An efficient segmentation method to price American Put options

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Abstract

A segmentation strategy to price different groups of American standard Put options with different methods is presented and discussed. The method, which exploits the properties of the odd waves of the BI adjusted evaluations introduced by Gaudenzi and Pressacco, proves to be very efficient in particular, to price critical in the money options.

1. Introduction

The purpose of this paper is twofold: a methodological critique to the statistical procedures usually applied to test the efficiency of pricing methods of standard American options, and on the basis of this critique, to propose a new pricing method which is able to greatly enhance the precision speed efficiency of the best estimation methods known, that is the BBSR of Broadie – Detemple [3] and the BIR of Gaudenzi – Pressacco [5], which are both binomial based tree methods.

The best practice to verify efficiency of methods to price American Put options (or other types of exotic options lacking closed formulas) is to evaluate the Mean Relative Error (MRE) and/or the Root Mean Squared Relative Error (RMSRE) on a large sample of options selected from a population with convenient parameters.

As we shall see later, even very large samples do not grant reliability of the results. Indeed, it happens that the magnitude of errors and their volatility are strongly correlated with the relevance of the early exercise opportunity (with the deepness of the American quality of the options); and if the percentage of options belonging to the decisive critical groups is too small, the results of the sample quite likely give unreliable volatile results, depending of the length of the tree and on the key parameters of the population (time to maturity and risk free interest rate). We will provide empirical evidence of these facts.

Before going on, we clarify that a good index of the American quality of an option is the ratio between (a reliable estimate of) the American price of the option and the Black – Scholes [1] price of the European twin, henceforth the characteristic ratio (C.R.) of an option. And we will use the C.R. as a proper basis to divide American options in different groups.

This is the bridge to the second goal of our paper: in principle, any group of options could be priced according to the estimation method which best fits the subgroup: the road to leave a uniform strategy for any option in favour of a segmentation strategy is open.

In Gaudenzi – Pressacco [5] a segmentation based on applying longer trees than the standard to critical options, that is those with C.R. greater than 1.5, has been proposed, showing promising results in term of precision speed trade off.

Here on the contrary, we are going to propose a different segmentation strategy with new methods which greatly enhance the precision for various groups of IN and OUT options; the most important being the one of IN options with C.R. greater than 1.4 which are responsible of the relevant part of the errors. As we shall see, this class of options is priced exploiting an unexpected characteristic of the waves of the odd BI, Binomial Interpolated adjusted evaluations introduced by Gaudenzi – Pressacco. For these options there are high frequency waves, whose local maxima are from the very early (for low values of the length of the tree) quite stable and very close to the true American value, so that making recourse to the local maxima, we will obtain, without increasing the computational time, estimates much more precise than those of BIR and BBSR. This is undoubtedly the main result of the paper. Minor improvements of the efficiency will be presented for other groups of options. The plan of the paper is as follows: chapter 2 gives a short recall of BBSR and

BIR; chapter 3 offers a discussion of a precision test of BIR and BBSR on a random sample of 5.000 options. A different sample of 1.000 options with parameters of the population chosen so as to increase the relevance of the American quality of the options is given in chapter 4. The new segmentation strategy is presented in chapter 5 along with numerical results that confirm its great efficiency.

2. BBSR and BIR

The BBSR method [3] is a binomial based method to price standard American options, which applies Richardson extrapolation to couples of adjusted binomial values. Given the length, n of the binomial tree, BBS(n) denotes the value obtained through the application of the ordinary binomial backward procedure [4], modified in that at every node of the last but one step of the tree, the Black – Scholes (BS) value replaces the usual binomial continuation value. The Richardson extrapolation is then applied to couples of BBS values of the same parity; more precisely given n even, to BBS(n), BBS($2n$) or to BBS(n – 1), BBS($2n$ –1). We denote by BBSR($2n$) or respectively BBSR($2n$ – 1) the values obtained through this procedure. For instance, BBSR(200) is obtained applying Richardson extrapolation to the couple BBS(100), BBS(200).

It is important to note that the replacement of a BS value at the last but one step, makes almost negligible the differences between BBS(n) and BBS(n – 1), so as even or odd evaluations give (almost) the same results and the parity differences disappear. This in turn implies the same consequence also for the BBSR even and odd evaluations.

Also the BIR method is a binomial based method which applies the same Richardson extrapolation logic to couples of adjusted binomial values BI(n), BI($2n$) or respectively BI(n – 1), BI($2n$ – 1). But the adjustment follows a strategy inspired by the desire to escape from the evaluation bias induced by the so called strike specification error. To reach this goal we compute, given n, a set of ten pure binomial values of ten options having computational strikes corresponding exactly to the ten nodes of the tree, lying symmetrically around the contractual strike. Intuitively, these evaluations are unaffected by strike specification errors, and may be used to interpolate at the contractual strike to obtain an adjusted binomial value BI(n), in turn unaffected by an error of this type. The sequences BI(n) are generally quite different according to the parity, but surprisingly this difference washes out when we pass to apply the Richardson extrapolation, thus obtaining BIR($2n$) values almost equivalent to BIR($2n$ – 1).

We do not enter in technical details concerning the many problems to be solved to reach computational efficiency of BIR (see [5]), but we want to remark here that the computational time needed to compute BIR(n) or BBSR(n) are quite similar for any n, except for the lowest values of n when BIR seems to be slightly faster. Then summing up, we may safely say that we can meaningfully compare the precision of BIR(n) and BBSR(n), provided the computational speed is, given n, almost the same.

To test the efficiency of BIR and BBSR in pricing American options, a standard test is usually applied, based on the computation of the (absolute value of) the relative errors, that is normalized deviations from a reliable benchmark (pure binomial at 24.000 steps), for any options of a large (at least 2.500) random sample of American options. The sample is selected randomly from a population with standard parameters, and the absolute values of the errors are summarized by the Mean Relative Error (MRE) and/or by the Squared Root of the Mean Quadratic Error (RMSRE). We cast serious doubts on the reliability of this procedure. A deeper analysis of the point follows in the next chapter.

3. An efficiency test of BIR and BBSR (5.000 options)

A first test of efficiency of BIR and BBSR was done using a sample of 5.000 options, originally studied in the dissertation of Ziani [6]. The options were selected randomly from a population with the following parameters: risk free interest rate uniform between 0 and 0.1; volatility uniform between 0.1 and 0.6; strike uniform between 70 and 100; time to maturity (years) uniform between 0 and 1 with probability 0.75 and uniform between 1 and 5 with probability 0.25; the initial price of the underlying, whose evolution is the classical lognormal is 100. Some of the selected options were discarded either because fully American (immediately exercisable) or because of their too low price, less than 0.50 (at BIR(200)). This way 4.196 were retained: 2.200 in the money and 1.996 out of the money. Then the options were grouped according to their C.R. (ratio between BIR(200) and the BS value of the European twin) in nine classes, as reported in Table 1.

Table 1

CLASS	C.R.	IN NUMBER	IN %	OUT NUMBER	OUT %
1	1,00 - 1,10	1708	77,6	1663	83,3
2	1,10 - 1,15	169	7,7	95	4,8
3	1,15 - 1,20	75	3,4	68	3,4
4	1,20 - 1,25	57	2,6	51	2,6
5	1,25 - 1,30	38	1,7	34	1,7
6	1,30 - 1,40	55	2,5	42	2,1
7	1,40 - 1,60	54	2,5	24	1,2
8	1,60 - 2,00	32	1,5	11	0,6
9	> 2,00	12	0,5	8	0,4
	TOTAL	**2200**	**100%**	**1996**	**100%**

Note that about 80% of the options belong to the first class with the lowest ratios, while on the other side, less than 5% of the options for IN and slightly more than 2% for OUT belong to the three classes with highest ratios. Table 2 reports, separately for IN and OUT options, the Global Relative Errors (GRE) of any group along with the respective Mean Relative Error for BIR(200) and BBSR(200).

Table 2

			IN		
CLASS	**N.**	**GRE BIR**	**GRE BBSR**	**MRE BIR**	**MRE BBSR**
1	1708	1.427.740	4.907.554	836	2.873
2	169	821.848	1.039.522	4.863	6.151
3	75	571.115	655.752	7.615	8.743
4	57	696.918	800.315	12.227	14.041
5	38	944.816	770.746	24.864	20.283
6	55	823.632	1.487.429	14.975	27.044
7	54	2.814.751	1.979.862	52.125	36.664
8	32	3.178.378	2.557.737	99.324	79.929
9	12	2.627.194	1.680.603	218.933	140.050
TOTAL	**2200**	**13.906.392**	**15.879.520**	**6.321**	**7.218**
			OUT		
CLASS	**N.**	**GRE BIR**	**GRE BBSR**	**MRE BIR**	**MRE BBSR**
1	1663	4.383.412	14.086.001	2.636	8.470
2	95	520.661	1.146.070	5.481	12.064
3	68	306.920	647.521	4.514	9.522
4	51	395.209	529.139	7.749	10.375
5	34	289.053	562.506	8.502	16.544
6	42	670.916	949.907	15.974	22.617
7	24	809.633	906.592	33.735	37.735
8	11	337.171	1.059.771	30.652	96.343
9	8	2.509.819	2.136.950	313.727	267.119
TOTAL	**1996**	**10.222.794**	**22.024.457**	**5.122**	**11.034**
		GRE BIR	**GRE BBSR**	**MRE BIR**	**MRE BBSR**
TOTAL	**4196**	**24.129.186**	**37.903.977**	**5.751**	**9.034**

Global Relative Error (GRE 100 - 200) and Mean Relative Error (MRE 100 – 200), times 10^{-8}

Some comments: a) the total error GRE BIR coming from 4.196 options is less than 2/3 that of BBSR, but is less than 1/2 for the OUT and 0,88 for the IN, which seems to imply at first sight that BIR is surely much better than BBSR to price OUT options, but only slightly better to price IN options; b) 62% of the global error of the IN options for BIR comes from the less than 5% options with greatest C.R., while respectively 25% of the global OUT error comes from the less than 0,5% of the options of the extreme class with the biggest C.R. On the other side, 56% of the total (IN + OUT) BBSR error comes from the first two classes with lowest C.R.; c) looking more carefully at the MRE values, they seem to denote a clear superiority of BIR in the first groups, especially in class 1, both for IN and OUT options. For the other groups the behaviour is different for IN, where there is a strong superiority of BBSR in the upper classes, while for OUT BBSR prevails only in class 9; d) despite the variability of the MRE values, there is a character of the distribution of errors common to BIR and BBSR: a strong correlation between the C.R. and the MRE. Except for a very few cases the average error is a monotone increasing function of the C.R.

Comments and conclusions would be very different once we use, as a measure of efficiency, RMSRE instead of MRE.

To appreciate this claim let's look at the following table.

Table 3

100 - 200	**IN**		**OUT**	
CLASS	**MRE BIR / MRE BBSR**	**RMSRE BIR / RMSRE BBSR**	**MRE BIR / MRE BBSR**	**RMSRE BIR / RMSRE BBSR**
1	0,29	0,44	0,31	0,39
2	0,79	0,85	0,45	0,51
3	0,87	0,94	0,47	0,55
4	0,87	0,74	0,75	0,78
5	1,23	1,21	0,51	0,59
6	0,55	0,62	0,71	0,75
7	1,42	1,38	0,89	0,81
8	1,24	1,21	0,32	0,32
9	1,56	1,68	1,17	0,98
ALL	**0,88**	**1,41**	**0,46**	**0,85**

It may be immediately seen that the overall index of quadratic errors (RMSRE) is much more in favour of BBSR than the index of average errors (MRE). This comes from the errors of the upper classes which are largely the most influential on the overall index and where we find the highest ratios of the sample.

To evaluate the structural reliability of these results, we look for data coming from the same test applied to two other different lengths of the tree, doubling or respectively multiplying by four the original (100 – 200) steps of the tree. Keeping account of the small number of OUT options belonging to classes 8 and 9, we have chosen to put together in a single group (8 for the future) these options.

Table 4

200 - 400	IN		OUT	
CLASS	**MRE BIR / MRE BBSR**	**RMSRE BIR / RMSRE BBSR**	**MRE BIR / MRE BBSR**	**RMSRE BIR / RMSRE BBSR**
1	0,39	0,51	0,36	0,40
2	0,76	0,87	0,33	0,46
3	0,80	0,78	0,45	0,50
4	1,37	1,62	0,74	0,77
5	1,01	0,94	0,65	0,66
6	1,07	0,91	0,75	0,72
7	0,79	0,87	1,03	1,04
8	1,08	0,95	1,25	1,20
9	1,11	1,02	///	///
ALL	**0,84**	**0,98**	**0,52**	**1,01**

Table 5

400 - 800	IN		OUT	
CLASS	**MRE BIR / MRE BBSR**	**RMSRE BIR / RMSRE BBSR**	**MRE BIR / MRE BBSR**	**RMSRE BIR / RMSRE BBSR**
1	0,82	0,75	0,71	0,63
2	0,75	0,74	0,41	0,46
3	0,92	1,01	0,44	0,47
4	1,10	1,24	0,51	0,55
5	0,79	0,67	0,71	0,71
6	0,85	0,72	0,94	0,90
7	1,10	1,14	0,86	0,88
8	0,66	0,66	1,10	1,61
9	0,61	0,59	///	///
ALL	**0,79**	**0,68**	**0,74**	**1,37**

Comments: a) OUT 200 – 400: at the MRE level things are more or less as in the shorter tree but at RMSRE there is a draw; b) OUT 400 – 800: once more on average the best results are that of BIR, but the quadratic error favours BBSR; a result totally driven by class 8 which confirms its over helming importance. Note that it is the only class where BBSR beats BIR in quadratic errors. Thus less than 20 options out of 1996 decide the precision of the method; c) IN 200 – 400: while class 1 (remember with almost 80% of the options) confirms the strong superiority of BIR, things change dramatically in the other classes, where there is more equilibrium except for class 4, where BBSR largely dominates. Equilibrium prevails at the overall level; d) IN 400 – 800: here, results are completely reversed compared with those of the short tree. While the superiority of BIR in class 1 diminishes, in the upper classes there is now an unequivocal superiority of BIR both at the MRE and at the RMSRE level. This determines an overall large superiority of BIR.

The overall results are summarized once more in Table 6.

Table 6

	IN		OUT	
	MRE BIR / MRE BBSR	**RMSRE BIR / RMSRE BBSR**	**MRE BIR / MRE BBSR**	**RMSRE BIR / RMSRE BBSR**
100 - 200	0,88	1,41	0,46	0,85
200 - 400	0,84	0,98	0,52	1,01
400 - 800	0,79	0,68	0,74	1,37

It is not easy to give a synthetic comment of these results. An increase of the length of the trees seems to act in favour of BIR in the case of IN options, but just the opposite happens for OUT options.

At the MRE level anyway, BIR dominates uniformly and unequivocally both for IN and OUT options. At RMSRE level on the contrary, there is a puzzling behaviour: short tree IN and long OUT in favour of BBSR; short OUT and long IN in favour of BIR; in the middle break – even both for IN and for OUT.

4. A test of efficiency on 1.000 options

As we said the results are discouraging. When the efficiency is measured by RSMRE the ranking between BIR and BBSR seems to depend (randomly?) on the type of the option (IN or OUT) and on the length of the trees. Moreover a few options, in turn characterized by high volatile errors, seem to play a decisive role in the ranking.

We decided then to pursue another test on a sample of 1.000 options (500 IN and 500 OUT) selected randomly from a different population. Indeed, some of the parameters were changed so as to enhance the American quality of the options: more precisely the risk free rate is now uniform between 0.04 and 0.10, and time to maturity is uniform between 1 and 5 (with probability 1). Moreover the strike is maintained uniform between 100 and 130 for the IN and between 70 and 100 for the OUT. No options were discarded among the OUT, while among the IN 451 survived as not immediately exercisable. The grouping according to the C.R. is given by the following table.

Table 7

CLASS	CR	IN NUMBER	IN %	OUT NUMBER	OUT %
1	1,00 - 1,10	71	15,7	143	28,6
2	1,10 - 1,15	64	14,2	122	24,4
3	1,15 - 1,20	69	15,3	66	13,2
4	1,20 - 1,25	44	9,7	48	9,6
5	1,25 - 1,30	47	10,4	52	10,4
6	1,30 - 1,40	50	11,1	28	5,6
7	1,40 - 1,60	41	9,1	31	6,2
8	1,60 - 2,00	39	8,6	8	1,6
9	> 2,00	26	5,8	2	0,4
	TOTAL	**451**	**100%**	**500**	**100%**

Table 8 reports, separately for IN and OUT options, the Global Relative Errors (GRE) of any group along with the respective Mean Relative Error for BIR(200) and BBSR(200).

Table 8

	IN				
CLASS	**N.**	**GRE BIR**	**GRE BBSR**	**MRE BIR**	**MRE BBSR**
1	71	104.801	342.065	1.476	4.818
2	64	148.472	305.099	2.320	4.767
3	69	452.372	454.743	6.556	6.590
4	44	596.054	437.856	13.547	9.951
5	47	792.895	669.613	16.870	14.247
6	50	639.835	1.208.913	12.797	24.178
7	41	1.928.379	1.707.743	47.034	41.652
8	39	3.471.590	3.095.332	89.015	79.367
9	26	5.867.809	6.989.677	225.685	268.834
TOTAL	**451**	**14.002.207**	**15.211.041**	**31.047**	**33.727**

	OUT				
CLASS	**N.**	**GRE BIR**	**GRE BBSR**	**MRE BIR**	**MRE BBSR**
1	143	255.539	1.203.782	1.787	8.418
2	122	408.325	1.013.918	3.347	8.311
3	66	377.038	555.352	5.713	8.414
4	48	417.170	446.947	8.691	9.311
5	52	286.350	780.874	5.507	15.017
6	28	460.196	559.226	16.436	19.972
7	31	1.626.781	1.072.803	52.477	34.607
8	8	415.057	924.405	51.882	115.551
9	2	389.382	324.897	194.691	162.449
TOTAL	**500**	**4.635.838**	**6.882.204**	**9.272**	**13.764**

		GRE BIR	**GRE BBSR**	**MRE BIR**	**MRE BBSR**
TOTAL	**951**	**18.638.045**	**22.093.245**	**19.598**	**23.232**

Global Relative Error (GRE 100 - 200) and Mean Relative Error (MRE 100 – 200), times 10^{-8}

Comments: a large part of the total error comes from the IN options (75% for BIR 68% for BBSR). Moreover class 9 of the IN alone counts for more than the global OUT errors both for BIR and BBSR, which confirms the decisive role of the IN options with the largest C.R. More generally the IN options with C.R. > 1.4 cause over 60% of the total (IN + OUT) error of BIR, and over 53% of the total error of BBSR.

For a deeper information about errors let's look at the following tables, which report both for MRE and for RSMRE the ratios between BIR and BBSR errors.

Table 9

100 - 200	IN		OUT	
CLASS	**MRE BIR / MRE BBSR**	**RMSRE BIR / RMSRE BBSR**	**MRE BIR / MRE BBSR**	**RMSRE BIR / RMSRE BBSR**
1	0,31	0,34	0,21	0,26
2	0,49	0,49	0,40	0,49
3	0,99	0,94	0,68	0,67
4	1,36	1,07	0,93	0,87
5	1,18	1,14	0,37	0,40
6	0,53	0,50	0,82	0,86
7	1,13	1,11	1,52	1,30
8	1,12	1,02	0,45	0,54
9	0,84	0,94	1,20	1,05
ALL	**0,92**	**0,95**	**0,67**	**0,85**

A comparison with the analogue table 3 would reveal that there are big differences in the ratios in the classes 2, 4, 7 and 9 for the IN, and in class 7 for the OUT. The overall effect is still clearly in favour of BIR for the OUT and has become slightly in favour of BIR for the IN, an effect due largely to a dramatic change in the class 9.

Now we will offer, always with reference to our new sample of 1.000 options, the same ratios as before also for the other standard lengths of the tree.

Table 10

200 - 400	IN		OUT	
CLASS	**MRE BIR / MRE BBSR**	**RMSRE BIR / RMSRE BBSR**	**MRE BIR / MRE BBSR**	**RMSRE BIR / RMSRE BBSR**
1	0,27	0,31	0,31	0,36
2	0,56	0,59	0,31	0,38
3	0,63	0,64	0,40	0,48
4	1,05	1,21	0,72	0,75
5	1,09	1,05	0,90	0,84
6	1,18	1,10	0,73	0,63
7	1,03	1,01	0,80	0,84
8	1,24	1,23	1,16	1,21
9	0,90	0,79	///	///
ALL	**0,99**	**0,84**	**0,63**	**0,97**

Table 11

400 - 800	IN		OUT	
CLASS	**MRE BIR / MRE BBSR**	**RMSRE BIR / RMSRE BBSR**	**MRE BIR / MRE BBSR**	**RMSRE BIR / RMSRE BBSR**
1	0,59	0,56	0,56	0,52
2	0,42	0,44	0,46	0,51
3	0,75	0,74	0,44	0,49
4	1,14	1,92	0,54	0,61
5	1,11	1,00	0,56	0,60
6	1,05	1,08	0,95	0,99
7	1,30	1,08	0,93	1,07
8	1,23	1,29	0,62	0,71
9	1,15	1,25	///	///
ALL	**1,14**	**1,25**	**0,63**	**0,75**

Let's try to give a synthesis of this enormous and someway controversial amount of data regarding our two samples: it seems that for OUT options BIR is uniformly and unequivocally better except for options with C.R. greater than 1.40, where there is no clear ranking.

For IN options there are three layers: for C.R. lesser than 1.20 BIR is unequivocally better, between 1.20 and 1.40 BBSR seems to be uniformly even if slightly better; over 1.40 there is a big variability of the ranking, but in any case, this is the source of the main and dominant errors and, at the end of the story, the driver of the overall ranking.

It is surely worth then to concentrate our attention on the options with C.R. > 1.40, henceforth critical options, especially the IN ones. Indeed, as we shell see in the next chapter, we shall be able to present a new pricing method which strongly decreases the errors in comparison with both BIR and BBSR for these critical options.

5. A new efficient method to price critical options

A careful study of the characteristics of the curves of the BI and BBS adjusted binomial values (as a function of the length of the tree), suggested us a more sophisticated segmentation strategy to price critical options.

The sophistication lies in that we leave the standard intervals used up to this point, and look also for strategies different from the extrapolation BIR and BBSR. More precisely, we suggest to group options according to the following table:

Table 12

IN		OUT	
CLASS	**C.R.**	**CLASS**	**C.R.**
1	1,00 - 1,17	1	1,00 - 1,40
2	1,17 - 1,25	2	1,40 - 1,45
3	1,25 - 1,30	3	1,45 - 1,53
4	1,30 - 1,40	4	> 1,53
5	> 1,40	///	///

We found that for each one of these classes a different pricing method may be efficiently applied at least for trees of short or medium length (that is up to 400 steps). Precisely in addition to the standard BIR($2n$) and BBSR($2n$), we will use BI($2n - 1$), BBS($2n - 1$) and BIM($n - 1$).

To understand the meaning of these symbols, the idea is that to make comparisons more easy, the methods should spend, more or less, the same computational time of the reference one, which is the time needed to compute BIR or BBSR for a given extrapolation (say e.g. 100 – 200). After that, if BIR(*2n*) is the reference, BBS(*2n* – 1) (recall that it is more or less equal to BBS(*2n*)) and BI(*2n* – 1) are simply the adjusted odd binomial evaluations introduced in second paragraph; the time needed to compute BBS(*2n* – 1) or BI(*2n* – 1) is thus 4/5 of the reference time. BIM(*n* – 1), in turn, denotes the local maximum of the odd BI evaluations near (*n* – 1). Indeed, we checked empirically that the average number of steps needed to localize and compute BIM(*n* – 1) is about 5, which means a computational time close to the reference.

To understand the reasons to apply different pricing method to different groups of options, let's begin with IN options.

For weak American options, say with C.R. < 1.17, the adjusted BI curves display, with same minor waves, the same monotonic regular behaviour of the at the money European options, allowing thus to reach high speed, high precision pricing through Richardson extrapolation. But when the American quality of the option becomes relevant, with higher values of the C.R., the waves become increasingly more predominant, so that the extrapolation does no more work with the same precision speed efficiency. Apparently this means that binomial based methods should be satisfied either with high precision but with relatively low speed, or with high speed but at the expense of precision. Luckily this is not true: especially for strong American options (that is with C.R. > 1.3), that is those options where Richardson extrapolation gives relatively big errors, we found that a proper use of the odd evaluations adds a lot of precision to binomial based pricing without increasing the computational time. More precisely, we keep account of the fact that within the interval 1.3 – 1.4 the odd BI curves are high frequency and amplitude waves, oscillating around the true value so that a simple BI(*2n* – 1) evaluation gives results much more precise, than (the more time expensive) BIR or BBSR(*2n*). For very strong IN American options (that is with C.R. > 1.4), that is those responsible of the large part of the errors of our samples, the odd BI curves are high frequency and amplitude waves whose relative maxima are, from the very early quite stable and very close to the true American value. We verified empirically that the average number of computations needed to localize and compute BIM(n – 1) is more or less the one needed to compute BIR or BBSR($2n$), but especially for short trees, the evaluations of BIM(n – 1) are dramatically more precise than those obtained through BIR or BBSR. Keeping account of the fact that the critical options, as repeatedly said previously, are responsible of the most relevant portion of the errors, the new segmentation method reveals then much more efficient than BIR or BBSR. As for OUT options, the positive results obtained from segmentation are quite smaller. Indeed, while BIR proves to be surely the better method for C.R. < 1.40, BIR or BIM turn out to be advantageous only for OUT with C.R. between 1.40 – 1.45 or respectively 1.45 – 1.53, while for the options with the highest C.R., still responsible of the largest errors (among the OUT), the odd big waves follow an increasing trend, so that nothing better than the extrapolation can be made. As for the superiority of BIR or BBSR it remains an open question to be tested with larger samples of critical OUT options than the one found with our parameters.

An evidence of the results comparing the efficiency of the new segmentation strategy with the classical ones, both at the level of each of the new non standard intervals and at the overall level, for the 2.200 and the 451 samples previously introduced is given in the following tables. Short (100 – 200) and medium (200 – 400) trees will be examined; at the longer tree we found that, as expected, the traditional extrapolation strategies tend to recover the best efficiency.

Table 13

	MRE					
IN 451	**< 1.17**	**1.17 - 1.25**	**1.25 - 1.30**	**1.30 - 1.40**	**> 1.40**	**TOTAL**
BIR (100 - 200)	2.059	10.947	16.870	12.797	106.300	**31.047**
BBSR (100 - 200)	4.919	8.513	14.247	24.178	111.252	**33.727**
BIM (99)	///	///	///	///	57.895	///
BBS (199)	///	5.907	16.346	///	///	///
BI (199)	///	///	///	10.517	///	///
BIR + BIM	2.059	10.947	16.870	12.797	57.895	**19.670**
SEGMENTATION	2.059	5.907	14.247	10.517	57.895	**18.150**

	RMSRE					
IN 451	**< 1.17**	**1.17 - 1.25**	**1.25 – 1.30**	**1.30 - 1.40**	**> 1.40**	**TOTAL**
BIR (100 - 200)	2.593	12.671	20.383	15.539	184.554	**90.052**
BBSR (100 - 200)	5.607	11.964	17.931	30.968	192.560	**94.308**
BIM (99)	///	///	///	///	97.863	///
BBS (199)	///	7.437	18.402	///	///	///
BI (199)	///	///	///	13.324	///	///
BIR + BIM	2.593	12.671	20.383	15.539	97.863	**48.529**
SEGMENTATION	2.593	7.437	17.931	13.324	97.863	**48.140**

	MRE					
IN 2200	**< 1.17**	**1.17 - 1.25**	**1.25 - 1.30**	**1.30 - 1.40**	**> 1.40**	**TOTAL**
BIR (100 - 200)	1.334	10.334	24.864	14.975	87.962	**6.321**
BBSR (100 - 200)	3.290	11.818	20.283	27.044	63.451	**7.218**
BIM (99)	///	///	///	///	35.801	///
BBS (199)	///	6.655	12.406	///	///	///
BI (199)	///	///	///	10.486	///	///
BIR + BIM	1.334	10.334	24.864	14.975	35.801	**5.507**
SEGMENTATION	1.334	6.655	12.406	10.486	35.801	**5.218**

	RMSRE					
IN 2200	**< 1.17**	**1.17 - 1.25**	**1.25 - 1.30**	**1.30 - 1.40**	**> 1.40**	**TOTAL**
BIR (100 - 200)	3.071	12.743	30.250	21.173	138.851	**30.015**
BBSR (100 - 200)	4.423	16.874	25.029	33.895	92.914	**21.288**
BIM (99)	///	///	///	///	65.340	///
BBS (199)	///	8.660	14.056	///	///	///
BI (199)	///	///	///	14.082	///	///
BIR + BIM	3.071	12.743	30.250	21.173	65.340	**15.466**
SEGMENTATION	3.071	8.660	14.056	14.082	65.340	**14.790**

Table 14

IN 451	MRE					
	< 1.17	1.17 - 1.25	1.25 - 1.30	1.30 - 1.40	> 1.40	TOTAL
BIR (200 - 400)	852	2.555	6.059	10.106	44.309	**12.971**
BBSR (200 - 400)	1.808	3.141	5.543	8.549	43.399	**13.124**
BIM (199)	///	///	///	///	29.228	///
BBS (399)	///	2.900	7.033	///	///	///
BI (399)	///	///	///	5.780	///	///
BIR + BIM	852	2.555	6.059	10.106	29.228	**9.426**
SEGMENTATION	852	2.555	5.543	5.780	29.228	**8.893**

IN 451	RMSRE					
	< 1.17	1.17 - 1.25	1.25 - 1.30	1.30 - 1.40	> 1.40	TOTAL
BIR (200 - 400)	1.138	3.791	7.026	12.067	72.782	**35.632**
BBSR (200 - 400)	2.059	3.867	6.681	10.931	87.026	**42.454**
BIM (199)	///	///	///	///	52.241	///
BBS (399)	///	3.752	7.780	///	///	///
BI (399)	///	///	///	6.877	///	///
BIR + BIM	1.138	3.791	7.026	12.067	52.241	**25.807**
SEGMENTATION	1.138	3.752	6.681	6.877	52.241	**25.586**

IN 2200	MRE					
	< 1.17	1.17 - 1.25	1.25 - 1.30	1.30 - 1.40	> 1.40	TOTAL
BIR (200 - 400)	600	4.661	5.994	10.246	34.498	**2.616**
BBSR (200 - 400)	1.236	4.060	5.933	9.605	34.502	**3.128**
BIM (199)	///	///	///	///	17.029	///
BBS (399)	///	3.375	5.494	///	///	///
BI (399)	///	///	///	4.907	///	///
BIR + BIM	600	4.661	5.994	10.246	17.029	**1.857**
SEGMENTATION	600	3.375	5.494	4.907	17.029	**1.645**

IN 2200	RMSRE < 1.17	1.17 - 1.25	1.25 - 1.30	1.30 - 1.40	> 1.40	TOTAL
BIR (200 - 400)	1.257	7.795	7.734	12.642	52.262	**11.430**
BBSR (200 - 400)	1.786	5.572	8.247	13.905	53.012	**11.631**
BIM (199)	///	///	///	///	31.370	///
BBS (399)	///	4.087	6.343	///	///	///
BI (399)	///	///	///	5.948	///	///
BIR + BIM	1.257	7.795	7.734	12.642	31.370	**7.309**
SEGMENTATION	1.257	4.087	6.343	5.948	31.370	**6.894**

Table 15

OUT 500	MRE < 1.40	1.40 - 1.45	1.45 - 1.53	> 1.53	TOTAL
BIR (100 - 200)	4.083	51.438	52.963	71.594	**9.271**
BBSR (100 - 200)	///	28.446	39.685	95.760	**13.764**
BIM (99)	///	///	13.169	///	///
BI (199)	///	8.237	///	///	///
BIR + BBSR	4.083	28.446	39.685	95.760	**9.053**
BIR+BI+BIM+BBSR	4.083	8.237	13.169	95.760	**7.839**
BIR+BI+BIM+BIR	4.083	8.237	13.169	71.594	**7.114**

OUT 500	RMSRE < 1.40	1.40 - 1.45	1.45 - 1.53	> 1.53	TOTAL
BIR (100 - 200)	7.500	53.553	58.868	95.752	**22.167**
BBSR (100 - 200)	///	33.834	49.447	123.976	**26.225**
BIM (99)	///	///	19.862	///	///
BI (199)	///	9.981	///	///	///
BIR + BBSR	7.500	33.834	49.447	123.976	**24.619**
BIR+BI+BIM+BBSR	7.500	9.981	19.862	123.976	**22.926**
BIR+BI+BIM+BIR	7.500	9.981	19.862	95.752	**18.427**

	MRE				
OUT 1996	< 1.40	1.40 - 1.45	1.45 - 1.53	> 1.53	TOTAL
BIR (100 - 200)	3.362	33.977	37.080	120.684	**5.122**
BBSR (100 - 200)	///	30.465	33.863	140.974	**11.034**
BIM (99)	///	///	///	///	///
BI (199)	///	14.142	///	///	///
BIR + BBSR	3.362	30.465	33.863	140.974	**5.345**
BIR+BI+BIM+BBSR	3.362	14.142	42.655	140.974	**5.311**
BIR+BI+BIM+BIR	3.362	14.142	42.655	120.684	**5.057**

	RMSRE				
OUT 1996	< 1.40	1.40 - 1.45	1.45 - 1.53	> 1.53	TOTAL
BIR (100 - 200)	5.614	36.752	43.812	195.892	**22.939**
BBSR (100 - 200)	///	35.041	43.988	215.564	**26.899**
BIM (99)	///	///	///	///	///
BI (199)	///	23.368	///	///	///
BIR + BBSR	5.614	35.041	43.988	215.564	**25.042**
BIR+BI+BIM+BBSR	5.614	23.368	61.568	215.564	**25.148**
BIR+BI+BIM+BIR	5.614	23.368	61.568	195.892	**23.044**

Table 16

	MRE				
OUT 500	< 1.40	1.40 - 1.45	1.45 - 1.53	> 1.53	TOTAL
BIR (200 - 400)	1.830	10.092	11.815	24.686	**2.990**
BBSR (200 - 400)	///	10.314	14.051	25.330	**4.778**
BIM (199)	///	///	7.364	///	///
BI (399)	///	3.003	///	///	///
BIR + BBSR	1.830	10.314	14.051	25.330	**3.073**
BIR+BI+BIM+BBSR	1.830	3.003	7.364	25.330	**2.709**
BIR+BI+BIM+BIR	1.830	3.003	7.364	24.686	**2.690**

	RMSRE				
OUT 500	< 1.40	1.40 - 1.45	1.45 - 1.53	> 1.53	TOTAL
BIR (200 - 400)	2.650	12.884	14.494	37.785	**7.685**
BBSR (200 - 400)	///	11.645	16.883	33.456	**7.937**
BIM (199)	///	///	10.913	///	///
BI (399)	///	3.616	///	///	///
BIR + BBSR	2.650	11.645	16.883	33.456	**7.139**
BIR+BI+BIM+BBSR	2.650	3.616	10.913	33.456	**6.593**
BIR+BI+BIM+BIR	2.650	3.616	10.913	37.785	**7.260**

	MRE				
OUT 1996	< 1.40	1.40 - 1.45	1.45 - 1.53	> 1.53	TOTAL
BIR (200 - 400)	1.475	9.039	17.385	52.146	**2.216**
BBSR (200 - 400)	///	8.750	11.908	44.014	**4.269**
BIM (199)	///	///	///	///	///
BI (399)	///	5.645	///	///	///
BIR + BBSR	1.475	8.750	11.908	44.014	**2.088**
BIR+BI+BIM+BBSR	1.475	5.645	20.127	44.014	**2.116**
BIR+BI+BIM+BIR	1.475	5.645	20.127	52.146	**2.217**

	RMSRE				
OUT 1996	< 1.40	1.40 - 1.45	1.45 - 1.53	> 1.53	TOTAL
BIR (200 - 400)	2.246	10.611	20.938	70.540	**8.351**
BBSR (200 - 400)	///	10.621	13.898	59.691	**8.230**
BIM (199)	///	///	///	///	///
BI (399)	///	11.001	///	///	///
BIR + BBSR	2.246	10.621	13.898	59.691	**7.137**
BIR+BI+BIM+BBSR	2.246	11.001	31.470	59.691	**7.387**
BIR+BI+BIM+BIR	2.246	11.001	31.470	70.540	**8.501**

Comments: the values of the MRE or RMSRE give immediately an idea of the best strategy for any group of options. The segmentation strategy is the union of the best choices for each group.

By BIR + BIM we denote the strategy consisting in applying BIR for IN options with C.R. < 1.40 and BIM for the other IN ones. It is clear that this simple segmentation is able to provide evaluations much more efficient than the traditional non segmented ones and only a bit less efficient than a more sophisticated segmentation including BBS and BI for other intervals of C.R.

For OUT options the gain from the segmentation is much smaller, even if the local efficiency of BI and BIM is promising; but BIR and BBSR are dominant where really there are the big errors (C.R. > 1.53).

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