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Viable control of an epidemiological SIR model

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Abstract We consider vaccination control of the spread of an epidemy in a classical SIR model. Our approach aims at controlling the number infected at the peak. It differs from the widespread stationary vaccination control strategies, based upon having control reproductive number stricly less than one to ensure convergence, and also from cost minimization optimal control ones. Indeed, instead of aiming at an equilibrium or optimizing, we look for policies able to maintain the number of infected individuals below a threshold for all times. Thus doing, we focus both on transitories and on asymptotics, in a robust way. We provide a formulation of an epidemy management as a dynamic control under constraint problem, for which the constraint to maintain the number of infected individuals below a threshold for all times has to be achieved by a time-dependent vaccination strategy. The so-called viability kernel is the set of initial states for which such a vaccination policy exists. We give an expression of the viability kernel, and characterize viable policies. We exhibit policies that are both viable and asymptotic, in that they both control the maximum number infected at the peak and asymptotically drive the number of infected to zero.

Keywords control theory · viability · epidemiology · SIR model · vaccination control

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1 Introduction

We consider vaccination control of the spread of an epidemy in a classical SIR model. Our approach aims at controlling the number infected at the peak. It differs from the widespread stationary vaccination control strategies, based upon having control

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reproductive number stricly less than one to ensure convergence, and also from cost minimization optimal control ones.

In the SIR model (and in many other models), a significant quantity is the "basic reproductive number" $\mathcal{R}_0$ which depends on parameters such as the transmission rate, the death and birth rate, etc. Numerous works (see references in [26,17]) exhibit conditions on $\mathcal{R}_0$ such that the number of infected individuals tends towards zero. With this tool, different management strategies of the propagation of the infection – quaranteen, vaccination, etc. – are compared with respect to how they modify $\mathcal{R}_0$. Thus, strategies are compared as to their capacity to drive the number of infected towards zero, focusing on asymptotics rather than on the transitory phase.

Other works deal with the whole trajectory, as in dynamic optimization where strategies are compared with respect to intertemporal costs and benefits [27], [30], [25], [28], [13], etc. More recently, [22] studies controls that minimize the outbreak size (or infectious burden) under the assumption that there are limited control resources.

Our approach focuses both on transitories and asymptotics, in a robust way. Instead of aiming at an equilibrium or optimizing, we look for policies able to maintain the number of infected individuals below a threshold for all times. To our knowledge, this approach is new. We have only found it mentioned in passing in [27] as a constraint – bounding above the maximum number infected at the peak – in a dynamic optimization problem, solved numerically.

In this paper, we provide a formulation of an epidemy management as a dynamic control under constraint problem, for which the constraint to maintain the number of infected individuals below a threshold for all times has to be achieved by a time-dependent vaccination strategy.

Dynamic control under constraints problems refers to viability [3] or invariance [12] frameworks. In the control theory literature, problems of constrained control lead to the study of positively invariant sets, particularly ellipsoidal and polyhedral ones for linear systems (see[8], [20], [21] and the survey paper [9]); reachability of target sets or tubes for nonlinear discrete time dynamics is examined in [7]. Such a viability approach has been applied to models related to the sustainable management of fisheries [6], [4], [14], [32], [18], to viable strategies to ensure survival of some species [11], to secure the prey predator system [10], to value the contribution of biodiversity to ecosystem performance [5], to forest management [34], to livestock management [35], or to monetary policy control [29]. Different examples may be found in [16] for sustainable management applications.

In Section 2, we present a classical SIR model with vaccination, then formulate a control problem with constraint on the infected population abundance. In Section 3, we define and give the expression of the so-called viability kernel. This is the set of initial states for which exists a vaccination policy such that the solution of the SIR model satisfies the viability constraint consisting in maintaining the number of infected individuals below a threshold for all times. We discuss the implications of our results. Proofs are relegated in the Appendix.

2 Bounding up infected population with vaccination control

We present a classical SIR model with vaccination, then formulate a control problem with constraint on the infected population abundance. Our aim consists in identifying

conditions under which a vaccination strategy (time-varying newborn vaccination rate) exists, making in sort that this constraint is satisfied for all times.

We consider a population subdivided in four groups:

- S the number of *susceptibles*,
- I the number of *infected*,
- R the number of *removed individuals*,
- V the number of *vaccinated*.

We shall suppose that the total population

$$N = S + I + R + V$$

is constant. We consider the classical SIR model with newborn vacccination (see [23, 19] and the references there in), where t denotes time:

$$\begin{aligned}
\frac{\mathrm{d}S}{\mathrm{d}t} &= -\beta IS + \delta N(1-p) - \delta S\,,\\
\frac{\mathrm{d}I}{\mathrm{d}t} &= \beta IS - \nu I - \delta I\,,\\
\frac{\mathrm{d}R}{\mathrm{d}t} &= \nu I - \delta R\,,\\
\frac{\mathrm{d}V}{\mathrm{d}t} &= \delta N p - \delta V\,.
\end{aligned}$$

In the above equations, β is the *transmission rate*, δ the *birth and death rate*, ν the *recovery rate*, and p the *newborn vaccination rate*.

Notice that the two variables S_t and I_t satisfy a coupled controlled dynamical system:

$$\frac{\mathrm{d}S_t}{\mathrm{d}t} = -\beta I_t S_t + \delta N(1-p_t) - \delta S_t\,, \tag{2a}$$

$$\frac{\mathrm{d}I_t}{\mathrm{d}t} = \beta I_t S_t - \nu I_t - \delta I_t\,. \tag{2b}$$

From now on, the *state* variable is the couple (S, I), while the variable p, newborn vaccination rate, is the *control variable* varying in $[0, 1]$.

Let $0 < I_{max} \leq N$. Our aim consists in finding, if it exists, a piecewise continuous function $t \mapsto p_t$ (vaccination rate policy), such that the following so-called viability constraint is satisfied:

$$I_t < I_{max}\,, \quad \forall t \geq t_0\,. \tag{3}$$

The existence of such a vaccination rate policy depends crucially on the initial state (S_{t_0}, I_{t_0}) at initial time t_0. We shall now study the set of such initial states, also called the *viability kernel* [3].

3 Viability kernel and viable vaccination policies for the control of infected

We shall now define and give the expression of the viability kernel. Then, we shall provide viable policies, examine viable equilibria, and study how the viability kernel varies with the parameters. Doing this, we are going to cross a well known quantity, the *basic reproductive number* $\mathcal{R}_0$ (without vaccination):

$$\mathcal{R}_0 := \frac{\beta}{\delta + \nu} N\,. \tag{4}$$

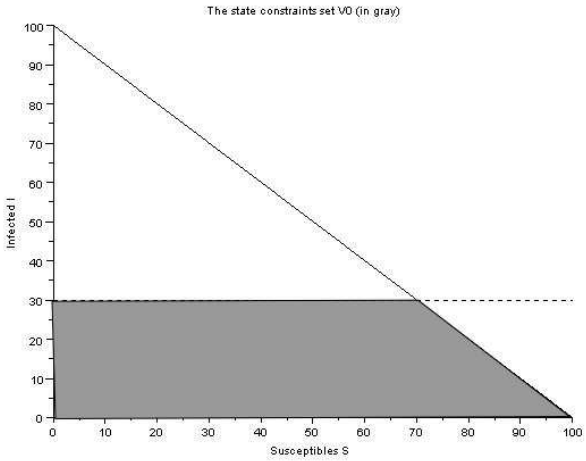

Fig. 1 The state constraints set $\mathbb{V}^0$ (in gray)

The so-called *control reproductive number* for a stationary vaccination rate p is $(1-p)\mathcal{R}_0$.

3.1 Expression of the viability kernel

Definition 1 The *viability kernel* $\mathbb{V}(I_{max})$ is the set of initial states (S_{t_0}, I_{t_0}) at initial time t_0 for which exists a vaccination rate policy $t \mapsto p_t \in [0,1]$, such that the solution of the dynamical system (2a)-(2b) satisfies the viability constraint (3).

First, let us recall that our study domain without constraints is the positively invariant set $\{(S,I) \mid S \geq 0, I \geq 0, S+I \leq N\}$. Second, the viability kernel $\mathbb{V}(I_{max})$ is necessarily included in the rectangle $[0,N] \times [0,I_{max}[$, because the initial point must satisfy the viability constraint (3). The so-called *state constraints set* is their intersection (see Figure 2)

$$\mathbb{V}^0 := \{(S,I) \mid S \geq 0, I_{max} > I \geq 0, S+I \leq N\} \,. \tag{5}$$

We put

$$S_{max} := \frac{N}{\mathcal{R}_0} = \frac{\delta+\nu}{\beta} \,. \tag{6}$$

Theorem 1 *The viability kernel $\mathbb{V}(I_{max})$ is either the whole state constraint set $\mathbb{V}^0$ or is strictly smaller (see Figure 1) depending on whether the upper bound I_{max} on the number of infected is high or low.*

- *When $I_{max} + S_{max} \geq N$, then $\mathbb{V}(I_{max}) = \mathbb{V}^0$ is the whole state constraint set.*
- *When $I_{max} + S_{max} < N$, then (see Figure 2)*

$$\mathbb{V}(I_{max}) = \mathbb{V}^0 \cap \{(S,I) \mid S_{max} \leq S \leq N \text{ and } I < \mathcal{I}(S)\} \tag{7}$$

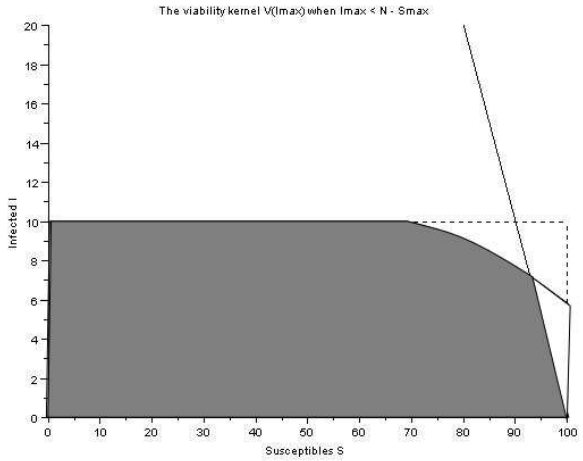

Fig. 2 The viability kernel $\mathbb{V}(I_{max})$ when $I_{max} + S_{max} < N$

is the domain of the whole state constraint set $\mathbb{V}^0$ *below the* vaccination barrier

$$\mathcal{B} := \{(S, \mathcal{I}(S)) \mid S_{max} \leq S \leq N\} \,. \tag{8}$$

This latter is given by the solution $S \in [S_{max}, N] \mapsto \mathcal{I}(S)$ *to the differential equation:*

$$0 = S(\beta\mathcal{I}(S) + \delta)\mathcal{I}'(S) + \beta\mathcal{I}(S)(S - S_{max}) \,, \tag{9a}$$

$$\mathcal{I}(S_{max}) = I_{max} \,. \tag{9b}$$

Any policy exhibiting maximal vaccination rate $p_t = 1$ *in the neighbourhood of the upper frontier* $[0, S_{max}] \times \{I_{max}\}$ *and of the vaccination barrier* $\mathcal{B} = \{(S, \mathcal{I}(S)) \mid S_{max} \leq S \leq N\}$ *is viable.*

3.2 Description of the vaccination barrier

The differential equation (9a) may be solved by separation of variables, giving

$$\frac{(\beta\mathcal{I} + \delta)}{\mathcal{I}} \mathrm{d}\mathcal{I} = \frac{-\beta S + \delta + \nu}{S} \mathrm{d}S \,,$$

and therefore the vaccination barrier $\mathcal{B}$ is also described by

$$\beta I + \delta \log I + \beta S - (\delta + \nu) \log S = \beta I_{max} + \delta \log I_{max} + \beta S_{max} - (\delta + \nu) \log S_{max} \,.$$

Notice that the differential equation (9a)–(9b) is the solution of the dynamical system (2a)-(2b) starting from the initial state (S_{max}, I_{max}) and with stationary control $p_t = 1$ (this is a well known result in viability theory [33,3]).

3.3 Maximum number infected at the peak in the stationary case

Our approach fixes first an infected threshold I_{max}, then looks for initial states such that the maximum number infected at the peak can remain below I_{max} by means of a *non-stationary* strategy. In [27], the opposite is done: one starts from a given initial state, applies a *stationary* vaccination rate $p_t \equiv p$, then identifies the maximum number infected at the peak. As shown below, it happens that both methods rely upon close differential equations, but with different initial conditions.

For fixed vaccination rate p, suppose that the following differential equation is well defined for all $S \in [0, N]$:

$$0 = \big(\delta N(1-p) + S(\beta \mathcal{I}_p(S) + \delta)\big)\mathcal{I}_p'(S) + \beta \mathcal{I}_p(S)(S - S_{max})\,, \tag{10a}$$

$$\mathcal{I}_p(S_{t_0}) = I_{t_0}\,. \tag{10b}$$

Starting from the initial state (S_{t_0}, I_{t_0}), we can show that the maximum number infected at the peak is given by $\max_{t \geq t_0} I_t = \mathcal{I}_p(S_{max})$. This is because the orbit of the dynamical system (2a)-(2b) starting from (S_{t_0}, I_{t_0}) is included in the 0-level set of the (Lyapunov) function $\mathcal{L}_p(S, I) = I - \mathcal{I}_p(S)$. Indeed, with classical notations, we put

$$\begin{aligned}\dot{\mathcal{L}_p}(S, I) &= \frac{\partial \mathcal{L}_p}{\partial I}(S, I, p)\big(\beta(S - S_{max}) - \mathcal{I}_p'(S)\big) + \frac{\partial \mathcal{L}_p}{\partial S}(S, I)\big(-\beta IS - \delta S + \delta N(1-p)\big) \\ &= I\beta(S - S_{max}) - \mathcal{I}_p'(S)\big(-\beta IS - \delta S + \delta N(1-p)\big)\,,\end{aligned}$$

and we deduce that $\dot{\mathcal{L}_p}(S, \mathcal{I}_p(S)) = 0$ by (10a). Now, we know that the peak is achieved when $\frac{dI}{dt}$ changes its sign, hence at $S = S_{max}$ by (2b). Thus, given an initial state (S_{t_0}, I_{t_0}), the maximum number infected at the peak is given by $\max_{t \geq t_0} I_t = \mathcal{I}_p(S_{max})$.

Notice that, by a comparison theorem [2], one can prove that $\mathcal{I}_p(S)$ is increasing with p. Therefore, the maximum number infected at the peak increases with vaccination rate, as suggested by intuition.

3.4 Relation with stationary vaccination control

We shall now compare our "non-stationary maximum peak" approach with the more traditional "stationary asymptotic" one.

For fixed vaccination rate p, the control reproductive number [19,17,24] is $\mathcal{R}_0(1-p)$. It is well known that, if $\mathcal{R}_0(1-p) < 1$, the equilibrium $((1-p)N, 0)$ is globally asymptotically stable, and the epidemy asympotically dies; on the contrary, if $\mathcal{R}_0(1-p) > 1$, susceptibles and infectives approach constant levels (see [23] for more details). Asymptotic control goes as follows [1,19]. For fixed vaccination rate p strictly above the *critical proportion of the population to be immunized*

$$p_c := 1 - \frac{1}{\mathcal{R}_0}\,, \tag{11}$$

one has $\mathcal{R}_0(1-p) < 1$. Therefore, stationary vaccination control $p_t \equiv p > p_c$ ensures that the number of infected I_t will tend to zero as time t goes on.

Our approach is different. We are not looking for a stationary vaccination rate policy to asymptotically achieve a steady state without infected. We first fix an infected threshold I_{max}, then look whether exist non-stationary vaccination rate policies such

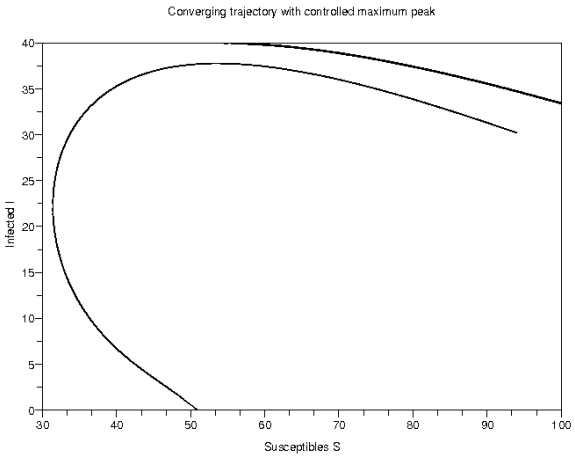

Fig. 3 Converging trajectory with controlled maximum peak

that the maximum number infected at the peak is less than I_{max}. Our results provides two types of information. First, we know for any initial state (S_{t_0}, I_{t_0}) if we can maintain or not the number of infected I_t below the threshold thanks to non-stationary vaccination control. Second, we know what type of vaccination strategy to implement. When (S_t, I_t) approaches the upper frontier $[0, S_{max}] \times \{I_{max}\}$ and the vaccination barrier $\mathcal{B} = \{(S, \mathcal{I}(S)) \mid S_{max} \leq S \leq N\}$, the vaccination must approach the maximal vaccination rate $p = 1$.

3.5 Viable and asymptotic non-stationary vaccination control strategies

We suggest the following vaccination control strategy aiming both at controlling the maximum number infected at the peak and at asymptotically driving the number of infected to zero. Assuming the initial state (S_{t_0}, I_{t_0}) to belong to the viability kernel $\mathbb{V}(I_{max})$,

- apply maximal vaccination rate control $p_t = 1$ in the vicinity of the upper frontier $[0, S_{max}] \times \{I_{max}\}$ and of the vaccination barrier $\mathcal{B} = \{(S, \mathcal{I}(S)) \mid S_{max} \leq S \leq N\}$,
- apply fixed vaccination rate $p_t = p$ where $p > p_c$ elsewhere within the viability kernel.

For example, one can use a strategy of the form

$$p_t = \max\{e^{-\lambda(\mathcal{I}(S_t) - I_t)}, \kappa p_c + 1 - \kappa\} \tag{12}$$

where $\lambda > 0$ and $0 < \kappa < 1$. An illustration is given in Figure 3.

3.6 Viable equilibria for fixed vaccination rate p

We here check that *viable equilibria*, namely those equilibria which respect the viability constraint (3), belong to the viability kernel $\mathbb{V}(I_{max})$, as is well known [3]. For fixed

vaccination rate p, the dynamical system (2a)-(2b) has the two following equilibria [15, 19,24] (when the following quantities are nonnegative):

$$(S_{max}, \frac{(\delta+\nu)(N(1-p)-S_{max})}{\beta S_{max}+\nu}) \text{ and } (N(1-p), 0)\ . \tag{13}$$

Now, we let vaccination rate p vary to obtain all the viable equilibria.

- If $S_{max} > N$, *i.e.* $\mathcal{R}_0 < 1$, the viable equilibria are the horizontal straight line $[0, N] \times \{0\}$.
- if $S_{max} \leq N$, *i.e.* $\mathcal{R}_0 \geq 1$, the viable equilibria are the horizontal straight line $[0, N] \times \{0\}$ and the vertical straight line $\{S_{max}\} \times [0, \frac{(\delta+\nu)(N-S_{max})}{\beta S_{max}+\nu}]$ when this latter upper bound is strictly less than I_{max}, or the vertical straight line $\{S_{max}\} \times [0, I_{max}[$ else.

One can check that these viable equilibria indeed belong to the viability kernel $\mathbb{V}(I_{max})$.

3.7 Sensitivity analysis

We examine now the shape of the vaccination barrier $\mathcal{B} = \{(S, \mathcal{I}(S)) | S_{max} \leq S \leq N\}$. We show in the Appendix that (9a) is equivalent to

$$\mathcal{I}'(S) = -1 + \frac{\delta S + \mathcal{I}(S)(\delta+\nu)}{(\beta\mathcal{I}(S)+\delta)S}\ . \tag{14}$$

Observing that $\mathcal{I}'(S) > -1$, we conclude that the vaccination barrier is above the straight line $\{(S, I) \mid S + I = S_{max} + I_{max}\}$, which has slope -1 and passes by (S_{max}, I_{max}).

- As the transmission rate β decreases, the slope of $\mathcal{I}$ increases, which results in an increase of the viability kernel by a comparison theorem [2]. Indeed, the smaller β, the less contagious the infection, so that it is easier to contain infected below a threshold if the spread of the disease is slow and weak.
- As the recovery rate ν increases, the slope of $\mathcal{I}$ increases, hence the viability kernel increases. Indeed, the larger ν, the less infection.
- The influence of the birth and death rate δ is ambiguous.
- The influence of $\mathcal{R}_0$ on the shape of the viability kernel is not clear. When $\mathcal{R}_0$ decreases, S_{max} increases and the rectangular left part of the viability kernel increases. However, we do not know how the curved right part moves since the vaccination barrier $\mathcal{B}$ depends both on $\mathcal{R}_0$ and on δ/β. Assuming that δ/β is fixed (only ν/β varies), we see that the vaccination barrier goes down when $\mathcal{R}_0$ decreases.

4 Conclusion

After having first fixed an infected threshold I_{max}, we have identified all initial states such that the maximum number infected at the peak can remain below I_{max} by means of a non-stationary strategy. We have also identified viable strategies, and given examples of strategies aiming both at controlling the maximum number infected at the peak and at asymptotically driving the number of infected to zero.

To our knowledge, this approach is new. The viable point of view in mathematical epidemiology seems not to have ever been studied before. We have presented the basic idea with a simple SIR model with vaccination rate as control. Nevertheless, on the one hand, our model can be made more realistic by putting an upper bound $p_{max} < 1$ on the control, avoiding full vaccination rate which is impossible or highly costly. This changes the vaccination barrier with a new frontier given by $\inf_{p\in[0,p_{max}]} \mathcal{H}(S, \mathcal{I}(S), -\mathcal{I}'(S), 1, p) = 0$ (where the Hamiltonian $\mathcal{H}$ is introduced in the proof in the Appendix) On the other hand, this approach can be extended to other models with other controls, such as isolation or quarantine.

A Appendix: simulations parameter values

Simulations are made with data from [15, p.44]:

$$N = 100\,, \quad \beta = 0.00\ 028\,, \quad \nu = 0.005\,, \quad \delta = 0.01\,.$$

B Appendix: proof of Theorem 1

We suppose that $I_{max} < \frac{\mathcal{R}_0 - 1}{\mathcal{R}_0} N$, *i. e.* $I_{max} + S_{max} < N$. Otherwise, the proof is easy and left to the reader.

The proof will consist of three Lemmas. It makes use of a geometric characterization of *viable sets*. A subset $\mathbb{V}$ of the state constraint set $\mathbb{V}^0$ in (5) is said to be a *viable set* if there exists $t \mapsto p_t \in [0, 1]$, such that the solution to the dynamical system (2a)-(2b) starting from any initial state $(S_{t_0}, I_{t_0}) \in \mathbb{V}$ at initial time t_0, remains within $\mathbb{V}$ for all times $t \geq t_0$. It may easily be seen that a union of viable sets still is a viable set. A Theorem by Aubin [3] states that the viability kernel is the largest viable set, that is, the union of all viable sets.

To the controlled dynamical system (2a)-(2b), we associate the *controlled vector field* g given by its two components:

$$g_S(S, I, p) = -\beta IS + \delta N(1 - p) - \delta S\,, \tag{15a}$$
$$g_I(S, I, p) = \beta IS - \nu I - \delta I\,. \tag{15b}$$

The scalar product of the controlled vector field g with a vector $n = \begin{pmatrix} n_S \\ n_I \end{pmatrix}$ is the so-called *Hamiltonian* [31]

$$\mathcal{H}(S, I, n_S, n_I, p) := g_S(S, I, p)n_S + g_I(S, I, p)n_I$$

with expression

$$\mathcal{H}(S, I, n_S, n_I, p) = (-\beta IS + \delta N(1 - p) - \delta S)n_S + \beta I(S - S_{max})n_I\,. \tag{16}$$

Lemma 1 *There exists a unique solution $S \in [S_{max}, N] \mapsto \mathcal{I}(S)$ to the differential equation* (9a)-(9b)*. This solution $\mathcal{I}$ is decreasing and strictly positive.*

Proof Notice that $I_{max} + S_{max} < N \Rightarrow S_{max} < N$.

In the neighbourhood of $S_{max} > 0$ and $\mathcal{I}(S_{max}) = I_{max} > 0$, the expression $(\beta\mathcal{I}(S) + \delta)S$ is strictly positive. Thus, we can write (9a) as

$$\mathcal{I}'(S) = -\frac{\mathcal{I}(S)(S - S_{max})}{(\mathcal{I}(S) + \delta/\beta)S}\,. \tag{17}$$

This differential equation has separable variables. We shall not follow the path to solve it this way, but we shall directly study the solution properties.

- By the Cauchy Lipschitz theorem applied to (17), there exists a local solution $\mathcal{I}$ of (9a) and (17) around $S_{max} > 0$.

- We shall now prove that the above local solution $\mathcal{I}$ is such that $\mathcal{I}(S) > 0$. Indeed, suppose that there exists $S_0 \geq S_{max}$ such that $\mathcal{I}(S_0) = 0$. In the neighbourhood of $S_0 > 0$, the expression $(\beta\mathcal{I}(S) + \delta)S$ is strictly positive so that (9a) and (17) are locally equivalent. Now, around $S_0 > 0$, we have two solutions, $S \mapsto \mathcal{I}(S)$ and $S \mapsto 0$, as can be checked on formulas (9a) or (17). By the Cauchy Lipschitz theorem, this may not happen by uniqueness. Therefore, no such $S_0 \geq S_{max}$ may exist, and thus $\mathcal{I}(S) > 0$ for all S such that the solution is well defined.
- Finally, we shall prove that the above local solution $\mathcal{I}$ is decreasing. Indeed, by examining (9a) or (17), we see that $\mathcal{I}'(S) < 0$.
- To conclude, the above unique local solution $\mathcal{I}$ of (9a) is decreasing and bounded below by 0. Therefore, it can be defined for all $S \geq S_{max}$.

Now, we shall prove that the intersection set

$$\mathbb{V} := \mathbb{V}^0 \cap \{(S, I) \mid S_{max} \leq S \leq N \text{ and } I < \mathcal{I}(S)\} \tag{18}$$

introduced in Theorem 1 is a viable domain.

Lemma 2 *The set $\mathbb{V}$ is a viable domain.*

Proof The proof consists in writing $\mathbb{V}$ as the union of viable domains $\mathbb{V}_\epsilon$ for all $\epsilon > 0$.

For this, we shall consider the following slightly modified version of the differential equation (9a)–(9b), where $\epsilon > 0$ is small enough:

$$-\epsilon\mathcal{I}_\epsilon(S) = S(\beta\mathcal{I}_\epsilon(S) + \delta)\mathcal{I}'_\epsilon(S) + \beta\mathcal{I}_\epsilon(S)(S - S_{max})\,, \tag{19a}$$

$$\mathcal{I}_\epsilon(S_{max} - \epsilon) = I_{max} - \epsilon\,. \tag{19b}$$

By the same proof as in Lemma 1, we can show that the above differential equation has a unique solution $S \in [S_{max} - \epsilon, N] \mapsto \mathcal{I}_\epsilon(S)$, strictly positive and strictly decreasing. We put

$$\mathbb{V}_\epsilon := \mathbb{V}^0 \cap \{(S, I) \mid S_{max} - \epsilon \leq S \leq N \text{ and } I < \mathcal{I}_\epsilon(S)\}\,. \tag{20}$$

By a comparison theorem [2], it can straightforwardly be seen that the solution $\mathcal{I}_\epsilon$ of (19a)–(19b) is below the solution $\mathcal{I}$ of (9a)–(9b). Hence $\mathbb{V}_\epsilon \subset \mathbb{V}$. By a continuity argument, $\mathbb{V}$ is the union of all $\mathbb{V}_\epsilon$ for $\epsilon > 0$.

Now, we prove that any $\mathbb{V}_\epsilon$ is a viable domain. Since the state constraint set $\mathbb{V}^0$ is strongly invariant, we can focus upon the frontier line $\{(S, I_{max} - \epsilon) \mid 0 \leq S < S_{max} - \epsilon\}$ and upon the frontier curve $\{(S, \mathcal{I}_\epsilon(S)) | S_{max} - \epsilon \leq S \leq N\}$. By examining the scalar product of the controlled vector field g with the normal vector at these two frontier curves, we shall prove the existence of a control $p \in [0, 1]$ such that the controlled vector field g is inward to the domain $\mathbb{V}_\epsilon$.

- All along the segment $\{(S, I_{max} - \epsilon) \mid 0 \leq S < S_{max} - \epsilon\}$, we have

  $$\mathcal{H}(S, I_{max}-\epsilon, n_S, n_I, p) = (-\beta(I_{max}-\epsilon)S+\delta N(1-p)-\delta S)n_S+\beta(I_{max}-\epsilon)I_{max}(S-S_{max})n_I\,.$$

  The outward normal cone to the segment is made of vectors $\begin{pmatrix} n_S \\ n_I \end{pmatrix} = \begin{pmatrix} 0 \\ n_I \end{pmatrix}$ with $n_I > 0$, so that

  $$\inf_{p\in[0,1]} \mathcal{H}(S, I_{max} - \epsilon, n_S, n_I, p) = \beta(I_{max} - \epsilon)(S - S_{max})n_I \leq -\beta\epsilon(I_{max} - \epsilon)n_I < 0$$

  because $S < S_{max} - \epsilon$. Therefore, the control $p = 1$ at the frontier $\{(S, I_{max} - \epsilon) \mid 0 \leq S < S_{max} - \epsilon\}$ is such that the controlled vector field g is inward to the domain $\mathbb{V}_\epsilon$.
- Along the frontier curve $\{(S, \mathcal{I}_\epsilon(S)) | S_{max} - \epsilon \leq S \leq N\}$, an outgoing normal vector is $\begin{pmatrix} n_S \\ n_I \end{pmatrix} = \begin{pmatrix} -\mathcal{I}'_\epsilon(S) \\ 1 \end{pmatrix}$. Therefore, the Hamiltonian (16) evaluated for this outgoing normal vector along this curve is given by

  $$\mathcal{H}(S, \mathcal{I}_\epsilon(S), -\mathcal{I}'_\epsilon(S), 1, p) = -\mathcal{I}'_\epsilon(S)(-\beta\mathcal{I}_\epsilon(S)S + \delta N(1 - p) - \delta S) + \beta\mathcal{I}_\epsilon(S)(S - S_{max})\,.$$

We have $\mathcal{I}'_\epsilon(S) < 0$, so that

$$\inf_{p\in[0,1]} \mathcal{H}(S, \mathcal{I}_\epsilon(S), n_S, n_I, p) = -\mathcal{I}'_\epsilon(S)(-\beta\mathcal{I}_\epsilon(S)S - \delta S) + \beta\mathcal{I}_\epsilon(S)(S - S_{max}) = -\epsilon\mathcal{I}_\epsilon(S) < 0\,,$$

because $\mathcal{I}_\epsilon$ is solution of (19a) and $\mathcal{I}_\epsilon(S) > 0$. Therefore, the control $p = 1$ at the frontier $\{(S, \mathcal{I}_\epsilon(S)) | S_{max} - \epsilon \leq S \leq N\}$ is such that the controlled vector field g is inward to the domain $\mathbb{V}_\epsilon$.

– At the common extremity $(S_{max}-\epsilon, I_{max}-\epsilon)$, the normal cone is $\begin{pmatrix} n_S \\ n_I \end{pmatrix} = a \begin{pmatrix} -\mathcal{I}'_\epsilon(S_{max} - \epsilon) \\ 1 \end{pmatrix} + b \begin{pmatrix} 0 \\ 1 \end{pmatrix}$, with $a \geq 0$ and $b \geq 0$ and $a + b > 0$. Therefore, the Hamiltonian (16) evaluated for such normal vectors at this common extremity $(S_{max} - \epsilon, I_{max} - \epsilon)$ is given by

$$\begin{aligned}\mathcal{H}(S_{max} - \epsilon, I_{max} - \epsilon, n_S, n_I, p) = \\ -a\mathcal{I}'_\epsilon(S_{max} - \epsilon)\big(- \beta(I_{max} - \epsilon)(S_{max} - \epsilon) + \delta N(1 - p) - \delta(S_{max} - \epsilon)\big) \\ -\beta\epsilon(a + b)(I_{max} - \epsilon)\,.\end{aligned}$$

Since $\mathcal{I}'_\epsilon(S) < 0$ and $a \geq 0$, we obtain that

$$\begin{aligned}&\inf_{p\in[0,1]} \mathcal{H}(S_{max} - \epsilon, I_{max} - \epsilon, n_S, n_I, p) = \\ &-a\mathcal{I}'_\epsilon(S_{max} - \epsilon)\big(- \beta(I_{max} - \epsilon)(S_{max} - \epsilon) - \delta(S_{max} - \epsilon)\big) - \beta\epsilon(a + b)(I_{max} - \epsilon) < 0\,,\end{aligned}$$

because $a+b > 0$. Therefore, the control $p = 1$ at the common extremity $(S_{max}-\epsilon, I_{max}-\epsilon)$ is such that the controlled vector field g is inward to the domain $\mathbb{V}_\epsilon$.

– At $(N, \mathcal{I}_\epsilon(N))$ the normal cone is $\begin{pmatrix} n_S \\ n_I \end{pmatrix} = a \begin{pmatrix} -\mathcal{I}'_\epsilon(N) \\ 1 \end{pmatrix} + b \begin{pmatrix} 1 \\ 0 \end{pmatrix}$ with $a \geq 0$ and $b \geq 0$, not both equal to 0. Therefore, the Hamiltonian (16) evaluated for such normal vectors at $(N, \mathcal{I}_\epsilon(N))$ is given by

$$\mathcal{H}(N, \mathcal{I}_\epsilon(N), n_S, n_I, p) = (b - a\mathcal{I}'_\epsilon(N))(-\beta\mathcal{I}_\epsilon(N)N - \delta pN) + a\beta\mathcal{I}_\epsilon(N)(N - S_{max})\,.$$

Since $\mathcal{I}'_\epsilon(N) < 0$ and $a \geq 0$, $b \geq 0$, we obtain that

$$\begin{aligned}\inf_{p\in[0,1]} \mathcal{H}(N, \mathcal{I}_\epsilon(N), n_S, n_I, p) &= (b - a\mathcal{I}'_\epsilon(N))(-\beta\mathcal{I}_\epsilon(N)N - \delta N) + a\beta\mathcal{I}_\epsilon(N)(N - S_{max}) \\ &= b(-\beta\mathcal{I}_\epsilon(N)N - \delta N) \\ &+ a(N(\beta\mathcal{I}_\epsilon(N) + \delta)\mathcal{I}'_\epsilon(N) + \beta\mathcal{I}_\epsilon(N)(N - S_{max})) \\ &= b(-\beta\mathcal{I}_\epsilon(N)N - \delta N) - a\epsilon\mathcal{I}_\epsilon(N) < 0\,,\end{aligned}$$

since $\mathcal{I}_\epsilon$ is a solution of (19a) and $\mathcal{I}_\epsilon > 0$. Therefore, the control $p = 1$ at $(N, \mathcal{I}_\epsilon(N))$ is such that the controlled vector field g is inward to the domain $\mathbb{V}_\epsilon$.

To conclude, we have shown that the control $p = 1$ applied all along the boundary of the set $\mathbb{V}_\epsilon$ is such that the controlled vector field g is inward to the domain $\mathbb{V}_\epsilon$. Thus, $\mathbb{V}_\epsilon$ is a viable set.

Lemma 3 *If a solution $t \mapsto (S_t, I_t)$ of the differential control system* (2a)-(2b) *starts from an initial state (S_{t_0}, I_{t_0}) outside the set $\mathbb{V}$ defined in* (18)*, it will violate the constraint* (3) *after a finite time.*

Proof We define a function $\mathcal{L}(S, I)$ on the rectangle $[S_{max}, N] \times [0, I_{max}]$:

$$\mathcal{L}(S, I) = I - \mathcal{I}(S)\,,$$

where $\mathcal{I}$ is the solution of (9a)-(9b). With this, the set $\mathbb{V}$ defined in (18) can be written as $\mathbb{V} = \{(S, I) | S_{max} \leq S \leq N, \mathcal{L}(S, I) < 0\}$. We introduce

$$\mathbb{D} := \{(S, I) | S_{max} \leq S \leq N, I > \mathcal{I}(S)\} = \{(S, I) | S_{max} \leq S \leq N, \mathcal{L}(S, I) \geq 0\} \qquad (22)$$

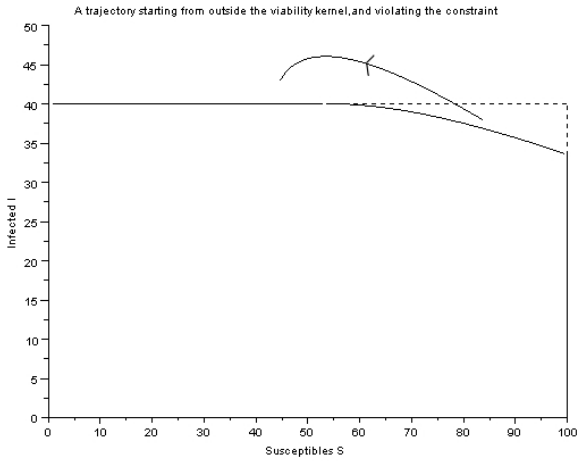

Fig. 4 A trajectory starting from outside the viability kernel, and violating the constraint

which is the complementary set of $\mathbb{V}$ in $\{(S,I)|S_{max} \leq S \leq N, I \geq 0\}$.

We shall now show that any trajectory starting from an initial state in $\mathbb{D}$ necessarily, on the one hand, remains within $\mathbb{D}$ ($\mathbb{D}$ is strongly invariant), and, on the other hand, violates the constraint (3). For this purpose, let us consider a solution $t \mapsto (S_t, I_t)$ of the differential control system (2a)-(2b) starting from an initial state $(S_{t_0}, I_{t_0}) \in \mathbb{D}$.

First, we shall show that $t \mapsto \mathcal{L}(S_t, I_t)$ is increasing. It is well known that $\frac{\mathrm{d}\mathcal{L}(I_t,S_t)}{\mathrm{d}t} = \dot{\mathcal{L}}(S_t, I_t, p_t)$, where

$$\begin{aligned}\dot{\mathcal{L}}(S,I,p) &= \frac{\partial \mathcal{L}}{\partial S}(S,I)g_S(S,I,p) + \frac{\partial \mathcal{L}}{\partial I}(S,I)g_I(S,I,p) \\ &= -\mathcal{I}'(S)(-\beta IS + \delta N(1-p) - \delta S) + I\beta(S - S_{max}) \\ &= \mathcal{I}'(S)(\beta IS + \delta S) - \mathcal{I}'(S)\delta N(1-p) + I\beta(S - S_{max})\ .\end{aligned}$$

Therefore, when $I \geq \mathcal{I}(S)$, we have that

$$\begin{aligned}\dot{\mathcal{L}}(S,I,p) &\geq \mathcal{I}(S)\beta(S - S_{max}) + \mathcal{I}'(S)(\beta\mathcal{I}(S)S + \delta S) - \mathcal{I}'(S)\delta N(1-p) \\ &\geq -\mathcal{I}'(S)\delta N(1-p) \geq 0\ ,\end{aligned}$$

because $\mathcal{I}'(S) < 0$ and $\mathcal{I}$ solves (9a). We have proved that $\mathcal{L}(S_t, I_t) \geq \mathcal{L}(S_{t_0}, I_{t_0}) > 0$ and therefore that (S_t, I_t) remains in the domain $\mathbb{D}$, hence outside $\mathbb{V}$.

Second, we shall prove that the constraint (3) is necessarily violated. Indeed, suppose the contrary: $I_t < I_{max}$ for all $t \geq t_0$. We put $L_{t_0} = \mathcal{L}(S_{t_0}, I_{t_0}) > 0$. We have

$$\mathcal{L}(S_t, I_t) \geq \mathcal{L}(S_{t_0}, I_{t_0}) \Rightarrow \mathcal{I}(S_t) < I_t - L_{t_0} < I_{max} - L_{t_0} = \mathcal{I}(S_{max}) - L_{t_0}\ ,$$

which implies that $S_t > \mathcal{I}^{-1}(\mathcal{I}(S_{max}) - L_{t_0})$. Therefore, $S_t - S_{max} > \kappa$, where $\kappa = \mathcal{I}^{-1}(\mathcal{I}(S_{max}) - L_{t_0}) - S_{max} > 0$ does not depend on time t. Now, according to (2b), $\dot{I} = \beta I(S_t - S_{max}) > \beta I\kappa$, and thus $I_t \geq I_{t_0} e^{\kappa(t-t_0)}$. Since $\kappa > 0$, this contradicts the initial assumption that $I_t < I_{max}$ for all $t \geq t_0$. Therefore, the constraint (3) is necessarily violated.

To conclude, we have proved, on the one hand, that the set $\mathbb{V}$ defined in (18) is a viable set and, on the other hand, that any trajectory starting from an initial state outside $\mathbb{V}$ violates the constraint (3) after a finite time. Therefore, this set $\mathbb{V}$ is the viability kernel $\mathbb{V}(I_{max})$.

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