Erasure-proof coding with fusion frames

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Abstract:

The main goal of this paper is the design of frames for transmitting vectors through a memoryless analog erasure channel. The channel transmits the frame coefficients perfectly or discards them, depending on the outcomes of Bernoulli trials with a failure probability q. For sufficiently small q, we construct frames which encode above a fixed non-zero rate and allow the receiver to recover part of the erased coefficients so that the remaining mean-square error vanishes as the frame size increases. We give examples for which the mean-square reconstruction error remaining after corrections are applied decays faster than any inverse power of the number of frame vectors.

1. Introduction

We are concerned with the linear transmission of vectors through a memoryless channel that either transmits a coefficient perfectly or discards it, in accordance with the outcomes of independent, identically distributed Bernoulli trials. The problem of reconstructing a vector in a finite-dimensional real or complex Hilbert space when not all of its frame coefficients are known has already received much attention in the literature [1–9]. However, many results focus on optimal performance for the smallest possible number of erased coefficients [4, 7–9], which is not typical for transmissions via a memoryless erasure channel. Other results on so-called maximally robust frames guarantee recovery from a certain fraction of lost frame coefficients [10], but this may involve inverting an arbitrarily ill-conditioned matrix.

The notion of a memoryless analog erasure channel is simply one that transmits each frame coefficient independently with a given success probability q and otherwise erases it, meaning it does not let the receiver access the coefficient. Within this error model for transmissions, we investigate the performance of fusion frames [11–13], previously also referred to as frames of subspaces [14] or weighted projections resolving the identity [15], which lend themselves to various methods of error correction. What makes the fusion frames useful for error correction purposes is that they have many subsets which are frames for their span. Thus, one can design hierarchical methods for error correction which make error estimates feasible.

The main result presented here is that for a fixed, sufficiently small erasure probability q, we design fusion frames such that their associated coding rate is bounded away from zero and the mean-square error remaining after error correction is applied decays faster than any polynomial in terms of the number of frame vectors.

The techniques for our results involve combinatorial elements similar to the construction of product codes initially investigated by Elias [16], together with some frame-specific arguments.

2. Preliminaries

Throughout the paper, we let $\mathcal{H}$ be a real or complex Hilbert space. Instead of expanding vectors in Hilbert spaces with orthonormal bases, many applications nowadays use frames, stable, non-unique (redundant) expansions, for various purposes. We first briefly recall the basic terminology, and refer the reader to [17] for further details.

Definition 1. *We call a family of vectors $\mathcal{F} = \{f_j\}_{j\in J}$ in $\mathcal{H}$ a* frame *if there exist constants $A, B > 0$ such that for all $x \in \mathcal{H}$ with $\|x\| = 1$, $A \leq \sum_{j\in J} |\langle x, f_j\rangle|^2 \leq B$. If we can choose $A = B$, then we say that the frame is A*-tight. *In case $A = B = 1$ we call $\mathcal{F}$ a* Parseval frame. *A frame is called* equal-norm *if there is a $c > 0$ such that all vectors have the norm $\|f_j\| = c$. With each frame $\mathcal{F}$, we associate the* analysis operator *$V : \mathcal{H} \to \ell^2(J)$, which maps a vector to its frame coefficients, $(Vx)_j = \langle x, f_j\rangle$.*

The fact that a vector is over-determined by its frame coefficients helps correct errors which may occur in the course of a transmission, or when frame coefficients are stored in an unreliable medium. A main goal of frame design is to optimize the performance of a frame given certain constraints. This could be, for example, the dimension of the Hilbert space and the number of frame vectors, or their ratio. In analogy with binary codes, we define a coding rate for a given frame.

Definition 2. *Let $\mathcal{H}$ be a Hilbert space of dimension d and $\mathcal{F}$ a frame for $\mathcal{H}$ consisting of n vectors. We say that $\mathcal{F}$ has a* coding rate *of $R = d/n$.*

The coding and error correction method we discuss hereafter relies on frames arising from tensor product constructions. These frames are a special type of a fusion frame, see e.g. [12–15].

Definition 3. *Given Hilbert spaces $\mathcal{H}_1, \mathcal{H}_2, \dots \mathcal{H}_m$ and tight frames $\mathcal{F}^{(i)} = \{f_j^{(i)}\}_{j \in J_i}$ for each $\mathcal{H}_i$, then the family of vectors $\mathcal{F} = \{f_{j_1}^{(1)} \otimes f_{j_2}^{(2)} \otimes \cdots \otimes f_{j_m}^{(m)} : j_i \in J_i$ for all $i\}$ is a tight frame for $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \cdots \otimes \mathcal{H}_m$. We call this frame $\mathcal{F}$ a tight* product frame.

Remark 1. *We note that if we fix all but one index, say the last, then the resulting set $f_{j_1}^{(1)} \otimes f_{j_2}^{(2)} \otimes \cdots \otimes f_{j_{m-1}}^{(m-1)} \otimes \mathcal{F}^{(m)}$ is a tight frame for its span. Therefore, $\mathcal{F}$ has a natural fusion frame architecture.*
Similarly, fixing only the first $m-k$ indices of the frame vectors in the tensor product would provide a tight frame for a subspace for any $0 \le k < m$. Moreover, there is a partial ordering on these tight frames for subspaces induced by the partial ordering of the subspaces they span.

3. Erasures and the mean-square error

A communication system is given by a frame $\mathcal{F}$ for a Hilbert space $\mathcal{H}$, and an error model for the transmission of frame coefficients. Our main error model assumes memoryless erasures, that is, the values of randomly selected frame coefficients become unknown in the course of transmission, in accordance with the outcomes of Bernoulli trials. In brief, frame coefficients are erased, independently of each other, with a fixed probability $q \ge 0$.
Depending on the implementation of decoding, the performance of a frame can be measured in different ways; we generally distinguish active error correction and blind reconstruction. When actively correcting erasures, one tries to fill in the values for the erased coefficients, and aims for a high probability of successfully restoring all lost coefficients. When blind reconstruction is used, one sets the missing coefficients to zero and reconstructs always in the same way. In this case, the usual goal is obtaining a small error norm, such as the mean-square error or the worst-case error.
In the present work we consider a combination of the two approaches. We measure the quality of error correction by the mean-square error that results from using the corrected coefficients with the possibly remaining, uncorrected erasures set to zero. The average in this mean-square error is taken over the random erasures and over random unit-norm input vectors. For simplicity, we consider input vectors which are independent of the erasures and uniformly distributed on the unit sphere of the Hilbert space.

Definition 4. *Let $\mathcal{F} = \{f_1, f_2, \dots f_n\}$ be a Parseval frame for a real or complex Hilbert space $\mathcal{H}$. The* blind reconstruction error *for an input vector $x \in \mathcal{H}$ and an erasure of frame coefficients with indices $K = \{j_1, j_2, \dots j_m\}$, $m \le n$, is given by*

$$\|V^*EVx - x\| = \|(V^*EV - I)x\|$$

where E is the diagonal $n \times n$ matrix with $E_{j,j} = 1$ if $j \notin K$ and $E_{j,j} = 0$ else. If the positive operator V^*EV has a bounded inverse, then we say that the corresponding erasure is correctible.

Remark 1. *If $\mathcal{F}$ is a Parseval frame then $(V^*EV - I)x = V^*(E-I)Vx$ and the inverse can be obtained from the norm-convergent Neumann series $(V^*EV)^{-1} = \sum_{n=0}^{\infty} (V^*(I-E)V)^n$. Applying this operator to the output of blind reconstruction gives perfect reconstruction of the input vector.*

Next, we define a measure for average reconstruction performance when probabilities for erasures are known. To this end, we average the square of the reconstruction error with the distribution of erasures and input vectors. Here and herafter, we denote the expectation of any random variable η with respect to the underlying probability measure $\mathbb{P}$ by $\mathbb{E}[\eta] = \int \eta d\mathbb{P}$.

Definition 5. *Let $\{\beta_j\}_{j \in J}$ be a family of binary ($\{0,1\}$-valued) random variables governed by a probability measure $\mathbb{P}$, and let Δ be the random diagonal matrix with entries $\Delta_{j,j} = \beta_j$. Moreover, let ξ be a random variable with values in the unit sphere $\{x \in \mathcal{H} : \|x\| = 1\}$ which is independent of the family $\{\beta_j\}$, and assume that the distribution of $U\xi$ is identical to that of ξ for any fixed unitary U. Given a Parseval frame $\mathcal{F}$ for a Hilbert space $\mathcal{H}$ with analysis operator V, we define the* mean-square error *by*

$$\sigma^2(V, \beta) = \mathbb{E}[\|V^*\Delta V\xi\|^2] \, .$$

There is a simple expression for the mean square error as the square of a weighted Frobenius norm of the Grammian VV^*.

Lemma 1. *Let $\{\beta_j\}_{j \in J}$ be as above, assume the family is identically distributed with probability $\mathbb{P}(\beta_1 = 1) = q$, and assume the joint distribution is such that $\mathbb{P}(\beta_j = \beta_{j'} = 1) = r$ for all $j \neq j'$. Let Δ be the random diagonal matrix with entries $\Delta_{j,j} = \beta_j$. If V is the analysis operator of a Parseval frame $\mathcal{F} = \{f_j\}_{j \in J}$ containing $n = |J|$ vectors in a Hilbert space of dimension d, then*

$$\sigma^2(V, \beta) = \frac{1}{d}\Big((q-r)\sum_{j=1}^{n} \|f_j\|^4 + r \sum_{j,l=1}^{n} |\langle f_j, f_l \rangle|^2\Big) \, .$$

4. Bounding the mean-square error for iterative decoding

This section describes how product frames can be used to trade an increase in block length of encoding for better error correction capabilities.
We first consider the simplest case in which $\mathcal{H}$ has two factors, $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$. Also, as preparation for our main theorem, we first consider packet erasures [15] instead of erasures for single frame coefficients. This means, we have a frame $\mathcal{F} = \mathcal{F}^{(1)} \otimes \mathcal{F}^{(2)}$ and a two-parameter family of random variables $\{\beta_{j,j'}\}$ which govern erasures of frame coefficients in such a way that either all coefficients belonging to some j' are erased or all of them are left intact. We compute the mean-square error for this error model.

Proposition 1. *Let $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2$ and let V_1 and V_2 be the analysis operators of Parseval frames $\mathcal{F}^{(1)} = \{f_j^{(1)}\}_{j \in J_1}$ and $\mathcal{F}^{(2)} = \{f_{j'}^{(2)}\}_{j' \in J_2}$ for $\mathcal{H}_1$ and $\mathcal{H}_2$ having dimension*

d_1 and d_2, respectively. Let $\{\beta_{j,j'} : j \in J_1, j' \in J_2\}$ be a two-parameter family of binary random variables which have probabilities $\mathbb{P}(\beta_{j,j'} = 1) = q$ and are distributed such that there is a family $\{\beta^{(2)}_{j'}\}_{j' \in J_2}$ and $\beta_{j,j'} = \beta^{(2)}_{j'}$ almost surely, regardless of j. The mean-square error for the frame $\mathcal{F}$ and this type of packet erasures reduces to that of $\mathcal{F}^{(2)}$,

$$\sigma^2(V_1 \otimes V_2, \beta) = \sigma^2(V_2, \beta^{(2)}) \, .$$

Next, we continue with three combinatorial lemmata. They prepare the main result which concerns the error correction capabilities of tight product frames. The main problem we wish to address with this result is the following: Given a fixed, sufficiently small erasure probability q, find frames such that their associated coding rate is bounded away from zero and the mean-square error remaining after error correction is applied decays fast in terms of the number of frame vectors.

We show hereafter that product frames of the form $\mathcal{F} = \mathcal{F}^{(1)} \otimes \cdots \otimes \mathcal{F}^{(m)}$, for which each factor $\mathcal{F}^{(i)}$ can correct up to two erased frame coefficients, satisfy the desired properties.

Lemma 2. *Let $n_1 \geq 3$ and let $\{\beta_1, \beta_2, \dots \beta_{n_1}\}$ be a family of independent, identically distributed random variables which take values in $\{0, 1\}$. Suppose $q_0 = \mathbb{P}(\beta_1 = 1)$ and let $q_1 = \mathbb{P}(\sum_{j=1}^{n_1} \beta_j \geq 3)$, then*

$$q_1 \leq \frac{1}{6} n_1^3 q_0^3 \, .$$

The probability estimated in this lemma is that of a packet of n_1 coefficients remaining corrupted after an error correction protocol has been applied which can correct any two erased coefficients.

By iteration, we obtain a simple consequence.

Lemma 3. *Let $\{n_i\}_{i=1}^m$ be the sizes of index sets $\{J_i\}_{i=1}^m$, with $n_i \geq 3$ for all $i \in \{1, 2, \dots m\}$. Assume there is an m-parameter family of binary, independent identically distributed random variables $\{\beta_{j_1, j_2, \dots j_m}\}$ and associated families $\{\beta^{(1)}_{j_2, j_3, \dots j_m}\}$, $\{\beta^{(2)}_{j_3, j_4, \dots j_m}\}$, $\dots \{\beta^{(m-1)}_{j_m}\}$ which are iteratively defined by $\beta^{(0)}_{j_1, j_2, \dots j_m} \equiv \beta_{j_1, j_2, \dots j_m}$ and*

$$\beta^{(k)}_{j_{k+1}, j_{k+2}, \dots j_m} = \begin{cases} 1, & \text{if } \sum_{j_k=1}^{n_k} \beta^{(k-1)}_{j_k, j_{k+1}, \dots j_m} \geq 3 \, , \\ 0, & \text{else.} \end{cases}$$

If $\mathbb{P}(\beta_{1,1,\dots 1} = 1) = q_0$, then the family $\{\beta^{(m-1)}_j\}$ is independent, identically distributed with $q_{m-1} = \mathbb{P}(\beta^{(m-1)}_j = 1)$ having the bound

$$q_{m-1} \leq 6^{-\frac{1}{2}(3^{m-1} - 1)} n_{m-1}^{3^1} n_{m-2}^{3^2} \cdots n_1^{3^{m-1}} q_0^{3^{m-1}} \, .$$

The probability computed in the above lemma is the probability of an erased block after applying erasure correction iteratively. The next lemma considers what happens when the error correction is applied to packets at the final level. Here, we deviate from the strategy of only reconstructing nontrivially when at most two packets are missing. Instead, we correct for missing packets and compute the probabilities for the residual mean-square error.

Lemma 4. *Let $\{\beta_1, \beta_2, \dots, \beta_n\}$, $n \geq 1$, be independent, identically distributed binary random variables with probability $\mathbb{P}(\beta_1 = 1) = q$. Let the random variables $\gamma_1, \gamma_2, \dots, \gamma_n$ be defined by $\gamma_j = \beta_j$ if $\sum_{j=1}^n \beta_j \geq 3$, and otherwise $\gamma_j = 0$ for all $j \in \{1, 2, \dots n\}$. Then, for any j,*

$$\mathbb{P}(\gamma_j = 1) \leq \frac{1}{6} n^3 q^4 ,$$

and for $j_1 \neq j_2$, we have

$$\mathbb{P}(\gamma_{j_1} = \gamma_{j_2} = 1) \leq n^2 q^4 .$$

These lemmata allow us to formulate an error bound for the remaining mean-square error for blind reconstruction after the error correction protocol has been applied.

Theorem 1. *Let $V = V_1 \otimes V_2 \otimes \cdots \otimes V_m$ be the analysis operator of a Parseval product frame $\mathcal{F} = \mathcal{F}^{(1)} \otimes \mathcal{F}^{(2)} \otimes \cdots \otimes \mathcal{F}^{(m)}$ for a Hilbert space $\mathcal{H} = \mathcal{H}_1 \otimes \mathcal{H}_2 \otimes \cdots \otimes \mathcal{H}_m$. Denote the dimension of each $\mathcal{H}_i$ by d_i and the number of frame vectors in $\mathcal{F}^{(i)}$ by n_i. Let $\{\beta_{j_1, j_2, \dots j_m}\}$ be an m-parameter family of binary independent, identically distributed random variables, define $\{\beta^{(m-1)}_j\}$ as above, and let $\gamma_{j_1, j_2, \dots j_m} = \beta^{(m-1)}_{j_m}$ if $\sum_{j_m=1}^{n_m} \beta^{(m-1)}_{j_m} \geq 3$ and $\gamma_{j_1, j_2, \dots j_m} = 0$ otherwise, then*

$$\sigma^2(V, \gamma) \leq \frac{1}{d_m} \Big((q_m - r_m) \sum_{j=1}^{n_m} \|f_j^{(m)}\|^4 + r_m \sum_{j,l=1}^{n_m} |\langle f_j^{(m)}, f_l^{(m)} \rangle|^2 \Big)$$

with

$$q_m = 6^{1 - 2 \cdot 3^{m-1}} n_m^3 n_{m-1}^{4 \cdot 3^1} n_{m-2}^{4 \cdot 3^2} \cdots n_1^{4 \cdot 3^{m-1}} q_0^{4 \cdot 3^{m-1}}$$

and

$$r_m \leq \frac{6}{n_m} q_m \, .$$

Corollary 1. *If $V = V_1 \otimes V_2 \otimes \cdots \otimes V_m$ and all V_i belong to equal-norm Parseval frames, then it is well known that $\|f_j^{(i)}\|^2 = \frac{d_i}{n_i}$ and by the Cauchy Schwarz inequality $|\langle f_j^{(i)}, f_l^{(i)} \rangle|^2 \leq d_i^2 / n_i^2$. Thus, we have*

$$\sigma^2(V, \gamma) \leq q_m \frac{d_m}{n_m} + r_m d_m \leq 7 q_m \frac{d_m}{n_m}$$

with

$$q_m = 6^{1 - 2 \cdot 3^{m-1}} n_m^3 n_{m-1}^{4 \cdot 3^1} n_{m-2}^{4 \cdot 3^2} \cdots n_1^{4 \cdot 3^{m-1}} q_0^{4 \cdot 3^{m-1}} \, .$$

Example 1. *Assume that an equal-norm product frame $\mathcal{F} = \mathcal{F}^{(1)} \otimes \cdots \otimes \mathcal{F}^{(m)}$ has $\mathcal{F}^{(i)}$ with $n_i = i^2 n_1$ vectors for each $i \in \{1, 2, \dots m\}$ and $n_1 \geq 3$. Let the dimension of the Hilbert space $\mathcal{H}_i$ spanned by $\mathcal{F}^{(i)}$ be*

$$\dim(\mathcal{H}_i) = i^2 n_1 - 2 \, ,$$

and assume the frame can correct any two erased coefficients. Examples of such frames are the harmonic ones, see e.g. [2].

The tensor product of these m Hilbert spaces, $\mathcal{H} = \otimes_{i=1}^m \mathcal{H}_i$, has dimension

$$\dim(\mathcal{H}) = (m!)^2 n_1^m \prod_{i=1}^m \Big(1 - \frac{2}{i^2 n_1}\Big) .$$

This means, the coding rate R is bounded, independently of m, by

$$R > \prod_{i=1}^m \Big(1 - \frac{2}{i^2 n_1}\Big) > (1 - \frac{2}{n_1})\Big(1 - \frac{2}{n_1}\sum_{i=2}^\infty \frac{1}{i^2}\Big)$$
$$= (1 - \frac{2}{n_1})\Big(1 - \frac{2}{6n_1}\Big(\frac{\pi^2}{6} - 1\Big)\Big) .$$

It is straightforward to check that $n_1 \geq 3$ ensures $R > 0$. The preceding theorem then states that after correcting erasures, the probability of an uncorrected block at the final level is

$$q_m \leq m^6 n_1^3 6^{1-2\cdot 3^{m-1}} q_0^{4\cdot 3^{m-1}} e^{4\sum_{k=1}^{m-1} 3^{m-k}\ln(k^2 n_1)}$$

and upon estimating the sum in the exponent with Jensen's inequality,

$$2\sum_{k=1}^{m-1} 3^{-k}\ln k \leq 2\sum_{k=1}^{\infty} 3^{-k}\ln k \leq \ln\frac{3}{2} ,$$

we have

$$q_m \leq m^6 n_1^3 6^{1-2\cdot 3^{m-1}} q_0^{4\cdot 3^{m-1}} e^{2(3^m-1)\ln n_1} e^{4\cdot 3^m \ln\frac{3}{2}} .$$

To achieve exponential decay of q_m in 3^m requires

$$-2\ln 6 + 4\ln q_0 + 6\ln n_1 + 12\ln\frac{3}{2} < 0 ,$$

which amounts to

$$\frac{27}{8\sqrt{6}} q_0 n_1^{3/2} < 1 .$$

Since $n_1 = 3$ is the smallest dimension to start the iteration, fast decay of the mean-square error needs $q_0 < 8\sqrt{2}/81 \approx 0.14$.
The number of transmitted frame coefficients is $(m!)^2 n_1^m$, so by Stirling's approximation $O(e^{(m+\frac{1}{2})\ln m + m\ln n_1})$, whereas by the preceding corollary the decay of the mean-square error is of order $O(e^{-c3^m})$, for a suitable $c > 0$. This implies that the mean-square error decays faster than any inverse power of the number of transmitted coefficients.

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