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Abstract In the present paper we address a relaxation theorem for a new integral functional of the calculus of variations defined on the space $W^{1,p}_0$ of functions whose gradient is an L^p-vector field with distributional divergence given by a Radon measure. The result holds for integrand of type $f(x,\Delta u)$ without any coerciveness condition, with respect to the second variable, and C^1-continuity assumptions with respect to the spatial variable x.

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1. Introduction

In a recent paper (see [2]) the authors have introduced a new functional of calculus of variations to preserve point-like and curve-like singularities in biological images corrupted by noise. More precisely the energy they deal with, was the following

$$\begin{aligned}\mathcal{F}(u) := &\int_\Omega f(\Delta u)dx + \int_\Omega f^\infty(\frac{d\mu^a}{d|\mu^a|})d|\mu^a| + \int_\Omega f^\infty(\frac{d\mu^0}{d|\mu^0|})d|\mu^0| + \int_\Omega g(\nabla u)dx \\ &+ \int_\Omega |u-u_0|^2dx, \end{aligned} \tag{1.1}$$

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where $\Omega \subset \mathbb{R}^N$, $N \in \{2,3\}$, $u \in \Delta\mathcal{M}^p(\Omega)$ with $\frac{2N}{N+2} \le p < \frac{N}{N-1}$. $\Delta\mathcal{M}^p(\Omega)$ is the space of $W_0^{1,p}$-functions whose gradient is an L^p-vector field with distributional divergence given by a Radon measure; the measures μ^a, μ^0 are given by the p-capacitary decomposition applied to the singular part of the divergence measure of ∇u, that is

$$\mathrm{Div}\nabla u = \Delta u dx + \mu^a + \mu^0;$$

$\frac{d\mu^a}{d|\mu^a|}$, $\frac{d\mu^0}{d|\mu^0|}$ are the Radon-Nikodym derivatives of the measures μ^a and μ^0 with respect to their total variation The restriction $p < \frac{N}{N-1}$ is needed to allow singularities on curves and points (see [3, 15] on this issue). The integrands f, g are convex functions and f^∞ is the recession function (see Section 2 for a precise definition of all these quantities and references on p-capacity). Finally u_0 is a given data. In [2], under suitable growth assumptions on the integrands f and g, an existence result was proven. It was also shown that functional (1.1) coincides with the lower semicontinuous envelope, with respect to $W_0^{1,p}(\Omega)$-weak convergence of the following functional

$$F(u) := \begin{cases} \int_\Omega f(\Delta u)dx + \int_\Omega g(\nabla u)dx + \int_\Omega |u-u_0|^2 dx & \text{on } W_0^{1,p,1}(\mathrm{Div};\Omega), \\ +\infty & \text{on } \Delta\mathcal{M}^p(\Omega) \setminus W_0^{1,p,1}(\mathrm{Div};\Omega). \end{cases}$$

$W_0^{1,p,1}(\mathrm{Div};\Omega)$ is the space of $W_0^{1,p}$-functions whose gradient is an L^p-vector field whose distributional divergence is a L^1-function.

For the applications it can be crucial to allow a dependence with respect to the spatial variable x in the integrand f. Indeed in the damaged image reconstruction problem one might like to emphasize the singularities contained in a given region of Ω by giving appropriate values to the integrand. Moreover such x-dependence, in most cases, does not satisfy strong regularity property, as strong differentiability for instance. Typical examples are integrand of type $f(x,\xi) = a(|x|)f(\xi)$. For instance when $\xi = \nabla u$ (i.e. in the BV-setting) anisotropic total variation has important applications in image processing problems such as edge linking (see [5]). Therefore it makes sense, even from an experimental point of view, to consider integrand of type $f(x, \Delta u)$. However in this work we have limited ourselves to a pure theoretical analysis. We refer to [2] for numerical applications in the isotropic case.

As a first step in view of integral representation formula, it is natural to investigate lower semicontinuity property when such an x-dependence is allowed. Indeed the lower semicontinuity of functional $\mathcal{F}$ corresponds to obtain the so called "lim inf" inequality (see subsection 2.6) for the relaxed functional of F.

Therefore in this paper we first study the lower semicontinuity, with respect to the $W_0^{1,p}(\Omega)$-weak convergence, of the following anisotropic version of functional (1.1):

$$\begin{aligned}\mathcal{F}(u) &= \int_\Omega f(x,\Delta u)dx + \int_\Omega f^\infty(x,\frac{d\mu^a}{d|\mu^a|})d|\mu^a| + \int_\Omega f^\infty(x,\frac{d\mu^0}{d|\mu^0|})d|\mu^0| + \int_\Omega g(\nabla u)dx \\ &+ \int_\Omega |u-u_0|^2dx. \end{aligned} \tag{1.2}$$

In order to prove $W_0^{1,p}(\Omega)$-weak lower semicontinuity result we use a successful technique, developed in these last years, to address the L^1-lower semicontinuity of integral functional (even if the integrand depends on the variable u) defined on the space $BV(\Omega)$. It permits to prove lower semicontinuity theorems by dropping the coerciveness assumptions and under weak differentiability requirements on the integrand $f(x,u,\nabla u)$. The main tools of this approach are a chain rule formula and an approximation result for a convex function due to E.De Giorgi (see Theorem 2.2). There is by now a vast literature on this topic. Without claiming of being exhaustive we refer to [14, 17] and references therein for an overview on this subject.

Very roughly speaking, by De Giorgi's Theorem, one can write functional $\mathcal{F}$ as a supremum of affine functionals involving the scalar product between certain coefficients (the so-called De Giorgi's coefficients) and the derivatives of u. Then if $\{u_n\} \subset \Delta\mathcal{M}^p(\Omega)$ is a sequence which converges to a function $u \in \Delta\mathcal{M}^p(\Omega)$, one can recover the convergence of the derivatives of u_n , by switching the derivatives to suitable test functions and so proving the continuity of those affine functionals and therefore the lower semicontinuity of $\mathcal{F}$.

In this paper we adapt this strategy to our different variational framework in order to obtain $W_0^{1,p}$-weak lower semicontinuity under weak regularity conditions on the integrand with respect to the spatial variable x. In particular a new Leibniz rule for the product between a proper scalar function b and $u \in \Delta\mathcal{M}^p(\Omega)$ is established. Besides in order to deal with the right duality involving the $W_0^{1,p}(\Omega)$-weak convergence, we assume that the integrand f belongs to $W^{1,p'}(\Omega)$ with $\nabla_x f \in L^{p'}_{\text{loc}}(\Omega\times\mathbb{R})$, $\frac{1}{p}+\frac{1}{p'}=1$ (see assumption (4.1)).

The last part of the paper is devoted to provide the so called "lim sup" inequality for the relaxed functional of F, which, combined with the "lim inf" inequality, gives the integral representation $SC^-F = \mathcal{F}$ (see subsection 2.6 and Theorem 5.1). The proof is based on an approximation result for lower semicontinuous functions contained in [13]. This result, under a suitable uniform lower semicontinuity condition with respect to the spatial variable (see Theorem 2.3), permits to write a convex integrand f as a supremum of functions which are split as a product of a function depending only on the spatial variable times a function only depending on the second variable. Then in order to prove "lim sup" inequality we can adapt the technique used in [2] for the isotropic functional (1.1). Moreover to attain the upper bound we need, as in the case of integral functional defined on BV-space, to require a linear growth from above on the integrand f.

The paper is organized as follows. Section 2 is devoted to notations, preliminary definitions and results. In Section 3 we address the new Leibniz rule formula. In section 4 we prove the lower semicontinuity result. Finally in section 5 we provide the integral representation formula.

2. Definition and main properties

2.1. **Distributional divergence and classical spaces.** In this subsection we recall the definition of the distributional space $L^{p,q}(\mathrm{Div};\Omega)$ and $\mathcal{DM}^p(\Omega)$, $1 \le p, q \le +\infty$, (see [1, 7, 8]). In all the paper $\Omega \subset \mathbb{R}^N$ is an open bounded set with Lipschitz boundary. $N \in \{2,3\}$ is the space dimension. $\mathcal{L}^n$ and dx will denote the Lebesgue measure on $\mathbb{R}^N$. The bracket $\langle \cdot, \cdot \rangle$ stands for the duality product in some distributional space.

Definition 2.1. *We say that* $U \in L^{p,q}(\mathrm{Div};\Omega)$ *if* $U \in L^p(\Omega;\mathbb{R}^N)$ *and if its distributional divergence* $\mathrm{Div}U \in L^q(\Omega)$*. If* $p = q$ *the space* $L^{p,q}(\mathrm{Div};\Omega)$ *will be denoted by* $L^p(\mathrm{Div};\Omega)$*.*

We say that a function $u \in W^{1,p}(\Omega)$ *belongs to* $W^{1,p,q}(\mathrm{Div};\Omega)$ *if* $\nabla u \in L^{p,q}(\mathrm{Div};\Omega)$*. We say that a function* $u \in W_0^{1,p}(\Omega)$ *belongs to* $W_0^{1,p,q}(\mathrm{Div};\Omega)$ *if* $\nabla u \in L^{p,q}(\mathrm{Div};\Omega)$*.*

Finally We say that a function $u \in W^{1,p}(\Omega)$ *belongs to* $W_{\mathrm{loc}}^{1,p,q}(\mathrm{Div};\Omega)$ *if* $\nabla u \in L^{p,q}(\mathrm{Div};A)$*, for every open set* $A \subset\subset \Omega$*.*

Definition 2.2. *For* $U \in L^p(\Omega;\mathbb{R}^N)$*,* $1 \le p \le +\infty$*, set*

$$|\mathrm{Div}U|(\Omega) := \sup\{\langle U, \nabla\varphi\rangle \ : \ \varphi \in C_0^\infty(\Omega),\ |\varphi| \le 1\}.$$

We say that U *is an* L^p*-Divergence measure field, i.e.* $U \in \mathcal{DM}^p(\Omega)$*, if*

$$\|U\|_{\mathcal{DM}^p(\Omega)} := \|U\|_{L^p(\Omega;\mathbb{R}^N)} + |\mathrm{Div}U|(\Omega) < +\infty.$$

We recall that $U \in L^p(\Omega;\mathbb{R}^N)$ belongs to $\mathcal{DM}^p(\Omega)$ if and only if there exists a Radon measure denoted by $\mathrm{Div}U$ such that

$$\langle U, \nabla\varphi\rangle = -\int_\Omega \mathrm{DivU}\varphi \quad \forall\varphi \in C_0^\infty(\Omega),$$

and the total variation of the measure $\mathrm{Div}U$ is given by $|\mathrm{Div}U|(\Omega)$

Let us recall the following classical result (see [8] Proposition 3.1).

Theorem 2.1. *Let* $\{U_h\}_h \subset \mathcal{DM}^p(\Omega)$ *be such that*

$$U_h \rightharpoonup U \quad \text{in } L^p(\Omega;\mathbb{R}^N), \text{ as } h \to +\infty \text{ for } 1 \le p < +\infty. \tag{2.1}$$

Then

$$\|U\|_{L^p(\Omega;\mathbb{R}^N)} \le \liminf_{h\to+\infty} \|U_h\|_{L^p(\Omega;\mathbb{R}^N)}, \quad |\mathrm{Div}U|(\Omega) \le \liminf_{h\to+\infty} |\mathrm{Div}U_h|(\Omega).$$

Finally we define the following space

$$\Delta\mathcal{M}^p(\Omega) := \{u \in W_0^{1,p}(\Omega),\ \nabla u \in \mathcal{DM}^p(\Omega)\}. \tag{2.2}$$

2.2. **p-capacity.** If $K \subset \mathbb{R}^N$ is a compact set and χ_K denotes its characteristic function, we define:

$$Cap_p(K, \Omega) = \inf\{\int_\Omega |\nabla f|^p dx,\ f \in C_0^\infty(\Omega), f \geq \chi_k\}.$$

If $U \subset \Omega$ is an open set and $K \subset U$ is a compact set, its p-capacity is given by

$$Cap_p(U, \Omega) = \sup_{K \subset U} Cap_p(K, \Omega).$$

Finally if $A \subset U \subset \Omega$ with A Borel set and U open, then

$$Cap_p(A, \Omega) = \inf_{A \subset U \subset \Omega} Cap_p(U, \Omega).$$

For general properties we refer the reader to [12, 16, 18].

It is known (see [9]) that given a Radon measure μ the following decomposition holds

$$\mu = \mu^a + \mu^0, \tag{2.3}$$

where the measure μ^a is absolutely continuous with respect to the p-capacity and μ^0 is singular with respect to the p-capacity, that is concentrated on sets with zero p-capacity. Besides it is also known (see [9]) that every measure which is absolutely continuous with respect to the p-capacity can be characterized as an element of $L^1 + W^{-1,p'}$, leading to the finer decomposition:

$$\mu = f - \text{Div}G + \mu^0, \tag{2.4}$$

where $G \in L^{p'}(\Omega; \mathbb{R}^2)$ with $\frac{1}{p} + \frac{1}{p'} = 1$ and $f \in L^1(\Omega)$. In particular if $u \in \Delta\mathcal{M}^p(\Omega)$ by applying the classical Radon-Nikodym decomposition together with (2.4) to the measure $\text{Div}\nabla u$ we have:

$$\text{Div}\nabla u = \Delta u dx + \mu^a + \mu^0 = \Delta u dx + f - \text{Div}G + \mu^0. \tag{2.5}$$

2.3. **Preliminary Lemmas.** We recall a classical result due to E.De Giorgi, G.Buttazzo and G.Dal Maso (see [11]).

Lemma 2.1. *Let μ be a positive Radon measure on an open set $\Omega \subset \mathbb{R}^N$. Consider a sequence $\{u_l\}$ of Borel measurable functions such that for every $l \in \mathbb{N}$, $u_l : \Omega \to [0, \infty]$. Then*

$$\int_\Omega \sup_l u_l\ d\mu = \sup_{l \in \mathbb{N}} \Big\{ \sum_{k=1}^{l} \int_{A_k} u_k\ d\mu \},$$

where $A_k \subset \Omega$ are open and pairwise sets disjoint with compact closure in Ω.

2.4. **Convex Functions.** We briefly recall the classical approximation theorem due to E.De Giorgi (see [10]). The result states that a given integrand $f(x,t)$ convex with respect to t, can be approximated by means of a sequence of affine functions. We state this result in the particular form we will use in the sequel. For more general statement we refer to [10].

Theorem 2.2. *Let $f : \Omega \times \mathbb{R} \to [0,+\infty)$, $(x,t) \mapsto f(x,t)$, be a Borel function convex with respect to t for all $x \in \Omega$. There exists a sequence $\{\xi_l\} \subset C_0^\infty(\mathbb{R})$ with $\xi_l \geq 0$ and $\int_{\mathbb{R}} \xi(t)dt = 1$, such that*

$$f(x,t) = \sup_{l \in \mathbb{N}} (\alpha_l(x) + \beta_l(x)t)^+,$$

where

$$\alpha_l(x) := \int_{\mathbb{R}} f(x,t)\big(2\xi_l(t) + \xi_l'(t)t\big)dt \tag{2.6}$$

$$b_l(x) := -\int_{\mathbb{R}} f(x,t)\xi_l'(t)dt. \tag{2.7}$$

It is worth noticing that the coefficients α_l and β_l explicitly depend on the function f. The explicit formulas permit to deduce regularity properties of the coefficients α_l and β_l from proper hypotheses satisfied by f.

We conclude this section with another approximation result for convex function contained in [13]. Also in this case we state the result in a simpler case and we refer the reader to Lemma 8 of [13] for a more general statement.

Theorem 2.3. *Let $f : \Omega \times \mathbb{R} \to [0,+\infty)$ be a lower semicontinuos function in (x,t) such that $f(x,\cdot)$ is convex for all $x \in \Omega$. Assume in addition that for all $\epsilon > 0$ and for all $x_0 \in \Omega$ there exists $\delta > 0$ such that*

$$f(x_0,t) \leq f(x,t) + \epsilon(1 + f(x,t)) \quad \text{for all } (x,t) \in \Omega \times \mathbb{R} \text{ such that } |x - x_0| < \delta. \tag{2.8}$$

Then there exist $\{a_k\} \subset C_0^\infty(\Omega)$ and $\{\psi_k\} \subset C_0^\infty(\Omega)$ satisfying, for all $k \in \mathbb{N}$, $0 \leq a_k \leq 1$, ψ_k convex satisfying $0 \leq \psi_k(t) \leq \Lambda_k(1 + |t|)$, for some $\Lambda_k \geq 0$, such that

$$f(x,t) = \sup_{k \in \mathbb{N}} a_k(x)\psi_k(t) \quad \text{for all } (x,t) \in \Omega \times \mathbb{R}.$$

2.5. **Functionals and their properties.** Let $f : \Omega \times \mathbb{R}$ be a Borel function convex in the second variable. We shall consider the following functionals defined on $\Delta\mathcal{M}^p(\Omega)$:

$$F(u) := \begin{cases} \int_\Omega f(x,\Delta u)dx + \int_\Omega g(\nabla u)dx + \int_\Omega |u - u_0|^2 dx & \text{on } W_0^{1,p,1}(\mathrm{Div};\Omega), \\ +\infty & \text{on } \Delta\mathcal{M}^p(\Omega) \setminus W_0^{1,p,1}(\mathrm{Div};\Omega); \end{cases} \tag{2.9}$$

$$
\begin{aligned}
\mathcal{F}(u): &= \int_\Omega f(x,\Delta u)dx + \int_\Omega f^\infty(x, \frac{d\mu^a}{d|\mu^a|})d|\mu^a| + \int_\Omega f^\infty(x,\frac{d\mu^0}{d|\mu^0|})d|\mu^0| + \int_\Omega g(\nabla u)dx \\
&+ \int_\Omega |u-u_0|^2 dx, \qquad (2.10)
\end{aligned}
$$

where f^∞ is the recession function given by $\lim_{t\to+\infty}\frac{f(x,t\xi)}{t}$, with $\xi\in\mathbb{R}$ and the measure μ^a and μ^0 are given by decomposition (2.5) applied to the measure $\mathrm{Div}\nabla u$. $\frac{d\mu^a}{d|\mu^a|}$, $\frac{d\mu^0}{d|\mu^0|}$ are the Radon-Nikodym derivatives of the measures μ^a and μ^0 with respect to their total variation. We recall that since f is convex f^∞ is a well defined Borel function convex in the second variable. Finally we assume the restriction $\frac{2N}{N+2}\le p<\frac{N}{N-1}$ in order to give sense to the L^2-fidelity term and to allow singularities on curves and points (see [3, 15]).

2.6. **Relaxation.** Let F be the functional defined in (2.9). For every $u\in\Delta\mathcal{M}^p(\Omega)$, we define the lower semicontinuous envelope or relaxed functional with respect to the $W_0^{1,p}$-weak convergence of F given by:

$$SC^-F(u) := \inf_{u_h\subset W_0^{1,p,1}(\mathrm{Div};\Omega)}\{\liminf_{h\to+\infty}F(u_h) \quad u_h\rightharpoonup u\}. \qquad (2.11)$$

Since we deal with $W_0^{1,p}$-weak convergence, functional (2.11) is characterized by the two following inequalities:

- for every $u\in\Delta\mathcal{M}^p(\Omega)$ and every $\{u_h\}_h\subset W_0^{1,p,1}(\mathrm{Div};\Omega)$, such that $u_h\to u$ weakly in $W_0^{1,p}(\Omega)$,

$$SC^-F(u)\le\liminf_{h\to+\infty}F(u_h), \qquad (2.12)$$

- for every $u\in\Delta\mathcal{M}^p(\Omega)$ there exists $\{\overline{u}_h\}_h\subset W_0^{1,p,1}(\mathrm{Div};\Omega)$, such that $\overline{u}_h\to u$ weakly in $W_0^{1,p}(\Omega)$,

$$SC^-F(u)\ge\limsup_{h\to+\infty}F(\overline{u}_h). \qquad (2.13)$$

For general properties of the relaxation we refer to [4, 6].

3. Leibniz Rule

For our purpose the Leibniz rule here below plays a crucial role.

Lemma 3.1. *Let $1 < p < \frac{N}{N-1}$. Let b be a scalar function belonging to $W^{1,p'}(\Omega)$, with $\frac{1}{p}+\frac{1}{p'}=1$. Then for every $u \in \Delta\mathcal{M}^p(\Omega)$ and for every $\phi \in C_0^\infty(\Omega)$, the following formula holds:*

$$\begin{aligned} &\int_\Omega b(x)\Delta u(x)\phi dx + \int_\Omega b(x)\frac{d\mu^a}{|d\mu^a|}(x)\phi(x)d|\mu^a| \\ +\; &\int_\Omega b(x)\frac{d\mu^0}{|d\mu^0|}(x)d|\mu^a| = -\int_\Omega \big(b(x)\nabla u(x)\big)\cdot\nabla\phi(x)dx \\ -\; &\int_\Omega \nabla b(x)\cdot\nabla u(x)\phi(x)dx \end{aligned}$$

The proof is based on approximation argument.

Proof. Let $\{\rho_\epsilon\}$ be a standard sequence of mollifiers. We set $b_\epsilon = b*\rho_\epsilon$. Then from Proposition 3.4 of [8] applied to the product between b_ϵ and ∇u we obtain in the sense of distribution:

$$\langle \mathrm{Div}(b_\epsilon\nabla u),\phi\rangle_{\mathcal{D}'(\Omega)} = \langle \nabla b_\epsilon\nabla u,\phi\rangle_{\mathcal{D}'(\Omega)} + \langle b_\epsilon \mathrm{Div}\nabla u,\phi\rangle_{\mathcal{D}'(\Omega)},$$

which writes as

$$\begin{aligned} &\int_\Omega b_\epsilon(x)\Delta u(x)\phi dx + \int_\Omega b_\epsilon(x)\frac{d\mu^a}{|d\mu^a|}(x)\phi(x)d|\mu^a| \\ +\; &\int_\Omega b_\epsilon(x)\frac{d\mu^0}{|d\mu^0|}(x)\phi(x)d|\mu^0| = -\int_\Omega \big(b_\epsilon(x)\nabla u(x)\big)\cdot\nabla\phi(x)dx \\ -\; &\int_\Omega \nabla b_\epsilon(x)\cdot\nabla u(x)\phi(x)dx \end{aligned} \tag{3.1}$$

Since $b \in W^{1,p'}(\Omega)$ and $p < \frac{N}{N-1}$, it is easy to check that $p' > N$. Then from classical Sobolev embedding it follows that b is continuous. Therefore we have that $b_\epsilon \to b$ everywhere as $\epsilon \to 0$. Then by the Lebesgue's dominated convergence theorem we obtain:

$$\lim_{\epsilon\to 0}\int_\Omega b_\epsilon(x)\Delta u(x)\phi dx = \int_\Omega b(x)\Delta u(x)\phi dx,$$

$$\lim_{\epsilon\to 0}\int_\Omega b_\epsilon(x)\frac{d\mu^a}{|d\mu^a|}(x)\phi(x)d|\mu^a| = \int_\Omega b(x)\frac{d\mu^a}{|d\mu^a|}(x)\phi(x)d|\mu^a|$$

and

$$\lim_{\epsilon\to 0}\int_\Omega b_\epsilon(x)\frac{d\mu^0}{|d\mu^0|}(x)\phi(x)d|\mu^0| = \int_\Omega b(x)\frac{d\mu^0}{|d\mu^0|}(x)\phi(x)d|\mu^0|.$$

We now focus on the right-hand side of identity (3.1). By taking into account that ∇b_ϵ converges $L^{p'}$-weakly to ∇b we infer

$$\lim_{\epsilon\to 0}\int_\Omega \big(b_\epsilon(x)\nabla u(x)\big)\cdot\nabla\phi(x)dx = \int_\Omega \big(b(x)\nabla u(x)\big)\cdot\nabla\phi(x)dx$$

and

$$\lim_{\epsilon\to 0}\int_\Omega \nabla b_\epsilon(x)\cdot\nabla u(x)\phi(x)dx = \int_\Omega \nabla b(x)\cdot\nabla u(x)\phi(x)dx.$$

Therefore by taking the limit as $\epsilon \to 0$ in (3.1) we achieve the proof. $\square$

4. Lower semicontinuity

In this section the lower semicontinuity result for functional (2.10) is addressed. The result is obtained by means of Theorem 2.2 and Lemma 3.1.

Let $\Omega \subset \mathbb{R}^N$ be a bounded open set. Let $f : \Omega \times \mathbb{R} \to [0, +\infty)$ be a Borel function, convex in the second variable, which satisfies the following condition:

$$\begin{cases} f(\cdot, t) \in W^{1,p'}(\Omega) \text{ with:} \\ \nabla_x f \in L^{p'}_{\mathrm{loc}}(\Omega \times \mathbb{R}). \end{cases} \tag{4.1}$$

Theorem 4.1. *Let $f : \Omega \times \mathbb{R} \to [0, +\infty)$ be a Borel function convex in the second variable satisfying (4.1). Let $g : \mathbb{R}^N \to [0, +\infty)$ be a convex function. Then functional (2.10) is lower semicontinuous on $\Delta\mathcal{M}^p(\Omega)$ with respect to the $W_0^{1,p}$-weak convergence.*

Proof.
Let us set

$$\mathcal{G}(u) := \int_\Omega f(x, \Delta u)dx + \int_\Omega f^\infty(x, \frac{d\mu^a}{d|\mu^a|})d|\mu^a| + \int_\Omega f^\infty(x, \frac{d\mu^0}{d|\mu^0|})d|\mu^0|.$$

By Theorem(2.2) there exists a sequence $\{\xi_l\} \subset C_0^\infty(\mathbb{R})$ with $\xi_l \geq 0$ and $\int_\mathbb{R} \xi_l dx = 1$ such that for any $(x, t) \in \mathbb{R}$ we have

$$f(x, t) = \sup_{l \in \mathbb{N}} \big(\alpha_l(x) + \beta_l(x)t\big)^+$$

and

$$f^\infty(x, t) = \sup_{l \in \mathbb{N}} \big(\beta_l(x)t\big)^+,$$

where, recalling (2.6) and (2.7)

$$\begin{aligned} \alpha_l(x) &= \int_\mathbb{R} f(x,)\big(2\xi_l(t) + \xi_l'(t)\big)dt \\ \beta_l(x) &= -\int_\mathbb{R} f(x, t)\xi_l'(t)dt. \end{aligned} \tag{4.2}$$

Then for every $u \in \Delta\mathcal{M}^p(\Omega)$, we have

$$\begin{aligned} \mathcal{G}(u) &= \int_\Omega \sup_{l \in \mathbb{N}} \big(\alpha_l(x) + \beta_l(x)\Delta u(x)\big)^+ dx + \int_\Omega \sup_{l \in \mathbb{N}} \big(\beta_l(x)\frac{d\mu^a}{d|\mu^a|}(x)\big)^+ d|\mu^a| \\ &+ \int_\Omega \sup_{l \in \mathbb{N}} \big(\beta_l(x)\frac{d\mu^0}{d|\mu^0|}(x)\big)^+ d|\mu^0|. \end{aligned} \tag{4.3}$$

Since the measures dx, $|\mu^a|$, $|\mu^0|$ are mutually singular we have:

$$\mathcal{G}(u) = \int_\Omega \sup_{l \in \mathbb{N}} \Big(\big(\alpha_l(x) + \beta_l(x)\Delta u(x)\big)^+ + \big(\beta_l(x)\frac{d\mu^a}{d|\mu^a|}(x)\big)^+ + \big(\beta_l(x)\frac{d\mu^0}{d|\mu^0|}(x)\big)^+\Big) d|\mathrm{Div}u|.$$

Hence, by applying Lemma 2.1, with $\mu = |\mathrm{Div}\nabla u|(\Omega)$, we obtain

$$\begin{aligned} \mathcal{G}(u) &= \sup_{l\in\mathbb{N}} \sum_{k=1}^{l} \Big(\int_{A_k} \big(\alpha_k(x) + \beta_k(x)\Delta u(x)\big)^+ dx + \int_{A_k} \big(\beta_k(x)\frac{d\mu^a}{d|\mu^a|}(x)\big)^+ d|\mu^a| \\ &+ \int_{A_k} \big(\beta_k(x)\frac{d\mu^0}{d|\mu^0|}(x)\big)^+ d|\mu^0|\Big), \end{aligned} \tag{4.4}$$

where $A_k \subset \Omega$ are open and pairwise disjoint. Let now for all k, η_k be a test function in $C_0^\infty(A_k)$ with $0 \le \eta \le 1$. As dx, $|\mu^a|$, $|\mu^0|$ are mutually singular measures

$$\begin{aligned} \mathcal{G}(u) &= \sup_{l\in\mathbb{N}} \sum_{k=1}^{l} \Big(\sup_{0\le\eta_k\le 1} \big(\int_\Omega \alpha_k(x)\eta_k(x)dx + \int_\Omega \beta_k(x)\eta_k(x)\Delta u(x)dx \\ &+ \int_\Omega \beta_k(x)\frac{d\mu^a}{d|\mu^a|}(x)\eta_k(x)d|\mu^a| + \int_\Omega \beta_k(x)\frac{d\mu^0}{d|\mu^0|}(x)\eta_k(x)d|\mu^0|\big)\Big). \end{aligned} \tag{4.5}$$

Let us define

$$\begin{aligned} H(u) &:= \int_\Omega \alpha_k(x)\eta_k(x) + \int_\Omega \beta_k(x)\Delta u(x)\eta_k(x)dx + \int_\Omega \beta_k(x)\frac{d\mu^a}{d|\mu^a|}(x)\eta_k(x)d|\mu^a| \\ &+ \int_\Omega \beta_k(x)\frac{d\mu^0}{d|\mu^0|}(x)\eta_k(x)d|\mu^0|. \end{aligned}$$

We are going to prove the continuity of $H(u)$ with respect to the $W_0^{1,p}(\Omega)$-weak topology , by applying Lemma 3.1 with β_k and u. Therefore, we need to check first that $\beta_k(x)$ satisfies the hypotheses of Lemma 3.1. For every test $\Phi \in C_0^\infty(\Omega;\mathbb{R}^N)$ we have that

$$\langle \nabla\beta_k(x), \Phi(x)\rangle_{\mathcal{D}'(\Omega)} = -\int_\Omega \beta_k(x)\mathrm{Div}\Phi(x)dx = -\int_\Omega \Big(\int_\mathbb{R} f(x,t)\xi_k'(t)dt\Big)\mathrm{Div}\Phi(x)dx.$$

Since $\nabla_x f \in L^{p'}(\Omega\times\mathbb{R})$ we can apply Fubini's theorem to get that

$$\begin{aligned} \langle \nabla\beta_k(x), \Phi(x)\rangle_{\mathcal{D}'(\Omega)} &= -\int_\mathbb{R} \xi_k'(t)dt \int_\Omega f(x,t)\mathrm{Div}\Phi(x)dx \\ &= \int_\mathbb{R} \xi_k'(t)dt \int_\Omega \nabla_x f(x,t)\cdot\Phi(x)dx. \end{aligned}$$

Hence, we conclude that

$$\langle \nabla\beta_k(x), \Phi(x)\rangle_{\mathcal{D}'(\Omega)} = \int_\Omega \Big(\int_\mathbb{R} \nabla_x f(x,t)\xi_k'(t)dt\Big)\cdot\Phi(x)dx,$$

and therefore we have the identification:

$$\nabla\beta_k = \int_\mathbb{R} \nabla_x f(x,t)\xi_k'(t)dt.$$

Finally since $\xi \subset C_0^\infty(\mathbb{R})$ and $\nabla_x f \in L^{p'}_{\mathrm{loc}}(\Omega\times\mathbb{R})$ we infer $\beta_k \in W^{1,p'}(\Omega)$.

We are now position of proving the continuity of $H(u)$. Let $\{u_n\} \subset \Delta M^p(\Omega)$ be a sequence $W_0^{1,p}(\Omega)$-weakly converging to $u \in \Delta\mathcal{M}^p(\Omega)$. By applying Lemma 3.1

$$\begin{aligned}\lim_{n\to+\infty} H(u_n) &= \\ &+ \int_\Omega \alpha_k(x)\eta_k(x)dx + \lim_{n\to+\infty} - \int_\Omega \big(\beta_k(x)\nabla u_n(x)\big)\cdot\nabla\eta_k(x)dx \\ &- \int_\Omega \nabla\beta_k(x)\cdot\nabla u_n(x)\eta_k(x)dx \\ &= \int_\Omega \alpha_k(x)\eta_k(x)dx + \int_\Omega \big(\beta_k(x)\nabla u(x)\big)\cdot\nabla\eta_k(x)dx \\ &- \int_\Omega \nabla\beta_k b(x)\cdot\nabla u(x)\eta_k(x)dx = H(u). \end{aligned} \tag{4.6}$$

where in the last equality we have used again Lemma 3.1.

Therefore (4.6) implies that $\mathcal{G}$, being the supremum of the sum of supremum of lower continuous functionals is lower semicontinuous itself. Then since g is convex and the term $\int_\Omega |u-u_0|^2$ is continuous, it easy to see that $\mathcal{F}$ is lower semicontinuous too.

5. RELAXATION

This section is devoted to the relaxation result. This result will be attained, once we will have proved inequalities (2.12) and (2.13). The first one is a consequence of the lower semicontinuity result proved in Theorem 4.1. In order to achieve upper bound (2.13) we will strengthen the assumptions on the integrand f, by requiring uniform lower semicontinuity condition with respect to x.

5.1. **Integral representation formula.** We will assume that the integrands $f : \Omega\times\mathbb{R} \to [0,+\infty)$ and $g : \mathbb{R}^N \to [0,+\infty)$ satisfy the following assumptions:

$$f(x,t) \le C_1(1+|t|) \quad \forall (x,t)\in\Omega\times\mathbb{R}, \tag{5.1}$$

where $0 < C_1 < +\infty$ is a constant;

$$g(\xi) \le C_2(1+|\xi|^p) \quad \forall \xi\in\mathbb{R}^N, \tag{5.2}$$

where $0 < C_2 < +\infty$ is a constant. Moreover from assumption (5.1) it follows that

$$f^\infty(x,t) \le C_1(|t|) \quad \forall (x,t)\in\Omega\times\mathbb{R}. \tag{5.3}$$

We also assume that for all $\epsilon > 0$ there exists $\delta > 0$ such that for all $x\in\Omega$

$$f(x,t) \le f(y,t) + \epsilon(1+f(y,t)) \quad \forall (y,t)\in\Omega\times\mathbb{R} \text{ such that } |x-y|\le\delta \tag{5.4}$$

which implies that for all $\epsilon > 0$ there exists $\delta > 0$ such that for all $x \in \Omega$

$$f^{\infty}(x,t) \leq f^{\infty}(y,t) + \epsilon f^{\infty}(y,t) \quad \forall (y,t) \in \Omega \times \mathbb{R} \text{ such that } |x-y| \leq \delta. \tag{5.5}$$

Remark 5.1. *Let us note that assumption (5.4) holds whenever the integrand f is coercive and satisfies the following stronger condition:*
for all $\epsilon > 0$ there exists $\delta > 0$ such that for all $x \in \Omega$

$$f(x,t) - f(y,t) < \epsilon t \quad \forall (y,t) \in \Omega \times \mathbb{R} \textit{ such that } |x-y| \leq \delta.$$

This is the case, for instance, if $f(x,t) = |x|t$.

Theorem 5.1. *Let $f : \Omega \times \mathbb{R} \to [0, +\infty)$ be a Borel function convex in the second variable satisfying (4.1). Assume that (5.1),(5.2) and (5.4) hold. Then we have*

$$\mathcal{F}(u) = SC^{-}F(u) \quad \forall u \in \Delta\mathcal{M}^{p}(\Omega).$$

Proof. Step one: We start by proving inequality (2.12).
Since $SC^{-}F$ is the greatest lower semicontinuous functional not greater than F, $\mathcal{F} \leq F$, and by Theorem 4.1 $\mathcal{F}$ is $W_0^{1,p}$-weak lower semicontinuous, we have $\mathcal{F}(u) \leq SC^{-}F(u)$ for all $u \in \Delta\mathcal{M}^p(\Omega)$. Therefore for every $u \in \Delta\mathcal{M}^p(\Omega)$ and every $\{u_h\}_h \subset W_0^{1,p,1}(\mathrm{Div};\Omega)$, such that $u_h \to u$ weakly in $W_0^{1,p}(\Omega)$,

$$\mathcal{F}(u) \leq SC^{-}F(u) \leq \liminf_{h \to +\infty} F(u_h).$$

Step two: We now prove inequality (2.13).
Let $u \in \Delta\mathcal{M}^p(\Omega)$. Let $\{u_h\} \subset C_0^{\infty}(\Omega)$ defined as $u_h = u * \rho_h$ with ρ_h a standard mollifying sequence, whose support is Ω_h. Note that as $f(\cdot,t) \in W^{1,p'}\Omega)$ with $p' > N$ ($p < \frac{N}{N-1}$) and $f(,t)$ is convex with linear growth, we have that f is continuous in (x,t). Then by taking into account assumption (5.4) we can apply Theorem 2.3 to the function f. So we have for all $(x,t) \in \Omega \times \mathbb{R}$

$$f(x,t) = \sup_k a_k(x)\psi_k(t), \quad f^{\infty}(x,t) = \sup_k a_k(x)\psi_k^{\infty}(t), \tag{5.6}$$

where ψ_k are convex functions. Therefore it follows that there exist $\{\alpha_k^l\}, \{\beta_k^l\} \subset \mathbb{R}$ such that for all $(x,t) \in \Omega \times \mathbb{R}$

$$f(x,t) = \sup_k a_k(x)\Big(\sup_l (\alpha_k^l + \beta_k^l t)\Big) \quad f^{\infty}(x,t) = \sup_k a_k(x)\Big(\sup_l \beta_k^l t\Big). \tag{5.7}$$

Then

$$a_k(x)\Big(\alpha_l^k + \beta_k^l \Delta u_h(x)\Big) = a_k(x)\Big(\int_{\Omega_h} \rho_h(x-y)\alpha_k^l + \beta_k^l \Delta u(y)dy$$
$$+ \int_{\Omega_h} \rho_h(x-y)\beta_k^l \frac{d\mu^a}{|d\mu^a|}(y)d|\mu^a| + \int_{\Omega_h} \rho_h(x-y)\beta_k^l \frac{d\mu^0}{|d\mu^0|}(y)d|\mu^0|\Big).$$

By taking into account (5.6) and (5.7) we get:

$$a_k(x)\Big(\alpha_l^k+\beta_k^l\Delta u_h(x)\Big)\le\int_{\Omega_h}\rho_h(x-y)f(x,\Delta u(y))dy$$
$$+\quad\int_{\Omega_h}\rho_h(x-y)f^\infty(x,\frac{d\mu^a}{|d\mu^a|}(y))d|\mu^a|+\int_{\Omega_h}\rho_h(x-y)f^\infty(x,\frac{d\mu^0}{|d\mu^0|}(y))d|\mu^0|.$$

Let δ be given by assumption (5.4). Then for h large enough we have, for $x,y\in\Omega_h$, that $|x-y|\le\frac{1}{h}\le\delta$. Hence from (5.4), (5.5) it follows that for all $\epsilon>0$

$$a_k(x)\Big(\alpha_l^k+\beta_k^l\Delta u_h(x)\Big)\le\int_{\Omega_h}\rho_h(x-y)f(y,\Delta u(y))dy$$
$$+\quad\int_{\Omega_h}\rho_h(x-y)f^\infty(y,\frac{d\mu^a}{|d\mu^a|}(y))d|\mu^a|+\int_{\Omega_h}\rho_h(x-y)f^\infty(y,\frac{d\mu^0}{|d\mu^0|}(y))d|\mu^0|$$
$$+\quad\epsilon\int_{\Omega_h}\rho_h(x-y)(1+f(y,\Delta u(y))dy+\epsilon\int_{\Omega_h}\rho_h(x-y)f^\infty(y,\frac{d\mu^a}{|d\mu^a|}(y))d|\mu^a|$$
$$+\quad\epsilon\int_{\Omega_h}\rho_h(x-y)f^\infty(y,\frac{d\mu^0}{|d\mu^0|}(y))d|\mu^0| \tag{5.8}$$

By taking the supremum firstly over l and over k on the left hand side of (5.8), performing an integration over Ω ,with respect to the variable x, and by taking into account the growth conditions (5.1) and (5.3) we get

$$\int_\Omega f(x,\Delta u_h(x))dx\le\int_\Omega dx\int_{\Omega_h}\rho_h(x-y)f(y,\Delta u(y))dy$$
$$+\quad\int_\Omega dx\int_{\Omega_h}\rho_h(x-y)f^\infty(y,\frac{d\mu^a}{|d\mu^a|}(y))d|\mu^a|$$
$$+\quad\int_\Omega dx\int_{\Omega_h}\rho_h(x-y)f^\infty(y,\frac{d\mu^0}{|d\mu^0|}(y))d|\mu^0|$$
$$+\quad\epsilon\Big(\int_\Omega dx\int_{\Omega_h}\rho_h(x-y)dy+\int_\Omega dx\int_{\Omega_h}\rho_h(x-y)d|\mathrm{Div}\nabla u|\Big)$$
$$\le\quad\int_\Omega dx\int_{\Omega_h}\rho_h(x-y)f(y,\Delta u(y))dy+\int_\Omega dx\int_{\Omega_h}\rho_h(x-y)f^\infty(y,\frac{d\mu^a}{|d\mu^a|}(y))d|\mu^a|$$
$$+\quad\int_\Omega dx\int_{\Omega_h}\rho_h(x-y)f^\infty(y,\frac{d\mu^0}{|d\mu^0|}(y))d|\mu^0|\ +\epsilon\mathcal{L}^n(\Omega)C_1(1+3|\mathrm{Div}\nabla u|(\Omega)). \tag{5.9}$$

Finally by applying Fubini's Theorem and taking into account that ϵ is arbitrary we conclude (5.10)

$$\limsup_{h\to+\infty}\int_\Omega f(x,\Delta u_h)dx\le\int_\Omega f(x,\Delta u)dx+\int_\Omega f^\infty(x,\frac{d\mu^a}{d|\mu^a|})d|\mu^a|+\int_\Omega f^\infty(x,\frac{d\mu^0}{d|\mu^0|})d|\mu^0|,$$

Finally, since ∇u_h converges in measure to ∇u, from the convexity of g together with (5.2), and Vitali's convergence theorem it follows that:

$$\lim_{h\to+\infty}\int_\Omega g(\nabla u_h)dx=\int_\Omega g(\nabla u)dx. \tag{5.11}$$

Then (5.10) and (5.11) imply

$$\limsup_{h\to+\infty} F(u_h) \le \mathcal{F}(u),$$

where we have used the fact that term $\int_\Omega |u-u_0|^2 dx$ is a continuous perturbation. So (2.12) and (2.13) are achieved. □

As a consequence of the relaxation result we obtain the following theorem.

Corollary 5.1. *Let $f : \Omega \times \mathbb{R} \to [0,+\infty)$ $g : \mathbb{R}^N \to [0,+\infty)$ be a convex functions satisfying (4.1), (5.1), (5.2) and (5.4). Assume that for every minimizing sequences $\{u_h\}_h \subset \Delta\mathcal{M}^p(\Omega)$ of $\mathcal{F}_h$ the following compactness property holds:*

$$\mathcal{F}_h(u_h) \le M \Rightarrow \exists \{u_{h_k}\}_k, u \subset \Delta\mathcal{M}^p(\Omega) \quad \text{with } u_{h_k} \rightharpoonup u \tag{5.12}$$

Then there exists a minimum $u \in \Delta\mathcal{M}^p(\Omega)$ of functional $\mathcal{F}$. Moreover the following equality holds

$$\inf_{u\in\Delta\mathcal{M}^p(\Omega)} F(u) = \min_{u\in\Delta\mathcal{M}^p(\Omega)} \mathcal{F}(u). \tag{5.13}$$

Proof. *By Theorem 4.1, functional $\mathcal{F}$ is lower semicontinuous with respect to the $W_0^{1,p}$-weak convergence. Then the existence of a minimum follows via the direct method of the calculus of variations.*

Finally thanks to growth conditions (5.1) and (5.2) the infimum is finite. Then property (5.13) can be achieved by standard arguments (see for instance [6]*).* □

Remark 5.2. *A sufficient condition to ensure property (5.12) is to assume the classical coercivity assumptions on the integrand f and g:*

$$f(x,t) \ge c_1|t| \quad \forall (x,t) \in \Omega \times \mathbb{R}, \tag{5.14}$$

where $0 < c_1 < +\infty$ is a constant;

$$g(\xi) \ge c_2|\xi|^p \quad \forall \xi \in \mathbb{R}^N, \tag{5.15}$$

where $0 < c_2 < +\infty$ is a constant. Indeed by taking into account Theorem 2.1 *it is not difficult to prove property (5.12).*

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