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Patterns in discretized parabolas and length estimation

(extended version)

Alain Daurat, Mohamed Tajine, and Mahdi Zouaoui

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Abstract. We estimate the frequency of the patterns in the discretization of parabolas, when the resolution tends to zero. We deduce that local estimators of length almost never converge to the length for the parabolas.

Keywords: Digital Curve, Pattern, Multigrid Convergence.

1 Introduction

Length estimation is an important domain of Image Analysis (see [1] for a review). In this paper, we will consider the problem of estimating the length of a curve from its discretizations at different resolutions. In particular we are interested in the comportment of estimators when the resolution tends to zero. We also restrict our study to special estimators called "local estimators" which consist in considering *patterns* which are pieces of fixed length of the discretized curve. Local estimators simply consist to fix a weight to each pattern and summing these weights to obtain the estimation of length (See Fig. 4 for illustration). So, if we want to study the estimated length by local estimators when the resolution tends to zero, we have at first to study the number of occurrences of a pattern of the discretization of digital curves. In fact an asymptotic result about the occurrence number of patterns for discretized general curves looks to be a quite hard problem, because the discretization process is not a continuous process (the integer part function is not continuous), so the estimation of the occurrence number of patterns cannot be deduced from Mathematical Analysis arguments, but by Number Theory arguments. The two first authors of this paper have already made this study for segments in [2]. In this paper we continue this work by considering another class of curves, the parabolas.

The paper is organized as follows: Section 2 describes the notations used in this paper, Section 3 will be devoted to the study of the frequency of patterns in parabolas, and finally Section 4 will apply the results of this study to the local estimators of length of parabolas. Appendix A contains the detailed proofs which are not in the main part of this paper.

2 Notations

In this section we precise the notations that will be used in all the paper.

- For $x \in \mathbb{R}$ we denote by $\lfloor x \rfloor$ (resp.$\lceil x \rceil$) the integer k such that $k \leq x < k+1$ (resp. $k-1 < x \leq k$).
- The fractional part of x is denoted $\langle x \rangle$ and is defined by $x = \lfloor x \rfloor + \langle x \rangle$.
- For $A, B \in \mathbb{Z}$, the discrete interval $\{A, A+1, \ldots, B-1, B\}$ is denoted $[\![A, B]\!]$.
- Let m be a positive integer. A pattern of size m is a function ω from $[\![0, m]\!]$ to $\mathbb{Z}$ such that $\omega(0) = 0$ and $\omega(k+1) \in \{\omega(k), \omega(k)+1\}$ (see Fig. 1). The set of patterns of size m is denoted $\mathcal{P}_m$.
- If X and Y are two real numbers such that $Y > 0$ then $X \bmod Y$ is the real number such that $0 \leq X \bmod Y < Y$ and $\frac{X - X \bmod Y}{Y} \in \mathbb{Z}$.
- For $r \in \mathbb{R}$ and $E \subset \mathbb{R}^2$, $rE = \{(rx, ry) | (x, y) \in E\}$.

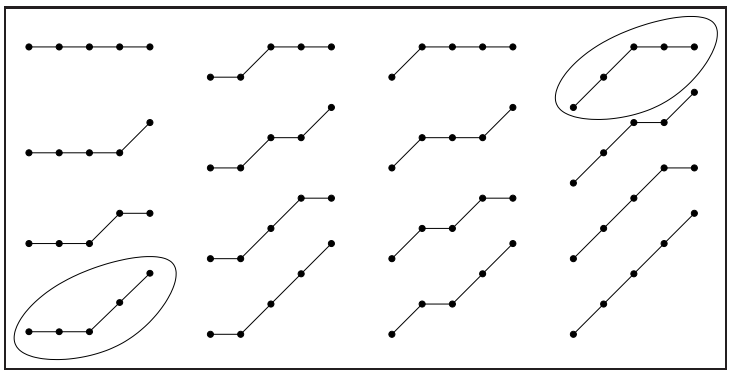

Fig. 1. The 16 patterns of size $m = 4$. Only the encircled one are not digital segments.

3 Frequency of patterns in Discrete Parabolas

Let $a, b \in \mathbb{R}$ such that $a < b$ and a derivable function $g : [a, b] \to \mathbb{R}$ which satisfies $0 \leq g'(x) \leq 1$ for all $x \in [a, b]$.

In all the following, for any $r > 0$ we use the notations:

- $A_r = \lceil \frac{a}{r} \rceil$, $B_r = \lfloor \frac{b}{r} \rfloor$, $N_r = B_r - A_r + 1$,
- $\mathcal{C}_r^g = r\{(X, Y) \in \mathbb{Z}^2 \mid A_r \leq X \leq B_r \text{ and } Y = \lfloor \frac{g(rX)}{r} \rfloor\}$.

The set $\mathcal{C}_r^g$ is the "naive" discretization of the graph of g at resolution r, and N_r is its number of points.

Let m be a positive integer. The pattern at position $X \in [\![A_r, B_r - m]\!]$ of size m of $\mathcal{C}_r^g$, denoted $\omega_{X,r,m}^g$ is defined by:

$$\omega_{X,r,m}^g(k) = \lfloor \frac{g(r(X+k))}{r} \rfloor - \lfloor \frac{g(rX)}{r} \rfloor.$$

The frequency of a pattern ω of size m in $\mathcal{C}_r^g$ is defined by:

$$F_r^g(\omega) = \frac{\mathrm{card}\{X \in [\![A_r, B_r - m]\!] \,|\, \omega_{X,r,m}^g = \omega\}}{N_r - m}.$$

The aim of this section is the study of $F_r^g(\omega)$ for some functions g. For this we will approximate the curve by its tangents which will also be discretized, so we need some notions about digital straight lines:

For $u \in [0,1]$ and $v \in [0,1)$, let us denote $s_m^{u,v}$ the pattern of size m defined by:

$$s_m^{u,v}(k) = \lfloor uk + v \rfloor.$$

A pattern of this form is called a digital segment of size m.

For any pattern ω,

$$PI(\omega) = \{(u,v) \in [0,1]^2 \,|\, \lfloor uk + v \rfloor = \omega(k) \text{ for any } k \in [\![0, m]\!]\},$$

$$\mathrm{pinf}_u(\omega) = \inf\{v \,|\, (u,v) \in PI(\omega)\}, \tag{1}$$

$$\mathrm{psup}_u(\omega) = \sup\{v \,|\, (u,v) \in PI(\omega)\}, \tag{2}$$

$$\begin{aligned} FL_u(\omega) &= 0 && \text{if } \{v \,|\, (u,v) \in PI(\omega)\} = \emptyset \\ &= \mathrm{psup}_u(\omega) - \mathrm{pinf}_u(\omega) && \text{otherwise.} \end{aligned}$$

$PI(\omega)$ is called the preimage of ω, it is nonempty if and only if ω is a digital segment. $FL_u(\omega)$ is intuitively the frequency of the pattern ω in the discretized straight lines of slope u. (See [2,3] for more details and [4] for the generalization to slopes of planes).

In all the paper, the considered curves are parabolas corresponding to the function $g(x) = \alpha x^2$. We distinguish two cases: the case α irrational and the case α rational. In Subsection 3.1, we will see that for α irrational, the frequency $F_r^g(\omega)$ converges, when r is rational and tends to zero, to a quantity which can be expressed by using the function $x \mapsto FL_{g'(x)}(\omega)$. In Subsection 3.2, we study the case α rational, but we do not succeed to prove a similar result as in the case α irrational. Nevertheless we obtain a weaker result (The Tangent Lemma).

3.1 Parabolas of equation $y = \alpha x^2$ with α irrational

In this subsection we consider curves $\mathcal{C}_r^g$ for g defined on $[a,b]$ by $g(x) = \alpha x^2$ with α irrational, and $0 \le a < b \le \frac{1}{2\alpha}$. This last hypothesis is needed to have $0 \le g'(x) \le 1$ for $x \in [a,b]$.

The main result of this subsection is the following:

Theorem 1. *If $g(x) = \alpha x^2$ with $\alpha \notin \mathbb{Q}$ then for any pattern ω we have*

$$F_r^g(\omega) \xrightarrow[\substack{r\to 0\\ r\in\mathbb{Q}}]{} \frac{1}{b-a}\int_a^b FL_{g'(x)}(\omega)\mathrm{d}x \tag{3}$$

The rest of this subsection is devoted to the proof of this Theorem.

The first needed lemma shows that the discretization of a curve and the discretization of its tangent are similar near the origin of the tangent and when the resolution tends to zero:

Lemma 1 (Tangent Lemma). *If g is defined on $[a,b]$ by $g(x) = \alpha x^2$ with α irrational, and $0 \le a < b \le \frac{1}{2\alpha}$ then*

$$\frac{\mathrm{card}\{X \in [\![A_r, B_r - m]\!] \,|\, \omega^g_{X,r,m} \neq s_m^{g'(rX),\langle\frac{g(rX)}{r}\rangle}\}}{N_r - m} \xrightarrow[\substack{r\to 0\\ r\in\mathbb{Q}}]{} 0.$$

Lemma 1 is illustrated by Fig. 2. In the last lemma we consider $r \in \mathbb{Q}$ because its proof needs the irrationality of αr.

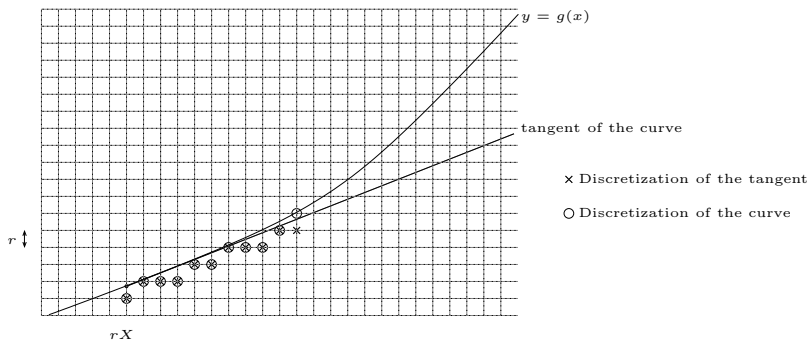

Fig. 2. Comparison of the discretization of the curve and the discretization of its tangent : here we have $\omega^g_{X,r,m} = s_m^{g'(rX),\langle\frac{g(rX)}{r}\rangle}$ for $m = 9$ but not for $m = 10$.

Before starting the proof of Lemma 1 we need one more notation and two other lemmas. For $X \in [\![A_r, B_r]\!]$, we define $P_{r,k}(X)$ by:

$$P_{r,k}(X) = \frac{g(rX)}{r} + g'(rX)k = \alpha r X^2 + 2\alpha k r X = \alpha r((X+k)^2 - k^2).$$

This definition is motivated by the following lemma:

Lemma 2. *If $r < \frac{1}{\alpha m^2}$, then for any X we have $\omega^g_{X,r,m} = s_m^{g'(rX),\langle\frac{g(rX)}{r}\rangle}$ if and only if for all $k \in [\![0,m]\!]$ we have $\langle P_{r,k}(X)\rangle < 1 - \alpha r k^2$.*

Proof.

$$\begin{aligned}\omega^g_{X,r,m}(k) &= \lfloor \frac{g(r(X+k))}{r} \rfloor - \lfloor \frac{g(rX)}{r} \rfloor \\ &= \lfloor \frac{g(rX)}{r} + g'(rX)k + \alpha r k^2 \rfloor - \lfloor \frac{g(rX)}{r} \rfloor\end{aligned}$$

We know that $\lfloor u+v \rfloor = \lfloor u \rfloor + \lfloor v \rfloor$ iff $\langle u \rangle + \langle v \rangle < 1$.

So $\omega^g_{X,r,m}(k) = \lfloor \frac{g(rX)}{r} + g'(rX)k \rfloor - \lfloor \frac{g(rX)}{r} \rfloor$ iff $\langle \frac{g(rX)}{r} + g'(rX)k \rangle + \langle \alpha r k^2 \rangle < 1$.

With the hypothesis $r < \frac{1}{\alpha m^2}$ we have $\langle \alpha r k^2 \rangle = \alpha r k^2$. But:

$$\lfloor \frac{g(rX)}{r} + g'(rX)k \rfloor - \lfloor \frac{g(rX)}{r} \rfloor = \lfloor \langle \frac{g(rX)}{r} \rangle + g'(rX)k \rfloor - \lfloor \langle \frac{g(rX)}{r} \rangle \rfloor = s_m^{g'(rX), \langle \frac{g(rX)}{r} \rangle}(k)$$

So $\omega^g_{X,r,m} = s_m^{g'(rX), \langle \frac{g(rX)}{r} \rangle}$ iff for all $k \in [\![0, m]\!]$ we have $\langle P_{r,k}(X) \rangle < 1 - \alpha r k^2$. □

Lemma 3. *Let I be an interval of $[0,1]$. We have $T_{r,k}(I) \xrightarrow[r \in \mathbb{Q}]{r \to 0} \mu(I)$ where*

$$T_{r,k}(I) = \frac{1}{N_r} \mathrm{card}\{X \in [\![A_r, B_r]\!] \,|\, \langle P_{r,k}(X) \rangle \in I\}$$

and $\mu(I)$ is the usual length of I.

The proof of this lemma uses Weyl's argument, as in the proof of Theorem 1 of [2] or Appendix A.1 of [4], but extended to the quadratic case following the same ideas as in [5, p6-7]. It is given in Appendix A.1.

Proof of Lemma 1. We recall that $\omega^g_{X,r,m} = s_m^{g'(rX), \langle \frac{g(rX)}{r} \rangle}$ iff for all $k \in [\![0, m]\!]$ we have $\langle P_{r,k}(X) \rangle < 1 - \alpha r k^2$ so:

$$\frac{\mathrm{card}\{X \in [\![A_r, B_r - m]\!] \,|\, \omega^g_{X,r,m} \neq s_m^{g'(rX), \langle \frac{g(rX)}{r} \rangle}\}}{N_r - m} \leq \frac{N_r}{N_r - m} \max_{k=0}^{m} T_{r,k}(I_{r,k})$$

where $I_{r,k} = [1 - \alpha r k^2, 1)$, so it is sufficient to show that $T_{r,k}(I_{r,k}) \xrightarrow[r \in \mathbb{Q}]{r \to 0} 0$ for any $k \in [\![0, m]\!]$. Let $\varepsilon > 0$, there exists $R_1 > 0$ such that for any $r < R_1$ we have $\alpha r k^2 < \frac{\varepsilon}{2}$. We know by Lemma 3 that there exists $R_2 > 0$ such that for any $r < R_2$ we have $T_{r,k}(I_{R_0,k}) \leq \mu(I_{R_0,k}) + \frac{\varepsilon}{2}$. If r is such that $r < \min(R_1, R_2)$, we have:

$$\begin{aligned}S_{r,k}(I_{r,k}) &\leq S_{r,k}(I_{R_1,k}) \\ &\leq \mu(I_{R_1,k}) + \frac{\varepsilon}{2} \\ &= \alpha r k^2 + \frac{\varepsilon}{2} \\ &\leq \frac{\varepsilon}{2} + \frac{\varepsilon}{2} = \varepsilon\end{aligned}$$

This finishes the proof of Lemma 1. □

Sketch of Proof of Theorem 1. We present here the main ideas of the proof of Theorem 1. The detailed proof is in Appendix A.2.

By using Lemma 1 we know that $F_r^g(\omega)$ has the same limit as r tends to zero as:

$$G_r^g(\omega) = \frac{\text{card}\{X \in [\![A_r, B_r - m]\!] \mid s_m^{g'(rX), \langle \frac{g(rX)}{r} \rangle} = \omega\}}{N_r - m}.$$

But $s_m^{x,y} = \omega$ is equivalent to $(x, \langle y \rangle) \in PI(\omega)$ so:

$$G_r^g(\omega) = \frac{\text{card}\{X \in [\![A_r, B_r - m]\!] \mid (g'(rX), \langle \frac{g(rX)}{r} \rangle) \in PI(\omega)\}}{N_r - m}$$

which has the same limit as $H_r(PI(\omega))$ where

$$H_r(E) = \frac{\text{card}\{X \in [\![A_r, B_r]\!] \mid (g'(rX), \langle \frac{g(rX)}{r} \rangle) \in E\}}{B_r - A_r + 1}.$$

By applying Lemma 3 with $k = 0$ to the piece of the curve $y = g(x)$ restricted to the domain $g'^{-1}(\alpha_1) \leq x \leq g'^{-1}(\alpha_2)$, we can prove:

$$H_r([\alpha_1, \alpha_2) \times I) \xrightarrow[r \in \mathbb{Q}]{r \to 0} \frac{g'^{-1}(\alpha_2) - g'^{-1}(\alpha_1)}{b - a} \mu(I)$$

So by approximating $PI(\omega)$ as the union of rectangles

$$\bigcup_{i=1}^{n} [y_{i-1}, y_i) \times [\text{pinf}_{y_i}(\omega), \text{psup}_{y_i}(\omega))$$

we approximate $H_r(PI(\omega))$ by:

$$\sum_{i=1}^{n} \frac{g'^{-1}(y_i) - g'^{-1}(y_{i-1})}{b - a} FL_{y_i}(\omega)$$

which is a Riemann sum for $\int_a^b FL_{g'(x)}(\omega) \mathrm{d}x$. □

Corollary 1. *If $g(x) = \alpha x^2$ with $\alpha \notin \mathbb{Q}$, then for any pattern ω which is not a digital segment we have*

$$F_r^g(\omega) \xrightarrow[r \in \mathbb{Q}]{r \to 0} 0.$$

Numerical Application: We illustrate Theorem 1 with an example. Consider the curve $\mathcal{C}$ defined $y = g(x) = \frac{1}{\sqrt{2}} x^2$ for x between $a = 0$ and $b = \frac{1}{\sqrt{2}}$, and the pattern ω of size $m = 3$ defined by $(\omega(0), \omega(1), \omega(2), \omega(3)) = (0, 1, 2, 2)$. We will compute the limit of the frequency of ω when the resolution tends to zero.

First we can compute easily $FL_\alpha(\omega)$ because $\alpha \mapsto FL_\alpha(\omega)$ is a continuous function which is affine between two m-Farey numbers (see [3]). So we deduce:

$$\begin{aligned} FL_\alpha(\omega) &= 0 && \text{if } \alpha \in [0, \frac{1}{2}] \\ &= 2\alpha - 1 && \text{if } \alpha \in [\frac{1}{2}, \frac{2}{3}] \\ &= 1 - \alpha && \text{if } \alpha \in [\frac{2}{3}, 1] \end{aligned}$$

Theorem 1 proves that:

$$\begin{aligned} F_r^g(\omega) \xrightarrow[\substack{r\to 0 \\ r\in\mathbb{Q}}]{} & \frac{1}{b-a}\int_a^b FL_{g'(x)}(\omega)\mathrm{d}x \\ =& \sqrt{2}\int_0^{\frac{1}{\sqrt{2}}} FL_{\sqrt{2}x}(\omega)\mathrm{d}x \\ =& \sqrt{2}\left(\int_{\frac{1}{2\sqrt{2}}}^{\frac{2}{3\sqrt{2}}} (2\sqrt{2}x-1)\mathrm{d}x + \int_{\frac{2}{3\sqrt{2}}}^{\frac{1}{\sqrt{2}}} (1-\sqrt{2}x)\mathrm{d}x\right) = \frac{1}{12} \end{aligned}$$

3.2 Parabolas of equation $y = \alpha x^2$ with α rational

Now we are interested in the case where α is rational. Theorem 1 can be generalized to α rational and to the irrational resolutions r because only the irrationality of αr is used in the proof of this theorem.

On the contrary, in the case α rational and rational resolutions, the only result we will prove in this subsection is about the Tangent Lemma. Moreover, we must impose some restrictions about the resolution r and the interval $[a, b]$ of definition of the parabola: $r\alpha = \frac{1}{p}$ where p is a prime number ; $a = 0$ and $b = \frac{1}{2\alpha}$. In all this subsection we suppose that these conditions are satisfied. Actually, we do not succeed to prove the Tangent Lemma in the general case for α rational.

Let $P_{r,k}(X) = r\alpha((X+k)^2 - k^2)$. We will prove that for any interval $I \subset [0, 1]$

$$\lim_{\substack{r\to 0 \\ \frac{1}{r\alpha} \text{ is prime}}} \frac{\mathrm{card}\{X \in [\![A_r, B_r - k]\!] \mid \langle P_{r,k}(x)\rangle \in I\}}{N_r - k} = \mu(I)$$

Definition 1. *Let p be a prime number and a be an integer number, we define the Legendre symbol $(\frac{a}{p})$ of a relatively to p by*

$$(\frac{a}{p}) = \begin{cases} 0 & \text{if } p \textit{ divides } a \\ 1 & \text{if } a \textit{ there exists } t \in \mathbb{Z} \textit{ such that } a \bmod p = t^2 \bmod p \\ -1 & \textit{otherwise.} \end{cases}$$

Property 1. (Pólya-Vinogradov inequality [6], [7, Chap. 23]) Let p be a prime number and M and N two positive integers then

$$|\sum_{n=M}^{M+N}(\frac{n}{p})| < \sqrt{p}\log(p)$$

Corollary 2. *Let $J = [\![M, M+N]\!]$ be an integer interval. Then*

$$|\frac{\text{card}(J)}{2} - \text{card}\{y \in I \,|\, (\frac{y}{p}) = 1\}| < \sqrt{p}\log(p).$$

Lemma 4. *Let α be a rational number and assume that $a = 0$ and $b = \frac{1}{2\alpha}$. Then, for any interval $I \subset [0,1]$*

$$\lim_{r\to 0}\frac{\text{card}\{X \in [\![A_r, B_r - k]\!] \,|\, \langle P_{r,k}(X)\rangle \in I\}}{N_r - k} = \mu(I)$$

where the limit is taken on the r such that $r > 0$ and $r\alpha = \frac{1}{p}$ where p is a prime number.

Proof. The function $X \mapsto X^2 \bmod p$ from $[\![1, \frac{p-1}{2}]\!]$ to $\{Y \,|\, (\frac{Y}{p}) = 1\}$ is a bijection.
Put $H_r = \dfrac{\text{card}\{X \in [\![A_r, B_r - k]\!] \,|\, \langle P_{r,k}(x)\rangle \in I\}}{N_r}$. Then

$$H_r = \frac{\text{card}\{X \in [\![k, \lfloor \frac{p}{2}\rfloor]\!] \,|\, \langle \frac{X^2-k^2}{p}\rangle \in I\}}{N_r - k} = \frac{2\text{card}\{X \in [\![k, \frac{p-1}{2}]\!] \,|\, \frac{X^2 \bmod p - k^2}{p} \in I\}}{p+1-2k} \quad (4)$$

Thus

$$\text{card}\{X \in [\![1, \frac{p-1}{2}]\!] \,|\, X^2 \bmod p \in J\} = \text{card}\{Y \in J \,|\, (\frac{Y}{p}) = 1\} \text{ where } J = pI + k^2.$$

So $|\text{card}\{X \in [\![k, \frac{p-1}{2}]\!] \,|\, X^2 \bmod p \in J\} - \text{card}\{Y \in J \,|\, (\frac{Y}{p}) = 1\}| < k$.
By using Pólya-Vinogradov inequality we have:

$$|\frac{\text{card}(J)}{2} - \text{card}\{X \in [\![1, \frac{p-1}{2}]\!] \,|\, X^2 \bmod p \in J\}| < \sqrt{p}\log(p)$$

So, $|\frac{\text{card}(J)}{2} - \text{card}\{X \in [\![k, \frac{p-1}{2}]\!] \,|\, X^2 \bmod p \in J\}| < \sqrt{p}\log(p) + k$.
By (4) we have

$$|\frac{2}{p+1-2k}\frac{\text{card}(J)}{2} - \frac{2}{p+1-2k}\text{card}\{X \in [\![k, \frac{p-1}{2}]\!] \,|\, X^2 \bmod p \in J\}| < \frac{2\sqrt{p}\log(p) + 2k}{p+1-2k}$$

Thus, $\lim_{r\to 0}\dfrac{\text{card}\{X \in [\![A_r, B_r - k]\!] \,|\, \langle P_{r,k}(X)\rangle \in I\}}{N_r - k} = \mu(I)$

□

Theorem 2 (Tangent Lemma for rational slopes and special rational resolutions). *For any rational $\alpha > 0$ and any a, b such that $0 \leq a < b \leq \frac{1}{2\alpha}$ we have*

$$\frac{\mathrm{card}\{X \in [\![A_r, B_r - m]\!] \,|\, \omega^g_{X,r,m} \neq s_m^{g'(rX), \langle \frac{g(rX)}{r} \rangle}\}}{N_r - m} \xrightarrow[\frac{1}{r\alpha} \text{ is prime}]{r \to 0} 0$$

Proof. The case where $a = 0$, $b = \frac{1}{2\alpha}$ can be proved exactly in the same way as for Lemma 1 by using Lemma 4 instead of Lemma 3. Consider the general case $[a, b] \subset [0, \frac{1}{2\alpha}]$. Then we know that:

$$\begin{aligned}
&\frac{\mathrm{card}\{X \in [\![A_r, B_r - m]\!] \,|\, \omega^g_{X,r,m} \neq s_m^{g'(rX), \langle \frac{g(rX)}{r} \rangle}\}}{N_r - m} \\
&\leq \frac{\lfloor \frac{1}{2\alpha r} \rfloor + 1 - m}{N_r - m} \cdot \frac{\mathrm{card}\{X \in [\![0, \lfloor \frac{b}{r} \rfloor - m]\!] \,|\, \omega^g_{X,r,m} \neq s_m^{g'(rX), \langle \frac{g(rX)}{r} \rangle}\}}{\lfloor \frac{1}{2\alpha r} \rfloor + 1 - m} \\
&\xrightarrow[\frac{1}{r\alpha} \text{ is prime}]{r \to 0} \frac{\frac{1}{2\alpha}}{b - a} \cdot 0 = 0 \qquad \text{because of the case } a = 0,\ b = \frac{1}{2\alpha}.
\end{aligned}$$

□

Unfortunately we do not successfully generalize Theorem 1 to rational α and some rational resolutions even if experimentally Theorem 1 seems to be true in all the cases.

4 Application to local estimators of length

A local estimator is given by a weight function p from the set $\mathcal{P}_m$ of patterns of size m to $\mathbb{R}$. The estimated length of the curve $y = g(x)$ where $g : [a, b] \to \mathbb{R}$ at resolution r is given by:

$$l(p, g, r) = r \sum_{k=0}^{\lfloor \frac{B_r - m - A_r}{m} \rfloor} p(\omega^g_{A_r + km, r, m}).$$

Theorem 3. *Let a, b such that $0 \leq a < b$. The length estimated by a local estimator of a parabola $y = \alpha x^2$, $x \in [a, b]$, ($\alpha \leq \frac{1}{2b}$) converges when the resolution r is rational and tends to zero, to the length of the parabola* only *for a finite number of irrational numbers α.*

Sketch of proof. For this proof we use the notation $[f(x)]_a^b = f(b) - f(a)$.

It is easy to see that:

$$l(p, g, r) - \frac{(b - a)}{m} \sum_{\omega \in \mathcal{P}_m} p(\omega) F'^g_r(\omega) \xrightarrow[r \to 0]{} 0$$

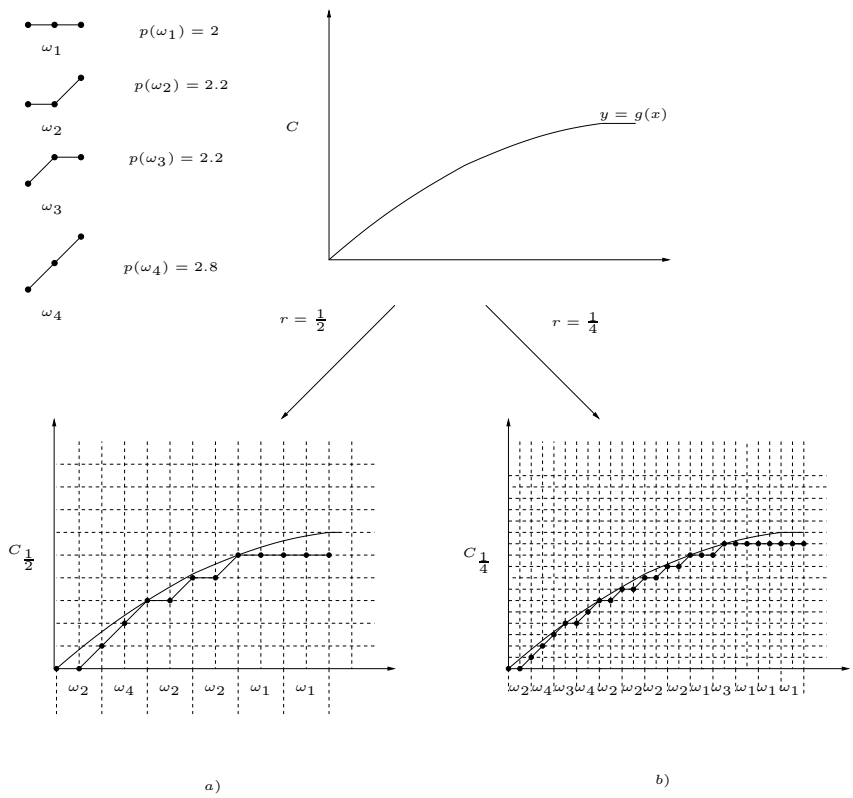

Fig. 3. Estimation of length of a curve from its discretization for two different resolutions: a) $l(p, g, \frac{1}{2}) = \frac{1}{2}(2p(\omega_1) + 3p(\omega_2) + 1p(\omega_4)) = 6.7$, b) $l(p, g, \frac{1}{4}) = \frac{1}{4}(4p(\omega_1) + 5p(\omega_2) + 2p(\omega_3) + 2p(\omega_4)) = 7.1$

where

$$F'^{g}_{r}(\omega) = \frac{\mathrm{card}\{X \in [\![A_r, B_r - m]\!] \cap (A_r + m\mathbb{Z}) \,|\, \omega^{g}_{X,r,m} = \omega\}}{\lfloor \frac{B_r - m - A_r}{m} \rfloor}.$$

Consider again the curve defined by $g(x) = \alpha x^2$ for $\alpha \notin \mathbb{Q}$. The proof of Theorem 1 can be extended to prove that:

$$F'^{g}_{r}(\omega) \xrightarrow[\substack{r \to 0 \\ r \in \mathbb{Q}}]{} \frac{1}{b-a} \int_a^b FL_{g'(x)}(\omega) \mathrm{d}x.$$

So,

$$l(p, g, r) \xrightarrow[\substack{r \to 0 \\ r \in \mathbb{Q}}]{} \frac{1}{m} \sum_{\omega \in \mathcal{P}_m} p(\omega) \int_a^b FL_{g'(x)}(\omega) \mathrm{d}x. \tag{5}$$

We know that $x \mapsto FL_x(\omega)$ is piecesewisely affine ([3]). From this property we deduce that we can partition the interval $[0, \frac{1}{2b}]$ in a finite number of intervals $(I_k)_{0 \leq k \leq n}$ such that $L_{est}(\alpha) = \lim_{\substack{r \to 0 \\ r \in \mathbb{Q}}} l(p, g, r)$ is of the form $\frac{A}{\alpha} + B\alpha + C$ on each interval I_k. (See Appendix A.3 for the details).

Let $L_{real}(\alpha)$ be the length of the parabola $\{(x, \alpha x^2) \mid x \in [a, b]\}$. We have:

$$L_{real}(\alpha) = \int_a^b \sqrt{1 + (2\alpha x)^2} \mathrm{d}x = \left[\frac{x\sqrt{1 + (2\alpha x)^2}}{2} + \frac{\arg\sinh(2\alpha x)}{4\alpha} \right]_a^b$$

Suppose that $L_{real}(\alpha) = L_{est}(\alpha)$ for an infinite number of irrational numbers α, then there exists an interval I_k of the previous partition of $[0, \frac{1}{2b}]$ such that $L_{real}(\alpha) = L_{est}(\alpha)$ for an infinite number of irrational numbers $\alpha \in I_k$. On I_k we know that $L_{est}(\alpha)$ has the form $\frac{A}{\alpha} + B\alpha + C$. The functions $\alpha \mapsto \alpha L_{real}(\alpha)$ and $\alpha \mapsto \alpha(\frac{A}{\alpha} + B\alpha + C)$ are holomorphic in a open set of $\mathbb{C}$ containing $[0, \frac{1}{2b}]$ and are equal for an infinite number of $\alpha \in I_k \subset [0, \frac{1}{2b}]$. So by Theorem on the zeros of holomorphic functions [8, Cha. 10] they are equal on $[0, \frac{1}{2b}]$. So:

$$\alpha L_{real}(\alpha) = A + B\alpha^2 + C\alpha \qquad \text{for all } \alpha \in [0, \frac{1}{2b}]$$

We have:

$$\begin{aligned} \frac{\partial(\alpha L_{real}(\alpha)}{\partial \alpha} &= b\sqrt{1 + (2\alpha b)^2} - a\sqrt{1 + (2\alpha a)^2} \\ &= b - a + 2(b^3 - a^3)\alpha^2 + o(\alpha^2) \qquad \text{when } \alpha \to 0 \end{aligned}$$

But $\frac{\partial(A + B\alpha^2 + C\alpha)}{\partial \alpha} = 2B\alpha + C$, so $2(b^3 - a^3) = 0$ which is impossible if $b > a$. So the hypothesis that $L_{real}(\alpha) = L_{est}(\alpha)$ for an infinite number of irrational α is absurd. □

Corollary 3. *Let a, b such that $0 \leq a < b$. The length estimated by a local estimator of a parabola $y = \alpha x^2$, $x \in [a, b]$, does* not *converge when the resolution is rational and tends to zero, to the length of the curve for almost all $\alpha \in [0, \frac{1}{2b}]$.*

Numerical application: Again, we take the curve $y = g(x) = \frac{1}{\sqrt{2}}x^2$ for x between $a = 0$ and $b = \frac{1}{\sqrt{2}}$. Suppose that we consider the local estimator Chamfer 5-7-11 ([9]) with $m = 2$, $p(000) = 2$, $p(001) = p(011) = \frac{22}{10}$, $p(012) = \frac{28}{10}$. With Equation (13), we can prove that the estimation of the length of the parabola given by this local estimator converges to $L_{est} = \frac{23}{40}\sqrt{2} \approx 0.813172$, this limit is different from the length of the parabola which is $\frac{1}{2} + \frac{1}{4}\sqrt{2}\log(1 + \sqrt{2}) \approx 0.811612$. Moreover Figure 4 shows how the length given by the estimator converges to its limit when the resolution tends to zero. It seems on this example that $l(p, g, r) - L_{est} = O(r)$.

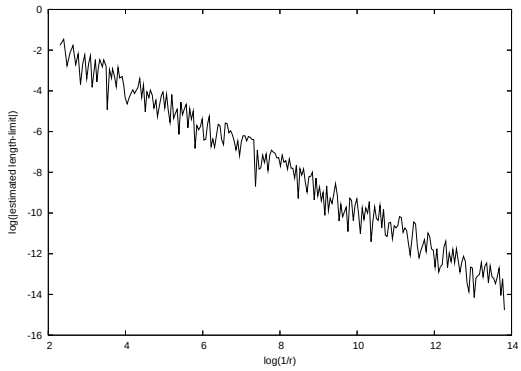

Fig. 4. Figure showing Convergence Speed of the Chamfer 5-7-11 to its limits for the parabola $y = \frac{1}{\sqrt{2}}x^2, \qquad 0 \leq x \leq \frac{1}{\sqrt{2}}$.

5 Conclusion and Perspectives

In this paper, we have proved some local properties of discretizations of parabolas: First we show that locally discretization of parabola and discretization of its tangent often coincide (Tangent Lemma: Lemmas 1 and Theorem 2). In particular, asymptotically, the local patterns of a discretized parabola are digital segments. From this, we also give an explicit formula for the limit of frequency of a pattern of a parabola when the resolution tends to zero (Theorem 1). This has the important consequence that we can know to what tend local estimators of length for the parabolas, moreover it can be proved that this limit is often different from the length of the curve.

This work mainly brings two perspectives:

- The extension of Formula (3), which gives the limit of the frequency of pattern when the resolution tends to zero, to more general curves, in particular to the curves $y = P(x)$ when P is a polynomial of degree greater than 2.
- The application of this work for recognition of curve by just looking at patterns. For example, if the frequencies of patterns of a curve does not satisfy Theorem 1 then it is not a parabola of equation $y = \alpha x^2$.

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A Proofs

Additional notations

- The distance between x and $\mathbb{Z}$ is denoted $\|x\|$. So $\|x\| = \min(\langle x\rangle, 1 - \langle x\rangle)$.
- If z is complex number $\overline{z}$ denotes its conjugate, and $\mathrm{Re}(z)$ denotes its real part.
- $\gcd(p, q)$ denotes the greatest common divisor of p and q.

A.1 Proof of Lemma 3

For any $f \in L^1([0, 1])$ we define:

$$S_{r,k}(f) = \frac{1}{N_r} \sum_{X=A_r}^{B_r} f(\langle P_{r,k}(X)\rangle).$$

so we have $T_{r,k}(I) = S_{r,k}(\chi_I)$ where χ_I is the indicator function of I. We denote $e(t) = e^{2\pi i t}$ and $e_c(t) = e(ct)$.

Sublemma 5 *For any $c \in \mathbb{Z} \setminus \{0\}$ we have $S_{r,k}(e_c) \xrightarrow[\substack{r\to 0\\ r\in\mathbb{Q}}]{} 0$.*

Proof.

$$\begin{aligned}
|S_{r,k}(e_c)|^2 &= \overline{S_{r,k}(e_c)} S_{r,k}(e_c) \\
&= \frac{1}{N_r^2} \sum_{A_r \le X, Y \le B_r} \overline{e_c(P_{r,k}(X))} e_c(P_{r,k}(Y)) \\
&= \frac{1}{N_r} + \frac{2}{N_r^2} \mathrm{Re}\left(\sum_{A_r \le X < Y \le B_r} e_c(P_{r,k}(Y) - P_{r,k}(X)) \right)
\end{aligned}$$

Let us pose $Y = X + h$, we have

$$\begin{aligned} P_{r,k}(Y) - P_{r,k}(X) &= r\alpha(X+h)^2 + 2\alpha k(X+H) - \alpha r X^2 - 2\alpha k r X \\ &= P_{r,k}(h) + 2\alpha r X h \end{aligned}$$

So:

$$\begin{aligned} |S_{r,k}(e_c)|^2 &= \frac{1}{N_r} + \frac{2}{N_r^2}\mathrm{Re}\left(\sum_{X=A_r}^{B_r-1}\sum_{h=1}^{B_r-X} e_c(P_{r,k}(h))e_c(2\alpha r X h)\right) \\ &= \frac{1}{N_r} + \frac{2}{N_r^2}\mathrm{Re}\left(\sum_{h=1}^{N_r-1}\sum_{X=A_r}^{B_r-h} e_c(P_{r,k}(h))e_c(2\alpha r X h)\right) \\ &= \frac{1}{N_r} + \frac{2}{N_r^2}\mathrm{Re}\left(\sum_{h=1}^{N_r-1}\left(e_c(P_{r,k}(h))\sum_{X=A_r}^{B_r-h} e_c(2\alpha r X h)\right)\right) \end{aligned}$$

Subsublemma 6 *For any $\beta \in \mathbb{R}\setminus\mathbb{Z}$, $u, v \in \mathbb{Z}$ such that $u \leq v$ we have:*

$$\left|\sum_{k=u}^{v} e(\beta k)\right| \leq \min(v-u+1, \frac{1}{2\|\beta\|}).$$

Proof.

$$\sum_{k=u}^{v} e(\beta k) = e(\beta u)\frac{1-e(\beta(v-u+1))}{1-e(\beta)}$$

so

$$\left|\sum_{k=u}^{v} e(\beta k)\right| \leq \frac{2}{|1-e(\beta)|}$$

But $|1-e(\beta)| = |e(\frac{\beta}{2})(e(-\frac{\beta}{2}) - e(\frac{\beta}{2}))| = 2|\sin(\pi\beta)|$. Moreover $\sin(\pi\beta) \geq 2\|\beta\|$ (because $\sin(\pi x) \geq 2x$ for $x \in [0, \frac{1}{2}]$). So:

$$\left|\sum_{k=u}^{v} e(\beta k)\right| \leq \frac{1}{2\|\beta\|}$$

With the clear inequality $|\sum_{k=u}^{v} e(\beta k)| \leq v-u+1$, it ends the proof of Subsublemma 6. □

As $\mathrm{Re}(x) \leq |x|$, Subsublemma 6 shows that:

$$|S_{r,k}(e_c)|^2 \leq \frac{1}{N_r} + \frac{2}{N_r^2}\sum_{h=1}^{N_r-1}\min(N_r, \frac{1}{2\|2\alpha r c h\|}) \tag{6}$$

Subsublemma 7 *Let $C_0 \geq 0$, $u \in \mathbb{R} \setminus \mathbb{Q}$, $N, p, q \in \mathbb{N}$ which satisfy $q \leq C_0 N$, $|uq - p| \leq \frac{1}{C_0 N}$, $\gcd(p,q) = 1$. We have*

$$\sum_{h=1}^{N-1} \min(N, \frac{1}{2\|uh\|}) \leq 4(1 + C_0) \left(\frac{N^2}{q} + N \log q \right).$$

Proof.

$$\sum_{h=1}^{N-1} \min(N, \frac{1}{2\|uh\|}) \leq \sum_{H=0}^{\lfloor \frac{N-1}{q} \rfloor} \sum_{h=qH+1}^{qH+q} \min(N, \frac{1}{2\|uh\|})$$

But:

$$\sum_{h=qH+1}^{qH+q} \min(N, \frac{1}{2\|uh\|}) = \sum_{h=1}^{q} \min(N, \frac{1}{2\|u(qH+h)\|})$$

$$\begin{aligned}
\|u(qH+h)\| &= \|(\frac{p+\varepsilon}{q})(qH+h)\| && \text{where } \varepsilon = uq - p \\
&= \|pH + \frac{p}{q}h + \varepsilon H + \frac{\varepsilon h}{q}\| \\
&= \|\frac{p}{q}h + \varepsilon H + \frac{\varepsilon h}{q}\|
\end{aligned}$$

Let $\gamma = (\varepsilon H) \bmod \left(\frac{1}{q}\right)$, and k_0 such that $\varepsilon H = \gamma + \frac{k_0}{q}$. We have: $(\frac{p}{q}h + \varepsilon H) \bmod 1 = \gamma + \frac{i_h}{q}$ where $i_h = (ph + k_0 q) \bmod q$. As $\gcd(p,q) = 1$, $\{i_h \mid h \in [\![1, q]\!]\} = [\![0, q-1]\!]$.

If $i_h < \frac{q}{2}$ then

$$\begin{aligned}
\|u(qH+h)\| &= \|\gamma + \frac{i_h}{q} + \frac{\varepsilon h}{q}\| \\
&\geq \|\gamma + \frac{i_h}{q}\| - \frac{|\varepsilon| h}{q} && \text{because } x \mapsto \|x\| \text{ is Lipschitz continuous with constant 1} \\
&= \gamma + \frac{i_h}{q} - \frac{|\varepsilon| h}{q} \\
&\geq \frac{i_h}{q} - |\varepsilon| && \text{as } h \leq q \\
&\geq \frac{i_h}{q} - \frac{1}{C_0 N}. && (7)
\end{aligned}$$

Similarly if $i_h \geq \frac{q}{2}$ then

$$\begin{aligned}
\|u(qH+h)\| &\geq \|\gamma + \frac{i_h}{q}\| - \frac{|\varepsilon| h}{q} \\
&= 1 - (\gamma + \frac{i_h}{q}) - \frac{|\varepsilon| h}{q} \\
&\geq \frac{q - 1 - i_h}{q} - \frac{1}{C_0 N}. && (8)
\end{aligned}$$

If we use the variable change $h \mapsto i_h$ for $i_h < \frac{q}{2}$, $h \mapsto q-1-i_h$ for $i_h \geq \frac{q}{2}$, the inequality $\min(N, \frac{1}{2\|u(qH+h)\|}) \leq N$ for $i_h = 0, 1, q-2, q-1$, and the inequalities (7),(8) for the other i_h, we deduce:

$$\sum_{h=1}^{q} \min(N, \frac{1}{2\|u(qH+h)\|}) \leq 4N + 2 \sum_{i=2}^{\lfloor \frac{q-1}{2} \rfloor} \frac{1}{2\left(\frac{i}{q} - \frac{1}{C_0 N}\right)}.$$

But $\frac{1}{1-x} \leq 1 + 2x$ for $x \in (0, \frac{1}{2}]$. So for $i \geq 2$ we have:

$$\begin{aligned}
\frac{1}{\frac{i}{q} - \frac{1}{C_0 N}} &= \frac{q}{i} \frac{1}{1 - \frac{q}{iC_0 N}} \\
&\leq \frac{q}{i}\left(1 + 2\frac{q}{iC_0 N}\right) \quad \text{because } \frac{q}{iC_0 N} \in (0, \frac{1}{2}] \text{ as } i \geq 2 \text{ and } q \leq C_0 N \\
&\leq 2\frac{q}{i}
\end{aligned}$$

So

$$\sum_{h=1}^{q} \min(N, \frac{1}{2\|u(qH+h)\|}) \leq 4N + 2q \sum_{i=2}^{\lfloor \frac{q-1}{2} \rfloor} \frac{1}{i}.$$

But:

$$\begin{aligned}
\sum_{i=2}^{\lfloor \frac{q-1}{2} \rfloor} \frac{1}{i} &= \sum_{i=2}^{\lfloor \frac{q-1}{2} \rfloor} \int_{i-1}^{i} \frac{1}{i} \mathrm{d}t \\
&\leq \sum_{i=2}^{\lfloor \frac{q-1}{2} \rfloor} \int_{i-1}^{i} \frac{1}{t} \mathrm{d}t \\
&= \int_{1}^{\lfloor \frac{q-1}{2} \rfloor} \frac{1}{t} \mathrm{d}t \\
&= \log(\lfloor \frac{q-1}{2} \rfloor)
\end{aligned}$$

So:

$$\begin{aligned}
\sum_{h=1}^{N-1} \min(N, \frac{1}{2\|uh\|}) &\leq \left(\lfloor \frac{N-1}{q} \rfloor + 1\right)\left(4N + 2q\log(\lfloor \frac{q-1}{2} \rfloor)\right) \\
&\leq \left(\frac{N}{q} + 1\right)(4N + 2q\log q) \\
&\leq \left(\frac{N}{q} + \frac{C_0 N}{q}\right)(4N + 2q\log q) \qquad \text{because } q \leq C_0 N \\
&\leq 4(1 + C_0)(N^2 + N\log q)
\end{aligned}$$

This finishes the proof of Subsublemma 7. □

In the following we suppose that r is rational. So αr is irrational, so by Dirichlet's principle ([11]) we know that there exist two coprime integers $q_r \leq \frac{2N_r}{\alpha c(b-a)}$ and p_r such that $|2\alpha r c q_r - p_r| \leq \frac{\alpha c(b-a)}{2N_r}$. By Subsublemma 7 with $C_0 = \frac{2}{\alpha c(b-a)}$ and Equation (6) we deduce:

$$\begin{aligned}|S_{r,k}(e_c)|^2 &\leq \frac{1}{N_r} + \frac{8(1+C_0)}{N_r^2}\left(\frac{N_r^2}{q} + N_r \log q_r\right)\\ &\leq \frac{1}{N_r} + \frac{8(1+C_0)}{q_r} + \frac{8(1+C_0)\log q_r}{N_r}\end{aligned}$$

As $N_r \xrightarrow[r\to 0]{} +\infty$ and $\frac{\log q_r}{N_r} \leq \frac{\log(C_0 N_r)}{N_r} \xrightarrow[r\to 0]{} 0$, it remains to prove that $q_r \xrightarrow[r\to 0]{} +\infty$.

We have:

$$\frac{p_r}{q_r} \geq 2\alpha r c - \frac{\alpha c(b-a)}{2N_r}$$

We have $N_r \geq (\frac{b}{r} - 1) - (\frac{a}{r} + 1) - 1 = \frac{b-a-r}{r}$. We suppose without loss of generality that $r \leq \frac{b-a}{2}$ so $N_r \geq \frac{b-a}{2r}$, so

$$\frac{p_r}{q_r} \geq 2\alpha r c - \alpha r c > 0,$$

so $p_r \geq 1$. We deduce:

$$\frac{1}{q_r} \leq \frac{p_r}{q_r} \leq 2\alpha r c + \frac{1}{N_r q_r} \leq 2\alpha r c + \frac{1}{N_r}$$

So $q_r \geq \frac{1}{2\alpha r c + \frac{1}{N_r}} \xrightarrow[r\to 0]{} +\infty$. This ends the proof of Sublemma 5. □

To finish the proof of Lemma 3 we must prove that $S_{r,k}(\chi_I) \xrightarrow[\substack{r\to 0\\ r\in\mathbb{Q}}]{} \mu(I)$ where χ_I is the indicator function of I. We have $S_{r,k}(e_0) = 1$ and by Sublemma 5 $S_{r,k}(e_c) \xrightarrow[\substack{r\to 0\\ r\in\mathbb{Q}}]{} 0$ for $c \neq 0$, so by Weyl's strategy (see for example [2] or [4, Appendix A.1]) we can prove Lemma 3. □

A.2 Proof of Theorem 1

Now let:

$$G_r^g(\omega) = \frac{\mathrm{card}\{X \in [\![A_r, B_r - m]\!] \mid s_m^{g'(rX),\langle \frac{g(rX)}{r}\rangle} = \omega\}}{N_r - m}.$$

By Lemma 1 we have

$$F_r^g(\omega) - G_r^g(\omega) \xrightarrow[\substack{r\to 0\\ r\in\mathbb{Q}}]{} 0. \tag{9}$$

We have by definition of $PI(\omega)$ (and because $\lfloor \langle \frac{g(rX)}{r} \rangle \rfloor = 0$):

$$G_r^g(\omega) = \frac{\mathrm{card}\{X \in [\![A_r, B_r - m]\!] \,|\, (g'(rX), \langle \frac{g(rX)}{r} \rangle) \in PI(\omega)\}}{N_r - m}$$

For any subset E of $\mathbb{R}^2$, we define:

$$H_r(E) = \frac{\mathrm{card}\{X \in [\![A_r, B_r]\!] \,|\, (g'(rX), \langle \frac{g(rX)}{r} \rangle) \in E\}}{N_r}$$

We have

$$\frac{N_r H_r(PI(\omega)) - m}{N_r - m} \leq G_r^g(\omega) \leq \frac{N_r H_r(PI(\omega))}{N_r - m}. \tag{10}$$

Suppose first that $E = [\alpha_1, \alpha_2) \times I$. The function g' is linear, so in particular is a bijection, we denote by g'^{-1} its reciprocal function. We suppose that $\alpha > 0$, so that g' is an increasing function. $(g'(rX), \langle \frac{g(rX)}{r} \rangle) \in E$ is equivalent to $g'(rX) \in [\alpha_1, \alpha_2)$ and $\langle \frac{g(rX)}{r} \rangle) \in I$. So

$$H_r(E) = \frac{\mathrm{card}\{X \in [\![\lceil \frac{g'^{-1}(\alpha_1)}{r} \rceil, \lfloor \frac{g'^{-1}(\alpha_2)}{r} \rfloor]\!] \,|\, \langle \frac{g(rX)}{r} \rangle \in I\}}{N_r}.$$

If we apply Lemma 3 with $k = 0$ to the piece of the curve $y = g(x)$ restricted to the domain $g'^{-1}(\alpha_1) \leq x \leq g'^{-1}(\alpha_2)$. we find:

$$\frac{N_r H_r(E)}{\lfloor \frac{g'^{-1}(\alpha_2)}{r} \rfloor - \lceil \frac{g'^{-1}(\alpha_1)}{r} \rceil + 1} \xrightarrow[r \in \mathbb{Q}]{r \to 0} \mu(I)$$

We deduce that:

$$H_r([\alpha_1, \alpha_2) \times I) \xrightarrow[r \in \mathbb{Q}]{r \to 0} \frac{g'^{-1}(\alpha_2) - g'^{-1}(\alpha_1)}{b - a} \mu(I) \tag{11}$$

In [10] it is shown that

$$PI(\omega) = \{(\alpha, \beta) \,|\, \alpha \in [\alpha_b, \alpha_e] \text{ and } \mathrm{pinf}_\alpha(\omega) \leq \beta < \mathrm{psup}_\alpha(\omega)$$

where $\alpha \mapsto \mathrm{pinf}_\alpha(\omega)$ and $\alpha \mapsto \mathrm{psup}_\alpha(\omega)$ are two piecewise affine functions which slope is between $-m$ and 0. [1] [2]

Let $n \in \mathbb{N} \setminus \{0\}$ and $y_i = \alpha_b + i \frac{\alpha_e - \alpha_b}{n}$ for $i \in [\![0, n]\!]$.

$PI(\omega)$ is approximated by an union of n rectangles:

$$\bigcup_{i=1}^{n} [y_{i-1}, y_i) \times [\mathrm{pinf}_{y_{i-1}}(\omega), \mathrm{psup}_{y_i}(\omega)) \subset PI(\omega) \subset \bigcup_{i=1}^{n} [y_{i-1}, y_i] \times [\mathrm{pinf}_{y_i}(\omega), \mathrm{psup}_{y_{i-1}}(\omega))$$

[1] In fact it is proved that $PI(\omega)$ is a triangle or a quadrangle.

[2] Formulas (1),(2) do not define pinf and psup for $\alpha = \alpha_b, \alpha_e$, we define these values just by continuity.

As $y_i - y_{i-1} \leq \frac{1}{n}$ and $\alpha \mapsto \mathrm{pinf}_\alpha(\omega)$ and $\alpha \mapsto \mathrm{psup}_\alpha(\omega)$ are two piecewise affine functions which slope is between $-m$ and 0, we have $\mathrm{pinf}_{y_{i-1}}(\omega) \leq \mathrm{pinf}_{y_i}(\omega) + \frac{m}{n}$ and $\mathrm{psup}_{y_{i-1}}(\omega) \leq \mathrm{psup}_{y_i}(\omega) + \frac{m}{n}$. So:

$$\bigcup_{i=1}^{n} [y_{i-1}, y_i) \times [\mathrm{pinf}_{y_i}(\omega) + \frac{m}{n}, \mathrm{psup}_{y_i}(\omega)) \subset PI(\omega) \subset \bigcup_{i=1}^{n} [y_{i-1}, y_i] \times [\mathrm{pinf}_{y_i}(\omega), \mathrm{psup}_{y_i}(\omega) + \frac{m}{n}) \quad (12)$$

Let

$$F_{n,r} = H_r \left(\bigcup_{i=1}^{n} [y_{i-1}, y_i) \times [\mathrm{pinf}_{y_i}(\omega) + \frac{m}{n}, \mathrm{psup}_{y_i}(\omega)) \right),$$

$$F'_{n,r} = H_r \left(\bigcup_{i=1}^{n} [y_{i-1}, y_i] \times [\mathrm{pinf}_{y_i}(\omega), \mathrm{psup}_{y_i}(\omega) + \frac{m}{n}) \right).$$

So by Equation (12):

$$F_{n,r} \leq H_r(PI(\omega)) \leq F'_{n,r}.$$

By summing equations of the form (11) we obtain:

$$F_{n,r} \xrightarrow[\substack{r \to 0 \\ r \in \mathbb{Q}}]{} \sum_{i=1}^{n} \frac{g'^{-1}(y_i) - g'^{-1}(y_{i-1})}{b-a} \left(FL_{y_i}(\omega) - \frac{m}{n} \right)$$

$$F'_{n,r} \xrightarrow[\substack{r \to 0 \\ r \in \mathbb{Q}}]{} \sum_{i=1}^{n} \frac{g'^{-1}(y_i) - g'^{-1}(y_{i-1})}{b-a} \left(FL_{y_i}(\omega) + \frac{m}{n} \right)$$

Let

$$F''_n = \sum_{i=1}^{n} \frac{g'^{-1}(y_i) - g'^{-1}(y_{i-1})}{b-a} FL_{y_i}(\omega).$$

$$\begin{aligned}
(\lim_{\substack{r \to 0 \\ r \in \mathbb{Q}}} F_{n,r}) - F''_n &= -\sum_{i=1}^{n} \frac{g'^{-1}(y_i) - g'^{-1}(y_{i-1})}{b-a} \frac{m}{n} \\
&= -\sum_{i=1}^{n} \frac{\frac{1}{2\alpha}(y_i - y_{i-1})}{b-a} \frac{m}{n} \\
&= -\frac{m(\alpha_e - \alpha_b)}{2\alpha(b-a)} \frac{1}{n}
\end{aligned}$$

Similarly

$$(\lim_{\substack{r \to 0 \\ r \in \mathbb{Q}}} F'_{n,r}) - F''_n = \frac{m(\alpha_e - \alpha_b)}{2\alpha(b-a)} \frac{1}{n}.$$

Let $z_i = g'^{-1}(y_i)$, we have

$$F''_n = \sum_{i=1}^{n} (z_i - z_{i-1}) \frac{FL_{g'(z_i)}(\omega)}{b-a}$$

which is a Riemann sum of $x \mapsto \frac{FL_{g'(x)}(\omega)}{b-a}$, but $t \mapsto FL_t(\omega)$ is continuous ([3]), so we have by [12, Chap. 5]:

$$F''_n \xrightarrow[n\to\infty]{} \frac{1}{b-a} \int_a^b FL_{g'(x)}(\omega) \mathrm{d}x$$

Let $\varepsilon > 0$, there exists N_1 such that for all $n > N_1$ we have

$$\left| F''_n - \frac{1}{b-a} \int_a^b FL_{g'(x)}(\omega) \mathrm{d}x \right| \leq \frac{\varepsilon}{3}.$$

There exists N_2 such that for any $n > N_2$ we have

$$\frac{m(\alpha_e - \alpha_b)}{2\alpha(b-a)} \frac{1}{n} < \frac{\varepsilon}{3}$$

Let $N = \max(N_1, N_2) + 1$. There exists $R_1 > 0$ such that for any rational $r < R_1$ we have:

$$|F_{N,r} - (F''_N - \frac{m(\alpha_e - \alpha_b)}{2\alpha(b-a)} \frac{1}{N})| \leq \frac{\varepsilon}{3}.$$

There exists $R_2 > 0$ such that for any rational $r < R_2$ we have:

$$|F'_{N,r} - (F''_N + \frac{m(\alpha_e - \alpha_b)}{2\alpha(b-a)} \frac{1}{N})| \leq \frac{\varepsilon}{3}.$$

Suppose that $r < \min(R_1, R_2)$.

We have:

$$\begin{aligned}
H_r(PI(\omega)) &\geq F_{n,r} \\
&\geq F''_N - \frac{m(\alpha_e - \alpha_b)}{2\alpha(b-a)} \frac{1}{N} - \frac{\varepsilon}{3} \\
&\geq F''_N - \frac{\varepsilon}{3} - \frac{\varepsilon}{3} \\
&\geq \frac{1}{b-a} \int_a^b FL_{g'(x)}(\omega) \mathrm{d}x - \frac{\varepsilon}{3} - \frac{\varepsilon}{3} - \frac{\varepsilon}{3} \\
&= \frac{1}{b-a} \int_a^b FL_{g'(x)}(\omega) \mathrm{d}x - \varepsilon
\end{aligned}$$

Similarly

$$\begin{aligned}
H_r(PI(\omega)) &\leq F'_{n,r} \\
&\leq \frac{1}{b-a} \int_a^b FL_{g'(x)}(\omega) \mathrm{d}x + \varepsilon,
\end{aligned}$$

so

$$H_r(PI(\omega)) \xrightarrow[\substack{r\to 0\\ r\in\mathbb{Q}}]{} \frac{1}{b-a}\int_a^b FL_{g'(x)}(\omega)\mathrm{d}x$$

With (9) and (10) this proves Theorem 1. □

A.3 Proof of Theorem 3

It is easy to see that:

$$l(p,g,r) - \frac{(b-a)}{m}\sum_{\omega\in\mathcal{P}_m} p(\omega){F'}_r^g(\omega) \xrightarrow[r\to 0]{} 0$$

where

$${F'}_r^g = \frac{\mathrm{card}\{X\in[\![A_r, B_r-m]\!]\cap(A_r+m\mathbb{Z})\,|\,\omega^g_{X,r,m}=\omega\}}{\lfloor\frac{B_r-m-A_r}{m}\rfloor}.$$

Consider again the curve defined by $g(x)=\alpha x^2$ for $\alpha\notin\mathbb{Q}$. The proof of previous section can be extended to prove:

$${F'}_r^g \xrightarrow[\substack{r\to 0\\ r\in\mathbb{Q}}]{} \frac{1}{b-a}\int_a^b FL_{g'(x)}(\omega)\mathrm{d}x$$

so

$$l(p,g,r) \xrightarrow[\substack{r\to 0\\ r\in\mathbb{Q}}]{} \frac{1}{m}\sum_{\omega\in\mathcal{P}_m} p(\omega)\int_a^b FL_{g'(x)}(\omega)\mathrm{d}x. \tag{5}$$

Let F_n the n-th Farey of order m.

We know that $x\mapsto FL_x(\omega)$ is affine on the intervals $[F_i, F_{i+1}]$ ([3]). So for each i there exists u_i, v_i we have $FL_x(\omega)=u_ix+v_i$.

Let k_α the smallest integer k such that $2\alpha a\le F_k$ and l_α the biggest integer l such that $F_l\le 2\alpha b$. We deduce that

$$\begin{aligned}\int_a^b FL_{g'(x)}(\omega)\mathrm{d}x &= \int_a^{\frac{F_{k_\alpha}}{2\alpha}}(u_{k_\alpha-1}(2\alpha x)+v_{k_\alpha-1})\mathrm{d}x + \sum_{i=k_\alpha}^{l_\alpha-1}\int_{\frac{F_i}{2\alpha}}^{\frac{F_{i+1}}{2\alpha}}(u_i(2\alpha x)+v_i)\mathrm{d}x + \\ &\qquad \int_{\frac{F_{l_\alpha}}{2\alpha}}^b (u_{l_\alpha}(2\alpha x)+v_{l\alpha})\mathrm{d}x \\ &= \left[u_{k_\alpha-1}\alpha x^2+v_{k_\alpha-1}x\right]_a^{\frac{F_{k_\alpha}}{2\alpha}} + \sum_{i=k_\alpha}^{l_\alpha-1}\left[u_i\alpha x^2+v_ix\right]_{\frac{F_i}{2\alpha}}^{\frac{F_{i+1}}{2\alpha}} + \left[u_{l_\alpha}\alpha x^2+v_{l_\alpha}x\right]_{\frac{F_{l_\alpha}}{2\alpha}}^b \\ &= \frac{A_{\alpha,\omega}}{\alpha} + B_{\alpha,\omega}\alpha + C_{\alpha,\omega}\end{aligned}$$

where

$$A_{\alpha,\omega} = \frac{1}{4}\left(u_{k_\alpha-1}F_{k_\alpha}^2 + 2v_{k_\alpha-1}F_{k_\alpha} + \sum_{i=k_\alpha}^{l_\alpha-1}\left[u_i x^2 + 2v_i x\right]_{F_i}^{F_{i+1}} - u_{l_\alpha}F_{l_\alpha}^2 - 2v_{l_\alpha}F_{l_\alpha}\right)$$
$$B_{\alpha,\omega} = u_{l_\alpha}b^2 - u_{k_\alpha-1}a^2$$
$$C_{\alpha,\omega} = v_{l_\alpha}b - v_{k_\alpha-1}a$$

As k_α and l_α are constant on the intervals which contains no reals $\frac{F_i}{2a}$ and $\frac{F_i}{2b}$, we can partition the interval $[0, \frac{1}{2b}]$ in a finite number of intervals $(I_k)_{0\le k\le n}$ such that the functions $\alpha \mapsto A_{\alpha,\omega}, B_{\alpha,\omega}, C_{\alpha,\omega}$ are constant on each interval I_k.

By (5), we have:

$$\lim_{\substack{r\to 0\\ r\in\mathbb{Q}}} l(p,g,r) = \frac{A_\alpha}{\alpha} + B_\alpha\alpha + C_\alpha \tag{13}$$

where

$$A_\alpha = \frac{1}{m}\sum_{\omega\in\mathcal{P}_m} p(\omega)A_{\alpha,\omega}, \quad B_\alpha = \frac{1}{m}\sum_{\omega\in\mathcal{P}_m} p(\omega)B_{\alpha,\omega}, \quad C_\alpha = \frac{1}{m}\sum_{\omega\in\mathcal{P}_m} p(\omega)C_{\alpha,\omega}$$

So $L_{est}(\alpha) = \lim_{\substack{r\to 0\\ r\in\mathbb{Q}}} l(p,g,r)$ is of the form $\frac{A}{\alpha} + B\alpha + C$ on each interval I_k.

Let $L_{real}(\alpha)$ the true length of the parabola curve $\{(x, \alpha x^2) \mid x \in [a,b]\}$. We have:

$$\begin{aligned} L_{real}(\alpha) &= \int_a^b \sqrt{1+(2\alpha x)^2}\mathrm{d}x \\ &= \left[\frac{x\sqrt{1+(2\alpha x)^2}}{2} + \frac{\arg\sinh(2\alpha x)}{4\alpha}\right]_a^b \end{aligned}$$

Suppose that $L_{real}(\alpha) = L_{est}(\alpha)$ for an infinite number of irrational α then there exists an interval I_k of the previous partition of $[0, \frac{1}{2b}]$ such that $L_{real}(\alpha) = L_{est}(\alpha)$ for an infinite number of irrational $\alpha \in I_k$. On I_k we know that $L_{est}(\alpha)$ has the form $\frac{A}{\alpha} + B\alpha + C$. The functions $\alpha \mapsto \alpha L_{real}(\alpha)$ and $\alpha \mapsto \alpha(\frac{A}{\alpha} + B\alpha + C)$ are holomorphic in a open set of $\mathbb{C}$ containing $[0, \frac{1}{2b}]$ and are equal for an infinite number of $\alpha \in I_k \subset [0, \frac{1}{2b}]$ so by Theorem on the zeros of holomorphic functions [8, Cha. 10] they are equal on $[0, \frac{1}{2b}]$. So:

$$\alpha L_{real}(\alpha) = A + B\alpha^2 + C\alpha \qquad \text{for all } \alpha \in [0, \frac{1}{2b}]$$

We have:

$$\begin{aligned} \frac{\partial(\alpha L_{real}(\alpha)}{\partial\alpha} &= b\sqrt{1+(2\alpha b)^2} - a\sqrt{1+(2\alpha a)^2} \\ &= b - a + 2(b^3 - a^3)\alpha^2 + o(\alpha^2) \qquad \text{when } \alpha \to 0 \end{aligned}$$

But $\frac{\partial(A+B\alpha^2+C\alpha)}{\partial\alpha} = 2B\alpha + C$, so $2(b^3 - a^3) = 0$ which is impossible if $b > a$. So the hypothesis that $L_{real}(\alpha) = L_{est}(\alpha)$ for an infinite number of irrational α is absurd. $\square$