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# Simulation of the spread of a viscous fluid using a bidimensional shallow water model

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## Abstract

In this paper we propose a numerical method to solve the Cauchy problem based on the viscous shallow water equations in an horizontally moving domain. More precisely, we are interested in a flooding and drying model, used to modelize the overflow of a river or the intrusion of a tsunami on ground. We use a non conservative form of the two-dimensional shallow water equations, in eight velocity formulation and we build a numerical approximation, based on the Arbitrary Lagrangian Eule-rian formulation, in order to compute the solution in the moving domain.

*Key words:* Shallow water equations, Free boundary problem, ALE discretization, Flooding and drying

*AMS Subject Classification :* 35Q30, 65M60, 76D03

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## 1 Introduction

The flooding and drying of a fluid on ground is a problem which has several applications in fluid mechanics such as coastal engineering, artificial lake filling, river overflow or sea intrusion on ground due to a tsunami. These works involves the modelling of the physical process near the triple contact line between the liquid (the water), the gas (the atmosphere) and the solid (the ground). The numerical simulation of this problem is very complex, due to the domain deformation and the triple contact line movement.

Some numerical approaches to solve this problem have been previously explored. For example, O. Bokhove [3] uses a conservative form of the shallow

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water equations associated to the discontinuous galerkin discretization. D. Yuan et al. [16] propose to use a non conservative form of the shallow water equations in total flow rate formulation and uses a finite difference scheme for the numerical computation.

In this paper, we use a two dimensional viscous shallow water model in eight velocity formulation. This model, called Stokes-like by P.L. Lions ([6], section 8.3) is, in some sense, intermediate between the semi-stationary model and the full model of compressible isentropic Navier-Stokes model. This model can be obtained by integrating vertically the so-called primitives equations of the ocean [5] with some hypothesis on the viscous terms. We have chosen this model because we can prove the existence of a solution under certain considerations as the smoothness of the initial conditions and an acceptable hypothesis on a boundary operator [8].

After the presentation of the model we describe the numerical method based on the demonstration of the convergence proposed in [8]. Finally, we present some numerical examples in two idealized domains and we give some perspectives for other studies.

### 1.1 Mathematical model

We assume that we know a continuous function  $H$  from  $\mathbb{R}^2$  to  $\mathbb{R}$  that represents the topography (the level of the ground with respect to a reference level).

We note  $\Omega_t$  (an open simply connected set of  $\mathbb{R}^2$ ) the horizontal domain occupied by the fluid at time  $t$  and  $\eta(t, x, y)$  the elevation of the fluid compared to a horizontal zero level. We set  $h(t, x, y)$  the height of the water column,  $h(t, x, y) = H(x, y) + \eta(t, x, y)$ .

The shallow water equations are based on a depth integration of an incompressible fluid conservation laws in a free surface-three dimensional domain. Governing equations for  $u$  and  $h$  can be obtained in the usual way ([1] for example) and the two-dimensional system can be written as follows [8,7,9]:

$$\frac{\partial u}{\partial t} + (u \cdot \nabla)u + g\nabla\eta - \nu\Delta u + C_d|u|u = f_1, \quad \text{in } \bigcup_{t \in [0, T]} \{t\} \times \Omega_t \quad (1)$$

$$\frac{\partial \eta}{\partial t} + \text{div}(uh) = 0, \quad \text{in } \bigcup_{t \in [0, T]} \{t\} \times \Omega_t \quad (2)$$

where  $g$  is the gravity constant,  $\nu$  the viscosity coefficient,  $C_d$  a drag coefficient.  $f_1$  represents external forcing (for example the wind stress). These equations are completed by initial conditions ( $u(t = 0) = u_0$ ,  $\eta(t = 0) = \eta_0$ ) and boundary conditions. In the following, we assume that the boundary of the

real domain is non-vertical. Hence the variation of the surface elevation at the boundary imposes a modification of the horizontal domain occupied by the fluid.

## 1.2 Boundary conditions

### 1.2.1 In a three dimensional domain

In many geophysical applications, the governing equations are solved in a domain  $[0, T] \times \Omega_t^3$  whereby  $\Omega_t^3 = \{(x, y, z) \in \mathbb{R}^3 / (x, y) \in \Omega_t, -H(x, y) < z < \eta(t, x, y)\}$  is a three dimensional moving domain. We denote by  $N = (n_1, n_2, n_3)$  the exterior unit normal to this domain, and we set  $N_h = (n_1, n_2)$  as its horizontal component. Moreover we denote  $V = (v_1, v_2, v_3)$  as the velocity of the fluid and  $v_h = (v_1, v_2)$  as its horizontal component.

In the following, we assume that the ground of the sea does not have a vertical wall. Then we have  $n_3 \neq 0$ . The impermeability boundary condition on the ground for the three dimensional problem imposes  $V(x, y, -H(x, y)) \cdot N = 0$ . Then, we have  $v_h \cdot N_h = -v_3 n_3$  on the ground and, since  $n_3 \neq 0$ , we obtain  $v_3 = -v_h \cdot \nabla H$ .

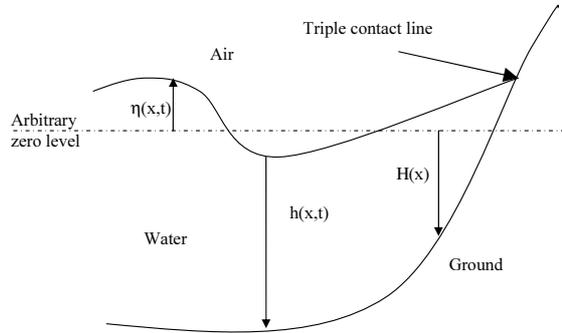


Fig. 1. Vertical section of the three dimensional domain

At the surface, vertical velocity is equal to the lagrangian derivative of the function  $\eta$ ,  $v_3 = \frac{\partial \eta}{\partial t} + v_h \cdot \nabla \eta$ .

We recall that we are interested in the effect of the overflow and we need to consider the problem in a moving domain that represents the domain actually occupied by the fluid. Then we need to impose a condition on the moving boundary. At the triple contact line (between air, ground and water, see figure 1), the horizontal velocity at the surface ( $v_h(x, y, \eta, t)$ ) is equal to the velocity at

the bottom ( $v_h(x, y, -H, t)$ ). Similarly, we have  $v_3(x, y, \eta, t) = v_3(x, y, -H, t)$ , coherent with the fact that the lagrangian derivative of  $h$  is equal to zero on this boundary.

This property is not verified if, during its displacement, a part of the triple contact line touches a vertical wall. In this case, the boundary cannot be represented by the line  $h = 0$ . This question is very interesting but not treated in this paper.

### 1.2.2 In a two dimensional domain

The mean velocity  $u$  used in the equations 1 and 2 is related to the previous three dimensional velocity by  $u = (u_1, u_2) = \frac{1}{h} \int_{-H}^{\eta} (v_1, v_2) dz$  where  $h \neq 0$ . The two dimensional domain  $\Omega_t$  is the projection of the three dimensional domain  $\Omega_t^3$  on the horizontal plane.

For the depth integrated model, we denote by  $\gamma_0$  the boundary of  $\Omega_0$  (at initial time). Assuming that  $\gamma_0$  is smooth enough, we define the deformed boundary as follows :  $\gamma_t = \{x' = x + d(x, t), x \in \gamma_0\}$ , where  $d$  represents the horizontal displacement  $d(x, t) = \chi(0, t, x) - x$  and where  $\chi(s, t, x)$  denotes the Lagrangian flow, i.e. the position at time  $t$  of the particle located at  $x$  at time  $s$ . With these notations, we have the natural condition  $\partial d(\chi(0, t, x), t) / \partial t = U(x, t)$  on  $\gamma_t$ , where  $U(x, t) = u(x + d(x, t), t)$ .

The continuity equation (2) can be written as follows:

$$h(\chi(x_0, 0, t), t) = h(x_0, 0) \exp \left( - \int_0^t \operatorname{div} u \right). \quad (3)$$

So, if at the initial time,  $h(x_0, 0) = 0$  then at all times  $h(\chi(x_0, 0, t), t) = 0$ . Moreover, if  $h(x_0, 0) \neq 0$  then  $h(\chi(x_0, 0, t), t) \neq 0$  at all times. We conclude that the boundary (and only the points of this boundary) at the initial time can define the boundary at all times. For example, if at initial time the domain is simply connected, the domain remains simply connected at all times  $t \geq 0$ .

We characterize the boundary motion  $\gamma_t$  by a condition on the normal component of the fluid stress tensor  $\sigma$  on the boundary

$$\sigma(x + d(x, t), t) \cdot n(x + d(x, t), t) |\det J|(x, t) = A \left( \frac{\partial U(x, t)}{\partial t} \right), \quad (4)$$

where  $A$  is a differential operator defined on  $\gamma_0$  taking into account the boundary forcing on the fluid (see [2] for example) and  $J$  is the jacobian matrix associated to the transformation  $x \mapsto \chi(x, 0, t)$ . In [8], the authors show that if the operator  $A$  is a Laplace-Beltrami operator ( $\int_{\gamma_0} A(v)v = \int_{\gamma_0} A^{1/2}(v)A^{1/2}(v) = \|v\|_{H^2(\gamma_0)}^2$ ) and if the data are small enough, the problem has a solution. The

condition on  $A$  is necessary to obtain the smoothness of the boundary and allows the use of the classical Sobolev injections [14,15]. More precisely, in this paper,  $D(A) = H^p(\gamma_0)$  with  $p = 2$  and currently, we are not able to prove an existence result for  $p < 2$ . But numerically, it is possible to obtain physical acceptable results with less smoothness on this operator (for example in [7] we have obtained results by taking  $A \equiv 0$ ).

It is not easy to characterize the operator  $A$ . It needs a good physical interpretation and give sufficient mathematical smoothness to prove the existence of a solution of our problem. Physically, this operator represents a friction condition in the triple contact line (between the fluid, the solid domain and the air). In the literature, a large number of physical approaches in the case of a wetting film of fluid or for a droplet [4] are found. Unfortunately, we cannot use directly these results. Firstly, we do not work at the microscopic scale (mainly governed by the Van Der Waals forces) and we need to take into account the effect of these phenomena at large scale. Secondly, the main effect studied for those phenomenon is in the vertical direction (for example for the capillary effect) but here, we use a depth integrated model.

## 2 The numerical method

### 2.1 The ALE equations

We use a numerical method based on the weak formulation of the problem in the domain  $\Omega_t$  depending on time. The ALE method that here we use, has been used to solve the Navier-Stokes equations in a moving domain [12].

We assume that at the initial time  $t_0$ , the fluid domain is covered by a regular mesh (for example, an usual finite element mesh). We also assume that on the boundary, each point of this mesh has the velocity of the fluid. The velocity of the internal points can be chosen arbitrarily, the only condition is to conserve the smoothness of the mesh at all times.

Then, at any time  $t \in [0, T]$ , the velocity  $c^t$  of the moving mesh can be defined by solving the following problem:

$$\Delta c^t = 0 \text{ in } \Omega_t \tag{5}$$

$$c^t = u(x, t) \text{ on } \gamma_t \tag{6}$$

and we consider the following mapping

$$\begin{aligned} c : \tilde{Q} &\rightarrow \mathbb{R}^2 \\ (x, t) &\mapsto c(x, t) = c^t(x), \end{aligned}$$

where  $\tilde{Q} = \cup_{t \in [0, T]} (\Omega_t \times \{t\})$ . The relation between the different domains  $\Omega_{t_i}$  is given by the mapping

$$\begin{aligned} \mathcal{C}(\cdot, t_1, t_2) : \Omega_{t_1} &\rightarrow \Omega_{t_2} \\ x_1 &\mapsto x_2 = \mathcal{C}(x_1, t_1, t_2) \end{aligned}$$

where  $\mathcal{C}(\cdot, t_1, t_2)$  is the characteristic curve from  $(x_1, t_1)$  to  $(x_2, t_2)$  corresponding to the velocity  $c^t$  in the space-time domain  $\tilde{Q}$ . For each time  $\tau$ , we denote by  $u_\tau$  and  $h_\tau$  the ALE velocity and the ALE thickness:

$$u_\tau(x, t) = u(\mathcal{C}(x, \tau, t), t), \quad h_\tau(x, t) = h(\mathcal{C}(x, \tau, t), t). \quad (7)$$

With these notations, we can write the ALE formulation of the problem

$$\frac{\partial u_\tau}{\partial t} + (u_\tau - c_\tau) \nabla u_\tau - \mu \Delta u_\tau + C_d u_\tau |u_\tau| + g \nabla h_\tau = O(t - \tau), \quad (8)$$

$$\frac{\partial h_\tau}{\partial t} + (u_\tau - c_\tau) \nabla h_\tau + h_\tau \operatorname{div} u_\tau = O(t - \tau). \quad (9)$$

In what follows, we will use a first order time discretization to solve this problem (see [10] for more details about second order schemes).

## 2.2 Time discretization

Let  $M \geq 1$ , we note  $\Delta t = \frac{T}{M}$  as the time step. For  $1 \leq n \leq M$ ,  $1 \leq m \leq M$ , we set  $\Omega^n = \Omega_{t^n}$  with boundary  $\gamma^n$  and

$$\begin{aligned} g_m^n(x) &= g_{t^m}(x, t^n) \text{ for } x \in \Omega^m, \\ g^n(x) &= g_n^n(x) = g(x, t^n) \text{ for } x \in \Omega^n. \end{aligned}$$

In order to ensure the positivity of  $h_{t^n}$ , the continuity equation is renormalised as follows

$$\frac{\partial}{\partial t} \log h_{t^n} + (u_{t^n} - c_{t^n}) \nabla \log h_{t^n} + \operatorname{div} u_{t^n} = 0. \quad (10)$$

With this renormalisation, we do not have conservation of the mass, but when we follow the evolution of the mass during a simulation (e.g. in the first test presented in the following section), only small variations of the mass quantity are observed.

We denote by  $U^n$  (respectively  $H^n$  and  $C^n$ ) the approximation of the exact solution  $u_{t^n}$  (respectively  $h_{t^n}$  and  $c_{t^n}$ ).

Then, we note  $\tilde{U}^n(x) = U^n(X^n(x, t^n))$  and  $\tilde{H}^n(x) = H^n(X^n(x, t^n))$  with  $X^n(x, \cdot)$  the characteristic curve solution of:

$$\begin{cases} \frac{\partial X^n}{\partial t}(x, t) = (U^n - C^n)(X^n(x, t)) \\ X^n(x, t^{n+1}) = x. \end{cases}$$

In our study, we approximate the foot of the characteristic by  $X^n(x, t^n) \simeq x - (U^n - C^n)(x)\Delta t$ . We set  $U_n^{n+1}$  (resp.  $H_n^{n+1}$ ) the approximation of  $u_{t^n}(x, t^{n+1})$  (resp.  $h_{t^n}(x, t^{n+1})$ ). Approximating the Lagrangian derivative by a first order Euler scheme, and using a linearised drag operator, we obtain the following:

$$U_n^{n+1} + \Delta t \left( g\nabla H_n^{n+1} + C_d U_n^{n+1} |\tilde{U}^n| - \nu \Delta U_n^{n+1} \right) = \tilde{U}^n, \quad \text{in } \Omega^n \quad (11)$$

$$\log H_n^{n+1} + \Delta t \operatorname{div} U_n^{n+1} = \log \tilde{H}^n \quad \text{in } \Omega^n, \quad (12)$$

$$\operatorname{div} U_n^{n+1} = A \left( \frac{\partial \tilde{U}^n}{\partial t} \right) \quad \text{on } \gamma^n, \quad (13)$$

$$\operatorname{curl} U_n^{n+1} = H_n^{n+1} = 0 \quad \text{on } \gamma^n. \quad (14)$$

Implicitly, taking into account the equation 3, we have  $H_n^{n+1} = 0$  on  $\gamma^n$ . This is an approximation at the first order in time of the initial shallow water problem.

### Remark 2.1

*This problem is solved by a fixed point technic. A first approximation of 12 is computed assuming  $U_n^{n+1} = \tilde{U}^n$  and the approximation of  $H_n^{n+1}$  is then used in 11 to obtain a first approximation of  $U_n^{n+1}$ . We repeat this operation in order to obtain convergence of the system. Prove of convergence can be found in [8].*

*We cannot easily increase the order of this scheme because the ALE formulation is only of the first order.*

The previous problem is solved on  $\Omega^n$  by using the spatial discretization proposed in the following section. After, we need to solve the problem given by equations 5 and 6 in order to compute the mesh velocity. Each point  $p_k$  with coordinates  $x_k$  of the mesh is then moved with the first order approximation  $x_k^{n+1} = x_k^n + \Delta t C^n(x_k^n, t^n)$ .

### 2.3 Spatial discretization

The spatial discretization of the previous problem is based on the Finite Element Galerkin method. In [8], an approach based on the Galerkin method with a special basis is proposed, but this approach cannot be applied easily if  $A \neq 0$ .

We note  $\{\phi_k\}$  the set of finite element functions of  $H^1(\Omega^n)$ . The weak formulation of our approximated problem is

$$\begin{aligned} & \int_{\Omega^n} U_n^{n+1}(x) \phi_k(x) + \nu \Delta t \int_{\Omega^n} \nabla U_n^{n+1}(x) \nabla \phi_k(x) \\ & - g \Delta t \int_{\Omega^n} H_n^{n+1}(x) \operatorname{div} \phi_k(x) + C_d \Delta t \int_{\Omega^n} U_n^{n+1}(x) |U_n^n(x)| \phi_k(x) \\ & + \int_{\gamma_t} \sigma(X + d(X, t), t) \cdot n(X + d(X, t), t) \gamma(\phi_k(x)) \\ & = \int_{\Omega^n} \tilde{U}^n(x) \phi_k(x) \end{aligned} \quad (15)$$

$$\log H_n^{n+1}(x) + \Delta t \operatorname{div} U_n^{n+1}(x) = \log \tilde{H}^n(x) \quad \text{in } \Omega^n \quad (16)$$

where the operator  $\gamma$  is the trace operator from  $\Omega^n$  to  $\gamma^n$ .

Then the equations (4) and (15) give

$$\begin{aligned} & \int_{\Omega^n} U_n^{n+1}(x) \phi_k(x) + \nu \Delta t \int_{\Omega^n} \nabla U_n^{n+1}(x) \nabla \phi_k(x) \\ & - g \Delta t \int_{\Omega^n} H_n^{n+1}(x) \operatorname{div} \phi_k(x) + C_d \int_{\Omega^n} U_n^{n+1}(x) |U_n^n(x)| \phi_k(x) \\ & + \int_{\gamma_0} A^{1/2} \left( \frac{\partial \tilde{U}}{\partial t} \right) A^{1/2} \left( \gamma(\tilde{\phi}_k(x_0)) \right) d\gamma_0 \\ & = \int_{\Omega^n} \tilde{U}^n(x) \phi_k(x) \end{aligned} \quad (17)$$

where  $\tilde{g}(x) = g(\chi(x, 0, t))$ .

We then use a first order discretization of the partial time derivative:

$$\begin{aligned} & \int_{\gamma_0} A^{1/2} \left( \frac{\partial \tilde{U}}{\partial t} \right) A^{1/2} \left( \gamma(\tilde{\phi}_k(x_0)) \right) d\gamma_0 \simeq \frac{1}{\Delta t} \int_{\gamma_0} A^{1/2} \left( \tilde{U}_n^{n+1}(x) \right) A^{1/2} \left( \gamma(\tilde{\phi}_k(x_0)) \right) d\gamma_0 \\ & - \frac{1}{\Delta t} \int_{\gamma_0} A^{1/2} \left( \tilde{U}_n^n(x) \right) A^{1/2} \left( \gamma(\tilde{\phi}_k(x_0)) \right) d\gamma_0. \end{aligned} \quad (18)$$

The first part of the right hand of this equation needs to be included on the left hand of the global problem and the second part on the right hand.

The global weak formulation is:

$$\begin{aligned}
& \int_{\Omega^n} U_n^{n+1}(x) \phi_k(x) + \nu \Delta t \int_{\Omega^n} \nabla U_n^{n+1}(x) \nabla \phi_k(x) - g \Delta t \int_{\Omega^n} H_n^{n+1}(x) \operatorname{div} \phi_k(x) \\
& \int_{\Omega^n} U_n^{n+1}(x) |U_n^n(x)| \phi_k(x) + \frac{1}{\Delta t} \int_{\gamma_0} A^{1/2}(\tilde{U}_n^{n+1}(x)) A^{1/2}(\gamma(\tilde{\phi}_k(x_0))) d\gamma_0 \\
& = \int_{\Omega^n} \tilde{U}^n(x) \phi_k(x) + \frac{1}{\Delta t} \int_{\gamma_0} A^{1/2}(\tilde{U}_n(x)) A^{1/2}(\gamma(\tilde{\phi}_k(x_0))) d\gamma_0, \quad (19) \\
& \log H_n^{n+1}(x) + \Delta t \operatorname{div} U_n^{n+1}(x) = \log \tilde{H}^n(x). \quad (20)
\end{aligned}$$

**Remark 2.2** On  $\gamma_t$ ,  $U(l, t) = \sum_k a_k \phi_k(l, t)$  where  $l$  is a point of  $\gamma_t$  and the sum is computed on all finite element functions. Then, on the initial boundary  $\gamma_0$ , we have the decomposition  $\tilde{U}(l_0, t) = \sum_k a_k \tilde{\phi}_k(l_0, t)$  with the same  $a_k$ .

**Remark 2.3** Formally, we can use two kinds of boundary conditions:

- (1) A condition of normal displacement of the boundary. We write  $u \cdot n = \frac{\partial d}{\partial t}$  where  $d$  is the normal displacement. With this condition, the well posed weak formulation associated to the diffusion operator is  $\int_{\Omega} \operatorname{div} u \operatorname{div} \phi + \int_{\Omega} \operatorname{curl} u \operatorname{curl} \phi - \int_{\gamma} \operatorname{div} u \phi \cdot n + \int_{\gamma} \operatorname{curl} u \phi \cdot \alpha(n)$  where  $\alpha(u_1, u_2) = (-u_2, u_1)$  and  $\operatorname{curl} u = \frac{\partial u_2}{\partial y} - \frac{\partial u_1}{\partial x}$ . The boundary condition is then imposed on  $h - \operatorname{div} u$  and  $\operatorname{curl} u$  since we do not want to impose a condition on  $u \cdot n$  or  $u \cdot \alpha(n)$ .
- (2) A condition on the displacement of the boundary  $u = \frac{\partial d}{\partial t}$  where  $d$  is a vector. With this condition, the well posed weak formulation associated to the diffusion operator is  $\int_{\Omega} \nabla u \nabla \phi - \int_{\gamma} \nabla u \phi \cdot n$  and the boundary condition is then imposed on  $\nabla u$  (more precisely, we need to take into account the trace tensor including condition on  $h$  in the boundary, but we assume  $h \equiv 0$  on this boundary).

**Remark 2.4** We can note that the friction term  $C_d u |u|$  is necessary to stabilize the flow. Indeed, if we use the following decomposition

$$(L^2(\Omega^n))^2 = \nabla H_0^1(\Omega^n) \oplus \operatorname{Curl} H_0^1(\Omega^n) \oplus \nabla \mathcal{H}(\Omega^n), \quad (21)$$

where  $\mathcal{H}$  is the intersection of  $H^1$  and the space of harmonic functions, we can see that the functions of  $\nabla \mathcal{H}(\Omega^n)$  are not controlled by the laplacian operator. We present in the following section a case with  $C_d = 0$  where we do not have damping of the flow.

## 2.4 First numerical test

The previous method is tested in order to simulate the behavior of a fluid in a simplified domain. We assume that the domain is axisymmetric using the function  $H(r) : r \mapsto ar^2 + 1$ . At the initial time  $t = 0$ , the fluid at rest touches

the wall ( $h(a, t = 0) = 0$ ) (see figure 2). Physically, our initial domain  $\Omega_0$  is a

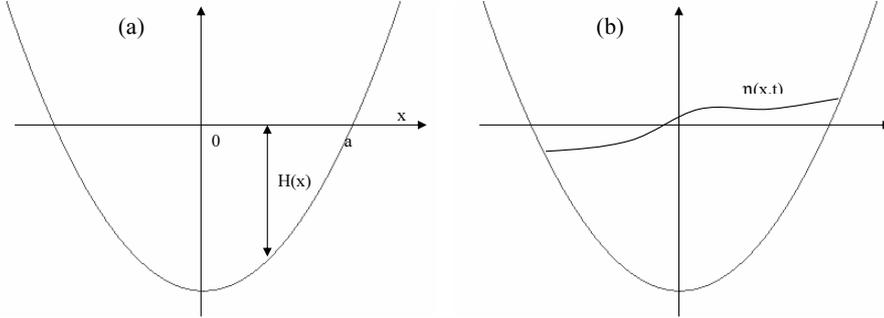


Fig. 2. Simplified domain - (a) at rest. (b) at an arbitrarily time.

disc with a radius of 130 meters. This domain is meshed with triangles (420 triangles for the mesh M1, 470 for the mesh M2 and 1002 for the mesh M3). The height of the column of water at the centre is 1 meter (since  $r = 0$  at the centre of our domain). Fluid viscosity is  $0.01m^2.s^{-1}$  and gravity coefficient  $g$  is assumed equal to  $1m.s^{-2}$  in order to amplify the elevation.

We apply a forcing at the surface of this fluid. This forcing is usually taken into account by applying the continuity of the horizontal stress tensor on the surface. So, for the three dimensional model,

$$\frac{\partial u_h}{\partial z} = CW_a|W_a|$$

where  $W_a$  is the wind velocity (at ten meters over the flow for the ocean for example) and  $C$  is a “drag coefficient” assumed to be constant. Using the vertical integration of the vertical operator  $\frac{\partial}{\partial z}\nu_z\frac{\partial u_h}{\partial z}$ , and assuming  $u_h(\eta) \simeq u$ , we obtain a forcing condition on the mean velocity  $f \propto \frac{W_a|W_a|}{H}$ .

We use here

$$\begin{cases} f = 1 & \text{if } 0 < t < 20 \text{ s} \\ f = 0 & \text{if } t > 20 \text{ s} \end{cases} \quad (22)$$

Hence, we observe an oscillation of the free surface of the water plan. At each oscillation, a part of the water moves on ground and a part of the ground is uncovered by this water (see figure 3). In the last part of the remark 2.4, we indicate that with  $C_d = 0$  a part of the flow component is not diffusive. More precisely, our solution is only a gradient. If we do not take into account boundary friction effects or bottom friction effects, a part of the fluid is not diffusive. To observe this effect, we plot (on figure 4) the level of a boundary point according to time. We can observe an accumulation of energy on

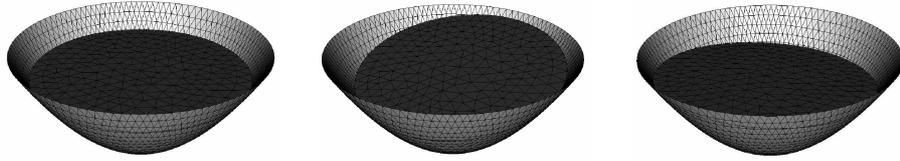


Fig. 3. Oscillation of the water

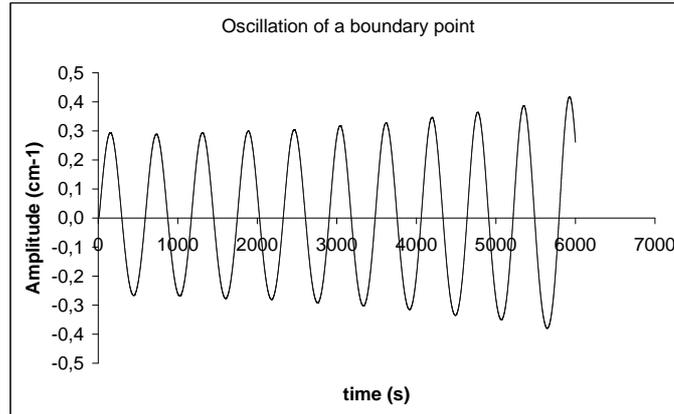


Fig. 4. Oscillations of the water without drag term

the fluid due to the numerical approximation, and this accumulation is not compensated by diffusive term.

In figure 5, we plot the variation of the same point taking into account the drag coefficient  $C_d|u|u$ . With this term, all the modes of the fluid are diffusive and, if we stop the forcing, global energy of the fluid vanishes.

In figure 6, we plot the time evolution of the kinetic energy of the flow for the different meshes (M1, M2 and M3). We only have a little difference for all these meshes but a very expensive cost for the mesh M3. After the forcing phase (20 seconds), we can observe the transformation of the kinetic energy in potential energy and conversely. When the kinetic energy vanishes, potential energy is maximal. If the drag coefficient is assumed to be equal to zero, the amplitude of the oscillation does not decrease.

Finally, figure 7 represents the evolution of the mass of fluid compared to the initial mass. Even if we do not have a rigorous conservation of the mass, due to the renormalisation of the mass equation, the variation of the mass is very little.

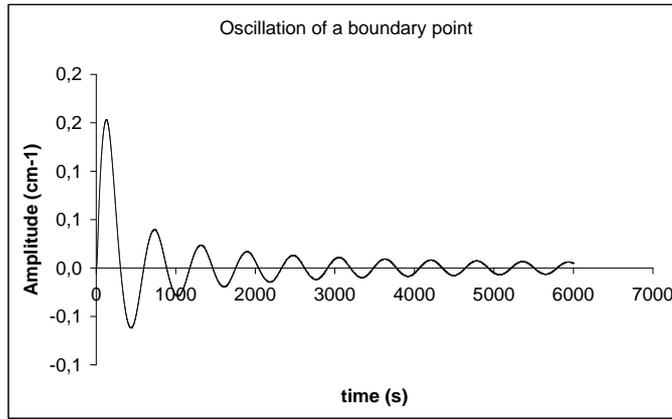


Fig. 5. Oscillations of the water with a drag term

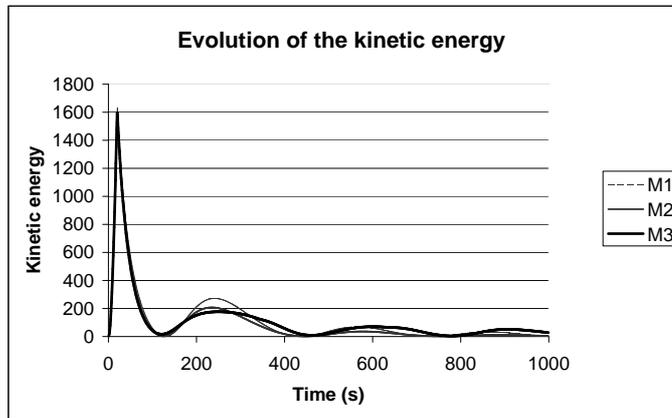


Fig. 6. Evolution of the solution for M1, M2 and M3 meshes

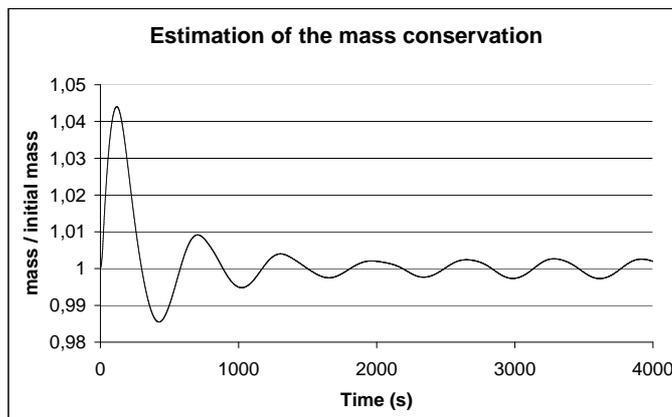


Fig. 7. Evolution of the mass in comparison with initial mass

## 2.5 Second numerical test

In this second numerical experiment, we use a more complex domain to test our numerical method. We assume that the domain is axisymmetric using the function  $H(r) : r \mapsto ar^3 + br^2 + cr + d$ .  $b, c$  and  $d$  are chosen in order to at the distance  $1/a$  of the centre of the fluid domain, the fluid touches the wall (see figure 8). We use the same external forcing as for the first experiment (22).

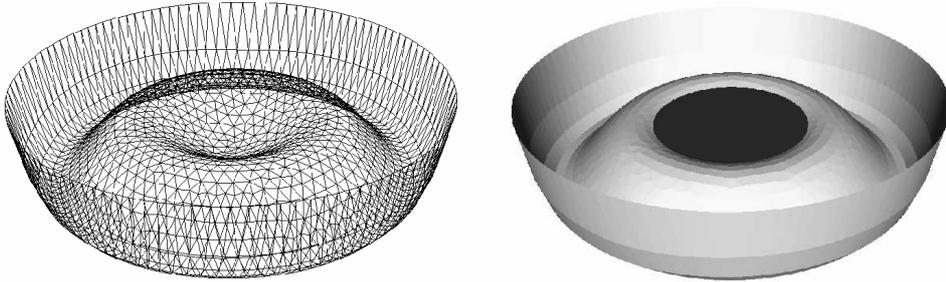


Fig. 8. Mesh of the ground and initial position of the water

But, due to the specific form of the topography, a part of the fluid goes to the external crown.

We present some results of the simulation in figure 9. Recalling that the shallow water equations are based in the continuum mechanic, it cannot cut this domain with a finite energy. Hence, even if the thickness of the layer of water is very low, we conserve a thin film of water between all the parts of the domain occupied by the fluid. We recall that the computation is only made



Fig. 9. Evolution of the fluid domain

in the moving two-dimensional domain  $\Omega_t$ . Mathematical study proves that

the main difficulty is to conserve the smoothness of the boundary. This result can be observed in this simulation because, at the final time, we have contact between two parts of the boundary (see last part of the figure 9). Since we do not use a “good” boundary operator, we do not have sufficient smoothness on the boundary. We observe then a change of connectedness and a hole appears in the fluid domain.

### 3 Conclusion

The presented work suggests that capillarity effect needs to be incorporated into the evolution equations, and more precisely in the triple contact line. Theoretical results presented in [8] give sufficient smoothness for the capillarity term used in equation 4.

The main difficulty is to describe the capillarity effects on the depth integrated model and in a dynamical system. A possible approach is to test some boundary operator and to compare numerical and experimental result. This approach will be proposed in future studies.

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