlinear observer design with linear error dynamics [15, 16, 22, 24]), where the unknown inputs act only on the last dynamics of each triangular form.

In the case of sliding mode observers, an estimate of the system states and the unknown inputs can be obtained in finite time. The finite time convergence property of sliding mode observers can be useful in observation problems such as observer-based controller design for nonlinear systems (for a large class of systems, the observer can be designed separately from the controller and the separation principle does not need to be proved) or in applications that require fast estimations of some unknown inputs like fault detection and identification or on-line parameter identification. Finite time variable structure observers can also be of interest to solve the observation problem of some class of hybrid systems or the problem of observability singularities that may occur in nonlinear systems.

This work aims at developing a method allowing for state and unknown input estimation for a class of nonlinear systems subject to unknown inputs when usual sliding mode observers can not be achieved. First, the canonical observable form usually considered for the design of robust finite time sliding mode observers is recalled in Section 2 and necessary and sufficient conditions leading to such a form are given. In Section 3, a higher order sliding mode observer is introduced and finite time convergence of the observer is proved using Lyapunov stability arguments. Then, in Section 4, the class of nonlinear systems for which such a sliding mode observer can be designed is enlarged using nonlinear transformations that put the system in an extended triangular observable form. Lastly, as a way of illustration, the problem of actuator fault detection and identification for an aircraft is discussed in Section 5.

2 A triangular observation form for nonlinear systems with unknown inputs

Consider, on an open set U, the nonlinear system:

$$\begin{cases} \dot{\chi} = f(\chi) + g(\chi)w = f(\chi) + \sum_{i=1}^{m} g_i(\chi)w_i \\ y = h(\chi) = [h_1(\chi), .., h_p(\chi)]^T \end{cases} \tag{1}$$

where $\chi \in U \subset \mathbb{R}^n$ is the state vector, $y \in \mathbb{R}^p$ is the output vector and $w = [w_1, ..., w_m] \in \mathbb{R}^m$ is the unknown input vector. The vector fields f, $g_1, ..., g_m$, and the functions $h_1, ..., h_p$, are assumed to be sufficiently smooth on U [1]. Without loss of generality, it is assumed that $p \geq m$, and that the distribution $G = \text{span}\{g_1, \ldots, g_m\}$ and the codistribution $\text{span}\{dh_1, \ldots, dh_p\}$ are nonsingular on U. Define the unknown input characteristic indexes $\{\rho_1, ..., \rho_p\}$ (see [21]) such that, for $1 \leq i \leq p$, for all $\chi \in U$:

$$\begin{array}{l} L_{g_j} L_f^k h_i(\chi) = 0, \text{ for } k < \rho_i - 1, \text{ and for all } 1 \leq j \leq m, \\ L_{g_j} L_f^{\rho_i - 1} h_i(\chi) \neq 0, \text{ for at least one } 1 \leq j \leq m, \end{array} .$$

The system (1), with $w = 0$, is supposed to be locally weakly observable on U. Then, there exist p integers $(\nu_1, \nu_2, \ldots, \nu_p)$ called the observability indices of system (1) (see [15] for a definition), that satisfy $\nu_1 + \cdots + \nu_p = n$, and such that the change of coordinates

$$\phi = \begin{pmatrix} h_1, & ..., & L_f^{\nu_1 - 1} h_1, & ..., & h_p, & ..., & L_f^{\nu_p - 1} h_p \end{pmatrix}^T,$$

after a suitable reordering of the state components, transforms locally the system (1) into

$$\dot{x}_i = A_{\delta_i} x_i + H_{\delta_i} V_{\delta_i}(x, \eta, w) \tag{2}$$
$$\dot{\eta} = a(x, \eta) + b(x, \eta)w \tag{3}$$
$$y_i = C_{\delta_i} x_i$$

with $x = \left(x_1^T, ..., x_p^T\right)^T, \; x_i \in \mathbb{R}^{\delta_i}, \quad \delta_i = \min(\nu_i, \rho_i), \quad 1 \leq i \leq p,$

$$V_{\delta_i}(x, \eta, w) = L_f^{\delta_i} h_i(x, \eta) + \sum_{j=1}^{m} L_{g_j} L_f^{\delta_i - 1} h_i(x, \eta) w_j \in \mathbb{R}$$

$$A_{\delta_i} = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \in \mathbb{R}^{\delta_i \times \delta_i}$$

$$H_{\delta_i} = \begin{pmatrix} 0 & 0 & \ldots & 1 \end{pmatrix}^T \in \mathbb{R}^{\delta_i}, \; C_{\delta_i} = \begin{pmatrix} 1 & 0 & \ldots & 0 \end{pmatrix} \in \mathbb{R}^{1 \times \delta_i}$$

[1] It is assumed that for all original states and all bounded $w_i \in C^0$ the Cauchy problem has a unique solution. For more details, see the book of Professor Khalil [17], page 67.

and a, b are smooth vector fields on U.
If $\delta_i = \nu_i$, i.e.:

$$\nu_i \leq \rho_i \text{ for all } 1 \leq i \leq p, \tag{4}$$

the system (1) is transformed into:

$$\begin{aligned} \dot{x}_i &= A_{\nu_i} x_i + H_{\nu_i} V_{\nu_i}(x, w) \\ y_i &= C_{\nu_i} x_i \end{aligned} \tag{5}$$

with $\sum_{j=1}^{m} L_{g_j} L_f^{\nu_i - 1} h_i(x) w_j \neq 0$ if and only if $\nu_i = \rho_i$. Finite time sliding mode observers for the form (5) can be found in the literature[2]. For instance, one can refer to the works [2, 8, 9] for design methods based on first order sliding mode algorithms. However, the observer dimension increases drastically due to the numerous filters used to avoid chattering problems. This can be overcome by designing higher order sliding mode based finite time observers [10, 20] based on the iterative use of the super twisting algorithm (a second order sliding mode algorithm) and the convergence can be proved on the features of homogeneous differential inclusions. Hereafter, an arbitrary order sliding mode observer based on a step-by-step procedure, also involving the super twisting algorithm, is designed. Due to the particular structure of the observer, finite time stability of the observation error is shown using Lyapunov stability arguments.

3 A finite time step-by-step sliding mode observer

Consider a SISO nonlinear system in triangular observable similar to a subsystem of (5):

$$\begin{aligned} \dot{x} &= A_\nu x + H_\nu V_\nu(x, w) \\ y &= C_\nu x \end{aligned} \tag{6}$$

where $x = [x_1, \cdots, x_\nu]^T \in \mathbb{R}^\nu$ is the state vector, $y \in \mathbb{R}$ is the output and $w \in \mathbb{R}$ is some unknown input. Assume that the state of the system is uniformly bounded, i.e. for all $t > 0$, $|x_i(t)| < d_i$. The unknown inputs as well as its first time derivative are also assumed to be ultimately bounded. A 2ν-dimensional

[2] One can find other finite time observers based on numerical approaches [7, 14] or algebraic methods [3].

observer is designed as follows:

$$\begin{array}{rcl}
\dot{\hat{x}}_1 & = & z_1 + \lambda_1 \left| x_1 - \hat{x}_1 \right|^{1/2} \operatorname{sign}(x_1 - \hat{x}_1) \\
\dot{z}_1 & = & \alpha_1 \operatorname{sign}(x_1 - \hat{x}_1) \\
\dot{\hat{x}}_2 & = & z_2 + \lambda_2 \left| z_1 - \hat{x}_2 \right|^{1/2} \operatorname{sign}(z_1 - \hat{x}_2) \\
\dot{z}_2 & = & \alpha_2 \operatorname{sign}(z_1 - \hat{x}_2) \\
 & \vdots & \\
\dot{\hat{x}}_{\nu-1} & = & z_{\nu-1} + \lambda_{\nu-1} \left| z_{\nu-2} - \hat{x}_{\nu-1} \right|^{1/2} \operatorname{sign}(z_{\nu-2} - \hat{x}_{\nu-1}) \\
\dot{z}_{\nu-1} & = & \alpha_{\nu-1} \operatorname{sign}(z_{\nu-2} - \hat{x}_{\nu-1}) \\
\dot{\hat{x}}_{\nu} & = & z_{\nu} + \lambda_{\nu} \left| z_{\nu-1} - \hat{x}_{\nu} \right|^{1/2} \operatorname{sign}(z_{\nu-1} - \hat{x}_{\nu}) \\
\dot{z}_{\nu} & = & \alpha_{\nu} \operatorname{sign}(z_{\nu-1} - \hat{x}_{\nu})
\end{array} \quad (7)$$

Define the observation errors as: $e_i = x_i - \hat{x}_i$ and $\xi_i = x_{i+1} - z_i$ for $i = 1, .., \nu$ where $x_{\nu+1} \triangleq V_\nu(x, w)$. The observer gains λ_i and α_i are positive scalars yet to be defined. The observation error dynamics is given by:

$$\begin{array}{rcll}
\dot{e}_1 & = & \xi_1 - \lambda_1 \left| e_1 \right|^{1/2} \operatorname{sign}(e_1) & (8) \\
\dot{\xi}_1 & = & x_3 - \alpha_1 \operatorname{sign}(e_1) & (9) \\
\dot{e}_2 & = & \xi_2 - \lambda_2 \left| e_2 - \xi_1 \right|^{1/2} \operatorname{sign}(e_2 - \xi_1) & \\
\dot{\xi}_2 & = & x_4 - \alpha_2 \operatorname{sign}(e_2 - \xi_1) & \\
 & \vdots & & \\
\dot{e}_{\nu-1} & = & \xi_{\nu-1} - \lambda_2 \left| e_{\nu-1} - \xi_{\nu-2} \right|^{1/2} \operatorname{sign}(e_{\nu-1} - \xi_{\nu-2}) & \\
\dot{\xi}_{\nu-1} & = & x_{\nu+1} - \alpha_{\nu-1} \operatorname{sign}(e_{\nu-1} - \xi_{\nu-2}) & \\
\dot{e}_{\nu} & = & \xi_{\nu} - \lambda_{\nu} \left| e_{\nu} - \xi_{\nu-1} \right|^{1/2} \operatorname{sign}(e_{\nu} - \xi_{\nu-1}) & \\
\dot{\xi}_{\nu} & = & \dot{x}_{\nu+1} - \alpha_{\nu} \operatorname{sign}(e_{\nu} - \xi_{\nu-1}) &
\end{array}$$

Consider the two first equations (8)-(9) and note $\psi = \begin{bmatrix} \psi_1 \\ \psi_2 \end{bmatrix} = \begin{bmatrix} \left| e_1 \right|^{1/2} \operatorname{sign}(e_1) \\ \xi_1 \end{bmatrix}$. One has:

$$\begin{array}{rcl}
\dot{\psi} & = & \left| \psi_1 \right|^{-1} \begin{pmatrix} -\lambda_1 & 1 \\ -\alpha_1 & 0 \end{pmatrix} \psi + \begin{bmatrix} 0 \\ x_3 \end{bmatrix} \\
 & = & \left| \psi_1 \right|^{-1} \left(M\psi + \begin{bmatrix} 0 \\ \left| \psi_1 \right| x_3 \end{bmatrix} \right).
\end{array} \quad (10)$$

Define the candidate Lyapunov function $V = \psi^T P \psi$ where $P = \begin{pmatrix} p_1 & p_3 \\ p_3 & p_2 \end{pmatrix}$ is a symmetric positive definite matrix. The time derivative of V along the solution of (10) is given by:

$$\dot{V} = \left| \psi_1 \right|^{-1} \left(\psi^T \left(M^T P + PM \right) \psi + 2\psi^T P \begin{bmatrix} 0 \\ \left| \psi_1 \right| x_3 \end{bmatrix} \right)$$

where

$$2\psi^T P \left[\begin{array}{c} 0 \\ |\psi_1| \, x_3 \end{array} \right] = 2x_3 \, |\psi_1| \, [p_3\psi_1 + p_2\psi_2] \, .$$

Since, for all $\varepsilon > 0$,

$$p_2 \, |\psi_1| \, |\psi_2| \leq \frac{\varepsilon}{2}\psi_1^2 + \frac{p_2^2}{2\varepsilon}\psi_2^2,$$

one has

$$2\psi^T P \left[\begin{array}{c} 0 \\ |\psi_1| \, x_3 \end{array} \right] \leq k_1\psi_1^2 + k_2\psi_2^2$$

with $k_1 = d_3 \, (2 \, |p_3| + \varepsilon)$ and $k_2 = d_3 \frac{p_2^2}{\varepsilon}$. Thus

$$\dot{V} \leq |\psi_1|^{-1} \, \psi^T \left(M^T P + PM + \left(\begin{array}{cc} k_1 & 0 \\ 0 & k_2 \end{array} \right) \right) \psi$$

Because M is a Hurwitz matrix, the observer gains λ_1 and α_1 can be chosen such that the matrix

$$-Q = M^T P + PM + \left(\begin{array}{cc} k_1 & 0 \\ 0 & k_2 \end{array} \right)$$

is negative definite. Then

$$\dot{V} \quad \leq \quad - |\psi_1|^{-1} \, \psi^T Q \psi \leq - |\psi_1|^{-1} \, \lambda_{\min} \{Q\} \, \|\psi\|^2 \, .$$

Note that V is not differentiable for $\psi_1 = 0$. However, Lyapunov and Lasalle Theorems still apply since $\lim_{\psi \longrightarrow 0} \dot{V} \leq 0$. Since

$$\|\psi\|^2 \geq \frac{V}{\lambda_{\max} \{P\}} \text{ and } |\psi_1| \leq \|\psi\| \leq \left(\frac{V}{\lambda_{\min} \{P\}} \right)^{\frac{1}{2}},$$

one gets:

$$\dot{V} \leq - \left(\frac{\lambda_{\min} \{P\}}{V} \right)^{\frac{1}{2}} \frac{\lambda_{\min} \{Q\}}{\lambda_{\max} \{P\}} V = -\lambda_{\min} \{P\}^{\frac{1}{2}} \frac{\lambda_{\min} \{Q\}}{\lambda_{\max} \{P\}} V^{\frac{1}{2}}$$

and this proves the finite time convergence of ψ to zero, i.e. the finite time convergence of e_1 and ξ_1 to zero.

Thus, after a finite time $\xi_1 = 0$ and the observer dynamics is given by:

$$\begin{array}{rcl} \dot{e}_1 & = & 0 \\ \dot{e}_2 & = & \xi_2 - \lambda_2 \, |e_2|^{1/2} \operatorname{sign}(e_2) \\ \dot{\xi}_2 & = & x_4 - \alpha_2 \operatorname{sign}(e_2) \\ & \vdots & \\ \dot{e}_{\nu-1} & = & \xi_{\nu-1} - \lambda_2 \, |e_{\nu-1} - \xi_{\nu-2}|^{1/2} \operatorname{sign}(e_{\nu-1} - \xi_{\nu-2}) \\ \dot{\xi}_{\nu-1} & = & x_{\nu+1} - \alpha_{\nu-1} \operatorname{sign}(e_{\nu-1} - \xi_{\nu-2}) \\ \dot{e}_\nu & = & \xi_\nu - \lambda_\nu \, |e_\nu - \xi_{\nu-1}|^{1/2} \operatorname{sign}(e_\nu - \xi_{\nu-1}) \\ \dot{\xi}_\nu & = & \dot{x}_{\nu+1} - \alpha_\nu \operatorname{sign}(e_\nu - \xi_{\nu-1}) \end{array}$$

In a similar way, it can be shown that the trajectories converge in finite time onto $\{e_2 = \xi_2 = 0\}$. So, one has in finite time $e_i = x_i - \hat{x}_i = 0$ and $\xi_i = x_{i+1} - z_i = 0$ for $i = 1, .., \nu$. In the final step of the procedure, $\xi_\nu = x_{\nu+1} - z_\nu = 0$ and one gets a finite time estimate of $V_\nu(x, w) = z_\nu$. For MIMO systems (5), the observer (7) can be generalized as follows:

$$\begin{aligned}
\dot{\hat{x}}_{i,1} &= z_{i,1} + \lambda_{i,1} \left|x_{i,1} - \hat{x}_{i,1}\right|^{1/2} \text{sign}(x_{i,1} - \hat{x}_{i,1}) \\
\dot{z}_{i,1} &= \alpha_{i,1}\text{sign}(x_{i,1} - \hat{x}_{i,1}) \\
\dot{\hat{x}}_{i,2} &= z_{i,2} + \lambda_{i,2} \left|z_{i,1} - \hat{x}_{i,2}\right|^{1/2} \text{sign}(z_{i,1} - \hat{x}_{i,2}) \\
\dot{z}_{i,2} &= \alpha_{i,2}\text{sign}(z_{i,1} - \hat{x}_2) \\
&\vdots \\
\dot{\hat{x}}_{i,\nu_i-1} &= z_{i,\nu_i-1} + \lambda_{i,\nu_i-1} \left|z_{i,\nu_i-2} - \hat{x}_{i,\nu-1}\right|^{1/2} \text{sign}(z_{i,\nu_i-2} - \hat{x}_{i,\nu_i-1}) \\
\dot{z}_{i,\nu_i-1} &= \alpha_{i,\nu_i-1}\text{sign}(z_{i,\nu_i-2} - \hat{x}_{i,\nu_i-1}) \\
\dot{\hat{x}}_{i,\nu_i} &= z_{i,\nu_i} + \lambda_{i,\nu_i} \left|z_{i,\nu_i-1} - \hat{x}_{i,\nu_i}\right|^{1/2} \text{sign}(z_{i,\nu_i-1} - \hat{x}_{i,\nu_i}) \\
\dot{z}_{i,\nu_i} &= \alpha_{i,\nu_i}\text{sign}(z_{i,\nu_i-1} - \hat{x}_{i,\nu_i})
\end{aligned} \tag{11}$$

where $y_i = x_{i,1}$ and $i \in \{1, ..., p\}$. After a finite time, all $\xi_{i,j}$ have converged to zero and consequently $z_{i,j} = x_{i,j+1}$ for all $i \in \{1, .., p\}$ and $j \in \{1, ..., \nu_i - 1\}$. It is also to possible to estimate the unknown inputs because one has:

$$z_{\nu_i} = V_{\nu_i}(x, w) = L_f^{\nu_i} h_i(x) + \sum_{j=1}^{m} L_{g_j} L_f^{\nu_i - 1} h_i(x) w_j, \quad 1 \leq i \leq p \tag{12}$$

The relations (12) can be rewritten as:

$$\begin{pmatrix} L_{g_1} L_f^{\nu_1-1} h_1(x) & \dots & L_{g_m} L_f^{\nu_1-1} h_1(x) \\ \vdots & & \vdots \\ L_{g_1} L_f^{\nu_p-1} h_p(x) & \dots & L_{g_m} L_f^{\nu_p-1} h_p(x) \end{pmatrix} w = \begin{pmatrix} z_{\nu_1} - L_f^{\nu_1} h_1(x) \\ \vdots \\ z_{\nu_p} - L_f^{\nu_p} h_p(x) \end{pmatrix}$$

Since the distribution span $\{g_1(x), \ldots, g_m(x)\}$ is nonsingular for all $x \in U$, the matrix

$$\Lambda(x) = \begin{pmatrix} L_{g_1} L_f^{\nu_1-1} h_1(x) & \dots & L_{g_m} L_f^{\nu_1-1} h_1(x) \\ \vdots & & \vdots \\ L_{g_1} L_f^{\nu_p-1} h_p(x) & \dots & L_{g_m} L_f^{\nu_p-1} h_p(x) \end{pmatrix}$$

has full column rank because:

$$\Lambda(x) = \begin{pmatrix} dL_f^{\nu_1-1} h_1(x) \\ \vdots \\ dL_f^{\nu_p-1} h_p(x) \end{pmatrix} \begin{pmatrix} g_1(x) & \dots & g_m(x) \end{pmatrix}$$

Thus, an estimate of the unknown input is given by:

$$\tilde{w} = \Lambda^{+}(\hat{x}) \begin{pmatrix} z_{\nu_1} - L_f^{\nu_1} h_1(\hat{x}) \\ \vdots \\ z_{\nu_p} - L_f^{\nu_p} h_p(\hat{x}) \end{pmatrix},$$

where Λ^{+} is a well defined pseudo-inverse of Λ.

4 A generalized triangular observable form

It may happen that the condition (4) is not satisfied. Here is provided an observation scheme that allows for the finite time estimation of both the state and the unknown inputs of (1) even if $\nu_j > \rho_j$ for at least a j in $\{1, ..., p\}$. Consider again the general form (2-3). Applying the given finite time observer (7) to the subsystems (2), for $1 \leq i \leq p$: (i) ξ_i can be estimated in finite time; (ii) one can also recover in finite time the last component V_{δ_i} of each subsystem of (2). The problem is to recover the remaining state η. Denote:

$$V(x) = \begin{pmatrix} V_{\delta_1}(x, w) \\ V_{\delta_2}(x, w) \\ \vdots \\ V_{\delta_p}(x, w) \end{pmatrix} = \begin{pmatrix} L_f^{\delta_1} h_1(x) \\ L_f^{\delta_2} h_2(x) \\ \vdots \\ L_f^{\delta_p} h_p(x) \end{pmatrix} + \Gamma_\delta(x) w$$

where

$$\Gamma_\delta(x) = \begin{pmatrix} L_{g_1} L_f^{\delta_1 - 1} h_1(x) & ... & L_{g_m} L_f^{\delta_1 - 1} h_1(x) \\ \vdots & & \vdots \\ L_{g_1} L_f^{\delta_p - 1} h_p(x) & ... & L_{g_m} L_f^{\delta_p - 1} h_p(x) \end{pmatrix}.$$

Let $\pounds$ be the commutative algebra of the measured outputs and their successive Lie derivatives up to order δ_i: $\pounds = \text{span}\{h_1, ..., L_f^{\delta_1 - 1} h_1, ..., h_p, ..., L_f^{\delta_p - 1} h_p\}$ and let $d\pounds$ be the codistribution: $d\pounds = \text{span}\{dh_1, ..., dL_f^{\delta_1 - 1} h_1, ..., dh_p, ..., dL_f^{\delta_p - 1} h_p\}$. Assume there exists a $1 \times p$ row vector $K(x) = (k_1(x), ..., k_p(x)) \neq 0$, $k_i \in \pounds$ for $1 \leq i \leq p$, such that:

$$K(x)\Gamma_\delta(x) = 0 \text{ for all } x \in U \tag{13}$$

and set: $\bar{y} = \bar{h}(x) = K(x)V(x) = \sum_{i=1}^{p} k_i(x) L_f^{\delta_i} h_i(x)$. Note that $\bar{y}$ is an available information (after a finite time) and is not affected by the unknown inputs. Therefore, if $d\pounds + \text{span}\left\{d\bar{h}\right\} \nsubseteq d\pounds$, $\bar{y}$ can be used as an additional output. Then, let $\bar{\rho}_i$ and $\bar{\nu}_i$ be the unknown input characteristic indexes and the observability indices of (1) with respect to the extended output $\left[y^T, \bar{y}\right]^T$. If $\bar{\nu}_i \leq \bar{\rho}_i$ for all

$1 \leq i \leq p+1$, the system (1) can be transformed into:

$$\begin{aligned}
\dot{\zeta}_i &= A_{\bar{\nu}_i}\zeta_i + H_{\bar{\nu}_i}V_{\bar{\nu}_i}(\tilde{\zeta}, w) \text{ for } 1 \leq i \leq p \\
y_i &= C_{\bar{\nu}_i}\zeta_i \\
\dot{\bar{\zeta}} &= A_{\bar{\nu}_{p+1}}\bar{\zeta} + H_{\bar{\nu}_{p+1}}V_{\bar{\nu}_{p+1}}(\tilde{\zeta}, w) \\
\bar{y} &= C_{\bar{\nu}_{p+1}}\bar{\zeta}
\end{aligned}$$

where $\tilde{\zeta} = \left(\zeta_1^T, ..., \zeta_p^T, \bar{\zeta}^T\right)^T \in \mathbb{R}^n$. Then, it is possible to recover both the state and the unknown inputs in finite time.

The discussion above can be recursively generalized as follows. Assume that the condition (4) is not still satisfied with the extended output obtained with the solutions of (13). On the basis of this new output, the corresponding matrix $\Gamma_{\bar{\delta}}$ can be computed and another set of fictitious outputs can eventually be found. One can iterate this procedure until the condition (4) is fulfilled for a new extended output. Then, the original system can be put into an extended block triangular observable form[3]:

$$\begin{aligned}
\dot{\zeta}_i^1 &= A_{\nu_i^1}\zeta_i^1 + H_{\nu_i^1}V_{\nu_i^1}(\zeta, w) \\
y_i^1 &= y_i = C_{\nu_i^1}\zeta_i^1, \qquad 1 \leq i \leq p^1 = p \\
\dot{\zeta}_i^2 &= A_{\nu_i^2}\zeta_i^2 + H_{\nu_i^2}V_{\nu_i^2}(\zeta, w) \\
y_i^2 &= C_{\nu_i^2}\zeta_i^2, \qquad 1 \leq i \leq p^2 \\
&\vdots \\
\dot{\zeta}_i^{k^*} &= A_{\nu_i^{k^*}}\zeta_i^{k^*} + H_{\nu_i^{k^*}}V_{\nu_i^{k^*}}(\zeta, w) \\
y_i^{k^*} &= C_{\nu_i^{k^*}}\zeta_i^{k^*}, \qquad 1 \leq i \leq p^{k^*}
\end{aligned} \tag{14}$$

where the integers ν_i^j are the observability indices of the system (1) with the new outputs y_i^j. The first subsystem is fed by the original outputs of the system. A finite time observer is designed to estimate the state of this subsystem and to provide in finite time the knowledge of the fictitious outputs y_i^2, $1 \leq i \leq p^2$. Then, the state of the second triangular observable form can be estimated as well as the fictitious outputs y_i^3. Thus, one can recursively obtain in finite time the whole state of the system as well as the unknown inputs. An algorithm that states if the system can be transformed into (14) and that provides the integers p^j, ν_i^j and the auxiliary outputs y_i^j $(j = 1, ..., k^*)$ was given in [4].

[3]Systems that admit such a form belong to the class of left invertible systems with trivial zero dynamics (see [3]).

5 Actuator fault detection for an aircraft

As a way of illustration, consider the 9 order dynamical model of an aircraft given in [13], page 268:

$$\begin{pmatrix} \dot\psi \\ \dot\vartheta \\ \dot\phi \end{pmatrix} = \begin{pmatrix} 0 & \frac{\sin(\phi)}{\sin(\vartheta)} & \frac{\cos(\phi)}{\sin(\vartheta)} \\ 0 & \cos(\phi) & -\sin(\phi) \\ 1 & \sin(\phi)\tan(\vartheta) & \cos(\phi)\tan(\vartheta) \end{pmatrix} \begin{pmatrix} p^* \\ q^* \\ r^* \end{pmatrix} := M(\psi,\vartheta,\phi) \begin{pmatrix} p^* \\ q^* \\ r^* \end{pmatrix}$$

$$\begin{pmatrix} \dot p \\ \dot q \\ \dot r \end{pmatrix} = J^{-1}S(p,q,r)J \begin{pmatrix} p \\ q \\ r \end{pmatrix} + J^{-1}T$$

$$\begin{aligned} \dot V &= \frac{-F_D}{m} - g\sin(\vartheta) \\ \dot\alpha &= q - \frac{q^*}{\sin\beta} - (p\cos(\alpha) + \sin(\alpha))\tan(\beta) \\ \dot\beta &= r^* + p\sin(\alpha) - r\cos(\alpha) \end{aligned}$$

The state variables ψ, ϑ and ϕ are the yaw, the pitch and the roll angles, respectively, and characterize the attitude of the aircraft with respect to the wind axes. p, q, r are the velocities of the aircraft with respect to a reference frame fixed with the aircraft. V is the speed of the aircraft and α, β are the angle of attack and the sideslip angle, respectively. The relations between p^*, q^*, r^*, the velocity angles expressed with respect to the wind axes, and p, q, r are the following:

$$\begin{aligned} p^* &= p\cos(\alpha)\cos(\beta) + (q - \dot\alpha)\sin(\beta) + r\sin(\alpha)\cos(\beta) \\ q^* &= \frac{1}{mV}(F_L - mg\cos(\vartheta)\cos(\phi)) \\ r^* &= \frac{1}{mV}(-F_S + mg\cos(\vartheta)\sin(\phi)) \end{aligned}$$

The vector T of the external torques can be approximated as follows:

$$T = V \begin{pmatrix} a_{12}r + a_{13}p \\ a_{23}q \\ a_{32}r + a_{33}p \end{pmatrix} + V^2 \begin{pmatrix} a_{11}\sin(\beta) \\ a_{21} + a_{22}p \\ a_{31}\sin(\beta) \end{pmatrix} + V^2 \begin{pmatrix} b_{12}\cos(\beta) & 0 & b_{13}\cos(\beta) \\ 0 & b_{22}\cos(\alpha) & 0 \\ 0 & 0 & b_{33}\cos(\beta) \end{pmatrix} \begin{pmatrix} \delta_a \\ \delta_e \\ \delta_r \end{pmatrix}$$

where δ_a, δ_e and δ_r are the deflections of the aileron, the elevator and the rudder, respectively. The external forces (F_D, F_L, F_S) are function of the relative aircraft speed and δ_P is the setting of the throttle:

$$\begin{pmatrix} F_D \\ F_L \\ F_S \end{pmatrix} = V^2 \begin{pmatrix} c_{11} + c_{12}\cos(\alpha) \\ c_{21} + c_{22}\sin(2\alpha) \\ c_{31}\sin(2\beta) \end{pmatrix} + P \begin{pmatrix} -\cos(\alpha)\cos(\beta) \\ \sin(\alpha) \\ \cos(\alpha)\cos(\beta) \end{pmatrix} \delta_P$$

The matrices J and S are given by:

$$J = \begin{pmatrix} I_x & 0 & -I_{xz} \\ 0 & I_y & 0 \\ -I_{xz} & 0 & I_z \end{pmatrix}, \quad S(p,q,r) = \begin{pmatrix} 0 & r & -q \\ -r & 0 & p \\ q & -p & 0 \end{pmatrix}$$

m is the mass of the aircraft, g is the gravity constant, the $a_{i,j}$, $b_{i,j}$ and c_{ij} are aerodynamical parameters and P is the maximum of thrust.

The available measurements are the speed V and the angles $(\vartheta, \psi, \phi, \alpha, \beta)$ and the control inputs are $u_1 = \delta_P$ and $u_2 = (\delta_a, \delta_e, \delta_r)^T$. After some manipulations, the dynamical model of the aircraft can be expressed in the form:

$$\begin{aligned} \dot{X}_1 &= F_1(X_1, X_2) + G_1(X_1, X_2)u_1 \\ \dot{X}_2 &= F_2(X_1, X_2, X_3) + G_2(X_1, X_2)u_1 \\ \dot{X}_3 &= F_3(X_1, X_2, X_3) + G_3(X_1, X_2, X_3)u_2 \\ y_1 &= X_1 \\ y_2 &= X_2 \end{aligned} \tag{15}$$

where

$$\begin{aligned} X_1 &= (V, \vartheta, \psi)^T := (x_{1,1}, x_{1,2}, x_{1,3})^T \\ X_2 &= (\phi, \alpha, \beta)^T := (x_{2,1}, x_{2,2}, x_{2,3})^T \\ X_3 &= (p, q, r)^T := (x_{3,1}, x_{3,2}, x_{3,3})^T \end{aligned}$$

For the sake of place, the exact expression of the vector fields are not given here. Nevertheless, it is important to note that F_1, G_1 and G_2 are only function of the measured states X_1 and X_2.

The problem of state observation in presence of actuator fault and actuator fault detection and isolation can be seen as follows:

$$\begin{aligned} \dot{X}_1 &= F_1(X_1, X_2) + G_1(X_1, X_2)(u_1 + w_1) \\ \dot{X}_2 &= F_2(X_1, X_2, X_3) + G_2(X_1, X_2)(u_1 + w_1) \\ \dot{X}_3 &= F_3(X_1, X_2, X_3) + G_3(X_1, X_2, X_3)(u_2 + w_2) \\ y_1 &= X_1 \\ y_2 &= X_2 \end{aligned} \tag{16}$$

where the unknown inputs w_i stand for defect in the actuators. For this system, one has $\rho_i = 1, \quad 1 \leq i \leq 6$. This implies that there is at least a j in $\{1, ..., 6\}$ such that $\nu_j > \rho_j$. Thus, the condition (4) is not satisfied and the aforementioned method can not be applied. Roughly speaking, w_1 appears in the output derivative before all the state components appear independently. However, the system can be transformed into an extended triangular form so that a step-by-step higher order sliding mode observer can be designed to estimate actuator faults. From (16), one has:

$$V(x) = \begin{pmatrix} F_1(X_1, X_2) \\ F_2(X_1, X_2, X_3) \end{pmatrix} + \Gamma_1 \begin{pmatrix} u_1 + w_1 \\ u_2 + w_2 \end{pmatrix}$$

with

$$\Gamma_1 \quad = \quad \begin{pmatrix} (G_1(X_1,X_2))_1 & 0 & 0 & 0 \\ (G_1(X_1,X_2))_2 & 0 & 0 & 0 \\ (G_1(X_1,X_2))_3 & 0 & 0 & 0 \\ (G_2(X_1,X_2))_1 & 0 & 0 & 0 \\ (G_2(X_1,X_2))_2 & 0 & 0 & 0 \\ (G_2(X_1,X_2))_3 & 0 & 0 & 0 \end{pmatrix}.$$

Thus, it is possible to find a $K(X)$ with all its components in

$$\pounds = span\{x_{1,1}, x_{1,2}, x_{1,3}, x_{2,1}, x_{2,2}, x_{2,3}\}$$

such that (13) is satisfied. For instance, one can choose:

$$K(X) \quad = \quad \begin{pmatrix} -(G_2(X_1,X_2))_1 & 0 & 0 & (G_1(X_1,X_2))_1 & 0 & 0 \\ -(G_2(X_1,X_2))_2 & 0 & 0 & 0 & (G_1(X_1,X_2))_1 & 0 \\ -(G_2(x_1,x_2))_3 & 0 & 0 & 0 & 0 & (G_1(X_1,X_2))_1 \end{pmatrix}.$$

Then, the auxiliary output $\bar{y}$ is equal to:

$$\bar{y} = (G_1(X_1,X_2))_1 F_2(X_1,X_2,X_3) - G_2(X_1,X_2)(F_1(X_1,X_2))_1$$

It can be shown that the codistribution

$$d\bar{\pounds} = span\{dx_{1,1}, dx_{1,2}, dx_{1,3}, dx_{2,1}, dx_{2,2}, dx_{2,3}, d(\bar{y})_1, d(\bar{y})_2, d(\bar{y})_3\}$$

is generically full rank [4] i.e. $dim\{d\pounds\} = 9$. Thus, using the change of coordinates $\zeta^1 = (X_1^T, X_2^T)^T$ and $\zeta^2 = \bar{y}$, one obtains a triangular form similar to (14) and the state can be estimated in finite time using the observer (11). One also has the knowledge of:

$$\bar{V} = \begin{pmatrix} F_1(X_1,X_2) \\ F_2(X_1,X_2,X_3) \\ \bar{F}(X_1,X_2,X_3) \end{pmatrix} + \Gamma_2 \begin{pmatrix} u_1 + w_1 \\ u_2 + w_2 \end{pmatrix}$$

where $\bar{F} = L_{F_1}\bar{y} + L_{F_2}\bar{y} + L_{F_3}\bar{y}$ and

$$\Gamma_2 = \begin{pmatrix} \Gamma_1 \\ \frac{P\cos(x_{2,2})\cos(x_{2,3})}{m} \frac{\partial F_2(x_1,x_2,.)}{\partial X_3}|_{x_3} G_3(x_1,x_2,x_3) \end{pmatrix}$$

Since $G_3(X_1,X_2,X_3)$ and $\frac{\partial F_2}{\partial X_3}$ are generically full rank, it is possible to recover generically w_1 and w_2.

[4]Singularities appear when $\frac{P\cos(x_{2,2})\cos(x_{2,3})}{m} = 0$ or $\frac{\partial F(X_1,X_2,.)}{\partial X_3}$ is not full rank.

6 Conclusion

In this paper, iterative methods have been used for the design of a step by step higher order sliding mode observer and for an iterative algorithm in order to analyze if the state and the unknown inputs of a nonlinear system can be estimate in finite time. Here, no studies with respect to measurement noise and parameters uncertainties were given because such a study is linked to the control algorithm which is out of the scope of the paper. This point of view of deriving at the same time both the control algorithm and the observer is strongly motivated by one of the most recent paper of Professor Khalil [1]. In this paper, the authors show that measurement noise influences on the observer based control behavior are drastically less than their influences on the observed state.

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