On Continuum Modelling the Interphase Layers in Certain Two-Phase Elastic Solids

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In this paper we propose a continuum model of a moving transition layer smoothly separating two different solid phases of a certain elastic material continuum and we investigate some necessary conditions to its existence and propagation. An application of this model to the analysis of wave propagation in micro-damaged media with open- and closed-micro cracked phases is shown.

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1. Introduction

In the literature great interest is paid to the study of propagation of discontinuity surfaces in continuous media. Such surfaces are introduced as models of a large variety of phenomena (cf. for instance [10]) including various kind of phase transitions. When these phenomena need more detailed description the modelization is improved endowing the discontinuity surfaces with more sophisticated structure, i.e. introducing suitable surface fields and postulating for them some balance equations. Such an approach (for a more careful discussion see [6], [11], [12]) sometimes leads to some models which are questionable from both the logical and physical point of view. On the other hand the passage from one phase to the other may be not always abrupt. In this case the interface has to be regarded as a region to be modeled as a three-dimensional interphase layer (see [6] and also the review paper [8] on mushy regions arising in phase transitions) in which we can assume to deal with a certain mixture of both phases. This circumstance occurs also in some particular cases of solid-solid phase transitions described for instance in [9]. Hence the question arises about the description of the properties of the material constituting the interphase layer and about the conditions under which this layer can exist and propagate.

In this paper we propose a continuum model of a moving transition layer smoothly separating two different solid phases of a certain elastic material continuum and we investigate some necessary conditions to its existence and propagation. The term *phase* means here state of a considered continuum, characterized by a given constitutive equation. The differences in material behaviour in different states of the solid body do not arise only because of structural phase transitions (e.g. martensitic transformations, described on the micro-level by discrete models in [1–3]) but can be also caused by the existence of different strain energy functions related to appropriate different classes of local deformations. Such a situation occurs, for example, if we consider a damaged medium in which micro-cracks are in different regions (phases) all open, all closed or partly open and partly closed (cf. [4]): this last possibility characterizing the interfacial region, in which a phase transition takes place.

The basic idea of our approach is that the phase transition occurs in a certain *thin non-material* (or following the terminology of TRUESDELL [13] *non-substantial*) *moving layer* (the *interphase layer*), separating different phases of the body, in which the strain energy function on a macro-level is not well defined. The main hypothesis is that the interphase layer – on a micro-level – is constituted by an arrangement of two different phases forming a definite banded quasi-periodic microstructure. After scaling this microstructure down we arrive at the macro-model of the interphase layer representing a certain fine mixture of both phases. To this end we adapt the micro-macro modelling procedure proposed in [5] for composite materials (the similarity between composite material structure and fine phase mixtures was already noted in [3]). Hence on the macro-level the interphase layer is modelled as a shell-like region of the medium, moving independently of the body motion. By means of an averaging procedure of this region over its thickness (cf. [6]) this model in special cases can be simplified obtaining as model for the interface a discontinuity surface, eventually endowed with material properties.

Intending to postpone the modelling of more complex phenomena, in this paper we will restrict ourselves to describe those in which purely mechanical notions are needed (e.g. diffusion less solid-solid phase transformations): non-mechanical notions as temperature or entropy are excluded from the present analysis.

The main results of this contribution can be stated as follows:

i) the material behaviour of the interphase layer depends not only on the material properties of both phases but also on the propagation speed of the layer,
ii) not every propagation speed of the interphase layer is possible,
iii) when the interphase layer does not propagate then the material properties of the interphase layer are similar to the macro-properties of a micro-layered two component medium (cf. [7], [14]–[16]).

2. Basic assumptions

We will assume that a continuum description of the motion of the considered body B is possible, so that it is possible to establish in which phase a substantial particle of B presents itself.

2.1 Motion

Let B be placed in the reference position $B_R \subseteq E$ where E is the set of places, whose tangent vector space is TE. In the time interval $\mathrm{I} := [\mathrm{t}_0, \mathrm{t}_f]$ a motion p maps $B_R \times \mathrm{I}$ into E. Therefore

$$\forall \mathrm{t} \in \mathrm{I}, \qquad p(\cdot\,, \mathrm{t}) : B_R \to E\,. \tag{2.1}$$

We will use the following notation:

$$p_t := p(\cdot\,, \mathrm{t})\,, \qquad f_t := \nabla p_t \in \mathrm{Lin}\,(\mathrm{TE})\,, \qquad B_t := p(B_R, \mathrm{t})\,,$$

where Lin (TE) is the set of invertible linear transformations mapping TE into TE. When this will not lead to misunderstanding we will skip the index t in the introduced notations.

2.2 Strain energy

We will assume that two strain energy functions $W_R^+(X, \cdot)$ and $W_R^-(X, \cdot)$ defined in disjoined subsets Φ^+, Φ^- of Lin (TE), respectively, are assigned to every substantial point X in B_R describing the behaviour of the two possible phases (denoted by $+$ and $-$) for the considered continuum.

We will assume that at the micro-level almost every material point X, $X \in B_R$, is present only in one of these two phases. This means that either $\nabla p_t(X) \in \Phi^+$ or $\nabla p_t(X) \in \Phi^-$ (but not both) holds throughout the body except possibly at some points, lines or surfaces in B_R.

However in certain regions the sets of the substantial points which present themselves in the phase $+$ or $-$ can have a very complicated structure, being, for instance, composed by many disjoined connected parts whose length dimensions can be neglected on the macro-level. We will refrain from the detailed description of all phenomena occurring during the change of phase. Therefore we will pass to a macro-description of phase change and we will introduce the concept of a certain ideal mixture of both phases constituting what will be called a non-substantial interphase layer occupying respectively in the referential and spatial descriptions the regions $B_R^0(\mathrm{t})$ and B_t^0. To be more precise we will assume that:

i) The region B_R and consequently spatial region B_t can be – on a macro-level and at every time instant t – partitioned into three mutually disjoint subregions (which in general are non-substantial) $B_R^0(\mathrm{t})$ and $B_R^{\pm}(\mathrm{t})$ such that

$$\bar{B}_R = \bar{B}_R^0(\mathrm{t}) \cup \bar{B}_R^-(\mathrm{t}) \cup \bar{B}_R^+(\mathrm{t})\,, \qquad \bar{B}_t = \bar{B}_t^0 \cup \bar{B}_t^- \cup \bar{B}_t^+\,. \tag{2.2}$$

In these regions the considered elastic material is present in different *phases*: the interphase layer occupies the region $B_R^0(\mathrm{t})$ separating the regions $B_R^{\mp}(\mathrm{t})$ occupied by the phases $+$ and $-$ respectively.

ii) Some macro-descriptors can be defined in $B_R^0(\mathrm{t})$ which allow the description of some of the more relevant interfacial phenomena in terms of fields smoothly varying from one phase to the other.

2.3 Transition layer assumptions

We will assume, rephrasing [6], that:

2.3.1) The interfacial region $B_R^0(\mathrm{t})$ can be obtained by *shifting along the normal* a given surface Σ_t. Referring to [6] for more details, for seek of self containment, we recall that this means that if the surface Σ_t has parametric representation $\mathbf{r}^0(\theta, \mathrm{t})$, $\theta = (\theta^1, \theta^2) = \Pi$ there exists a surface scalar field δ_t such that

$$\forall X \in B_R^0(\mathrm{t})\; \exists!\, (\theta, \theta^3)/X = \mathbf{r}^0(\theta, \mathrm{t}) + \theta^3 \mathbf{n}(\theta, \mathrm{t})\,; \tag{2.3}$$

where $\mathbf{n}$ represents the normal vector field to Σ_t and $\theta^3 \in [-\delta_t(\theta), \delta_t(\theta)]$. The set obtained fixing in (2.3) θ^3 and letting θ vary we will call θ^3-shifted surface from Σ_t.

As a consequence of this assumption we can define a scalar vector field $\zeta(X, \mathrm{t})$ and a unit normal field $\mathbf{n}(X, \mathrm{t})$ in $B_R^0(\mathrm{t})$ mapping every $X \in B_R^0(\mathrm{t})$ into the unique θ^3 and normal (to Σ_t) vector corresponding to it. We remark that, if c_n is the field of the normal speed of the surface Σ_t, then the following equality holds

$$\frac{\partial \zeta}{\partial \mathrm{t}}\,(X, \mathrm{t}) = c_n(X, \mathrm{t})\,. \tag{2.4}$$

2.3.2) The boundary of $B_R^0(\mathrm{t})$ can be partitioned as follows:

$$\partial B_R^0(\mathrm{t}) = \Sigma_t^+ \cup \Sigma_t^-\,,$$

where Σ_t^+ and Σ_t^- are the respectively δ_t-shifted and $(-\delta_t)$-shifted surfaces from Σ_t representing the surfaces dividing $+$ and $-$ phases, respectively, from the interphase (transition) layer.

2.4 Assumption on the microstructure inside the interphase layer

2.4.1) ε-micro-layers

In $B^0_R(t)$ coexist the two phases + and −. More precisely we will assume that – on a micro-level – the interfacial region is constituted by thin micro-layers of constant thickness ε each of which is partitioned into two sub layers of varying thickness (denoted respectively ε^+ and ε^-) filled with phase + and – respectively. These micro layers and sub layers are assumed to be obtained translating $\boldsymbol{\Sigma}_t$ along the normal. We will assume that

$$\varepsilon = \frac{\delta_t}{n} \quad \text{with} \quad n \gg 1 ,$$

and hence in our description of the microstructure of the interphase layer all terms of an order higher than ε will be neglected.

2.4.2) Some auxiliary definitions

For the time being assume t, θ, and ε fixed. When not causing misunderstanding we will simply use the symbol δ for the introduced thickness $\delta_t(\theta)$.

Let $\varepsilon^+(\cdot)$, $\varepsilon^-(\cdot)$ be linear functions defined on the interval $[-\delta, \delta]$ such that

$$\varepsilon^+(-\delta) = \varepsilon^-(\delta) = 0 ; \qquad \varepsilon^+(\delta) = \varepsilon^-(-\delta) = \varepsilon .$$

Obviously we have that

$$\forall \theta^3 \in [-\delta, \delta] , \qquad \varepsilon^+(\theta^3) + \varepsilon^-(\theta^3) = \varepsilon .$$

Also we define on $[-\delta, \delta]$ the functions

$$\varphi^+(\theta^3) := \frac{\varepsilon^-(\theta^3)}{\varepsilon} ; \qquad \varphi^-(\theta^3) := \frac{\varepsilon^+(\theta^3)}{\varepsilon} .$$

We partition the interval $[-\delta, \delta]$ into the n disjoined intervals, of constant length ε,

$$I_i := (\theta^3_i - \varepsilon^-(\theta^3_i), \theta^3_i + \varepsilon^+(\theta^3_i)) .$$

We can now define the function $h^\varepsilon : [-\delta, \delta] \to \mathbb{R}$ as the real-valued continuous function which

i) in every subinterval $(\theta^3_i - \varepsilon^-(\theta^3_i), \theta^3_i)$ $(\theta^3_i, \theta^3_i + \varepsilon^+(\theta^3_i))$ of I_i $(i = 1, \ldots, n)$ is linear with angular coefficients respectively $-\varphi^+('\theta^3_i)$ and $\varphi^-(''\theta^3_i)$ (for some $'\theta^3_i$ and $''\theta^3_i$ in I_i)

ii) when $\theta^3 = \theta^3_i$ it attains the values $\dfrac{\varepsilon^-(\theta^3_i)\, \varepsilon^+(\theta^3_i)}{\varepsilon}$.

We remark that the choice of $'\theta^3_i$ and $''\theta^3_i$ can be made in order to assure the continuity of h^ε at points separating the intervals I_i. It is easy to see that neglecting terms of an order ε we have that

$$\varphi^+(\theta^3_i) \cong \varphi^+('\theta^3_i) ; \qquad \varphi^-(\theta^3_i) \cong \varphi^-(''\theta^3_i) \qquad i = 1, \ldots, n .$$

The function h^ε will be called micro-shape function and – as will be explained below – plays an important rôle in the modelling of the interphase layer.

In the sequel we shall also use the functions σ^+, σ^- defined in $[-\delta, \delta]$ and given by

$$\sigma^+(\theta^3) := \frac{\varepsilon^+(\theta^3)}{\varepsilon} ; \qquad \sigma^-(\theta^3) := \frac{\varepsilon^-(\theta^3)}{\varepsilon} ,$$

which for $\theta^3 = \theta^3_i$ represent the percentage of the + and – phase inside the layer descibed by the interval I_i included in $[-\delta, \delta]$.

2.4.3) Kinematic assumptions

The motion p of the body B inside the transition layer can be represented as the sum of a macroscopic part (i.e. independent of ε)

$$\forall t \in \mathrm{I} , \qquad P(\cdot\,, t) : B_R \to E ,$$

and of a superimposed small disturbance D^ε

$$\forall t \in \mathrm{I} , \qquad D^\varepsilon(\cdot\,, t) : B_R \to T(E)$$

caused by the micro-inhomogeneity of the transition layer which has the following form:

$$\forall X \in B^0_R(t) , \qquad D^\varepsilon(X, t) := h^\varepsilon(X, t)\, Q(X, t) .$$

In this last equation Q, which we will call *descriptor of interfacial layer micro-structure*, models the micro-state of the considered substantial particle and h^ε is a suitable micro-shape function constant on every θ^3-shifted surface from $\boldsymbol{\Sigma}_t$.

2.5 *Fine phase mixture* hypothesis

The macro-model of the interphase layer will be obtained – after using the kinematic assumption 2.4.3) – by means of the limiting passage $\varepsilon \to 0$ in all relations describing the mechanical behaviour of the media constituting the interphase layer.

We will call the macro-model of the interphase layer obtained using the hypotheses introduced in this section *interfacial fine phase mixtures.* It has to be emphasized that, due to the properties of function h^ε, in this model all relevant quantities smoothly vary in the passage from one phase to the other.

The validity of the above formulated modelling hypotheses needs to be tested by means of the comparison of the subsequent theoretical predictions with experimental evidence. We are aware of the fact that they restrict the scope of applicability of our model, which could be improved in some particular instances introducing more sophisticated micro-shape functions and/or more descriptors of micro-structure.

3. Evolution equations

As we have stated before, we assume in this section that all quantities infinitesimal of order equal or greater than ε are negligible.

This hypothesis restricts our treatment to interfaces for which the hypothesis of fine phase mixture holds.

Consequently ε^+ and ε^- are negligible. This is not the case for the relative thicknesses $\sigma^\pm$ and for the strain modulators $\varphi^\pm$ which remain constant when ε tends to zero.

The model resulting after this limit passage will introduce for the interfacial region $B_R^0(t)$ a microstructured continuum with the internal variable Q. Every substantial *macro*-particle of this continuum contains both phases + and −, in the proportion controlled by the macro-fields $\sigma^\pm$. The state of every macro-particle is determined by its macro-strain tensor F and by the vector Q.

The evolution equations for macro-displacements and for Q are obtained postulating the principle of stationary action for motions belonging to the class specified by the hypotheses of the previous section and neglecting in the expression of the action all quantities infinitesimal together with ε.

3.1 The action functional

We start expressing micro-deformation tensor f and the velocity field v in the referential description in terms of macro-deformation tensor $F := \nabla P$, the descriptor Q, the macro-velocity field $V := \dot{P}$ and the propagating speed of the interfacial layer. Using the Hypotheses of subsection 2.4 and recalling that F and V are the derivatives with respect X and time t of the function P, we obtain (quantities infinitesimal with ε are neglected)

$$f = F + \left\{ \begin{matrix} -\varphi^+ \\ \varphi^- \end{matrix} \right\} Q \otimes \mathbf{n} =: \left\{ \begin{matrix} f^+(F, Q) \\ f^-(F, Q) \end{matrix} \right\}, \tag{3.1}$$

$$v = V + \left\{ \begin{matrix} -\varphi^+ \\ \varphi^- \end{matrix} \right\} V \cdot \mathbf{n}; \qquad Q =: \left\{ \begin{matrix} v^+(V, Q) \\ v^-(V, Q) \end{matrix} \right\}, \tag{3.2}$$

where the symbol { } means that the equality holds respectively in the + or − phases.

We can now introduce the action functional $\mathscr{A}$ for the interphase layer occupying in the reference configuration the region $B_R^0(t)$:

$$\mathscr{A} = \int_{t_0}^{t_f} (\mathscr{K} - \mathscr{P})\, dt, \tag{3.3}$$

the kinetic energy $\mathscr{K}$ and the potential $\mathscr{P}$ being defined – on a micro-level – as follows (dV_R and dA_R are respectively the volume and surface elements in the reference configuration):

$$\mathscr{K} = \int_{B_R^0(t)} \varrho_R v^2\, dV_R; \qquad \mathscr{P} = \int_{B_R^0(t)} W_R(X, f)\, dV_R + \int_{\partial B_R^0(t)} t_R \cdot p\, dA_R, \tag{3.4}$$

where the strain energy W_R is a function of the material point X and the strain f, it is equal to $W_R^+(X, \cdot)$ or to $W_R^-(X, \cdot)$ if X belongs respectively to phase + or −, ϱ_R is the mass density in the referential description and t_R are the external contact forces applied on the boundary of the interfacial transition layer, which we assume are independent of the micro-structure of the layer.

Once the hypothesis of fine phase mixture is used neglecting all terms infinitesimal with ε, with simple algebra one obtains the following expression for the action functional

$$\mathscr{A} = \int_{B_R^0(t)} \tfrac{1}{2} \varrho_R V^2 + \tfrac{1}{2} \varrho_R \psi c_n^2 Q^2\, dV_R + \int_{B_R^0(t)} \langle W_R \rangle (X, F, Q)\, dV_R + \int_{\partial B_R^0(t)} t_R \cdot P\, dA_R, \tag{3.5}$$

where c_n is defined in eq. (2.4), $\psi(\theta^3) := \sigma^+(\theta^3)\,\sigma^-(\theta^3)$, we have assumed that the referential mass densities $\varrho_R^\pm$ of phase + and − are equal, and we have used the following denotations:

$$\varrho_R = \varrho_R^+ = \varrho_R^- ; \tag{3.6}$$

$$\langle W_R\rangle\,(X, F, Q) := \sigma^+(X)\,W_R^+(X, f^+(F, Q)) + \sigma^-(X)\,W_R^-(X, f^-(F, Q)) . \tag{3.7}$$

The evolution equations for macro-placement P and the descriptor Q inside $B_R^0(t)$ are obtained assuming that the first variation of the action $\mathscr{A}$ determined by equation (3.5) is vanishing.

1) Equation of motion

$$\operatorname{Div} S_R - \varrho_R \frac{dV}{dt} = 0 \quad \text{in} \quad B_R^0(t) \quad \text{and} \quad S_R N = t_R \quad \text{on} \quad \partial B_R^0(t) , \tag{3.8}$$

where the vector field N is the unit normal to $\partial B_R^0(t)$ and the (Piola-Kirchhoff) macro-stress-tensor S_R is determined by the macro-constitutive relation

$$S_R(X, F, Q) = \frac{\partial\langle W_R\rangle}{\partial F} . \tag{3.9}$$

2) Propagation condition

$$\frac{\partial\langle W_R\rangle}{\partial Q} + \varrho_R \psi c_n^2 Q = 0 \quad \text{in} \quad B_R^0(t) . \tag{3.10}$$

We remark that equations (3.8)−(3.10) describe the behaviour of the interfacial continuum if the motion of the interfacial region $B_R^0(t)$ is known *a priori*. The equations governing the motion of this region has to be found modelling its specific physical nature, as is done, in the particular instance treated there, in sect. 4.

It has to be emphasized that:

i) in the proposed macro-model the properties of the continuum constituting the interphase layer can be easily determined once the constitutive properties of phases + and − and the micro-structure functions $\sigma^\pm$ are known;

ii) since c_n is the relative velocity of the interfacial layer with respect to the referential space then the condition (3.10) is Galilean invariant;

iii) equations (3.10) is an algebraic relation among Q, F, and c_n^2. The definition of average strain energy implies that using it one can express Q in terms of F and c_n^2 and therefore that the strain energy can be regarded as a function $\langle\tilde{W}_R\rangle$ of the macro-strain F and the speed c_n. On the other hand the physically admissible domain for a strain energy function is included in the set of independent variables for which the energy is positive: in particular this implies that some restrictions have to be expected on possible speeds c_n. In the next section we find, in the case of transition layer in linear elastic solids, an explicit form of such a restriction.

iv) When we deal with a substantial layer the the speed c_n is vanishing. The equations (3.8)−(3.10) reduces to those found in [5] where some macro-models of micro-laminated materials are considered.

4. An application

In this section we want to apply the model developed up to now to the study of the propagation of a particular interfacial layer.

More precisely we consider the micro-cracked material investigated in [4]: the considered solid body is assumed damaged because of the presence of planar micro-cracks (parallel to the laminae interfaces) whose planes are characterized by a regular field **n** of unit vectors.

We assume that the region B_R can be partitioned into three regions: i.e. the region in which the micro-cracks are open (phase +), that in which the micro-cracks are closed (phase −), and the interphase layer in which the micro-cracks are partly closed and partly open.

In the interphase layer a macro-strain F and one macro-descriptor Q, following the procedure showed in the previous section, are introduced.

In the considered instance the partitioning at the macro-level can be determined by means of the *phase defining function* and *thresholds*

$$\gamma : (X, F) \in B_R \times \operatorname{Lin}(TE) \mapsto \gamma_X(F) = \mathbf{n}(X) \cdot F^T F \mathbf{n}(X) - 1 ; \qquad (\gamma_-, \gamma_+) \in \mathbb{R}^+ \times \mathbb{R}^+ .$$

Indeed we have that:

$$\begin{cases} B_R^+(t) := \{X \in B_R / \gamma_X(\nabla P(X,t)) > \gamma_+\} , & B_t^+ := p_t(B_R^+(t)) , \\ B_R^-(t) := \{X \in B_R / \gamma_X(\nabla P(X,t)) < -\gamma_-\} , & B_t^- := p_t(B_R^-(t)) , \\ B_R^0(t) := \{X \in B_R / \gamma_X(\nabla P(X,t)) \in (-\gamma_-, \gamma_+)\} , & B_t^0 := p_t(B_R^0(t)) . \end{cases}$$

The physical interpretation of the function γ_X and the constants (γ_-, γ_+) is very easy: in the + phase micro-cracks are open, in the − phase micro-cracks are closed, and in order to open or to close micro-cracks the stress needs a component along **n** which respectively corresponds to the thresholds γ_+ and γ_-. In points X in which

$$\gamma_X(f) \in (-\gamma_-, \gamma_+),$$

we are in the part of the body where phase transition – on a macro-level – occurs: i.e. in the part of the body where the knowledge of the strain F and the normal **n** is not sufficient to establish whether the micro-cracks are closed or open. For the points in $\mathcal{B}^0_R(t)$ the knowledge of the values of γ does not allow the unique determination of the strain energy which in this sense is not well-defined (cf. [4]) at the macro-level until more the detailed description of the structure of the interfacial layer allowed by the introduction of the vector field Q is added to our model. Indeed the average strain energy $\langle W_R \rangle$ is not only a function of macro-strain F but also a function of the macro-descriptor Q.

We will limit ourselves to the case of a linear elastic micro-cracked material in which the vector field **n** is constant and to the study of interphase layers propagation in the direction of **n**.

Therefore we will restrict our consideration to uniaxial strain states. In this case the strain energy functions W_R^+ and W_R^- (respectively the strain energy in the medium with open cracks and with closed cracks) and the sets of admissible local deformations are given, in terms of the infinitesimal strain e, by

$$\begin{cases} \Phi^+ := \{e/\mathbf{n}\cdot e\mathbf{n} \geq \mathrm{e}_+\}, & \mathrm{W}_R^+(e) = \frac{1}{2}\,\mathrm{E}^+(\mathbf{n}\cdot e\mathbf{n})^2, \\ \Phi^- := \{e/\mathbf{n}\cdot e\mathbf{n} \leq -\mathrm{e}_-\}, & \mathrm{W}_R^-(e) = \frac{1}{2}\,\mathrm{E}^-(\mathbf{n}\cdot e\mathbf{n})^2, \end{cases} \tag{4.1}$$

where:
the positive constants $\mathrm{e}_\pm$, which we can call *strain thresholds*, play the rôle, in the case of linear elasticity, of the phase defining thresholds;
the constants $\mathrm{E}^\pm$ are positive and verify the relationship

$$\mathrm{E}^- > E^+, \tag{4.2}$$

which is simply physically interpreted recalling that once the cracks are open the stiffness of the material is lower than in the case of closed cracks.

In the model discussed in [4] one can assume that the traction $\mathcal{S}$ required to open (close) all micro-cracks is given by $\mathcal{S} = \mathrm{E}^+\mathrm{e}_+ = \mathrm{E}^-\mathrm{e}_-$.

In the sequel we will study the propagation of a disturbance of displacement in the direction of **n**. In other words we will study the propagation of a longitudinal plane wave of displacements, opening (or closing) the micro-cracks, in which the phase change occurs inside a thick layer whose micro-structure verifies the hypotheses listed in the previous section.

We will use the following denotation and hypotheses:
i) U is the component of macro-displacement along **n**, $\mathrm{U}_{'n}$ its derivative in the direction of **n**,
ii) we introduce only one micro-shape function and one vectorial macro-descriptor,
iii) Q is the component of the macro-descriptor vector along **n**,
iv) the interface layer is the union of planes orthogonal to **n**,
v) S is the normal stress $\mathbf{n}\cdot \mathrm{S_R}\mathbf{n}$.

The average strain energy for the considered continuum and displacement fields is easily evaluated once equation (3.7) is recalled

$$\langle \mathrm{W_R}\rangle = \tfrac{1}{2}\,\tilde{\mathrm{E}}(\mathrm{U}_{'n})^2 + [\mathrm{E}]\,\psi \mathrm{Q}\mathrm{U}_{'n} + \tfrac{1}{2}\,\hat{\mathrm{E}}\psi\mathrm{Q}^2 \tag{4.4}$$

with $\psi := \sigma^+\sigma^-$ and where

$$\tilde{\mathrm{E}} := \mathrm{E}^+\sigma^+ + \mathrm{E}^-\sigma^-; \qquad [\mathrm{E}] := \mathrm{E}^- - \mathrm{E}^+; \qquad \hat{\mathrm{E}} := \mathrm{E}^+\sigma^- + \mathrm{E}^-\sigma^+. \tag{4.5}$$

Therefore we obtain, using (3.9), the following expression for the normal stress S in terms of Q and $\mathrm{U}_{'n}$:

$$\mathrm{S} = \tilde{\mathrm{E}}\mathrm{U}_{'n} + \psi[\mathrm{E}]\,\mathrm{Q}, \tag{4.6}$$

while the propagation condition (3.10) becomes

$$\hat{\mathrm{E}}\mathrm{Q} + [\mathrm{E}]\,\mathrm{U}_{'n} - \varrho_R c_n^2 \mathrm{Q} = 0. \tag{4.7}$$

Equation (4.7) allows to express Q in terms of $\mathrm{U}_{'n}$ and c_n^2

$$\mathrm{Q} = -\frac{[\mathrm{E}]}{(\hat{\mathrm{E}} - \varrho_R c_n^2)}\,\mathrm{U}_{'n}, \tag{4.8}$$

so that S can be expressed in terms of $\mathrm{U}_{'n}$ and c_n^2 as follows:

$$\mathrm{S} = \left(\tilde{\mathrm{E}} - \psi\,\frac{[\mathrm{E}]^2}{(\hat{\mathrm{E}} - \varrho_R c_n^2 \varphi)}\right)\mathrm{U}_{'n}. \tag{4.9}$$

We will call *interphase longitudinal modulus* the quantity

$$\mathrm{E}(\mathrm{c}_\mathrm{n}^2) := \left(\tilde{\mathrm{E}} - \psi \frac{[\mathrm{E}]^2}{(\hat{\mathrm{E}} - \varrho_\mathrm{R}\mathrm{c}_\mathrm{n}^2)}\right), \tag{4.10}$$

which depends on the coordinate θ^3 inside the interfacial layer because in it the fields (see equation (4.5)) $\sigma^\pm$ appear.

The interphase longitudinal modulus must be positive everywhere inside the interphase layer. This physical condition implies the following restrictions on the propagation speed c_n (recall condition 4.2):

$$\varrho_\mathrm{R}\mathrm{c}_\mathrm{n}^2 < \mathrm{E}^+ \quad \text{or} \quad \varrho_\mathrm{R}\mathrm{c}_\mathrm{n}^2 > \mathrm{E}^- . \tag{4.11}$$

The field U has to be found solving the hyperbolic equation which is implied by (3.8),

$$\mathrm{S}_{,\mathrm{n}} - \varrho_\mathrm{R}\frac{\mathrm{d}^2\mathrm{U}}{\mathrm{dt}^2} = 0 \quad \text{in} \quad \mathcal{B}_\mathrm{R}^0(\mathrm{t}), \tag{4.12}$$

where the expression (4.9) for S has to be substituted.

To (4.11) some boundary conditions must be added, together with some free-moving boundary conditions determining the position of Σ_t and the scalar δ_t. In particular we must assume that the following conditions must hold on the surfaces Σ_t^+ and Σ_t^-, which in the considered case are planar, together with Σ_t:

$$\begin{cases} \mathrm{U}_{,\mathrm{n}}(\xi(\mathrm{t}) + \delta_\mathrm{t}, \mathrm{t}) = \mathrm{e}_+ , \\ \mathrm{U}_{,\mathrm{n}}(\xi(\mathrm{t}) + \delta_\mathrm{t}, \mathrm{t}) = -\mathrm{e}_- , \end{cases} \tag{4.13}$$

where $\xi(\mathrm{t})$ represents the distance of Σ_t from a reference plane.

Moreover the displacements and normal stresses will be assumed to be continuous on Σ_t^+ and Σ_t^-.

$$\mathrm{U} \quad \text{and} \quad \mathrm{S} \quad \text{do not jump on} \quad \Sigma_\mathrm{t} . \tag{4.14}$$

Concerning condition (4.10) we remark that

i) interphase longitudinal modulus reduces to the effective modulus found in [4] when c_n is vanishing;

ii) when the speed c_n tends to infinity the interphase longitudinal modulus tends to $\tilde{\mathrm{E}}$;

iii) when $\|\mathrm{c}_\mathrm{n}\|$ tends to $\sqrt{\frac{\mathrm{E}^+}{\varrho}}$ or to $\sqrt{\frac{\mathrm{E}^-}{\varrho}}$ respectively from lower or from higher values then respectively on Σ_t^- and on Σ_t^+ the interphase longitudinal modulus vanishes.

The free moving boundary problem (4.12)–(4.14) seems to be interesting enough to deserve an accurate mathematical analysis. In particular it will be interesting to determine the set of initial conditions for which condition (4.11) is verified for all subsequent time instant and to examine under which condition the interphase layer initially present disappears in a finite time, or when it does not disappear, eventually growing indefinitely.

5. Conclusions

In this paper we prove that for the family of interphase layer whose structure is described by the hypotheses listed in sect. 3,

i) the material behaviour of the medium filling the interphase layer depends not only on the material properties of both phases but also on the propagation speed of the layer,

ii) if one assumes that the interphase longitudinal modulus has to be always positive not every propagation speed of the interphase layer is possible,

iii) when the interphase layer does not propagate in the referential description then the material properties of the substantial particles belonging to it are similar to the effective properties of a micro-laminated two component medium (cf. [14]–[16]).

Moreover we particularize the general model introduced in sect. 3 to study the one dimensional propagation in the direction of the layering of an interphase layer in the microcracked solid studied in [4].

We find a free moving boundary problem for the boundaries delineating the interphase layer and the field of longitudinal displacements which seems to deserve mathematical interest.

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