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Panuvuth Lakhonchai [(1)], Jouni Sampo [(2)] and Songkiat Sumetkijakan [(1)]

(1) Department of Mathematics, Chulalongkorn University, Phyathai Road, Patumwan, Bangkok 10330, Thailand.
(2) Department of Applied Mathematics, Lappeenranta University of Technology, Lappeenranta, Finland.
panuvuth@hotmail.com, jouni.sampo@lut.fi, songkiat.s@chula.ac.th

Abstract:

Using Hart Smith's, curvelet, and shearlet transforms, we investigate L^2 functions with sufficiently smooth background and present here sufficient and necessary conditions, which include the special case with 1-dimensional singularity line. Specifically, we consider the situation where regularity on a line in a non-parallel direction is much lower than directional regularity along the line in a neighborhood and how this is reflected in the behavior of the three transforms.

1. Introduction

Wavelet transforms, both continuous and discrete, have proved to be a very efficient tool in detecting point singularities. However, due to its isotropic scaling, wavelet transforms are not ideal tools in detecting one-dimensional singularities like singularity lines or curves. Recently, wavelet-like transforms with parabolic scaling, such as Hart Smith's and curvelet transforms, were introduced and applied successfully in edge detection. Our goal is then to investigate how these transforms can be used in detecting point, line, and curve singularities. New necessary and new sufficient conditions for an $L^2(\mathbb{R}^2)$ function to possess Hölder regularity, uniform and pointwise, with exponent $\alpha > 0$ are given. Similar to the characterization of Hölder regularity by the continuous wavelet transform, the conditions here are in terms of bounds of the Smith and curvelet transforms across fine scales. However, due to the parabolic scaling, the sufficient and necessary conditions differ in both the uniform and pointwise cases, with larger gap in pointwise regularities. Naturally, global conditions for pointwise singularities can be weakened. We then investigate functions with sufficiently smooth background in one direction and potential singularity in the perpendicular (non-parallel) direction. Specifically, sufficient and necessary conditions, which include the special case with one-dimensional singularity line, are derived for pointwise Hölder exponent. Inside their "cones" of influence, these conditions are practically the same, giving near-characterization of direction of singularity.

2. Directional Regularity

We shall restrict our definition to a real-valued function f of two variables. Generalization to a function of several variables is straightforward. For a given positive exponent α not in $\mathbb{N}$, its pointwise, uniform, and directional Hölder (or Lipschitz) regularities are defined as follows. Fix a point $\boldsymbol{u} \in \mathbb{R}^2$ at which regularity is under investigation. f is said to be *pointwise Hölder regular with exponent α at $\boldsymbol{u}$*, denoted by $f \in C^\alpha(\boldsymbol{u})$, if there exists a polynomial $P_{\boldsymbol{u}}$ of degree less than α and a constant $C = C_{\boldsymbol{u}}$ such that for all $\boldsymbol{x}$ in a neighborhood of $\boldsymbol{u}$

$$|f(\boldsymbol{x}) - P_{\boldsymbol{u}}(\boldsymbol{x}-\boldsymbol{u})| \le C\|\boldsymbol{x}-\boldsymbol{u}\|^\alpha. \tag{1}$$

If there exists a uniform constant C so that for all $\boldsymbol{u}$ in an open subset Ω of $\mathbb{R}^2$ there is a polynomial $P_{\boldsymbol{u}}$ of degree less than α such that (1) holds for all $\boldsymbol{x} \in \Omega$, then we say that f is *uniformly Hölder regular with exponent α on Ω* or $f \in C^\alpha(\Omega)$. The *uniform Hölder exponent* of f on Ω is defined to be

$$\alpha_l(\Omega) := \sup\{\alpha \colon f \in C^\alpha(\Omega)\}, \tag{2}$$

and the *pointwise Hölder exponent* is defined in an analogous manner. Following [9], the *local Hölder exponent* of f at $\boldsymbol{u}$ is defines as

$$\alpha_l(\boldsymbol{u}) = \lim_{n\to\infty} \alpha_l(I_n).$$

where $\{I_n\}_{n\in\mathbb{N}}$ is a family of nested open sets in $\mathbb{R}^2$, i.e. $I_{n+1} \subset I_n$, with intersection $\cap_n I_n = \{\boldsymbol{u}\}$.
In order to define directional regularity, let $\boldsymbol{v} \in \mathbb{R}^d$ be a fixed unit vector representing a direction and $\boldsymbol{u}$ be a point in $\mathbb{R}^d$. f is said to be *pointwise Hölder regular with exponent α at $\boldsymbol{u}$ in the direction $\boldsymbol{v}$*, denoted by $f \in C^\alpha(\boldsymbol{u};\boldsymbol{v})$, if there exist a constant $C = C_{\boldsymbol{u},\boldsymbol{v}}$ and a polynomial $P_{\boldsymbol{u},\boldsymbol{v}}$ of degree less than α such that

$$|f(\boldsymbol{u}+\lambda\boldsymbol{v}) - P_{\boldsymbol{u},\boldsymbol{v}}(\lambda)| \le C|\lambda|^\alpha \tag{3}$$

holds for all λ in a neighborhood of $0 \in \mathbb{R}$. We next define directional regularity on a set $\Omega_1 \subseteq \mathbb{R}^2$. Let Ω_2 be an open neighborhood of Ω_1 representing a set on which the Hölder estimate holds. Then f is said to be in $C^\alpha(\Omega_1,\Omega_2;\boldsymbol{v})$ if there exists a constant $C = C_{\boldsymbol{v}}$ so that for all $\boldsymbol{u} \in \Omega_1$ there is a polynomial $P_{\boldsymbol{u},\boldsymbol{v}}$ of degree less than α such that (3) holds for all $\lambda \in \mathbb{R}$ with $\boldsymbol{u}+\lambda\boldsymbol{v} \in \Omega_2$. If $\Omega_1 = \Omega_2$, then we denote $C^\alpha(\Omega_1,\Omega_2;\boldsymbol{v})$ simply by $C^\alpha(\Omega_1;\boldsymbol{v})$. Of course, the *directional pointwise and uniform Hölder exponents* could be defined in the same way as (2). In the pointwise case, this directional

Hölder exponent measures one-dimensional regularity of f at $\boldsymbol{u}$ on the line passing through $\boldsymbol{u}$ and parallel with $\boldsymbol{v}$. See [5]. For $C^{\alpha}(\Omega_1, \Omega_2; \boldsymbol{v})$, the set Ω_1 in our context of line singularity will usually be a line and $\boldsymbol{v}$ points in a direction that is nonparallel with the line. In this situation, $f \in C^{\alpha}(\Omega_1, \Omega_2; \boldsymbol{v})$ has a ridge along the line provided that hte regularity in the direction of the line is sufficiently high. See Theorem 4.

3. Three Transforms with Parabolic Scaling

3.1 Hart Smith Transform

Originally defined in [10], the Hart Smith transform was described in [1, 2] as follows. For a given $\varphi \in L^2(\mathbb{R}^2)$, we define

$$\varphi_{a\boldsymbol{b}\theta}(\boldsymbol{x}) = a^{-\frac{3}{4}}\varphi\left(\mathrm{D}_{\frac{1}{a}}\mathrm{R}_{-\theta}\left(\boldsymbol{x}-\boldsymbol{b}\right)\right),$$

for $\theta \in [0, 2\pi)$, $\boldsymbol{b} \in \mathbb{R}^2$, and $0 < a < a_0$, where a_0 is a fixed coarsest scale, $\mathrm{D}_{\frac{1}{a}} = \operatorname{diag}\left(\frac{1}{a}, \frac{1}{\sqrt{a}}\right)$, and $R_{-\theta}$ is the matrix affecting planar rotation of θ radians in clockwise direction. Hart Smith transform can then be defined as

$$\overline{\Gamma}_f\left(a, \boldsymbol{b}, \theta\right) := \left\langle \varphi_{a\boldsymbol{b}\theta}, f \right\rangle.$$

This gives a true affine transform that uses parabolic scaling. For each scale a and direction θ, let us define the norm

$$\|\boldsymbol{v}\|_{a,\theta} := \left\|D_{\frac{1}{a}}R_{-\theta}\boldsymbol{v}\right\| \qquad \text{for } \boldsymbol{v} \in \mathbb{R}^2.$$

We define vector $\boldsymbol{v}_\theta := R_\theta(0,1)^T$ so that $\boldsymbol{v}_\theta$ is parallel to the major axis of the ellipse $\|\boldsymbol{v}\|_{a,\theta} = 1$.

Reconstruction Formula [10, 1, 2]
There exists a Fourier multiplier M of order 0 so that whenever $f \in L^2(\mathbb{R}^2)$ is a high-frequency function supported in frequency space $\|\boldsymbol{\xi}\| > \frac{2}{a_0}$, then, in $L^2(\mathbb{R}^2)$

$$\begin{aligned} f &= \int_0^{a_0}\int_0^{2\pi}\int_{\mathbb{R}^2} \left\langle \varphi_{a\boldsymbol{b}\theta}, Mf \right\rangle \varphi_{a\boldsymbol{b}\theta}\, d\boldsymbol{b}\, d\theta\, \frac{da}{a^3} \qquad (4)\\ &= \int_0^{a_0}\int_0^{2\pi}\int_{\mathbb{R}^2} \left\langle \varphi_{a\boldsymbol{b}\theta}, f \right\rangle M\varphi_{a\boldsymbol{b}\theta}\, d\boldsymbol{b}\, d\theta\, \frac{da}{a^3}. \end{aligned}$$

3.2 Continuous Curvelet Transform

Following Candès and Donoho[1, 2], the continuous curvelet transform (CCT) is defined in the polar coordinates (r, ω) of the Fourier domain. Let W be a positive real-valued C^∞ function supported inside $\left(\frac{1}{2}, 2\right)$, called a *radial window*, and let V be a real-valued C^∞ function supported on $[-1, 1]$, called an *angular window*, for which the following admissibility conditions hold:

$$\int_0^\infty W(r)^2\frac{dr}{r} = 1 \quad \text{and} \quad \int_{-1}^1 V(\omega)^2\, d\omega = 1. \qquad (5)$$

At each scale a, $0 < a < a_0$, $\gamma_{a\boldsymbol{0}0}$ is defined by

$$\widehat{\gamma_{a\boldsymbol{0}0}}\left(r\cos(\omega), r\sin(\omega)\right) = a^{\frac{3}{4}}W\left(ar\right)V\left(\omega/\sqrt{a}\right)$$

for $r \geq 0$ and $\omega \in [0, 2\pi)$. For each $0 < a < a_0$, $\boldsymbol{b} \in \mathbb{R}^2$, and $\theta \in [0, 2\pi)$, a *curvelet* $\gamma_{a\boldsymbol{b}\theta}$ is defined by

$$\gamma_{a\boldsymbol{b}\theta}(\boldsymbol{x}) = \gamma_{a\boldsymbol{0}0}\left(R_\theta\left(\boldsymbol{x}-\boldsymbol{b}\right)\right), \quad \text{for } \boldsymbol{x} \in \mathbb{R}^2. \qquad (6)$$

The continuous curvelet transform of $f \in L^(\mathbb{R}^2)$ is

$$\Gamma_f\left(a, \boldsymbol{b}, \theta\right) = \left\langle \gamma_{a\boldsymbol{b}\theta}, f \right\rangle$$

for $0 < a < a_0$, $\boldsymbol{b} \in \mathbb{R}^2$, and $\theta \in [0, 2\pi)$.
The admissibility conditions (5) and the polar coordinate design of curvelets yield the following:

Reconstruction formula [2]
There exists a bandlimited purely radial function Φ such that for all $f \in L^2(\mathbb{R}^2)$,

$$f = \tilde{f} + \int_0^{a_0}\int_0^{2\pi}\int_{\mathbb{R}^2} \left\langle \gamma_{a\boldsymbol{b}\theta}, f \right\rangle \gamma_{a\boldsymbol{b}\theta}\, d\boldsymbol{b}\, d\theta\, \frac{da}{a^3}, \qquad (7)$$

where $\tilde{f} = \int_{\mathbb{R}^2}\left\langle \Phi_{\boldsymbol{b}}, f \right\rangle \Phi_{\boldsymbol{b}}\, d\boldsymbol{b}$ and $\Phi_{\boldsymbol{b}}(\boldsymbol{x}) = \Phi(\boldsymbol{x}-\boldsymbol{b})$.
For analysis of singularities of f, the low frequency part $\tilde{f}$ is not an issue as it is always C^∞. Unlike Smith transform, curvelet transform does not use a true affine parabolic scaling as a slightly different generating function $\gamma_{a\boldsymbol{0}0}$ is used at each scale $a > 0$.

3.3 Continuous Shearlet Transform

We will follow mainly the definitions and notations in G. Kutyniok and D. Labate[6]. Let $\psi_1, \psi_2 \in L^2(\mathbb{R})$ and $\psi \in L^2(\mathbb{R}^2)$ be given by

$$\hat{\psi}(\xi_1, \xi_2) = \hat{\psi}_1(\xi_1)\hat{\psi}_2\left(\frac{\xi_2}{\xi_1}\right), \quad \xi_1 \neq 0, \xi_2 \in \mathbb{R}, \qquad (8)$$

where ψ_1 satisfies the admissibility condition and $\hat{\psi}_1 \in C_0^\infty(\mathbb{R})$ with $\operatorname{supp}\hat{\psi}_1 \subset [-2, -\frac{1}{2}] \cup [\frac{1}{2}, 2]$ while $\hat{\psi}_2 \in C_0^\infty(\mathbb{R})$ with $\operatorname{supp}\hat{\psi}_2 \subset [-1, 1]$, $\hat{\psi}_2 > 0$ on $(-1, 1)$, and $\|\psi\|_2 = 1$. Given such a *shearlet function* ψ, a *continuous shearlet system* is the family of functions $\psi_{as\boldsymbol{t}}$, $a \in \mathbb{R}^+$, $s \in \mathbb{R}$, $\boldsymbol{t} \in \mathbb{R}^2$, where

$$\psi_{as\boldsymbol{t}} = a^{-\frac{3}{4}}\psi\left(\mathrm{D}_a^{-1}\mathrm{B}_s^{-1}(\cdot - \boldsymbol{t})\right)$$

where B_s is the *shear matrix* $\begin{pmatrix} 1 & -s \\ 0 & 1 \end{pmatrix}$ and D_a is the diagonal matrix $\begin{pmatrix} a & 0 \\ 0 & \sqrt{a} \end{pmatrix}$. The *continuous shearlet transform* of f is then defined for such $(a, s, \boldsymbol{t})$ by

$$\mathcal{SH}_\psi f(a, s, \boldsymbol{t}) = \left\langle f, \psi_{as\boldsymbol{t}} \right\rangle.$$

Many properties of the continuous shearlet are more evident in the frequency domain. So we note here that each $\hat{\psi}_{as\boldsymbol{t}}$ is supported on the set

$$\left\{(\xi_1, \xi_2) \colon \frac{1}{2a} \leq |\xi_1| \leq \frac{2}{a}, \left|\frac{\xi_2}{\xi_1} - s\right| \leq \sqrt{a}\right\}.$$

Reconstruction Formula [6]
Let $\psi \in L^2(\mathbb{R}^2)$ be a shearlet function. Then, for all $f \in L^2(\mathbb{R}^2)$,

$$f = \int_{\mathbb{R}^2}\int_{\mathbb{R}}\int_{\mathbb{R}^+} \left\langle \psi_{as\boldsymbol{t}}, f \right\rangle \psi_{as\boldsymbol{t}}\, \frac{da}{a^3}\, ds\, d\boldsymbol{t} \text{ in } L^2. \qquad (9)$$

If $\operatorname{supp}\hat{f} \subset C = \left\{(\xi_1,\xi_2)\colon |\xi_1| \geq 2 \text{ and } \left|\frac{\xi_2}{\xi_1}\right| \leq 1\right\}$, then

$$f = \int_{\mathbb{R}^2}\int_{-2}^{2}\int_0^1 \langle \psi_{ast}, f\rangle \psi_{ast} \frac{da}{a^3}\, ds\, d\boldsymbol{t} \text{ in } L^2. \quad (10)$$

Even though the second reconstruction formula (10) is valid only for functions with frequency support in the union C of two infinite horizontal trapezoids, it has the advantage that the integral involves only scales a and shear parameters s in bounded sets. A complementary shearlet system $\psi_{ast}^{(v)}$ can be similarly defined so that one has a reconstruction formula which is valid for f with $\operatorname{supp}\hat{f} \subset C^{(v)} = \left\{(\xi_1,\xi_2)\colon |\xi_2| \geq 2 \text{ and } \left|\frac{\xi_2}{\xi_1}\right| > 1\right\}$. Finally, every $f \in L^2(\mathbb{R}^2)$ can be decomposed into three functions with frequency supports in C, $C^{(v)}$, and $W = [-2,2]^2$. The former two functions can then be reconstructed from ψ_{ast} and $\psi_{ast}^{(v)}$ respectively, while the latter is C^∞. Therefore, regularity analysis can be carried out by considering the continuous shearlet transform with respect to these two shearlet systems. For more details, see [6].

4. Common Properties of the Transforms

We shall suppose from this point onward that $\hat{\varphi} \in C^\infty$ and that there exist $C_1' > C_1' > 0$ and $C_2 > 0$ such that $\operatorname{supp}(\hat{\varphi}) \subset ([-C_1', -C_1] \cup [C_1, C_1']) \times [-C_2, C_2]$. This assumption ensures that all our three kernel functions, Hart Smith, curvelet, and shearlet functions, have Fourier supports away from the Y-axis, which in turns results in crucial properties needed to prove our main results.

4.1 Vanishing Directional Moments

A function f of two variables is said to have an *L-order vanishing directional moments along a direction $\boldsymbol{v} = (v_1, v_2)^T \neq \mathbf{0}$* if

$$\int_{\mathbb{R}} b^n f(b\boldsymbol{v}+\boldsymbol{w})db = 0, \quad \text{for all } \boldsymbol{w} \in \mathbb{R}^2 \text{ and } 0 \leq n < L.$$

Lemma 1: Let $\boldsymbol{v} = (v_1, v_2)^T$ be a unit vector.

1. There exists $C < \infty$ (independent of $a, \boldsymbol{b}$ and θ) such that if $|\theta + \arctan(\frac{v_1}{v_2})| \geq C\sqrt{a}$ then the curvelet functions $\gamma_{\boldsymbol{b}a\theta}$ and the Smith functions $\varphi_{a\boldsymbol{b}\theta}$ and $M\varphi_{a\boldsymbol{b}\theta}$ have vanishing directional moments of any order $L < \infty$ along the direction $\boldsymbol{v}$.

2. If $\left|s + \frac{v_1}{v_2}\right| > \sqrt{a}$ then the shearlet functions ψ_{ast} have vanishing directional moments of any order $L < \infty$ along the direction $\boldsymbol{v}$. Here, if v_2 is 0 then $\frac{v_1}{v_2}$ are treated as ∞ so that the assumed inequality holds for all $a \in (0,1)$ and $s \in [-2,2]$, hence ψ_{ast} has vanishing directional moments of any order $L < \infty$ along the direction $\boldsymbol{v} = (v_1, 0)$.

4.2 Smoothness and Decay Properties

Lemma 2: For each $N = 1, 2, \ldots$ there is a constant C_N such that for all $\boldsymbol{x} \in \mathbb{R}^2$ and $\nu \in \mathbb{N}_0^2$

$$|\partial^\nu \gamma_{a\boldsymbol{b}\theta}(\boldsymbol{x})| \leq \frac{C_N a^{-3/4-|\nu|}}{1 + \left\|D_{\frac{1}{a}} R_{-\theta}(\boldsymbol{x}-\boldsymbol{b})\right\|^{2N}} \quad (11)$$

and

$$|\partial^\nu \psi_{ast}(\boldsymbol{x})| \leq \frac{C_N a^{-3/4-|\nu|}(\sqrt{a}+|s|)^{\nu_2}}{1 + \left\|\mathrm{D}_{1/a}\mathrm{B}_{-s}(\boldsymbol{x}-\boldsymbol{t})\right\|^{2N}}. \quad (12)$$

Moreover, (11) also holds for functions $\varphi_{a\boldsymbol{b}\theta}$ and $M\varphi_{a\boldsymbol{b}\theta}$.

5. Singularity Lines

Let $\phi_{a\boldsymbol{b}\theta}$ denote any of the $\gamma_{a\boldsymbol{b}\theta}$, $\varphi_{a\boldsymbol{b}\theta}$, or $M\varphi_{a\boldsymbol{b}\theta}$. Let us quote the following results.[8, 7]

Theorem 1: Let $f \in L^2(\mathbb{R}^2)$, $\boldsymbol{u} \in \mathbb{R}^2$, and assume that $\alpha > 0$ is not an integer. If there exist $\alpha' < 2\alpha$, $\theta_0 \in [0, 2\pi]$, and $A, C < \infty$ such that $|\langle \phi_{a\boldsymbol{b}\theta}, f\rangle|$ is bounded by

$$\begin{cases} Ca^{\alpha+\frac{5}{4}}\left(1 + \left\|\frac{\boldsymbol{b}-\boldsymbol{u}}{a^{1/2}}\right\|^{\alpha'}\right), & \text{if } |\theta-\theta_0| \geq A\sqrt{a} \\ Ca^{\alpha+\frac{3}{4}}\left(1 + \left\|\frac{\boldsymbol{b}-\boldsymbol{u}}{a^{1/2}}\right\|^{\alpha'}\right), & \text{if } |\theta-\theta_0| \leq A\sqrt{a} \end{cases}$$

for all $a \in (0, a_0)$, $\boldsymbol{b} \in \mathbb{R}^2$, and $\theta \in [0, 2\pi)$, then $f \in C^\alpha(\boldsymbol{u})$.

Theorem 2: Let $f \in L^2(\mathbb{R}^2)$, $\boldsymbol{u} \in \mathbb{R}^2$, and assume that $\alpha > 0$ is not an integer. If there exist $\alpha' < 2\alpha$, $-2 \leq s_0 \leq 2$, and $C, C' < \infty$ such that, for each $0 < a < 1$, $-2 \leq s \leq 2$, and $\boldsymbol{t} \in \mathbb{R}^2$, $|\langle \psi_{ast}, P_{C_1} f\rangle|$ is bounded by

$$\begin{cases} Ca^{\alpha+\frac{5}{4}}\left(1 + \left\|\frac{\boldsymbol{t}-\boldsymbol{u}}{a^{1/2}}\right\|^{\alpha'}\right), & \text{if } |s-s_0| > C'\sqrt{a}, \\ Ca^{\alpha+\frac{3}{4}}\left(1 + \left\|\frac{\boldsymbol{t}-\boldsymbol{u}}{a^{1/2}}\right\|^{\alpha'}\right), & \text{if } |s-s_0| \leq C'\sqrt{a}, \end{cases} \quad (13)$$

and

$$\left|\left\langle \psi_{ast}^{(v)}, P_{C_2} f\right\rangle\right| \leq Ca^{\alpha+\frac{5}{4}}\left(1 + \left\|\frac{\boldsymbol{t}-\boldsymbol{u}}{a^{1/2}}\right\|^{\alpha'}\right), \quad (14)$$

then $f \in C^\alpha(\boldsymbol{u})$. Similar statement holds if the inequality (13) holds for $\left\langle \psi_{ast}^{(v)}, P_{C_2} f\right\rangle$ and the inequality (14) holds for $\langle \psi_{ast}, P_{C_1} f\rangle$.

Theorem 3 Let f be bounded with local Hölder exponent $\alpha \in (0,1]$ at point $\boldsymbol{u}$ and $f \in C^{2\alpha+1+\varepsilon}(\mathbb{R}^2, \boldsymbol{v}_{\theta_0})$ for some $\theta_0 \in [0, 2\pi)$ with any fixed $\varepsilon > 0$. Then there exist $\alpha' \in [\alpha-\varepsilon, \alpha]$ and $A, C < \infty$ such that for $a > 0$ and $\boldsymbol{b} \in \mathbb{R}^2$, $|\langle \phi_{a\boldsymbol{b}\theta}, f\rangle|$ is bounded by

$$\begin{cases} Ca^{\alpha+\frac{5}{4}}, & \text{if } |\theta-\theta_0| \geq A\sqrt{a}, \\ Ca^{\alpha'+\frac{3}{4}}\left(1 + \left\|\frac{\boldsymbol{b}-\boldsymbol{u}}{a}\right\|^{\alpha'}\right), & \text{if } |\theta-\theta_0| \leq A\sqrt{a}. \end{cases}$$

For $s_0 \in [-2,2]$ and $\boldsymbol{u} = (u_1, u_2) \in \mathbb{R}^2$, let $\Gamma_{\boldsymbol{u}}$ denote the vertical line passing though $\boldsymbol{u}$ and $\Gamma_{\boldsymbol{u},s_0}$ denote the line passing through $\boldsymbol{u}$ with slope $-\frac{1}{s_0}$. Observe that we may write $\Gamma_{\boldsymbol{u}} = \Gamma_{\boldsymbol{u},0}$ so that $(x_1, x_2) \in \Gamma_{\boldsymbol{u},s_0}$ if and only if $x_1 = -s_0(x_2 - u_2) + u_1$. Recall that if $\Gamma \subseteq \mathbb{R}^2$ and $\rho > 0$, then $\Gamma(\rho)$ is the ρ-neighborhood of Γ, i.e. the set of all points whose distance to Γ is less than ρ.

Theorem 4 Let $f \in C^\alpha(\Gamma_{\boldsymbol{u},s_0}, \Gamma_{\boldsymbol{u},s_0}(\rho)\,; (1,0))$ and bounded for some $\alpha \in (0,1]$, $\boldsymbol{u} \in \mathbb{R}^2$, $s_0 \in [-2,2]$ and $\rho > 1$. Suppose also that f is in $C^{2\alpha+1+\varepsilon}(\Gamma_{\boldsymbol{u},s_0}(\rho)\,; \mathrm{B}_{s_0}(0,1))$ for some fixed $\varepsilon > 0$. Then there exists $C < \infty$ such that if $0 < a < a_0 < 1$ and $\boldsymbol{t} \in \Gamma_{\boldsymbol{u}}(r)$ with $r < \rho/2$ and $s \in [-2,2]$, the continuous shearlet transform $\langle \psi_{as\boldsymbol{t}}, f \rangle$ is bounded in magnitude by

$$\begin{cases} Ca^{\alpha+\frac{5}{4}}, & \text{if } |s - s_0| > \sqrt{a}, \\ Ca^{\alpha+\frac{3}{4}} \left(1 + \left|\dfrac{d_{s_0}(\boldsymbol{t},\boldsymbol{u})}{a}\right|^\alpha\right), & \text{if } |s - s_0| \le \sqrt{a}, \end{cases}$$

where $d_{s_0}(\boldsymbol{t},\boldsymbol{u}) = |t_1 + s_0 t_2 - u_1 - s_0 u_2|$ denotes the distance between the parallel lines with slope $-\frac{1}{s_0}$ (vertical line if $s_0 = 0$) and passing through $\boldsymbol{t}$ and $\boldsymbol{u}$ respectively.

Edge analysis has been done successfully using the continuous shearlet transform ([11, 4, 3, 6]). They consider the shearlet transform of the characteristic function of a set with piecewise smooth boundary and found that, at a regular boundary point $\boldsymbol{t}$, the shearlet transform decays like $a^{3/4}$ if $s = s_0 = \pm\frac{v_1}{v_2}$ and decays rapidly at other $s \neq s_0$, where $\boldsymbol{v} = (v_1, v_2)$ is the normal vector of the boundary curve at $\boldsymbol{t}$. Since this characteristic function has Hölder exponent 0 (bounded and discontinuous) at any boundary point in the normal direction, this decay rate of $a^{3/4}$ at $s = s_0 = 0$ agrees with that of Theorem 4. However, when $s_0 \neq 0$ the two directions in Theorem 4 along which regularity is assumed are not perpendicular. More comparisons of our results and the aforementioned work are needed.

References:

[1] Emmanuel J. Candès and David L. Donoho. Continuous curvelet transform. I: Resolution of the wavefront set. *Appl. Comput. Harmon. Anal.*, 19(2):162–197, 2005.

[2] Emmanuel J. Candès and David L. Donoho. Continuous curvelet transform. II: Discretization and frames. *Appl. Comput. Harmon. Anal.*, 19(2):198–222, 2005.

[3] K. Guo, Labate D., and W-Q. Lim. Edge analysis and identification using the continuous shearlet transform. *Appl. Comput. Harmon. Anal.*, 2008. In Press.

[4] K. Guo and D. Labate. Characterization and analysis of edges using the continuous shearlet transform. 2008. Preprint.

[5] S. Jaffard. Multifractal functions: Recent advances and open problems. *Menuscript*, 2004.

[6] G. Kutyniok and D. Labate. Resolution of the wavefront set using continuous shearlets. *Trans. AMS.*, 105(1):157–175, 2007.

[7] P. Lakhonchai, J. Sampo, and S. Sumetkijakan. Shearlet transforms and hölder regularities. 2009. Preprint.

[8] J. Sampo and S. Sumetkijakan. Estimations of Hölder regularities and direction of singularity by Hart Smith and curvelet transforms. *Journal of Fourier Analysis and Applications*, 15(1):58–79, 2009.

[9] S. Seuret and J. Lévy Véhel. The local Hölder function of a continuous function. *Appl. Comput. Harmon. Anal.*, 13(3):263–276, 2002.

[10] Hart F. Smith. A Hardy space for Fourier integral operators. *J. Geom. Anal.*, 8(4):629–653, 1998.

[11] S. Yi, D. Labate, G.R. Easley, and H. Krim. Edge detection and processing using shearlets. 2008. Preprint.