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Global solutions to rough differential equations with unbounded vector fields*

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Abstract

We give a sufficient condition to ensure the global existence of a solution to a rough differential equation whose vector field has a linear growth. This condition is slightly weaker than the ones already given and may be used for geometric as well as non-geometric rough paths with values in any suitable (finite or infinite dimensional) space. For this, we study the properties the Euler scheme as done in the work of A.M. Davie.

Keywords: controlled differential equations, rough paths, Euler scheme, global solution to differential equation, rough differential equation.
AMS Classification: 60H10, 65C30

1 Introduction

Initiated a decade ago, the theory of rough paths imposed itself as a convenient tool to define stochastic calculus with respect to a large class of stochastic processes out of the range of semi-martingales (fractional Brownian motion, ...) and also allows one to define pathwise stochastic differential equations [6, 7, 9, 10, 14, 16, 17].

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The idea is that one may define integrals of differential forms along irregular paths or solutions of differential equations driven by irregular paths by extending properly such paths in a suitable non-commutative space. These extensions encode in some sense the "iterated integrals" of the paths. Let us denote by x the driving path which lives in a tensor space and Lie group $T(\mathbb{R}^d) := \mathbb{R} \oplus \mathbb{R}^d \oplus (\mathbb{R}^d \otimes \mathbb{R}^d)$ and that projects onto some continuous path of finite p-variations on $\mathbb{R}^d$ with $p \in [2, 3)$.

The goal of this article is to study the rough differential equation (RDE)

$$y_t = y_0 + \int_0^t f(y_s)\,\mathrm{d}x_s \tag{1}$$

when f is not bounded.

The special case of a linear vector field f was studied for example in the articles [1, 6, 8, 11, 15].

Standard results asserts the existence of a solution to (1) when f is bounded with a bounded derivative which is γ-Hölder continuous, $2+\gamma > p$. Uniqueness and continuity of the Itô map $x \mapsto y$ follows when f is bounded, twice differentiable with bounded derivatives and $\nabla^2 f$ is γ-Hölder continuous with $2+\gamma > p$ [6, 9, 10, 14, 16, 17]. One knows from [2] that these conditions on f are essentially sharp.

Providing an alternative approach to the one of T. Lyons based on a fixed point theorem, A.M. Davie studied in [2] the Euler scheme defined by

$$y_{i+1}^n = y_i^n + f(y_i^n)x_{t_i,t_{i+1}}^1 + f \cdot \nabla f(y_i^n)x_{t_i,t_{i+1}}^2$$

where $x_{s,t} = x_s^{-1} \otimes x_t$ and x^1 (resp. x^2) is the projection of x onto $\mathbb{R}^d$ (resp. $\mathbb{R}^d \otimes \mathbb{R}^d$). With conditions only on the regularity of the vector field, local existence is shown, and a condition is given to get global existence.

In [5, 6], P. Friz and N. Victoir used this idea to provide an alternative construction of the solutions of RDE that relies on sub-Riemannian geodesics and hence of geometric rough paths. In [6, Exercise 10.61], they show that it is not necessary that the vector field is bounded, provided that f is Lipschitz continuous, ∇f is γ-Hölder continuous and $f \cdot \nabla f$ is also γ-Hölder continuous.

In [11], we have also studied the existence of a global solution, by using the approach on fixed point theorem. Global existence was proved but only under some sub-linear conditions on the growth of the vector field.

If $1 \leqslant p < 2$, then existence does not requires the boundedness of the vector field only γ-Hölder regularity with $1 + \gamma > p$ [12].

If $d = 1$, H. Doss [4] and H. Sussmann [18] have shown that a solution to an equation of type (1) may be defined by considering the solution to the ODE $\mathrm{d}z_t = f(z_t)\,\mathrm{d}t$ and setting $y_t = z_{x_t}$. Hence, global existence holds under a linear growth condition on f as it follows easily from an application of the Gronwall inequality.

However, the situation is more intricate for $p \geqslant 2$.

If x is a p-rough path (a rough path of finite p-variation) with $p \in [2,3)$, then for any function φ of finite $p/2$-variations with values in $\mathbb{R}^d \otimes \mathbb{R}^d$, $z = x + \varphi$ is also a p-rough path. In addition, as shown in [13] for the general case, the solution to $y_t = y_0 + \int_0^t f(y_s)\,\mathrm{d}z_s$ is also solution to

$$y_t = y_0 + \int_0^t f(y_s)\,\mathrm{d}x_s + \int_0^t f \cdot \nabla f(y_s)\,\mathrm{d}\varphi_s.$$

Consider any rough path z living above the path 0 on $\mathbb{R}^d$ with $\varphi(t) = ct$ for a matrix c (if c is anti-symmetric, then such a rough path is the limit of a sequence of smooth paths lifted in the tensor space by their iterated integrals). Then y is solution to

$$y_t = y_0 + \int_0^t f \cdot \nabla f(y_s) c\,\mathrm{d}s.$$

Thus, explosion may occurs according to the behavior of $f \cdot \nabla f$. In particular, f may have a bounded derivative, but $f \cdot \nabla f$ may grow faster that linearly, which may lead to an explosion.

Example 1 (M. Gubinelli). Consider the solution y of the RDE $y_t = a + \int_0^t f(y_s)\,\mathrm{d}x_s$ living in $\mathbb{R}^2$ and driven by the rough path $x_t = (1, 0, (1 \otimes 1)t)$ with values in $1 \oplus \mathbb{R} \oplus (\mathbb{R} \otimes \mathbb{R})$. This rough path lies above the constant path at $0 \in \mathbb{R}$ and has only a pure area part which proportional to t. Then y is also a solution to $y_t = a + \int_0^t (f \cdot \nabla f)(y_s)\,\mathrm{d}s$ (See [13]). The vector field $f \in \mathbb{R}^2 \to \mathrm{L}(\mathbb{R}, \mathbb{R}^2)$ given by

$$f(\xi) = (\sin(\xi_2)\xi_1, \xi_1),\ \ \xi = (\xi_1, \xi_2) \in \mathbb{R}^2$$

has a linear growth but

$$(f \cdot \nabla f)(\xi) = (\sin^2(\xi_2)\xi_1 + \xi_1^2 \cos(\xi_2), \sin(\xi_2)\xi_1).$$

Take the initial point $a = (a_1, 0)$ with $a_1 > 0$. Then $(y_t)_2 = 0$ and $(y_t)_1 = a_1 + \int_0^t (y_s)_1^2\,\mathrm{d}s$ so that $(y_t)_1 \to +\infty$ in finite time. This proves that explosion may occur in a finite time.

Using the ideas from [2], we show the existence of a global solution to (1) when f has a bounded derivative and is such that $f \cdot \nabla f$ is γ-Hölder continuous. This leads to slightly weaker conditions than the one of P. Friz and N. Victoir in [6]. In addition, we are not bound to use geometric rough

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paths. While the above-mentioned approach relies on the sub-Riemannian geometry (and then in some finite, Euclidean structure), our approach relies on generic computations and could be used in any suitable setting, finite or infinite-dimensional when possible.

Besides, we show that if f is twice differentiable and is such that $F := f \cdot \nabla f$ is bounded and differentiable with a derivative which is γ-Hölder continuous, then then solution is unique and the Itô map $x \mapsto y$ is locally Lipschitz continuous. The rate of convergence of the Euler scheme is also studied, as well as the distance between two Euler schemes under this regularity condition. Agina, these computations ara valid for a whatever the Banach space the drive x lives in, whatever his dimension.

Let us note that we are not in the same setting as the one introduced by T. Lyons, so that we use the notion of solution introduced by A.M. Davie in [2]. We show finally that the two notions of solutions coincide when ∇f is γ-Hölder continuous. This notion of solution is also the one used in the framework proposed by M. Gubinelli (See [7] and subsequent works).

2 Notations and hypotheses

Let ω be a control. By this, we mean a function defined from $\Delta^2 := \{0 \leqslant s \leqslant t \leqslant T\}$ to $\mathbb{R}_+$ which is continuous close to its diagonal and super-additive

$$\omega(s,r) + \omega(r,t) \leqslant \omega(s,t),\ 0 \leqslant s < r < t \leqslant T.$$

For the sake of simplicity, we assume here that ω is continuous and that $\omega(s,t) > 0$ as soon as $t > s$.

Let x be a path with values in $T(\mathbb{R}^d) = \mathbb{R} \oplus \mathbb{R}^d \oplus (\mathbb{R}^d \otimes \mathbb{R}^d)$. We set $x_{s,t} = x^{-1} \otimes x_t$. The part of x in $\mathbb{R}^d$ is denoted by x^1 and the part in $\mathbb{R}^d \otimes \mathbb{R}^d$ is denoted by x^2. Then x is a *rough path of finite p-variation controlled by ω* when the quantity

$$\|x\| := \sup_{0 \leqslant s < t \leqslant T} \max \left\{ \frac{|x^1_{s,t}|}{\omega(s,t)^{1/p}}, \frac{|x^2_{s,t}|}{\omega(s,t)^{2/p}} \right\}$$

is finite for a fixed p.

If $\omega(s,t) = t - s$, then we work indeed with paths that are $1/p$-Hölder continuous.

Throughout all this article, we consider only the case where $p \in [2,3)$.

For a path y with values in $\mathbb{R}^m$, we set

$$\|y\| := \sup_{0 \leqslant s < t \leqslant T} \frac{|x^1_{s,t}|}{\omega(s,t)^{1/p}}.$$

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A vector field is an application f which is linear from $\mathbb{R}^m$ to $\mathrm{L}(\mathbb{R}^d, \mathbb{R}^m)$, the space of linear applications from $\mathbb{R}^d$ to $\mathbb{R}^m$. With indices, we set $f(x) = e_i f^i_j(x) \widehat{e}^*_j$, where $\{e_i\}_{i=1,\dots,m}$ is the canonical basis of $\mathbb{R}^m$, and $\{\widehat{e}^*_j\}_{j=1,\dots,d}$ is the dual of the canonical basis of $\mathbb{R}^d$. We set

$$f \cdot \nabla f(x) := \sum_{i=1}^{m} e_i \sum_{k=1}^{d} \frac{\partial f^i_j}{\partial x_k}(x) f^k_\ell(x) \widehat{e}^*_\ell \otimes \widehat{e}^\star_j,$$

which means that $f \cdot \nabla f$ is an application from $\mathbb{R}^m$ to $\mathrm{L}(\mathbb{R}^d \otimes \mathbb{R}^d, \mathbb{R}^m)$.

Hypothesis 1. The function f is continuously differentiable from $\mathbb{R}^m$ to $\mathrm{L}(\mathbb{R}^d, \mathbb{R}^m)$ and is such that ∇f is bounded and $F(x) := f(x) \cdot \nabla f(x)$ is γ-Hölder continuous with norm $H_\gamma(F)$ from $\mathbb{R}^m$ to $L(\mathbb{R}^m \otimes \mathbb{R}^d, \mathbb{R}^m)$.

Note that this hypothesis is slightly different from the one given usually to prove existence to solutions of RDE where it is assumed that ∇f is γ-Hölder continuous.

Hypothesis 2. We assume that $p \in [2, 3)$ and that $\theta := (2+\gamma)/p$ is greater than 1.

Definition 1 (Solution in the sense of Lyons)**.** We call by a *solution to* (1) *in the sense of Lyons* the projection onto $\mathbb{R}^m$ of a rough path z of finite p-variations controlled by ω with values in $T(\mathbb{R}^m \oplus \mathbb{R}^d)$ which solves the following equation

$$z_t = z_0 + \int_0^t g(z_s)\, \mathrm{d}z_s, \tag{2}$$

where z projects onto $T(\mathbb{R}^d)$ as x, onto $\mathbb{R}^m$ as y, and g is the differential form $g(y, x) = \mathrm{d}x + f(y)\, \mathrm{d}x$. The integral in (2) shall be understood as the "rough integral", that is as an integral in the sense of rough path.

Note that the definitions of z involves the "cross-iterated integrals" between x and y, and requires that the derivative of f is γ-Hölder continuous.

Under Hypothesis 1, it is not compulsory that ∇f is γ-Hölder continuous, so that we shall use another notion of solution, since we cannot use a fixed point theorem that relies on the definition of a rough integral.

3 Solution in the sense of Davie

The notion of solution of (1) we use is the notion of solution in the sense of Davie, introduced in [2].

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Definition 2 (Solution in the sense of Davie)**.** A solution of (1) in the sense of Davie is a continuous path y from $[0,T]$ to $\mathbb{R}^m$ of finite p-variation controlled by ω such that for some constant L,

$$|y_t - y_s - \mathfrak{D}(s,t)| \leqslant L\omega(s,t)^\theta,\ \forall (s,t) \in \Delta^2 \tag{3}$$

with

$$\mathfrak{D}(s,t) = f(y_s)x^1_{s,t} + F(y_s)x^2_{s,t}.$$

The next propositions, which assume the existence of a solution in the sense of Davie, will be proved below in Section 4

Proposition 1. *Let y be a solution of (1) in the sense of Davie under Hypotheses 1 and 2. Then $\mathfrak{D}(s,t)$ is an almost rough path whose associated rough path is y and for a family of partitions $\left\{\{t^n_i\}_{i=0,\dots,n}\right\}_{n\in\mathbb{N}^*}$ of $[0,t]$ whose meshes decrease to 0,*

$$y_t = y_0 + \lim_{n\to\infty} \sum_{i=0}^{n} \mathfrak{D}(t^n_i, t^n_{i+1}). \tag{4}$$

Proposition 2 (Boundedness of the solution)**.** *Let y be a solution to (1) in the sense of Davie under Hypotheses 1 and 2. Then $\|y\|$ and $\|y\|_\infty$ are bounded by some constants that depend only on $\|\nabla f\|_\infty$, $H_\gamma(F)$, $|f(y_0)|$, $|y_0|$, $\|x\|$, $\omega(0,T)$, γ and p. More precisely, Then there exist some constants C_1 depending only on $\|\nabla f\|_\infty$, $H_\gamma(F)$, $\|x\|$, p and γ and C_2 depending only on $|f(0)|$, $\|\nabla f\|_\infty$, $H_\gamma(F)$, $\|x\|$, p, γ such that*

$$\sup_{t\in[0,T]} |y_t| \leqslant R(T)|y_0| + C_2(R(T)-1)$$

with

$$R(T) \leqslant \exp(C_1 \max\{\omega(0,T)^{1/p}, \omega(0,T)\}).$$

4 Proofs of Propositions 1 and 2

Let us set for $0 \leqslant s \leqslant r \leqslant t \leqslant T$,

$$\mathfrak{D}(s,r,t) := \mathfrak{D}(s,t) - \mathfrak{D}(s,r) - \mathfrak{D}(r,t).$$

Then the main idea of the proofs (as well as the proofs of the other theorems) are the following: First, we find a function $C(\|y\|, L, T)$ such that $|\mathfrak{D}(s,r,t)| \leqslant C(\|y\|, L, T)\omega(s,t)^\theta$ for all $(s,t) \in \Delta^2$. From the sewing Lemma (See for example [10, Theorem 5, p. 89]), after having shown that y is the

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rough path corresponding to the almost rough path $(\mathfrak{D}(s,t))_{(s,t)\in\Delta^2}$, we get that for some universal constant M,

$$|y_t - y_s - \mathfrak{D}(s,t)| \leqslant MC(\|y\|, L, T)\omega(s,t)^\theta.$$

Then, after having estimated $|\mathfrak{D}(s,t)\| \leqslant C'(\|y\|, L, T)\omega(s,t)^{1/p}$, we get an inequality of type

$$\|y\| \leqslant MC(\|y\|, L, T)\omega(0,T)^{\theta-1/p} + C'(\|y\|, L, T).$$

A careful examination of the functions $C(\|y\|, L, T)$ and $C'(\|y\|, L, T)$ shows that indeed L itself depends on y_0, $\|y\|$ and T and that

$$\|y\| \leqslant A(y_0, T) + B(T)\|y\|$$

with $B(T)$ decreasing to 0 as T decreases to 0. Then, choosing T small enough implies that $\|y\|$ is bounded in small time and then for any time T using the arguments presented in Appendix. This idea is the central one used for example in [11, 12].

Let us set

$$B_1 := \sup_{t\in[0,T]} |F(y_t)|,\ B_2(s,t) := y_t - y_s - f(y_s)x^1_{s,t},$$
$$B_3(a,b) := f(b) - f(a),\ B_4(a,b) := F(b) - F(a),$$
$$\text{and } B_5(s,t) := f(y_t) - f(y_s) - F(y_s)x^1_{s,t}$$

as well as

$$\mu := \omega(0,T)^{1/p}.$$

Remark 1. The quantity μ will be used to denote the "short time" horizon and is the central quantity for getting our estimates.

Since F is γ-Hölder continuous,

$$B_1 \leqslant |F(y_0)| + H_\gamma(F)\|y\|^\gamma \mu^\gamma.$$

If y is a solution in the sense of Davie with a constant L,

$$|B_2(s,t)| \leqslant L\omega(s,t)^\theta + B_1\|x\|\omega(s,t)^{2/p}.$$

Since f is Lipschitz continuous,

$$|B_3(a,b)| \leqslant \|\nabla f\|_\infty |b-a|.$$

In addition,

$$|B_4(a,b)| \leqslant H_\gamma(F)|b-a|^\gamma.$$

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Finally,

$$
\begin{aligned}
f(y_t) - f(y_s) &= \int_0^1 \nabla f(y_s + \tau y_{s,t}) y_{s,t} \,\mathrm{d}\tau \\
&= \int_0^1 \nabla f(y_s + \tau y_{s,t}) f(y_s) x^1_{s,t} \,\mathrm{d}\tau + \int_0^1 \nabla f(y_s + \tau y_{s,t}) B_2(s,t) \,\mathrm{d}\tau \\
&= -\int_0^1 \nabla f(y_s + \tau y_{s,t})(f(y_s + \tau y_{s,t}) - f(y_s)) x^1_{s,t} \,\mathrm{d}\tau \\
&\quad + \int_0^1 F(y_s + \tau y_{s,t}) x^1_{s,t} \,\mathrm{d}\tau + \int_0^1 \nabla f(y_s + \tau y_{s,t}) B_2(s,t) \,\mathrm{d}\tau \\
&= \int_0^1 F(y_s + \tau y_{s,t}) x^1_{s,t} \,\mathrm{d}\tau + \int_0^1 \nabla f(y_s + \tau y_{s,t})(B_2(s,t) - B_3(y_s, y_s + \tau y_{s,t}) x^1_{s,t}) \,\mathrm{d}\tau \\
&= F(y_s) x^1_{s,t} + \int_0^1 B_4(y_s, y_s + \tau y_{s,t}) x^1_{s,t} \,\mathrm{d}\tau \\
&\quad + \int_0^1 \nabla f(y_s + \tau y_{s,t})(B_2(s,t) - B_3(y_s, y_s + \tau y_{s,t}) x^1_{s,t}) \,\mathrm{d}\tau. \quad (5)
\end{aligned}
$$

This proves that

$$
\begin{aligned}
|B_5(s,t)| \leqslant{} & H_\gamma(F) \|y\|^\gamma \|x\| \omega(s,t)^{(1+\gamma)/p} + \|\nabla f\|_\infty^2 \|x\| \|y\| \omega(s,t)^{2/p} \\
& + \|\nabla f\|_\infty L \omega(s,t)^\theta + \|\nabla f\|_\infty |F(y_0)| \|x\| \omega(s,t)^{2/p} \\
& + H_\gamma(F) \|y\|^\gamma \omega(0,T)^{\gamma/p} \|x\| \omega(s,t)^{2/p}.
\end{aligned}
$$

Lemma 1. *For any* $0 \leqslant s \leqslant r \leqslant t \leqslant T$,

$$
|\mathfrak{D}(s,r,t)| \leqslant (C_3(\mu)\|y\|^\gamma + C_4(\mu)\|y\| + C_5(\mu) L + C_6(\mu, y_0)) \omega(s,t)^\theta
$$

with

$$
\begin{aligned}
C_3(\mu) &:= H_\gamma(F)\big(\|x\|^2(1+\mu^{1+\gamma}) + \|x\|\big), \\
C_4(\mu) &:= \|\nabla f\|_\infty^2 \|x\|^2 \mu, \\
C_5(\mu) &:= \|\nabla f\|_\infty \|x\| \mu \\
\text{and } C_6(\mu, y_0) &:= \|\nabla f\|_\infty |F(y_0)| \|x\|^2 \mu^{1-\gamma} \leqslant |f(y_0)| \|\nabla f\|_\infty^2 \|x\|^2 \mu^{1-\gamma}.
\end{aligned}
$$

Proof. With $x^2_{s,t} = x^2_{s,r} + x^2_{r,t} + x^1_{s,r} \otimes x^1_{r,t}$,

$$
\mathfrak{D}(s,r,t) = (f(y_s) - f(y_r)) x^1_{r,t} + F(y_s) x^1_{s,r} \otimes x^1_{r,t} + (F(y_s) - F(y_r)) x^2_{r,t}.
$$

Hence

$$
|\mathfrak{D}(s,r,t)| \leqslant |B_5(s,r)| \|x\| \omega(s,t)^{1/p} + |B_4(y_s, y_r)| \|x\| \omega(s,t)^{2/p}.
$$

This proves the result. □

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Lemma 2. *For any* $(s,t)\in\Delta^2$,

$$|\mathfrak{D}(s,t)|\leqslant(C_7(\mu,y_0)+C_8(\mu)\|y\|^\gamma+C_9(\mu)\|y\|)\omega(s,t)^{1/p}$$

with

$$\begin{aligned}C_7(\mu,y_0)&:=(|f(y_0)|+|F(y_0)|\mu)\|x\|\leqslant|f(y_0)|(1+\|\nabla f\|_\infty\mu)\|x\|,\\ C_8(\mu)&:=H_\gamma(F)\|x\|\mu^\gamma,\\ \text{and } C_9(\mu)&:=\|\nabla f\|_\infty\|x\|\mu.\end{aligned}$$

Proof. This follows from

$$\begin{aligned}|\mathfrak{D}(s,t)|\leqslant|f(y_0)|\|x\|\omega(s,t)^{1/p}&+\|y\|\|x\|\|\nabla f\|_\infty\omega(s,t)^{2/p}\\&+|F(y_0)|\|x\|\omega(s,t)^{2/p}+\|y\|^\gamma\|x\|H_\gamma(F)\omega(s,t)^{(1+\gamma)/p}.\end{aligned}$$

This proves the Lemma. □

We have now all the required estimates to prove Propositions 1 and 2.

Proof of Proposition 1. It follows from Lemma 1 that $\{\mathfrak{D}(s,t)\}_{(s,t)\in\Delta^2}$ is an almost rough path. From the sewing lemma (See for example [10, Theorem 5, p. 89]), there exists a path $\{z_t\}_{t\in[0,T]}$ as well as a constant M depending only on θ such that

$$|z_t-z_s-\mathfrak{D}(s,t)|\leqslant M\omega(s,t)^\theta,\ \forall(s,t)\in\Delta^2. \tag{6}$$

This function is unique in the class of functions satisfying (6) and with (3), z is equal to y. Equality (4) follows from the very construction of z. □

Proof of Proposition 2. We assume first that y is a solution in the sense of Davie with constant L. Let us note that

$$|y_t-y_s|\leqslant|y_t-y_s-\mathfrak{D}(s,t)|+|\mathfrak{D}(s,t)|\leqslant L\omega(s,t)^\theta+|\mathfrak{D}(s,t)|.$$

With Lemma 2, if $\mu=\omega(0,T)^{1/p}$ is small enough such that

$$C_8(\mu)\leqslant\frac{1}{4}\text{ and }C_9(\mu)\leqslant\frac{1}{4}, \tag{7}$$

then

$$\|y\|\leqslant L\mu^{1+\gamma}+C_7(\mu,y_0)+\frac{1}{4}\|y\|+\frac{1}{4}\|y\|^\gamma.$$

Since $\gamma\leqslant 1$, if $\|y\|\geqslant 1$, then $\|y\|^\gamma\leqslant\|y\|$ and then for μ small enough (note that the choice of μ does not depend on y_0),

$$\|y\|\leqslant 2\max\{1,L\mu^{1+\gamma}+C_7(\mu,y_0)\}. \tag{8}$$

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Since $C_7(\mu, y_0)$ decreases with μ, the boundedness of $\|y\|$ and $\|y\|_\infty$ also hold, with a different constant, for any time T by applying Proposition 6 in Appendix.

Now, if y is a solution in the sense of Davie with a constant L, it is also a solution in the sense of Davie with the constant

$$L' := \sup_{(s,t)\in\Delta^2,\ s\neq t} \frac{|y_t - y_s - \mathfrak{D}(s,t)|}{\omega(s,t)^\theta}.$$

From the sewing Lemma, there exists a universal constant M depending only on θ such that

$$L' \leqslant M \sup_{(s,t)\in\Delta^2,\ s\neq t,\ r\in(s,t)} \frac{|\mathfrak{D}(s,r,t)|}{\omega(s,t)^\theta}.$$

From Lemma 1 and the inequalities $a^\gamma \leqslant 1 + a$ for $a \geqslant 0$ and $\gamma \in [0,1]$ as well as $|f(y_0)| \leqslant |f(0)| + \|\nabla f\|_\infty |y_0|$,

$$\begin{aligned} L' \leqslant & M(C_3(\mu)\|y\|^\gamma + C_4(\mu)\|y\| + C_5(\mu)L' + C_6(\mu, y_0)) \\ \leqslant & C_{10}(\mu) + C_{13}(\mu)\|y\| + MC_5(\mu)L' + C_{12}(\mu)|y_0|, \end{aligned}$$

with

$$\begin{aligned} C_{10}(\mu) &:= MC_3(\mu) + |f(0)|C_{11}(\mu), \\ C_{11}(\mu) &:= M\|\nabla f\|_\infty^2 \|x\|^2 \mu^{1-\gamma}, \\ C_{12}(\mu) &:= \|\nabla f\|_\infty C_{11}(\mu), \\ \text{and } C_{13}(\mu) &:= M(C_3(\mu) + C_4(\mu)). \end{aligned}$$

If

$$MC_5(\mu) \leqslant 1/2, \tag{9}$$

then

$$L' \leqslant 2C_{10}(\mu) + 2C_{13}(\mu)\|y\| + 2C_{12}(\mu)|y_0|.$$

With (8), under conditions (7) and (9) on μ,

$$\|y\| \leqslant 2 + 4C_{10}(\mu)\mu^{1+\gamma} + 4C_{13}(\mu)\mu^{1+\gamma}\|y\| + 4C_{12}(\mu)|y_0|\mu^{1+\gamma} + C_7(\mu, y_0),$$

Under the additional condition that

$$4C_{13}(\mu)\mu^{1+\gamma} \leqslant \frac{1}{2}, \tag{10}$$

we get that

$$\|y\| \leqslant C_{14}(\mu) + C_{15}(\mu)|y_0|$$

with

$$C_{14}(\mu) := 4 + 8C_{10}(\mu)\mu^{1+\gamma} + 2|f(0)|(1 + \|\nabla f\|_\infty \mu)\|x\|,$$
$$C_{15}(\mu) := 8C_{12}(\mu)\mu^{1+\gamma} + 2\|\nabla f\|_\infty (1 + \|\nabla f\|_\infty \mu)\|x\|.$$

Due to the dependence of the constants with respect to μ, conditions (7), (9) and (10) hold true if $\mu \leqslant K$ for a constant K depending only on $\|\nabla f\|_\infty$, $H_\gamma(F)$, $\|x\|$, p and γ.

Indeed, (8) also holds for $y_{|[S,S']}$ on any time interval $\omega(S,S')$ provided that $\omega(S,S')^{1/p}$ is small enough. The result follows from Proposition 7 in Appendix. □

5 Existence of a solution

The existence of a solution is proved thanks to the Euler scheme, which allows one to study to define a family of paths that is uniformly bounded with the uniform norm.

Let us fix a partition $\{t_i\}_{i=0,\dots,n}$ of $[0,T]$ and let us set $x_{i,j} := x_{t_i}^{-1} \otimes x_{t_j}$ and $\omega_{i,j} := \omega(t_i, t_j)$.

Let us consider the Euler scheme

$$y_{i+1} = y_i + f(y_i)x^1_{i,i+1} + F(y_i)x^2_{i,i+1},\ i = 0, \dots, n-1$$

as well as the family $\{y_{i,j}\}_{0 \leqslant i < j \leqslant n}$ defined by

$$y_{i,j} := f(y_i)x^1_{i,j} + F(y_i)x^2_{i,j}.$$

We set

$$\|y\|_{\star,0,n} := \sup_{0 \leqslant i < j \leqslant n} \frac{|y_j - y_i|}{\omega_{i,j}^{1/p}}. \tag{11}$$

Lemma 3. *If $\mu := \omega_{0,n}^{1/p}$ is small enough so that*

$$C_5(\mu) = \|\nabla f\|_\infty \|x\|\mu \leqslant \frac{1-K}{4} \text{ with } K := 2^{1-\theta} < 1 \tag{12}$$

and

$$L := 4\frac{C_6(\mu, y_0) + C_3(\mu)\|y\|^\gamma_{\star,0,n} + C_4(\mu)\|y\|_{\star,0,n}}{1-K}$$

then

$$|y_{i,k} - y_k - y_i| \leqslant L\omega^\theta_{i,k} \tag{13}$$

for all $0 \leqslant i \leqslant k \leqslant n$.

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Proof. The proof of this Lemma follows along the lines the one of Lemma 2.4 in [2] and relies on an induction on $k-i$.

Clearly, (13) is true for $k=i+1$. Fix $m>1$ and let us assume that (13) holds for any $i<k$ such that $k-i<m$.

Let us choose $i<k$ such that $k-i=m$, $m\geqslant 2$. Let j be the index such that $\omega_{i,j}\leqslant\omega_{i,k}/2$ and $\omega_{i,j+1}>\omega_{i,k}/2$. Since ω is super-additive, $\omega_{j+1,k}\leqslant\omega_{i,k}/2$ and then

$$\omega_{i,j}^{\theta}+\omega_{j+1,k}^{\theta}\leqslant 2^{1-\theta}\omega_{i,k}^{\theta}=K\omega_{i,k}^{\theta}. \tag{14}$$

We set

$$y_{i,j,k}:=y_{i,k}-y_{i,j}-y_{j,k}.$$

For j as above, since $y_{j,j+1}-y_j-y_{j+1}=0$, and using (14),

$$\begin{aligned}|y_{i,k}-y_k+y_i|&\leqslant|y_{i,j,k}|+|y_{i,j}-y_j+y_i|+|y_{j,k}-y_k+y_j|\\&\leqslant|y_{i,j,k}|+|y_{j,j+1,k}|+|y_{j+1,k}-y_k+y_{j+1}|+|y_{i,j}-y_i+y_j|\\&\qquad+|y_{j,j+1}-y_{j+1}+y_j|\leqslant|y_{i,j,k}|+|y_{j,j+1,k}|+LK\omega_{i,k}^{\theta}.\end{aligned}$$

Since $x_{i,k}^2=x_{i,j}^2+x_{j,k}^2+x_{i,j}^1\otimes x_{j,k}^1$,

$$y_{i,j,k}=(f(y_j)-f(y_i))x_{j,k}^1+(F(y_j)-F(y_i))x_{j,k}^2+F(y_i)x_{i,j}^1\otimes x_{j,k}^1.$$

Using the same computations as (5),

$$\begin{aligned}y_{i,j,k}&=(f(y_i)-f(y_j))x_{i,j}^1-F(y_i)x_{i,j}^1\otimes x_{j,k}^1+(F(y_i)-F(y_j))x_{j,k}^2\\&=\int_0^1(F(y_i+\tau(y_j-y_i))-F(y_i))x_{i,j}^1\otimes x_{j,k}^1\,\mathrm{d}\tau\\&\quad+\int_0^1\nabla f(y_i+\tau(y_j-y_i))(y_j-y_i-y_{i,j}+F(y_i)x_{i,j}^2)x_{i,j}^1\,\mathrm{d}\tau\\&\quad-\int_0^1\int_0^1\tau\nabla f(y_i+\tau(y_j-y_i))\nabla f(y_i+\rho\tau(y_j-y_i))(y_j-y_i)\,\mathrm{d}\rho\,\mathrm{d}\tau x_{i,j}^1\otimes x_{j,k}^1\\&\quad+(F(y_i)-F(y_j))x_{j,k}^2.\end{aligned} \tag{15}$$

We then face the same estimates as the one in the proof of Lemma 1, where we replace the fact that y is a solution in the sense of Davie with a constant L by our induction hypothesis on $|y_{i,j}-y_j+y_i|$. Then

$$|y_{i,j,k}|\leqslant(C_3(\mu)\|y\|_{\star,0,n}^{\gamma}+C_4(\mu)\|y\|_{\star,0,n}+C_5(\mu)L+C_6(\mu,y_0))\omega_{i,k}^{\theta}.$$

The results follows from our choice of μ and L. □

The next lemma is the equivalent of Proposition 2 for the Euler scheme.

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Lemma 4. *For n such that $t_n = T$, $\|y\|_{\star,0,n}$ is bounded by a constant that depends only on $\omega(0,T)$, $\|x\|$, $\|\nabla f\|_\infty$, $H_\gamma(F)$, γ, p and $|f(0)|$.*

Proof. Using Lemma 2,

$$|y_{i,j}| \leqslant (C_7(\mu, y_0) + C_8(\mu)\|y\|^\gamma_{\star,0,n} + C_9(\mu)\|y\|_{\star,0,n})\omega^{1/p}_{i,j}$$

and then with Lemma 3,

$$\|y\|_{\star,0,n} \leqslant (C_7(\mu, y_0) + C_8(\mu)\|y\|^\gamma_{\star,0,n} + C_9(\mu)\|y\|_{\star,0,n}) \\ + 4\mu^{1+\gamma}\frac{C_6(\mu, y_0) + C_3(\mu)\|y\|^\gamma_{\star,0,n} + C_4(\mu)\|y\|_{\star,0,n}}{1-K}.$$

In addition to (12), we choose μ small enough so that

$$C_8(\mu) + 4(1-K)^{-1}\mu^{1+\gamma}C_3(\mu) \leqslant \frac{1}{4} \tag{16}$$

and

$$C_9(\mu) + 4(1-K)^{-1}\mu^{1+\gamma}C_3(\mu) \leqslant \frac{1}{4}. \tag{17}$$

Since $\gamma \leqslant 1$, $\|y\|^\gamma_{\star,0,n} \leqslant \|y\|_{\star,0,n}$ when $\|y\|_{\star,0,n} \leqslant 1$. Hence,

$$\|y\|_{\star,0,n} \leqslant \max\{1, 2C_7(\mu, y_0) + 8\mu^{1+\gamma}C_6(\mu, y_0))\}.$$

This proves that for a choice of μ (or equivalently T or n) small enough depending only on $\|x\|$, $\|\nabla f\|_\infty$, $H_\gamma(F)$, γ and p, then $\|y\|_{\star,0,n}$ is bounded by a constant that depends only on $\|x\|$, $\|\nabla f\|_\infty$, $H_\gamma(F)$, γ, p, $|f(y_0)|$ and $|F(y_0)|$. However, $|F(y_0)|$ and $|f(y_0)|$ are bounded by some constants that depends only on $|f(0)|$ and $\|\nabla f\|_\infty$.

The result is proved by finding a sequence $n_0 = 0 < n_1 < \cdots < n_N$ such that $\omega_{n_1,n_{i+1}} \leqslant \mu^p$ with $t_{n_N} = T$ and μ satisfying (12), (16) and (17). Since ω is continuous close to its diagonal, there exists such a finite number N of intervals, and this number depends only on the choice of μ, and then on $\|x\|$, $\|\nabla f\|_\infty$, $H_\gamma(F)$, γ and p (and not on y_0 nor $f(0)$). Finally, it is easily shown that

$$\|y\|_{\star,0,n_N} \leqslant N^{p-1}\sum_{i=1}^{N-1}\|y\|_{\star,n_i,n_{i+1}},$$

which proves the result by applying the result on the successive time intervals $[t_{n_i}, t_{n_{i+1}}]$ and replacing y_0 by y_{t_i}. □

Finally, is order to interpolate the Euler scheme and to get a good control, we shall add an hypothesis on ω, which is trivially satisfied in the case of $\omega(s,t) = t - s$, that is for Hölder continuous rough paths.

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Hypothesis 3. We assume that there exists a continuous, increasing function φ such that

$$\frac{\omega(s,t)}{\varphi(t)-\varphi(s)} \text{ and } \frac{\varphi(t)-\varphi(s)}{\omega(s,t)}$$

are bounded for $0 \leqslant s < t \leqslant T$.

Proposition 3. *Under Hypotheses 1, 2 and 3, there exists a least a solution in the sense of Davie to* (1)*.*

Proof. From the family $\{y_i\}$, we construct a path from $[0,T]$ to $\mathbb{R}^m$ by

$$y_t = y_i + \frac{\varphi(t)-\varphi(t_i)}{\varphi(t_{i+1})-\varphi(t_i)} y_{i,i+1},\ t \in [t_i, t_{i+1}]. \tag{18}$$

From standard computations, there exists a constant C_{16} which depends only on p and the lower and upper bounds of $\omega(s,t)/(\varphi(t)-\varphi(s))$ such that

$$\|y\| \leqslant C_{16}\|y\|_{\star,0,n}.$$

With Lemma 4, we have a uniform bound in $\|y\|_{\star,0,n}$, and then on the constant L when μ (or n) is small enough. Hence, for any partition satisfying Hypothesis 3, the path y has a p-variation which is bounded by a constant that does not depend on the choice of the partition.

Now, let y^n be a family of paths constructed along an increasing family of partitions Π^n whose meshes decrease to 0.

Then there exists a subsequence $\{y^{n_k}\}_{k\geqslant 1}$ of $\{y^n\}_{n\in\mathbb{N}^*}$ which converges in q-variation for $q > p$ to some path y of finite p-variation.

For any (s,t) in $\cap_{n\geqslant 0}\Pi^n$,

$$|y_t - y_s - f(y_s)x^1_{s,t} - F(y_s)x^2_{s,t}| \leqslant L\omega(s,t)^\theta.$$

Since $\cap_{n\geqslant 0}\Pi^n$ is dense in $[0,T]$ and y is continuous, this proves that y is the solution to (13) in the sense of Davie, at least when T is small enough.

The passage from a solution on $[0,T]$ with T small enough to a global solution is done by using the arguments of Lemma 7, Lemma 8 and Proposition 6. □

6 Distance between two solutions and uniqueness

We now consider a more stringent assumption than Hypothesis 1.

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Hypothesis 4. The function f is twice continuously differentiable from $\mathbb{R}^m$ to $L(\mathbb{R}^d,\mathbb{R}^m)$ and is such that ∇f, $\nabla^2 f$ are bounded and $F(x) := f(x)\cdot\nabla f(x)$ is such that ∇F is γ-Hölder continuous with constant $H_\gamma(\nabla F)$ from $\mathbb{R}^m$ to $L(\mathbb{R}^m\otimes\mathbb{R}^d,\mathbb{R}^m)$.

We consider two rough paths u and v of finite p-variation controlled by ω, $p\in[2,3)$, as well as two vector fields f and g satisfying Hypothesis 4. Let y and z be respectively some solutions to $y_t = y_0 + \int_0^t f(y_s)\,\mathrm{d}u_s$ and $z_t = z_0 + \int_0^t g(z_s)\,\mathrm{d}v_s$.

We have seen that y and z remain is a ball of radius R that depends only on $\|\nabla f\|_\infty$, $\|\nabla g\|_\infty$, $\|\nabla F\|_\infty$, $\|\nabla G\|_\infty$ (since F and $G = g\cdot\nabla g$ are Lipschitz continuous), $\|u\|$, $\|v\|$, y_0, z_0, $|f(0)|$, $|g(0)|$, $\omega(0,T)$, γ and p.

Definition 3. We say that a constant C satisfies Condition (S) if it depends only on the above quantities, as well as $H_\gamma(\nabla F)$, $H_\gamma(\nabla G)$, $\|\nabla^2 f\|_\infty$ and $\|\nabla^2 g\|_\infty$.

We then set for some functions h, h',

$$\delta_R(h,h') = \sup_{z\in B(0,R)} |h(z)-h'(z)|.$$

We also set

$$\delta(u,v) = \|u-v\| \text{ and } \delta(y,z) = \sup_{0\leqslant s<t\leqslant T} \frac{|y_{s,t}-z_{s,t}|}{\omega(s,t)^{1/p}}.$$

Theorem 1. *Under Hypotheses 2 and 4, there exists some constant C_{17} satisfying Condition* (S) *such that for all $(s,t)\in\Delta^2$,*

$$\begin{aligned}|y_{s,t}-z_{s,t}| \leqslant C_{17}\omega(s,t)^{1/p}(&|y_0-z_0| \\ &+\delta(u,v)+\delta_R(f,g)+\delta_R(F,G)+\delta_R(\nabla f,\nabla g)+\delta_R(\nabla F,\nabla G)).\end{aligned} \quad (19)$$

The proof of the following corollary is then immediate from the previous estimate.

Corollary 1. *Under Hypotheses 2 and 4, there exists a unique solution in the sense of Davie to* (1).

Theorem 1 proves that the Itô map which sends x to the unique solution to (1) is locally Lipschitz continuous in y_0, f and x.

The next lemma is the main estimate of the proof and show why extra regularity shall be assumed on F.

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Lemma 5. *Let h be a function of class $\mathcal{C}^1$ from $\mathbb{R}^m$ to $\mathbb{R}$ such that ∇h is γ-Hölder and bounded. Then for all a, b, c, d in $\mathbb{R}^m$,*

$$|h(a) - h(b) + h(c) - h(d)| \\ \leqslant |a - b - c + d| \|\nabla h\|_\infty + H_\gamma(\nabla h)(|a - b|^\gamma + |c - d|^\gamma)|b - d|.$$

Proof. The result follows from

$$\begin{aligned} &|h(a) - h(b) + h(c) - h(d)| \\ &= \left| \int_0^1 \nabla h(c + \tau(a - c))(a - c)\,\mathrm{d}\tau - \int_0^1 \nabla h(d + \tau(b - d))(b - d)\,\mathrm{d}\tau \right| \\ &= \left| \int_0^1 \nabla h(c + \tau(a - c))(a - c - b + d)\,\mathrm{d}\tau \right| \\ &+ \left| \int_0^1 \big(\nabla h(c + \tau(a - c)) - \nabla h(d + \tau(b - d))\big)(b - d)\,\mathrm{d}\tau \right| \\ &\leqslant |a - b - c + d| \|\nabla h\|_\infty + H_\gamma(\nabla h) \int_0^1 |(1 - \tau)(c - d) + \tau(a - b)|^\gamma |b - d|\,\mathrm{d}\tau, \end{aligned}$$

since $(x + y)^\gamma \leqslant x^\gamma + y^\gamma$ for $x, y \geqslant 0$ and $\gamma \in [0, 1]$. □

Let us denote by $\mathfrak{d}$ the operator which, applied to an expression involving y, f and u, takes the difference between this expression and the similar expression with y replaced by z, f replaced by g and u replaced by v. For example,

$$\mathfrak{d}(f(y_s)x^1_{s,t}) = f(y_s)u^1_{s,t} - g(z_s)v^1_{s,t}.$$

If $\alpha(y, f, u)$ and $\beta(y, f, u)$ are two expressions, then

$$\mathfrak{d}(\alpha(y, f, u)\beta(y, f, u)) = \mathfrak{d}(\alpha(y, f, u))\beta(y, f, u) + \alpha(z, g, v)\mathfrak{d}(\beta(y, f, u)). \quad (20)$$

Proof of Theorem 1. In the proof, we assume without loss of generalities that $\gamma < 1$.

Let us choose a constant A such that

$$|\mathfrak{d}(\mathfrak{D}(s, r, t))| \leqslant A\omega(s, t)^\theta \quad (21)$$

and

$$|\mathfrak{d}(\mathfrak{D}(s, t))| \leqslant A\omega(s, t)^{1/p}. \quad (22)$$

From the sewing lemma on the difference of two almost rough paths (See for example [10, Theorem 6, p. 95]), there exists some universal constant M (depending only on θ) such that

$$|\mathfrak{d}(y_{s,t} - \mathfrak{D}(s, t))| \leqslant MA\omega(s, t)^\theta. \quad (23)$$

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Our aim is to obtain some estimate on A.

Let us note first that

$$\begin{aligned}|\mathfrak{d}F(y_s)u^2_{s,t}| \leqslant (|F(y_s)-F(z_s)|+|F(z_s)-G(z_s)|)\|u\|\omega(s,t)^{2/p}\\ +(|G(z_0)|+\|\nabla G\|_\infty\|z\|\mu)\delta(u,v)\omega(s,t)^{2/p}.\end{aligned}$$

Since $|y_s-z_s|\leqslant \delta(y,z)\mu+|y_0-z_0|$, with Lemma 5,

$$|F(y_s)-F(z_s)|\leqslant \|\nabla F\|_\infty(\delta(y,z)\mu+|y_0-z_0|),$$

then for some constant C_{18} satisfying Condition (S),

$$|\mathfrak{d}F(y_s)u^2_{s,t}| \leqslant C_{18}(\delta_R(F,G)+|y_0-z_0|+\mu\delta(y,z)+\delta(u,v))\omega(s,t)^{2/p}.$$

With (23),

$$\begin{aligned}|\mathfrak{d}B_2(s,t)| &= |\mathfrak{d}(y_{s,t}-f(y_s)u^1_{s,t})| = |\mathfrak{d}(y_{s,t}-\mathfrak{D}(s,t)+F(y_s)u^2_{s,t})|\\ &\leqslant MA\omega(s,t)^\theta+|\mathfrak{d}F(y_s)u^2_{s,t}|\\ &\leqslant MA\omega(s,t)^\theta+C_{18}(\delta_R(F,G)+|y_0-z_0|+\mu\delta(y,z)+\delta(u,v))\omega(s,t)^{2/p}.\end{aligned}$$

Since f is differentiable, for $\tau\in[0,1]$,

$$f(\tau y_t+(1-\tau)y_s)-f(y_s)=\int_0^1\nabla f(y_s+\rho(\tau y_t+(1-\tau)y_s))\tau y_{s,t}\,\mathrm{d}\rho. \quad (24)$$

Hence

$$\begin{aligned}|f(\tau y_t+(1-\tau)y_s)-f(y_s)-g(\tau y_t+(1-\tau)y_s)+g(y_s)|\\ \leqslant \delta_R(\nabla f,\nabla g)\|y\|\omega(s,t)^{1/p}. \quad (25)\end{aligned}$$

With (25) and (24) for $\tau\in[0,1]$,

$$\begin{aligned}\mathfrak{d}(f(\tau y_t+(1-\tau)y_s)-f(y_s)) \leqslant \delta_R(\nabla f,\nabla g)\|y\|\omega(s,t)^{1/p}\\ +2\|\nabla^2 f\|_\infty(\delta(y,z)\mu+|y_0-z_0|)\|y\|\omega(s,t)^{1/p}+\delta(y,z)\|\nabla f\|_\infty\omega(s,t)^{1/p}.\end{aligned}$$

With Lemma 5 and (24)-(25) applied to F and G, for $\tau\in[0,1]$,

$$\begin{aligned}|\mathfrak{d}B_4(y_s,y_s+\tau y_{s,t})| &= \mathfrak{d}(F(\tau y_t+(1-\tau)y_s)-F(y_s))\\ &\leqslant \delta_R(\nabla F,\nabla G)\|y\|\omega(s,t)^{1/p}+2\|\nabla F\|_\infty\delta(y,z)\omega(s,t)^{1/p}\\ &\quad +2H_\gamma(\nabla F)(\|y\|^\gamma+\|z\|^\gamma)\omega(s,t)^{\gamma/p}(\mu\delta(y,z)+|y_0-z_0|).\end{aligned}$$

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Besides,

$$\mathfrak{d}(\nabla f(y_s+\tau y_{s,t}))\leqslant \delta_R(\nabla f,\nabla g)+\|\nabla^2 f\|_\infty(\delta(y,z)\mu+|y_0-z_0|).$$

In addition,

$$|\mathfrak{d}(F(y_0))|\leqslant \delta_R(F,G)+\|\nabla F\|_\infty|y_0-z_0|.$$

Combining all these estimates using (20) with the ones given in Section 4 in some lengthy computations,

$$\begin{aligned}|\mathfrak{d}(\mathfrak{D}(s,r,t))|\leqslant C_{19}\omega(s,t)^\theta\big(MA\mu+\mu\delta(y,z)+\delta(y,z)\mu^{1-\gamma}\\ +|z_0-y_0|+\delta(u,v)+\delta_R(F,G)+\delta_R(\nabla F,\nabla G)+\delta_R(\nabla f,\nabla g)\big),\end{aligned}\tag{26}$$

where C_{19} satisfies Condition (S) (Note that C_{19} decreases with T).

We have also

$$\begin{aligned}|\mathfrak{d}(\mathfrak{D}(s,t))|&\leqslant \delta_R(f,g)(\|y\|\mu+|y_0|)\|u\|\omega(s,t)^{1/p}\\&+(\|\nabla f\|_\infty+\|\nabla F\|_\infty)(\mu\delta(y,z)+|y_0-z_0|)\|u\|(1+\mu)\omega(s,t)^{1/p}\\&+(\|g\|_\infty+\|G\|_\infty)\delta(u,v)\omega(s,t)^{2/p}+\delta_R(F,G)\|u\|\omega(s,t)^{2/p}\\&\leqslant C_{20}(\delta_R(F,G)+\delta_R(f,g)+\delta(u,v)+|y_0-z_0|+\mu\delta(y,z))\omega(s,t)^{1/p},\end{aligned}$$

where C_{20} satisfies Condition (S) and decreases when T decreases. With (23),

$$\begin{aligned}|\mathfrak{d}y_{s,t}|\leqslant |\mathfrak{d}(y_{s,t}-\mathfrak{D}(s,t))|+|\mathfrak{d}\mathfrak{D}(s,t)|\leqslant MA\omega(s,t)^\theta\\ +C_{20}(\delta_R(F,G)+\delta_R(f,g)+\delta(u,v)+|y_0-z_0|+\mu\delta(y,z))\omega(s,t)^{1/p}.\end{aligned}\tag{27}$$

Let us choose A such that an equality holds in either (21) or (22). If $|\mathfrak{d}(y_{s,t})|=A$, then from (27),

$$A\leqslant MA\mu^{1+\gamma}+C_{20}(\delta_R(F,G)+\delta_R(f,g)+\delta(u,v)+|y_0-z_0|+\mu\delta(y,z)).$$

If $|\mathfrak{d}(\mathfrak{D}(s,r,t))|=A$, then from (26),

$$\begin{aligned}A\leqslant C_{19}\big(MA\mu+\mu\delta(y,z)+\delta(y,z)\mu^{1-\gamma}\\ +|z_0-y_0|+\delta(u,v)+\delta_R(F,G)+\delta_R(\nabla F,\nabla G)+\delta_R(\nabla f,\nabla g)\big).\end{aligned}$$

In any case, we may choose μ small enough in function of C_{19} or of M (which depends only on θ) such that

$$\begin{aligned}A\leqslant 2C_{19}\big(\mu\delta(y,z)+\delta(y,z)\mu^{1-\gamma}\\ +|z_0-y_0|+\delta(u,v)+\delta_R(F,G)+\delta_R(\nabla F,\nabla G)+\delta_R(\nabla f,\nabla g)\big)\\ +2C_{20}(\delta_R(F,G)+\delta_R(f,g)+\delta(u,v)+|y_0-z_0|+\mu\delta(y,z)).\end{aligned}$$

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Injecting this inequality in (27), we get that

$$\delta(y,z) \leqslant C_{21}(B + (\mu + \mu^{2-\gamma} + \mu^{2(1-\gamma)})\delta(y,z) + |y_0 - z_0|)$$

with

$$B := \delta_R(F,G) + \delta_R(f,g) + \delta(u,v) + \delta_R(\nabla f, \nabla g) + \delta_R(\nabla F, \nabla G)$$

and C_{21} depends only on C_{19}, C_{20}, M and $\omega(0,T)$.

Again, choosing μ small enough in function of C_{19}, C_{20}, M and $\omega(0,T)$ gives the required bound in $\delta(y,z)$ in short time. As the choice of μ does not depend on $|y_0 - z_0|$ and μ satisfies Condition (S), Proposition 7 may be applied to $y - z$. □

7 Distance between two Euler schemes

Let us give two rough paths u and v of finite p-variations, as well as a partition $\{t_i\}_{i=0}^n$ of $[0,T]$.

We use the same notations and conventions as in Section 5. Again, we set $\mu := \omega_{0,n}^{1/p}$.

For $0 \leqslant i < j \leqslant n$, we set

$$z_{i+1} = z_i + g(z_i)v^1_{i,i+1} + G(z_i)v^2_{i,i+1},\ z_j = z_i + g(z_i)v^1_{i,j} + G(z_i)v^2_{i,j},$$

For a family $\{\varepsilon_{i,j}\}_{i,j=1,\dots,n}$ such that $\sup_{0\leqslant i\leqslant j\leqslant n} |\varepsilon_{i,j}|/\omega^\theta_{i,j}$ is finite, we set

$$\begin{aligned} y_{i+1} &= y_i + f(y_i)u^1_{i,i+1} + F(y_i)u^2_{i,i+1} + \varepsilon_{i,i+1},\ i = 0,\dots,n-1 \\ y_j &= y_i + f(y_i)u^1_{i,j} + F(y_i)u^2_{i,j} + \varepsilon_{i,j},\ 0 \leqslant i < j \leqslant n. \end{aligned} \tag{28}$$

In addition, define

$$\alpha := \sup_{i=0,\dots,n-1} \frac{|\varepsilon_{i,i+1}|}{\omega^\theta_{i,i+1}}.$$

Here, there are three cases of interest: (a) Both z and y are given by some Euler schemes and then $\varepsilon_{i,j} = 0$ for all $0 \leqslant i < j \leqslant n$. (b) The path y is a solution to $y_t = y_0 + \int_0^t f(y_s)\,\mathrm{d}u_s$ and then $\varepsilon_{i,j} = y_j - y_i - f(y_i)u^1_{i,j} - F(y_i)u^2_{i,j}$, while $v = u$ and $g = f$. (c) While $v = u$ and $g = f$, the family $\{y_i\}_{i=0,\dots,n}$ is given by the Euler scheme with respect to a partition $\{t'_i\}_{i=0,\dots,m} \subset \{t_i\}_{i=0,\dots,n}$.

We assume that z and y belong to the ball of radius R and that $\|z\|_{\star,0,n}$ and $\|y\|_{\star,0,n}$ (defined by (11)) are bounded by R'. In any cases, R and R' depend only on $\|\nabla f\|_\infty$, $\|\nabla F\|_\infty$, $\|\nabla g\|_\infty$, $\|\nabla G\|_\infty$, $\|u\|$, $\|v\|$, $|f(0)|$, $|g(0)|$, $|y_0|$, $|z_0|$, $\omega(0,T)$, γ and p.

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Definition 4. We say that *a constant C satisfies condition* $(\mathrm{S_e})$ if it depends only on the quantities listed above as well as $H_\gamma(\nabla F)$, $H_\gamma(\nabla G)$, $\|\nabla^2 f\|_\infty$, $\|\nabla^2 g\|_\infty$ and $\sup_{0\leqslant i\leqslant j\leqslant n} |\varepsilon_{i,j}|/\omega_{i,j}^\theta$.

Theorem 2. *If f and g satisfies Hypothesis 4, then for some constant C_{22} satisfying Condition* $(\mathrm{S_e})$,

$$\|z-y\|_{\star,0,n} \leqslant C_{22}(\alpha + |z_0 - y_0| + \delta_R(f,g) + \delta_R(\nabla f, \nabla g) \\ + \delta_R(\nabla F, \nabla G) + \delta_R(F,G) + \delta(u,v)).$$

Proof. We set

$$A := \max_{0\leqslant i<j\leqslant n} \frac{|y_j - y_i - y_{i,j} - z_j - z_i - z_{i,j}|}{\omega_{i,j}^\theta}. \tag{29}$$

We have

$$|F(y_i) - F(z_i)| \leqslant \|\nabla F\|_\infty (|y_0 - z_0| + \|z-y\|_{\star,0,n}\mu)$$

and

$$|\nabla f(y_i + \tau(y_j - y_i)) - \nabla f(z_i + \tau(z_j - z_i))| \\ \leqslant 2\|\nabla^2 f\|_\infty (|y_0 - z_0| + \|z-y\|_{\star,0,n}\mu).$$

Besides, with Lemma 5,

$$|F(y_j) - F(y_i) - F(z_j) + F(z_i)| \leqslant \|\nabla F\|_\infty \|z-y\|_{\star,0,n} \omega_{i,j}^{1/p} \\ + H_\gamma(\nabla F)(\|y\|^\gamma + \|z\|^\gamma)(|y_0 - z_0| + \mu\|z-y\|_{\star,0,n})\omega_{i,j}^{\gamma/p}.$$

With (15) and considering the difference between $y_{i,j,k}$ and $z_{i,j,k}$ for $i<j<k$,

$$|y_{i,j,k} - z_{i,j,k}| \leqslant C_{23}\omega_{i,k}^\theta (|y_0 - z_0| + A\mu + (\mu + \mu^{1-\gamma})\|z-y\|_{\star,0,n} + B_1).$$

with C_{23} satisfying Condition $(\mathrm{S_e})$ and

$$B_1 := \delta(u,v) + \delta_R(F,G) + \delta_R(\nabla f, \nabla g) + \delta_R(\nabla F, \nabla G).$$

On the other hand, for $i = 0, \dots, n-1$, $z_{i+1} - z_i - z_{i,i+1} = 0$ and

$$|y_{i+1} - y_i - y_{i,i+1}| \leqslant \alpha\omega_{i,i+1}^\theta.$$

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For j given as in the proof of Lemma 3, we have for $K := 2^{1-\theta}$,

$$\begin{aligned}\frac{|y_k - y_i - y_{i,k} - z_k - z_i - z_{i,k}|}{\omega_{i,k}^\theta} \\ &\leqslant KA + \alpha + \frac{|y_{i,j,k} - z_{i,j,k}|}{\omega_{i,k}^\theta} + \frac{|y_{j,j+1,k} - z_{j,j+1,k}|}{\omega_{i,k}^\theta} \\ &\leqslant KA + \alpha + C_{24}(|y_0 - z_0| + A\mu + (\mu + \mu^{1-\gamma})\|z - y\|_{\star,0,n} + B_1)\end{aligned}$$

with $C_{24} = 2C_{23}$. From the definition of A (see (29)),

$$A \leqslant 2\alpha + KA + C_{24}(|y_0 - z_0| + A\mu + (\mu + \mu^{1-\gamma})\|z - y\|_{\star,0,n} + B_1). \quad (30)$$

On the other hand, since both f and F are Lipschitz continuous, for some constant C_{25} satisfying Condition ($\mathrm{S_e}$),

$$\begin{aligned}\|z - y\|_{\star,0,n} &\leqslant \sup_{0\leqslant i<j\leqslant n} \frac{|y_j - y_i - y_{i,j} - z_j + z_i + z_{i,j}|}{\omega_{i,j}^{1/p}} + \frac{|y_{i,j} - z_{i,j}|}{\omega_{i,j}^{1/p}} \\ &\leqslant A\mu^{1+\gamma} + C_{25}(|y_0 - z_0| + \mu\|z - y\|_{\star,0,n}) + B_2\end{aligned}$$

with

$$B_2 := C_{26}(\delta_R(F, G) + \delta_R(f, g) + \delta(u, v)),$$

for some constant C_{26} satisfying Condition ($\mathrm{S_e}$). If μ is small enough so that $C_{25}\mu \leqslant 1/2$, then

$$\|z - y\|_{\star,0,n} \leqslant 2C_{25}|y_0 - z_0| + 2A\mu^{1+\gamma} + 2B_2. \quad (31)$$

Injecting this in (30),

$$A \leqslant 2\alpha + KA + C_{27}(|y_0 - z_0| + AC_{28}(\mu)) + B_3$$

with a $C_{28}(\mu)$ satisfying Condition ($\mathrm{S_e}$) for fixed μ and that decreases to 0 as μ decreases to 0, C_{27} satisfying Condition ($\mathrm{S_e}$), and $B_3 = C_{29}(B_1 + B_2)$ for some constant C_{29} satisfying Condition ($\mathrm{S_e}$). For $K + C_{27}C_{28}(\mu) < (1 + K)/2 < 1$, then

$$A \leqslant \frac{2}{1 - K}(2\alpha + C_{27}|y_0 - z_0| + B_3).$$

Using the inequality on (31), this leads to the required inequality for a value of μ small enough. Thus usual arguments proves now that this is true for any time horizon T up to changing the constants. □

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8 Rate of convergence of the Euler scheme

Let us consider now the solution to $y_t = y_0 + \int_0^t f(y_s)\,\mathrm{d}x_s$ as well as the associated Euler scheme

$$e_{i+1} = e_i + f(e_i)x^1_{i,i+1} + F(e_i)x^2_{i,i+1},\ e_0 = y_0$$

when f satisfies Hypothesis 4.

We are willing to estimate the difference between y and e. Our key argument is provided by Theorem 2 above.

Proposition 4. *Assume Hypothesis 4 on f. For $\delta := \sup_{0\leqslant i<n}\omega_{i,i+1}$ and $p \in [2,3)$,*

$$\sup_{i=0,\dots,n} |e_i - y_i| \leqslant C_{30}\delta^{(3-p)/p-\eta}. \tag{32}$$

where C_{30} depends only on y_0, $|f(0)|$, $\|x\|$, $\|\nabla f\|_\infty$, $\|\nabla^2 f\|_\infty$, $\|\nabla F\|_\infty$, $H_\gamma(\nabla F)$, $\omega(0,T)$, p and γ.

It follows that the rate of convergence is smaller than $(3-p)/p$ and belongs to $(0,1/2)$. In addition, when p increases to 3, the rate of convergence decreases to 0. This rate is similar to the one given by A.M. Davie [2]. (See also [3] for the convergence of the Milstein scheme for the fractional Brownian motion).

Proof. Let us note first that since y and e remains bounded, if ∇F is γ-Hölder continuous, then it is locally γ'-Hölder continuous for any $\gamma' < \gamma$. Since the constraint $2 + \gamma' > p$ is in force, we set $\gamma' = p - 2 + \eta p$ for some $0 < \eta < (3-p)/p < 1/2$ and $\theta' := (2+\gamma')/p = 1 + \eta$.

Since F is Lipschitz continuous, then y is a solution in the sense of Davie with $\theta = 3/p$. This way,

$$|y_t - y_s - f(y_s)x^1_{s,t} - F(y_s)x^2_{s,t}| \leqslant L\omega(s,t)^{3/p}.$$

It follows that $\{y_i\}_{i=0}^n$ is solution to (28) with

$$\varepsilon_{i,i+1} = y_{i+1} - y_i - f(y_i)x^1_{i,i+1} - F(y_i)x^2_{i,i+1}.$$

Then

$$\alpha := \sup_{0\leqslant i<n} \frac{|\varepsilon_{i,i+1}|}{\omega_{i,i+1}^{\theta'}} \leqslant L\delta^{3/p-\theta'} = L\delta^{(3-p)/p-\eta}.$$

Applying Theorem 2 with our choice of θ' leads to

$$\|e - y\| \leqslant C_{31}\delta^{(3-p)/p-\eta},$$

where C_{31} depends only on y_0, $|f(y_0)|$, $\|x\|$, $\|\nabla f\|_\infty$, $\|\nabla^2 f\|_\infty$, $\|\nabla F\|_\infty$, $H_\gamma(\nabla F)$, $\omega(0,T)$, p and γ. Then (32) is immediate. □

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Let $\{\{t_i^n\}_{i=0,\dots,n}\}_{n\in\mathbb{N}^*}$ be an increasing family of partitions such that the mesh $\sup_{i=0,\dots,n}\omega(t_i^n,t_{i+1}^n)$ converges to 0. Let $\{\{e_i^n\}_{i=0,\dots,n}\}_{n\in\mathbb{N}^*}$ be the corresponding Euler schemes for a rough path x and a vector field f satisfies Hypothesis 4. With Lemmas 3, 4 and 8,

$$\sup_{n\in\mathbb{N}^*}\sup_{0\leqslant i<j\leqslant n}\frac{|e_i^n-e_j^n-f(e_i^n)x^1_{t_i^n,t_j}-F(e_i^n)x^2_{t_i^n,t_j^n}|}{\omega(t_i^n,t_{j+1}^n)^\theta}<+\infty$$

and with Proposition 3 under Hypothesis 3, the interpolation of the Euler scheme e^n constructed as in (18) has a p-variation norm $\|e^n\|$ which is bounded. Using the same proof as above, we get the following corollary.

Corollary 2. *Under Hypotheses 2, 3 and 4, the family of interpolated Euler schemes $\{e^n\}_{n\in\mathbb{N}^*}$ for a Cauchy sequence of the uniform norm and the q-variation norm for any $q>p$.*

Remark 2. Here the dimension d of the space plays no role so that this argument may be used for an infinite dimensional rough path. This Corollary allows one to define the solution to (1) as the limit of the sequence $\{e^n\}_{n\in\mathbb{N}^*}$.

9 Case of geometric rough paths

We now consider the case where x is a geometric rough path of finite p-variation controlled by ω and the vector field f satisfies Hypothesis 4. This means that there exists a sequence of rough paths x^n converging to x in p-variation such that x^n lives above a piecewise smooth path z^n in $\mathbb{R}^d$ and $x_t^n=1+z_t^n+\int_0^t(z_s^n-z_0^n)\otimes z_s^n\,\mathrm{d}s$. Such a path x^n is called a smooth rough path.

In order to simplify the analysis, we assume that $\omega(s,t)=t-s$ and then we are dealing with α-Hölder continuous paths with $\alpha=1/p$.

The core idea of P. Friz and N. Victoir was to consider a family of smooth rough paths $(\widetilde{x}^n)_{n\geqslant 1}$ such that, given a family of partitions $\{\{t_i^n\}_{i=0,\dots,n}\}_{n\in\mathbb{N}^*}$, $\widetilde{x}^n$ converges to x in the β-Hölder norm for any $\beta<\alpha$ and $\widetilde{x}^n_{t_i^n}=x_{t_i^n}$ for $i=0,\dots,n$.

For this, they used sub-Riemannian geodesics. In [10], we have provided an alternative construction using some segments and some loops.

Let $\widetilde{z}^n$ be the projection of $\widetilde{x}^n$ onto $\mathbb{R}^d$, and let y^n be the solution of the ODE $y_t^n=y_0+\int_0^t f(y_s^n)\,\mathrm{d}\widetilde{z}_s^n$. As $\widetilde{z}^n$ is piecewise smooth, one knows that this solution corresponds to the one of the solution of the RDE $y_t^n=y_0+\int_0^t f(y_s^n)\,\mathrm{d}\widetilde{x}_s^n$ in the sense of Davie or in the sense of Lyons (See Section 10 below).

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Let $\widetilde{e}^n$ be the Euler scheme associated to y^n with the partition $\{t_i^n\}_{i=0,\dots,n}$:

$$\widetilde{e}^n_{i+1} = \widetilde{e}^n_i + f(\widetilde{e}^n_i)\widetilde{x}^{1,n}_{t_i^n,t_{i+1}^n} + F(\widetilde{e}^n_i)\widetilde{x}^{2,n}_{t_i^n,t_{i+1}^n},\ i = 0,\dots,n,$$

and e^n the Euler scheme

$$e^n_{i+1} = e^n_i + f(e^n_i)x^{1,n}_{t_i^n,t_{i+1}^n} + F(e^n_i)x^{2,n}_{t_i^n,t_{i+1}^n},\ i = 0,\dots,n,$$

The key observation from P. Friz and N. Victoir is that from the very construction of $\widetilde{x}^n$, $\widetilde{x}^n_{t_i^n,t_{i+1}^n} = \widetilde{x}^n_{t_i^n,t_{i+1}^n}$ and then $\widetilde{e}^n = e^n$.

With Proposition 4, for some $\eta \in (0, 3\alpha - 1)$,

$$\|e^n - y^n\|_\star := \sup_{0\leqslant i<j\leqslant n} \frac{|e^n_j - e^n_i - y^n_j + y^n_i|}{\omega_{i,j}^{1/p}} = \|\widetilde{e}^n - y^n\| \leqslant C_{32}\delta^{3\alpha-1-\eta}$$

where C_{32} depends only on $\|x\|$, f, T, α and η, and $\delta := \sup_{i=0,\dots,n+1}\omega(t_i^n, t_{i+1}^n)$.

On the other hand with Theorem 1,

$$\|y^n - y^m\|_\star \leqslant \|y^n - y^m\| \leqslant C_{33}\|\widetilde{x}^m - \widetilde{x}^n\|,$$

where C_{33} depends only on $\|x\|$, f, T, α and η (when $\widetilde{x}^n$ is such that $\|\widetilde{x}^n\| \leqslant A\|x\|$ which is the case with the constructions mentioned above).

Since $(\widetilde{x}^n)_{n\in\mathbb{N}^*}$ is a Cauchy sequence in the space of β-Hölder continuous functions, $\beta < \alpha$, it follows that (y^n) is a Cauchy sequence and converges to some element y.

We then obtain the convergence of e^n to y in the sense that $\|e^n - y\|_\star$ decreases to 0 as n tends to infinity.

10 Solution in the sense of Davie and solution in the sense of Lyons

The notion of solution in the sense of Davie (Definition 2) is different of the solution in the sense of Lyons (Definition 1), as the iterated integrals of y and the cross-iterated integrals between y and x are not constructed, while they are part of the solution in the sense of Lyons.

However, once a solution in the sense of Davie is constructed, it is easy to construct a rough paths with values in $T_1(\mathbb{R}^d \oplus \mathbb{R}^m)$.

Lemma 6. *A solution y in the sense of Davie — which is a path with values in $\mathbb{R}^m$ — may be lifted to a rough path with values in $T_1(\mathbb{R}^d \oplus \mathbb{R}^m)$, as the rough path $\widetilde{y}$ associated to the almost rough path*

$$h_{s,t} := 1 + x_{s,t} + y_{s,t} + f(y_s)\otimes f(y_s)\cdot x^2_{s,t} + f(y_s)\otimes 1\cdot x^2_{s,t} + 1\otimes f(y_s)\cdot x^2_{s,t}. \quad (33)$$

Besides, the map $y \mapsto \widetilde{y}$ is locally Lipschitz continuous.

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Proof. It is easily checked that

$$|h_{s,r,t} - h_{s,r} \otimes h_{r,t}| \leqslant C_{34}\omega(s,t)^\theta$$

with $\theta > 1$ and then that it is an almost rough path. With Proposition 1, the associated rough path $\widetilde{y}$ projects onto (x,y) in $\mathbb{R}^d \oplus \mathbb{R}^m$. The local Lipschitz continuity of $y \mapsto \widetilde{y}$ follows from the same kind of computation as the one of the proof of Theorem 1. □

Proposition 5. *Let us assume that f is a vector field with a bounded derivative which is γ-Hölder continuous, such that a solution to* (1) *exists in the sense of Lyons. Let us assume also that $f \cdot \nabla f$ is γ-Hölder continuous, so that a solution to* (1) *exists in the sense of Davie, lifted as a rough path with values in $T_1(\mathbb{R}^d \oplus \mathbb{R}^m)$ as above. Then the two solutions coincide.*

Proof. Let y be a solution in the sense of Davie of (1), which is lifted as a rough path $\widetilde{y}$ using Lemma 6. Let us consider

$$u_{s,t} := f(y_s)x^1_{s,t} + \nabla f(y_s)\widetilde{y}^\times_{s,t},$$

where $\widetilde{y}^\times_{s,t}$ is the projection onto $\mathbb{R}^m \otimes \mathbb{R}^d$ of $\widetilde{y}_{s,t}$ (or roughly speaking, it is the cross-iterated integral between y and x).

Since $\widetilde{y}^\times_{s,t} = y_{s,r} \otimes x^1_{r,t}$,

$$\begin{aligned}
|u_{s,t} - u_{s,r} - u_{r,t}| &\leqslant |(f(y_r) - f(y_s))x^1_{r,t} - \nabla f(y_s)y_{s,r} \otimes x^1_{r,t})| \\
&\quad + |\nabla f(y_r) - \nabla f(y_s)||\widetilde{y}^\times_{s,r}| \\
&\leqslant \left|\int_0^1 (\nabla f(y_s + \tau y_{s,r}) - \nabla f(y_s))y_{s,r} \otimes x^1_{r,t}\,\mathrm{d}\tau\right| + H_\gamma(\nabla f)\|y\|^\gamma\|z\|\omega^\theta_{s,t} \\
&\leqslant H_\gamma(\nabla f)(\|y\|^{1+\gamma}\|x\| + \|y\|^\gamma\|z\|)\omega(s,t)^\theta.
\end{aligned}$$

Then $\{u_{s,t}\}_{(s,t)\in\Delta^2}$ is an almost rough path in $\mathbb{R}^m$ whose associated rough path v satisfies

$$|v_t - v_s - u_{s,t}| \leqslant C_{35}\omega(s,t)^\theta.$$

From the very construction, v is the rough integral $v_t = v_0 + \int_0^t g(z_s)\,\mathrm{d}z_s$ with z is a rough path lying above $(x,y,y^\times) \in (\mathbb{R} \oplus \mathbb{R}^d \oplus \mathbb{R}^d, \mathbb{R}^m, \mathbb{R}^m \otimes \mathbb{R}^d)$ and $g(y,x) = \mathrm{d}x + f(y)\,\mathrm{d}x$. Thus $y = v$ when $v_0 = y_0$. With Lemma 6, this is also true for the iterated integrals. This proves that the a solution in the sense of Davie is also a solution in the sense of Lyons.

If z is a solution in the sense of Lyons, by construction, for a constant L and all $(s,t) \in \Delta^2$,

$$\begin{aligned}
|z_t - z_s - f(z_s)x^1_{s,t} - \nabla f(z_s)z^\times_{s,t}| &\leqslant L\omega(s,t)^\theta \\
\text{and } |z^\times_{s,t} \leqslant 1 \otimes f(z_s)x^2_{s,t}| &\leqslant L\omega(s,t)^\theta,
\end{aligned}$$

where $z^\times$ lives in $\mathbb{R}^m \otimes \mathbb{R}^d$. Hence, it is immediate the a solution in the sense of Lyons is also a solution in the sense of Davie. $\square$

A From local to global theorems

We present here some results which allows one to pass from local estimates to global estimates and then show the existence for any horizon T provided some uniform estimates.

Lemma 7. *Let $\mu > 0$ be fixed. Then there exists a finite number of times $0 = T_0 < T_1 < \cdots < T_{N+1}$ with $T_N \leqslant T < T_{N+1}$ and $\omega(T_i, T_{i+1}) = \mu$.*

Proof. Let us extend ω on $D_+ := \{(s,t) \,|\, 0 \leqslant s \leqslant t\}$ by

$$\omega(s,t) = \begin{cases} \omega(s,T) + T - s & \text{if } s \leqslant T \leqslant t, \\ t - s & \text{if } T \leqslant s \leqslant t. \end{cases}$$

Let us note that ω is continuous on D_+, and that $\omega(s,t) \xrightarrow[t\to\infty]{} +\infty$. Then for any $\mu > 0$, for any $s \geqslant 0$, there exists a value $\tau(s)$ such that $\omega(s,\tau(s)) = \mu$, since $\omega(s,s) = 0$.

For $T > 0$ fixed, let us set $T_0 = 0$ and $T_{i+1} = \tau(T_i)$. Then there exists a finite number N such that $\omega(0, T_{N-1}) \leqslant \omega(0,T) \leqslant \omega(T_0, T_{N+1})$ and $\omega(T_i, T_{i+1}) = \mu$. This follows from the fact that

$$N\mu = \sum_{i=0}^{N-1} \omega(T_i, T_{i+1}) \leqslant \omega(0, T_N)$$

and then $\omega(0,T_N) \xrightarrow[N\to\infty]{} +\infty$. This proves the lemma. $\square$

For a path y and times $0 \leqslant S < S' \leqslant T$, let us set

$$\|y\|_{[S,S']} := \sup_{S\leqslant s<t\leqslant S'} \frac{|y_t - y_s|}{\omega(s,t)^{1/p}}.$$

Lemma 8. *Let us assume that a continuous path y on $[S,S'']$ is a solution in the sense of Davie on time interval $[S,S']$ and $[S',S'']$ respectively with constants L_1 and L_2. Then y is a solution in the sense of Davie on $[S,S'']$ with a new constant L_3 that depends only on L_1, L_2, $H_\gamma(F)$, $\|\nabla f\|_\infty$, $\|x\|$, $\|y\|_{|[S,S']}$, $\omega(0,T)$, γ and p.*

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Proof. First, it is classical that

$$\|y\|_{[S,S'']} \leqslant 2^{1-1/p} \max\{\|y\|_{[S,S']}, \|y\|_{[S',S'']}\},$$

so that y is of finite p-variation over $[S, S'']$. For $s \in [S, S']$ and $t \in [S', S'']$,

$$|y_t - y_s - \mathfrak{D}(s,t)| \leqslant |y_t - y_{S'} - \mathfrak{D}(S',t)| + |y_{S'} - y_s - \mathfrak{D}(s,S')| + |\mathfrak{D}(s,S',t)|.$$

Let us note that

$$|\mathfrak{D}(s,S',t)| \leqslant C_{36}\omega(s,t)^\theta$$

where C_{36} depends only on L_1, $\|y\|_{|[S,S']}$, $H_\gamma(F)$, $\|\nabla f\|_\infty$, $|F(y_s)|$, $\|x\|$, $\omega(0,T)$, γ and p. Hence, since $\omega(s,S')^\theta + \omega(S',t) \leqslant \omega(s,t)$,

$$|y_t - y_s - \mathfrak{D}(s,t)| \leqslant (\max\{L_1, L_2\} + C_{36})\omega(s,t)^\theta.$$

This proves the result. □

Proposition 6. *Let us assume that a solution to* (1) *exists on any time interval* $[S, S']$ *with a condition that* $\omega(S, S') \leqslant C_{37}$ *where* C_{37} *does not depend on* S*. Then a solution exists for any time* T*.*

Proof. Using the sequence $\{T_i\}_{i=0,\dots,N}$ of times given by Lemma 7, it is sufficient to solve (1) successively on $[T_i, T_{i+1}]$ with initial condition y_{T_i} and to invoke Lemma 8 to prove the existence of a solution in the sense of Davie in $[0, T]$. □

Proposition 7. *Let* y *be a path of finite* p*-variations such that for some constants* A*,* B *and* K*,*

$$\textit{when } \omega(S,S') \leqslant K,\ \|y\|_{[S,S']} \leqslant B|y_S| + A.$$

Then

$$\sup_{t\in[0,T]} |y_t| \leqslant R(T)|y_0| + (R(T)-1)\frac{A}{B} \tag{34}$$

with

$$R(T) = \exp(B(1+K^{-1})^{1-1/p} \max\{\omega(0,T), \omega(0,T)^{1/p}\}),$$

and the p*-variation norm of* $\|y\|_{[0,T]}$ *depends only on* A*,* B*,* $\omega(0,T)$*,* K *and* p*.*

Proof. Remark first that

$$\sup_{t\in[S,S']} |y_t| \leqslant |y_S|(1 + B\omega(S,S')^{1/p}) + A\omega(S,S')^{1/p}.$$

Let us choose an integer N such that $\mu := \omega(0,T)/N \leqslant K$ and construct recursively T_i with $T_0 = 0$ and $\omega(T_i, T_{i+1}) = \mu$. Then

$$|y_{T_{i+1}}| \leqslant |y_{T_i}|(1 + \mu^{1/p} B) + A\mu^{1/p}.$$

From a classical result easily proved by an induction,

$$|y_T| \leqslant \exp(N\mu^{1/p}B)|y_0| + \frac{A}{B}(\exp(N\mu^{1/p}B) - 1).$$

As this is true for any T, surely (34) holds.

Choosing N such that $\omega(0,T)/N \leqslant K$ and $\omega(0,T)/(N-1) > K$,

$$N \leqslant \frac{\omega(0,T)}{K} + 1.$$

This way

$$N\mu^{1/p} \leqslant \left(\frac{\omega(0,T)}{K} + 1\right)^{1-\frac{1}{p}} \omega(0,T)^{\frac{1}{p}} \leqslant \begin{cases} \widetilde{K}\omega(0,T)^{1/p} & \text{if } \omega(0,T) \leqslant 1, \\ \widetilde{K}\omega(0,T) & \text{if } \omega(0,T) > 1. \end{cases}$$

with $\widetilde{K} := (1 + K^{-1})^{1-1/p}$.

Finally,

$$\|y\|_{[0,T]} \leqslant N^{1-1/p} \max_{i=0,\dots,N-1} \|y\|_{[T_i,T_{i+1}]},$$

which proves the last statement □

References

[1] S. Aida, *Notes on proofs of continuity theorem in rough path analysis* (2006). Preprint of Osaka University.

[2] A.M. Davie, *Differential Equations Driven by Rough Signals: an Approach via Discrete Approximation*, Appl. Math. Res. Express. AMRX **2** (2007), Art. ID abm009, 40.

[3] A. Deya, A. Neuenkirch, and S. Tindel, *A Milstein-type scheme without Levy area terms for SDEs driven by fractional Brownian motion* (2010). arxiv:1001.3344.

[4] H. Doss, *Liens entre équations différentielles stochastiques et ordinaires*, Ann. Inst. H. Poincaré Sect. B (N.S.) **13** (1977), no. 2, 99–125.

[5] P. Friz and N. Victoir, *Euler Estimates of Rough Differential Equations*, J. Differential Equations **244** (2008), no. 2, 388–412.

[6] ———, *Multidimensional Stochastic Processes as Rough Paths. Theory and Applications*, Cambridge University Press, 2010.

[7] M. Gubinelli, *Controlling rough paths*, J. Funct. Anal. **216** (2004), no. 1, 86–140.

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[8] Y. Inahama and H. Kawabi, *Asymptotic expansions for the Laplace approximations for Itô functionals of Brownian rough paths*, J. Funct. Anal. **243** (2007), no. 1, 270–322.

[9] A. Lejay, *An introduction to rough paths*, Séminaire de probabilités XXXVII, Lecture Notes in Mathematics, vol. 1832, Springer-Verlag, 2003, pp. 1–59.

[10] ———, *Yet another introduction to rough paths*, Séminaire de probabilités XLII, Lecture Notes in Mathematics, vol. 1979, Springer-Verlag, 2009, pp. 1–101.

[11] ———, *On rough differential equations*, Electron. J. Probab. **14** (2009), no. 12, 341–364.

[12] ———, *Controlled differential equations as Young integrals: a simple approach*, 2009. Preprint.

[13] A. Lejay and N. Victoir, *On* (p, q)*-rough paths*, J. Differential Equations **225** (2006), no. 1, 103–133.

[14] T.J. Lyons, *Differential equations driven by rough signals*, Rev. Mat. Iberoamericana **14** (1998), no. 2, 215–310.

[15] T. Lyons and Z. Qian, *Flow of diffeomorphisms induced by a geometric multiplicative functional*, Probab. Theory Related Fields **112** (1998), no. 1, 91–119.

[16] ———, *System Control and Rough Paths*, Oxford Mathematical Monographs, Oxford University Press, 2002.

[17] T. Lyons, M. Caruana, and T. Lévy, *Differential Equations Driven by Rough Paths*, École d'été des probabilités de Saint-Flour XXXIV — 2004 (J. Picard, ed.), Lecture Notes in Mathematics, vol. 1908, Springer, 2007.

[18] H.J. Sussmann, *On the gap between deterministic and stochastic ordinary differential equations*, Ann. Probability **6** (1978), no. 1, 19–41.

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