Convergence Rate of the Causal Jacobi Derivative Estimator

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Abstract. Numerical causal derivative estimators from noisy data are essential for real time applications especially for control applications or fluid simulation so as to address the new paradigms in solid modeling and video compression. By using an analytical point of view due to Lanczos [9] to this causal case, we revisit n^{th} order derivative estimators originally introduced within an algebraic framework by Mboup, Fliess and Join in [14, 15]. Thanks to a given noise level δ and a well-suitable integration length window, we show that the derivative estimator error can be $\mathcal{O}(\delta^{\frac{q+1}{n+1+q}})$ where q is the order of truncation of the Jacobi polynomial series expansion used. This so obtained bound helps us to choose the values of our parameter estimators. We show the efficiency of our method on some examples.

Keywords: Numerical differentiation, Ill-posed problems, Jacobi orthogonal series

1 Introduction

There exists a large class of numerical derivative estimators which were introduced according to different scopes ([8, 18, 1, 22, 20, 16, 17]). When the initial discrete data are corrupted by a noise, numerical differentiation becomes an ill-posed problem. By using an algebraic method inspired by [6, 13, 10], Mboup, Fliess and Join introduced in [14, 15] real-time numerical differentiation by integration estimators that provide an effective response to this problem. Concerning the robustness of this method, [4, 5] give more theoretical foundations. These estimators extend those introduced by [9, 19, 23] in the sense that they use Jacobi polynomials. In [14], the authors show that the mismodelling due to the truncation of the Jacobi expansion can be improved by allowing a small time-delay in the derivative estimation. This time-delay is obtained as the product of the length of the integration window by the smallest root of the first Jacobi polynomial in the remainder of series expansion.

In [11], we extend to the real domain the parameter values of these Jacobi estimators. This allows us to decrease the value of this smallest root and consequently the time-delay estimation. In [12], we study for center derivative Jacobi estimators the convergence rate of these estimators.

Thanks to these results, we propose in this article to tackle the causal convergence rate case. We give an optimal convergence rate of these estimators depending on the derivative order, the noise level of the data and the truncation order. Moreover, we show that the estimators for the n^{th} order derivative of a smooth function can be obtained by taking n derivations to the zero-order estimator of the function. Hence, we can give a simple expression for these estimators, which is much easier to calculate than the one given in [12].

This paper is organized as follows: in Section 2 the causal estimators introduced in [14] are studied with extended parameters. The convergence rate of these estimators are then studied. Finally, numerical tests are given in Section 3. They help us to show the efficiency and the stability of this proposed estimators. We will see that the numerical integration error may also reduce this time-delay for a special class of functions.

2 Derivative Estimations by Using Jacobi Orthogonal Series

Let $f^{\delta} = f + \varpi$ be a noisy function defined on an open interval $I \subset \mathbb{R}$, where $f \in \mathcal{C}^n(I)$ with $n \in \mathbb{N}$ and ϖ be a noise[5] which is bounded and integrable with a noise level δ, *i.e.* $\delta = \sup_{x \in I} |\varpi(x)|$. Contrary to [19] where Legendre polynomials were used, we propose to use, as in [14, 15], truncated Jacobi orthogonal series so as to estimate the n^{th} order derivative of f. In this section, we are going to give a family of causal Jacobi estimators by using Jacobi polynomials defined on $[0, 1]$. From now on, we assume that the parameter $h > 0$ and we denote $I_h := \{x \in I; [x - h, x] \subset I\}$.

The n^{th} order Jacobi polynomials (see [21]) defined on $[0, 1]$ are defined as follows

$$P_n^{(\alpha,\beta)}(t) = \sum_{j=0}^{n} \binom{n+\alpha}{j} \binom{n+\beta}{n-j} (t-1)^{n-j} (t)^j \tag{1}$$

where $\alpha, \beta \in]-1, +\infty[$. Let g_1 and g_2 be two functions which belong to $\mathcal{C}([0, 1])$, then we define the scalar product of these functions by

$$\langle g_1(\cdot), g_2(\cdot) \rangle_{\alpha,\beta} := \int_0^1 w_{\alpha,\beta}(t) g_1(t) g_2(t) dt,$$

[5] More generally, the noise is a stochastic process, which is bounded with certain probability and integrable in the sense of convergence in mean square (see [11]).

where $w_{\alpha,\beta}(t) = (1-t)^{\alpha} t^{\beta}$ is a weighted function. Hence, we can denote its associated norm by $\|\cdot\|_{\alpha,\beta}$. We then have

$$\|P_n^{(\alpha,\beta)}\|_{\alpha,\beta}^2 = \frac{1}{2n+\alpha+\beta+1} \frac{\Gamma(\alpha+n+1)\Gamma(\beta+n+1)}{\Gamma(\alpha+\beta+n+1)\Gamma(n+1)}. \tag{2}$$

Let us recall two useful formulae (see [21])

$$P_n^{(\alpha,\beta)}(t) w_{\alpha,\beta}(t) = \frac{(-1)^n}{n!} \frac{d^n}{dt^n}[w_{\alpha+n,\beta+n}(t)] \quad \text{(the Rodrigues formula)}, \tag{3}$$

$$\frac{d}{dt}[P_n^{(\alpha,\beta)}(t)] = (n+\alpha+\beta+1) P_{n-1}^{(\alpha+1,\beta+1)}(t). \tag{4}$$

Let us ignore the noise ϖ for a moment. Since f is assumed to belong to $\mathcal{C}^n(I)$, we define the q^{th} ($q \in \mathbb{N}$) order truncated Jacobi orthogonal series of $f^{(n)}(x-ht)$ ($t \in [0,1]$) by the following operator: $\forall x \in I_h$,

$$D_{h,\alpha,\beta,q}^{(n)} f(x-th) := \sum_{i=0}^{q} \frac{\left\langle P_i^{(\alpha+n,\beta+n)}(\cdot), f^{(n)}(x-h\cdot) \right\rangle_{\alpha+n,\beta+n}}{\|P_i^{(\alpha+n,\beta+n)}\|_{\alpha+n,\beta+n}^2} \; P_i^{(\alpha+n,\beta+n)}(t). \tag{5}$$

We also define the $(q+n)^{th}$ order truncated Jacobi orthogonal series of $f(x-ht)$ ($t \in [0,1]$) by the following operator

$$\forall x \in I_h,\ D_{h,\alpha,\beta,q}^{(0)} f(x-th) := \sum_{i=0}^{q+n} \frac{\left\langle P_i^{(\alpha,\beta)}(\cdot), f(x-h\cdot) \right\rangle_{\alpha,\beta}}{\|P_i^{(\alpha,\beta)}\|_{\alpha,\beta}^2} \; P_i^{(\alpha,\beta)}(t). \tag{6}$$

It is easy to show that for each fixed value x, $D_{h,\alpha,\beta,q}^{(0)} f(x-h\cdot)$ is a polynomial which approximates the function $f(x-h\cdot)$. We can see in the following lemma that $D_{h,\alpha,\beta,q}^{(n)} f(x-h\cdot)$ is in fact connected to the n^{th} order derivative of $D_{h,\alpha,\beta,q}^{(0)} f(x-h\cdot)$. It can be expressed as an integral of f.

Lemma 1. *Let $f \in \mathcal{C}^n(I)$, then we have*

$$\forall x \in I_h,\ D_{h,\alpha,\beta,q}^{(n)} f(x-th) = \frac{1}{(-h)^n} \frac{d^n}{dt^n} \left[D_{h,\alpha,\beta,q}^{(0)} f(x-th) \right]. \tag{7}$$

Moreover, we have

$$\forall x \in I_h,\ D_{h,\alpha,\beta,q}^{(n)} f(x-th) = \frac{1}{(-h)^n} \int_0^1 Q_{\alpha,\beta,n,q,t}(\tau) f(x-h\tau) d\tau, \tag{8}$$

where $Q_{\alpha,\beta,n,q,t}(\tau) = w_{\alpha,\beta}(\tau) \sum_{i=0}^{q} C_{\alpha,\beta,n,i} P_i^{(\alpha+n,\beta+n)}(t) P_{n+i}^{(\alpha,\beta)}(\tau)$, *and* $C_{\alpha,\beta,n,i} = \frac{(\beta+\kappa+2n+2i+1)\Gamma(\beta+\alpha+2n+i+1)\Gamma(n+i+1)}{\Gamma(\beta+n+i+1)\Gamma(\alpha+n+i+1)}$ *with* $\alpha, \beta \in]-1,+\infty[$.

Proof. By applying n times derivations to (6) and by using (4), we obtain

$$\begin{aligned}
&\frac{d^n}{dt^n}\left[D^{(0)}_{h,\alpha,\beta,q} f(x-th)\right] \\
=&\sum_{i=0}^{q} \frac{\left\langle P^{(\alpha,\beta)}_{i+n}(\cdot), f(x-h\cdot)\right\rangle_{\alpha,\beta}}{\|P^{(\alpha,\beta)}_{i+n}\|^2_{\alpha,\beta}} \frac{d^n}{dt^n}\left[P^{(\alpha,\beta)}_{i+n}(t)\right] \\
=&\sum_{i=0}^{q} \frac{\left\langle P^{(\alpha,\beta)}_{i+n}(\cdot), f(x-h\cdot)\right\rangle_{\alpha,\beta}}{\|P^{(\alpha,\beta)}_{i+n}\|^2_{\alpha,\beta}} \frac{\Gamma(\alpha+\beta+2n+i+1)}{\Gamma(\alpha+\beta+n+i+1)} P^{(\alpha+n,\beta+n)}_{i}(t).
\end{aligned} \tag{9}$$

By applying two times the Rodrigues formula given in (3) and by taking n integrations by parts, we get

$$\begin{aligned}
&\left\langle P^{(\alpha+n,\beta+n)}_{i}(\cdot), f^{(n)}(x-h\cdot)\right\rangle_{\alpha+n,\beta+n} \\
=&\int_0^1 w_{\alpha+n,\beta+n}(\tau) P^{(\alpha+n,\beta+n)}_{i}(\tau)\, f^{(n)}(x-h\tau)d\tau \\
=&\int_0^1 \frac{(-1)^i}{i!} w^{(i)}_{\alpha+n+i,\beta+n+i}(\tau)\, f^{(n)}(x-h\tau)d\tau \\
=&\frac{1}{(-h)^n}\int_0^1 \frac{(-1)^{i+n}}{i!} w^{(n+i)}_{\alpha+n+i,\beta+n+i}(\tau)\, f(x-h\tau)d\tau \\
=&\frac{1}{(-h)^n}\int_0^1 \frac{(n+i)!}{i!} w_{\alpha,\beta}(\tau) P^{(\alpha,\beta)}_{n+i}(\tau)\, f(x-h\tau)d\tau.
\end{aligned}$$

Then, after some calculations by using (2) we can obtain

$$\begin{aligned}
&\frac{\left\langle P^{(\alpha+n,\beta+n)}_{i}(\cdot), f^{(n)}(x-h\cdot)\right\rangle_{\alpha+n,\beta+n}}{\|P^{(\alpha+n,\beta+n)}_{i}\|^2_{\alpha+n,\beta+n}} \\
=&\frac{\left\langle P^{(\alpha,\beta)}_{i+n}(\cdot), f(x-h\cdot)\right\rangle_{\alpha,\beta}}{(-h)^n\|P^{(\alpha,\beta)}_{n+i}\|^2_{\alpha,\beta}} \frac{\Gamma(\alpha+\beta+2n+i+1)}{\Gamma(\alpha+\beta+n+i+1)}.
\end{aligned} \tag{10}$$

Finally, by taking (5) and (9) we obtain

$$\forall x \in I_h,\ D^{(n)}_{h,\alpha,\beta,q} f(x-th) = \frac{1}{(-h)^n}\frac{d^n}{dt^n}\left[D^{(0)}_{h,\alpha,\beta,q} f(x-th)\right]. \tag{11}$$

The proof is complete. □

If we consider the noisy function f^{δ}, then it is sufficient to replace $f(x-h\cdot)$ in (8) by $f^{\delta}(x-h\cdot)$. In [14], for a given value $t_\tau \in [0,1]$, $D^{(n)}_{h,\alpha,\beta,q} f^{\delta}(x-t_\tau h)$ (with $\alpha,\beta \in \mathbb{N}$ and $q \leq \alpha+n$) was proposed as a point-wise estimate of $f^{(n)}(x)$ by admitting a time-delay $t_\tau h$. We assume here that these values α and β belong to $]-1,+\infty[$. This is possible due to the definition of the Jacobi polynomials. Contrary to [14], we do not have constraints on the value of the truncation order q. Moreover, the function $Q_{\alpha,\beta,n,q,t}$ is easier to calculate than the one given in [12]. Thus, we can define the extended point-wise estimators as follows.

Definition 1. *Let $f^{\delta} = f + \varpi$ be a noisy function, where $f \in C^n(I)$ and ϖ be a bounded and integrable noise with a noise level δ. Then a family of causal Jacobi estimators of $f^{(n)}$ is defined as*

$$\forall x \in I_h,\ D^{(n)}_{h,\alpha,\beta,q} f^{\delta}(x - t_\tau h) = \frac{1}{(-h)^n} \int_0^1 Q_{\alpha,\beta,n,q,t_\tau}(u) f^{\delta}(x - hu) du, \quad (12)$$

where $\alpha, \beta \in]-1,+\infty[$, $q, n \in \mathbb{N}$ and t_τ is a fixed value on $[0,1]$.

Hence, the estimation error comes from two sources : the remainder terms in the Jacobi series expansion of $f^{(n)}(x - h\cdot)$ and the noise part. In the following proposition, we study these estimation errors.

Proposition 1. *Let f^{δ} be a noisy function where $f \in \mathcal{C}^{n+1+q}(I)$ and ϖ be a bounded and integrable noise with a noise level δ. Assume that there exists $M_{n+1+q} > 0$ such that for any $x \in I$, $\left|f^{(n+q+1)}(x)\right| \leq M_{n+1+q}$, then*

$$\left\| D^{(n)}_{h,\alpha,\beta,q} f^{\delta}(x - t_\tau h) - f^{(n)}(x - t_\tau h) \right\|_{\infty} \leq C_{q,t_\tau} h^{q+1} + E_{q,t_\tau} \frac{\delta}{h^n}, \quad (13)$$

where $C_{q,t_\tau} = M_{n+1+q}\left(\frac{1}{(n+1+q)!}\int_0^1 \left|u^{n+1+q} Q_{\alpha,\beta,n,q,t_\tau}(u)\right| du + \frac{t_\tau^{q+1}}{(q+1)!}\right)$ and $E_{q,t_\tau} = \int_0^1 \left|Q_{\alpha,\beta,n,q,t_\tau}(u)\right| du$. Moreover, if we choose $h = \left[\frac{nE_{q,t_\tau}}{(q+1)C_{q,t_\tau}}\delta\right]^{\frac{1}{n+q+1}}$, then we have

$$\left\| D^{(n)}_{h,\alpha,\beta,q} f^{\delta}(x - t_\tau h) - f^{(n)}(x - t_\tau h) \right\|_{\infty} = \mathcal{O}(\delta^{\frac{q+1}{n+1+q}}). \quad (14)$$

Proof. By taking the Taylor series expansion of f at x, we then have for any $x \in I_h$ that there exists $\xi \in]x-h, x[$ such that

$$f(x - t_\tau h) = f_{n+q}(x - t_\tau h) + \frac{(-h)^{n+1+q} t_\tau^{n+1+q}}{(n+1+q)!} f^{(n+1+q)}(\xi), \quad (15)$$

where $f_{n+q}(x - t_\tau h) = \sum_{j=0}^{n+q} \frac{(-h)^j t_\tau^j}{j!} f^{(j)}(x)$ is the $(n+q)^{th}$ order truncated Taylor series expansion of $f(x - t_\tau h)$. By using (8) with $f_{n+q}(x - h\cdot)$ we obtain

$$f^{(n)}_{n+q}(x - t_\tau h) = \frac{1}{(-h)^n} \int_0^1 Q_{\alpha,\beta,n,q,t_\tau}(\tau) f_{n+q}(x - h\tau) d\tau. \quad (16)$$

Thus, by using (12) and (16) we obtain

$$\begin{aligned} & D^{(n)}_{h,\alpha,\beta,q} f(x - t_\tau h) - f^{(n)}_{n+q}(x - t_\tau h) \\ =& \frac{(-h)^{q+1}}{(n+1+q)!} \int_0^1 Q_{\alpha,\beta,n,q,t_\tau}(\tau) \tau^{n+1+q} f^{(n+1+q)}(\xi) d\tau. \end{aligned}$$

Consequently, if for any $x \in I$ $\left|f^{(n+1+q)}(x)\right| \leq M_{n+1+q}$, then by taking the q^{th} order truncated Taylor series expansion of $f^{(n)}(x-t_\tau h)$

$$f^{(n)}(x-t_\tau h) = f^{(n)}_{n+q}(x-t_\tau h) + \frac{(-h)^{1+q} t_\tau^{1+q}}{(1+q)!} f^{(n+1+q)}(\hat{\xi}),$$

we have

$$\begin{aligned}
&\left\| D^{(n)}_{h,\alpha,\beta,q} f(x-t_\tau h) - f^{(n)}(x-t_\tau h) \right\|_\infty \\
\leq &\left\| D^{(n)}_{h,\alpha,\beta,q} f(x-t_\tau h) - f^{(n)}_{n+q}(x-t_\tau h) \right\|_\infty + \left\| f^{(n)}_{n+q}(x-t_\tau h) - f^{(n)}(x-t_\tau h) \right\|_\infty \\
\leq &h^{q+1} M_{n+1+q} \left(\frac{1}{(n+1+q)!} \int_0^1 \left| \tau^{n+1+q} Q_{\alpha,\beta,n,q,t_\tau}(\tau) \right| d\tau + \frac{t_\tau^{q+1}}{(q+1)!} \right).
\end{aligned} \tag{17}$$

Since

$$\begin{aligned}
&\left\| D^{(n)}_{h,\alpha,\beta,q} f^\delta(x-t_\tau h) - D^{(n)}_{h,\alpha,\beta,q} f(x-t_\tau h) \right\|_\infty \\
&\quad = \left\| D^{(n)}_{h,\alpha,\beta,q} \left[f^\delta(x-t_\tau h) - f(x-t_\tau h) \right] \right\|_\infty \\
&\quad \leq \frac{\delta}{h^n} \int_0^1 \left| Q_{\alpha,\beta,n,q}(\tau) \right| d\tau,
\end{aligned}$$

by using (17) we get

$$\begin{aligned}
&\left\| D^{(n)}_{h,\alpha,\beta,q} f^\delta(x-t_\tau h) - f^{(n)}(x-t_\tau h) \right\|_\infty \\
&\quad \leq \left\| D^{(n)}_{h,\alpha,\beta,q} f^\delta(x-t_\tau h) - D^{(n)}_{h,\alpha,\beta,q} f(x-t_\tau h) \right\|_\infty \\
&\qquad + \left\| D^{(n)}_{h,\alpha,\beta,q} f(x-t_\tau h) - f^{(n)}(x-t_\tau h) \right\|_\infty \\
&\quad \leq C_{q,t_\tau} h^{q+1} + E_{q,t_\tau} \frac{\delta}{h^n},
\end{aligned}$$

where $C_{q,t_\tau} = M_{n+1+q} \left(\frac{1}{(n+1+q)!} \int_0^1 \left| \tau^{n+1+q} Q_{\alpha,\beta,n,q,t_\tau}(\tau) \right| d\tau + \frac{t_\tau^{q+1}}{(q+1)!} \right)$ and $E_{q,t_\tau} = \int_0^1 \left| Q_{\alpha,\beta,n,q,t_\tau}(\tau) \right| d\tau$.

Let us denote the error bound by $\psi(h) = C_{q,t_\tau} h^{q+1} + E_{q,t_\tau} \frac{\delta}{h^n}$. Consequently, we can calculate its minimum value. It is obtained for $h^* = \left[\frac{n E_{q,t_\tau}}{(q+1) C_{q,t_\tau}} \delta \right]^{\frac{1}{n+q+1}}$ and

$$\psi(h^*) = \frac{n+1+q}{q+1} \left(\frac{q+1}{n} \right)^{\frac{n}{n+1+q}} C_{q,t_\tau}^{\frac{n}{n+1+q}} E_{q,t_\tau}^{\frac{q+1}{n+1+q}} \delta^{\frac{q+1}{n+1+q}}. \tag{18}$$

The proof is complete. □

Let us mention that if we set $t_\tau = \theta_{q+1}$, the smallest root of the Jacobi polynomial $P^{(\alpha+n,\beta+n)}_{q+1}$ in (5), then $D^{(n)}_{h,\alpha,\beta,q} f(x-\theta_{q+1} h)$ becomes the $(q+1)^{th}$ order truncated Jacobi orthogonal series of $f^{(n)}(x-\theta_{q+1} h)$. Hence, we have the following corollary.

Corollary 1. *Let $f \in \mathcal{C}^{n+2+q}(I)$ where q is an integer. If we set $t_\tau = \theta_{q+1}$, the smallest root of the Jacobi polynomial $P_{q+1}^{(\alpha+n,\beta+n)}$ in (12) and we assume that there exists $M_{n+2+q} > 0$ such that for any $x \in I$, $\left|f^{(n+q+2)}(x)\right| \leq M_{n+2+q}$, then we have*

$$\left\| D_{h,\alpha,\beta,q}^{(n)} f^\delta(x - \theta_{q+1}h) - f^{(n)}(x - \theta_{q+1}h) \right\|_\infty \leq C_{q,\theta_{q+1}} h^{q+2} + E_{q,\theta_{q+1}} \frac{\delta}{h^n},$$

where $C_{q,\theta_{q+1}} = M_{n+2+q} \left(\frac{1}{(n+q+2)!} \int_0^1 \left|\tau^{n+q+2} Q_{\alpha,\beta,n,q,\theta_{q+1}}(\tau)\right| d\tau + \frac{t^{q+2}}{(q+2)!} \right)$ and $E_{q,\theta_{q+1}}$ is given in Proposition 1. Moreover, if we choose $\hat{h} = \left[\frac{n E_{q,\theta_{q+1}}}{(q+2) C_{q,\theta_{q+1}}} \delta \right]^{\frac{1}{n+q+2}}$, then we have

$$\left\| D_{h,\alpha,\alpha,q}^{(n)} f^\delta(x - \theta_{q+1}h) - f^{(n)}(x - \theta_{q+1}h) \right\|_\infty = \mathcal{O}(\delta^{\frac{q+2}{n+2+q}}).$$

Proof. If $t_\tau = \theta_{q+1}$, the smallest root of polynomial $P_{q+1}^{(\alpha+n,\beta+n)}$ in (5), then we have

$$\begin{aligned} &D_{h,\alpha,\beta,q}^{(n)} f(x - \theta_{q+1}h) \\ &\qquad = \sum_{i=0}^{q+1} \frac{\left\langle P_i^{(\alpha+n,\beta+n)}(\cdot), f^{(n)}(x - h\cdot) \right\rangle_{\alpha+n,\beta+n}}{\|P_i^{(\alpha+n,\beta+n)}\|_{\alpha+n,\beta+n}^2} P_i^{(\alpha+n,\beta+n)}(\theta_{q+1}). \end{aligned}$$

This proof can be completed by taking the $(n+q+1)^{th}$ order truncated Taylor series expansion of f as it was done in Proposition 1. □

The numerical calculation of $E_{q,\theta_{q+1}}$ for q, $n \in \mathbb{N}$ and $\alpha, \beta \in]-1, 10]$ shows that $E_{q,\theta_{q+1}}$ increases with respect to q. Hence, in order to reduce the noise influence it is preferable to choose q as small as possible. However, $C_{q,\theta_{q+1}}$ decreases with respect to q. A compromise consists in choosing $q = 2$. If we take $D_{h,\alpha,\beta,2}^{(n)} f^\delta(x - \theta_2 h)$ as an estimator of $f^{(n)}(x)$, then we produce a time-delay $\theta_2 h$. For this choice of θ_2, we have $D_{h,\alpha,\beta,1}^{(n)} f^\delta(x - \theta_2 h) = D_{h,\alpha,\beta,2}^{(n)} f^\delta(x - \theta_2 h)$. We can see that the estimators $D_{h,\alpha,\beta,2}^{(n)} f^\delta(x)$ do not produce a time-delay but $C_{1,\theta_2} < C_{2,0}$ and $E_{1,\theta_2} < E_{2,0}$. This generally introduces more important estimation errors. Consequently, so as to estimate $f^{(n)}(x)$ we use $D_{h,\alpha,\beta,1}^{(n)} f^\delta(x - \theta_2 h)$ which presents a time-delay.

3 Numerical Experiments

In order to show the efficiency and the stability of the previously proposed estimators, we give some numerical results in this section.

From now on, we assume that $f^\delta(x_i) = f(x_i) + c\varpi(x_i)$ with $x_i = T_s i$ for $i = 0, \cdots, 500$ ($T_s = \frac{1}{100}$), is a noisy measurement of $f(x) = \exp(\frac{-x}{1.2}) \sin(6x + \pi)$. The noise $c\varpi(x_i)$ is simulated from a zero-mean white Gaussian *iid* sequence by using the Matlab function 'randn' with STATE reset to 0. Coefficient c is adjusted in such a way that the signal-to-noise ratio $SNR = 10\log_{10}\left(\frac{\sum |y(t_i)|^2}{\sum |c\varpi(t_i)|^2}\right)$ is equal to $SNR = 22.2$dB (see, e.g., [7] for this well known concept in signal processing). By using the well known three-sigma rule, we can assume that the noise level for $c\varpi$ is equal to $3c$. We can see the noisy signal in Figure 1. We use the trapezoidal method in order to approximate the integrals in our estimators where we use $m+1$ discrete values. The estimated derivatives of f at $x_i \in I = [0,5]$ are calculated from the noise data $f^{\delta}(x_j)$ with $x_j \in [-x_i - h, x_i]$ where $h = mT_s$.

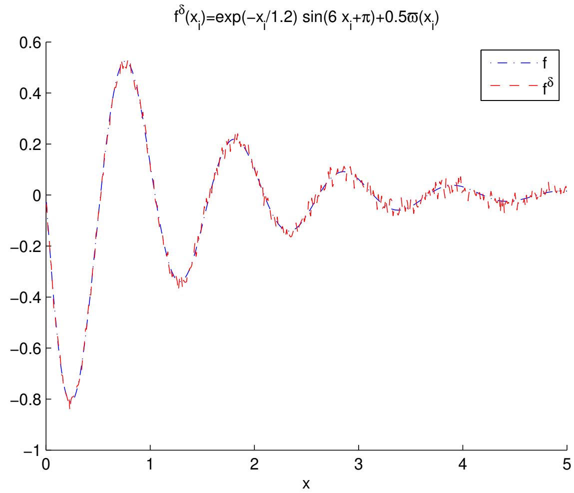

Fig. 1. Signal and noisy signal.

We can see the estimation results for the first order derivative of f in Figure 2. The corresponding estimation errors are given in Figure 3 and in Figure 4. We can see that the estimate given by the causal Jacobi estimator with integer parameters introduced in [14] (dash line), produces a time-delay of value $\theta_{q+1}h = 0.11$. The estimate given by the causal Jacobi estimator with extended parameters (dotted line) is time-delay free. Firstly, the root values θ_{q+1} for $q = 0, 1$ can be reduced with the extended negative parameters, so does the time-delay. Secondly, the numerical integration method with a negative value for β produces a numerical error which allows us to finally compensate this reduced time-delay. This last phenomena is due to the fact that the initial function f satisfies the following differential equation $f^{(2)} + kf = g$ where $k \in \mathbb{R}$ and g is a continuous function. Consequently, in the case of the first order derivative estimations, we can verify that this numerical error which depends on f may reduce the effect of the error due to the truncation in the Jacobi series expansion. This is due to the fact that the truncation error depends on $f^{(2)}$. Hence, the final total error is

$\mathcal{O}(g)$. Finally, since $D^{(1)}_{mT_s,\alpha,\beta,q} f^\delta(x_i - \theta_2 h)$ produces a time-delay of value $\theta_2 h$, we give in Figure 5 the errors $D^{(1)}_{mT_s,\alpha,\beta,q} f^\delta(x_i - \theta_2 h) - f^{(1)}(x_i - \theta_2 h)$.

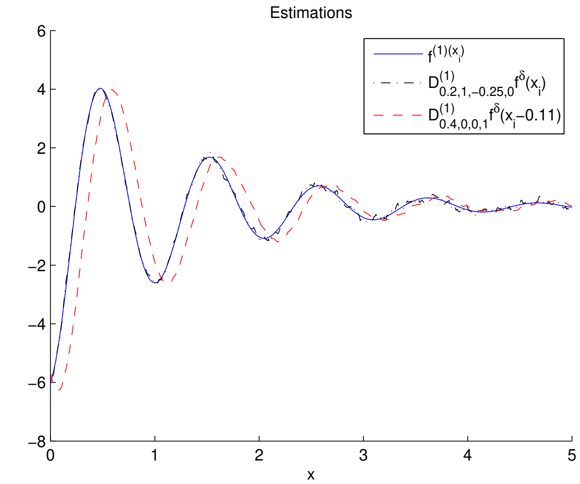

Fig. 2. Estimations by Jacobi estimators $D^{(1)}_{h,\alpha,\beta,q} f^\delta(x_i)$ with $h = 0.2, \beta = -0.25, \alpha = 1, q = 0$ and $D^{(1)}_{h,\alpha,\beta,q} f^\delta(x_i - \theta_2 h)$ with $h = 0.4, \alpha = \beta = 0, q = 1, \theta_2 = 0.276$.

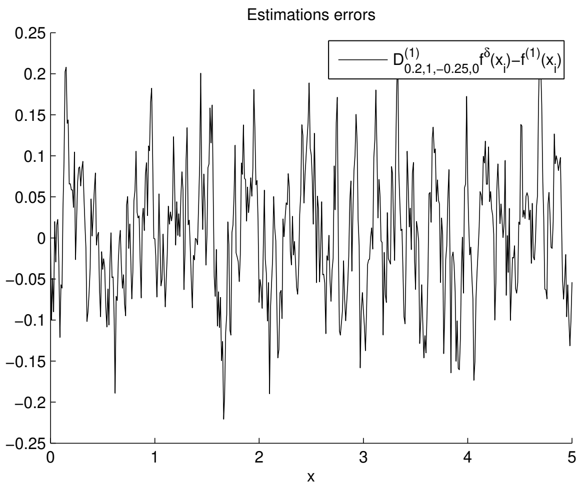

Fig. 3. $D^{(1)}_{h,\alpha,\beta,q} f^\delta(x_i) - f^{(1)}(x_i)$ with $h = 0.2, \beta = -0.25, \alpha = 1, q = 0$.

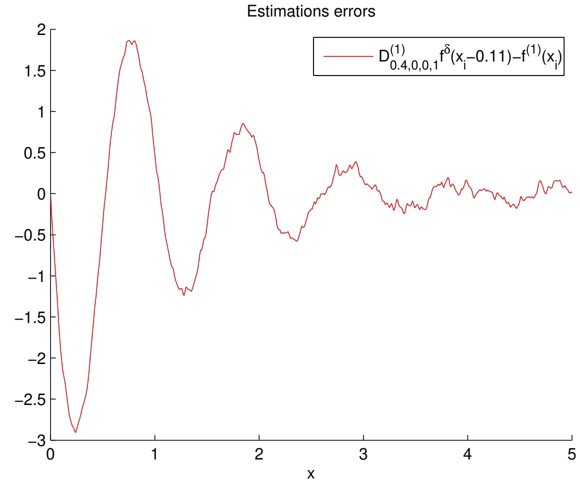

Fig. 4. $D^{(1)}_{h,\alpha,\beta,q}f^{\delta}(x_i-\theta_2 h)-f^{(1)}(x_i)$ with $h=0.4, \alpha=\beta=0, q=1, \theta_2=0.276$.

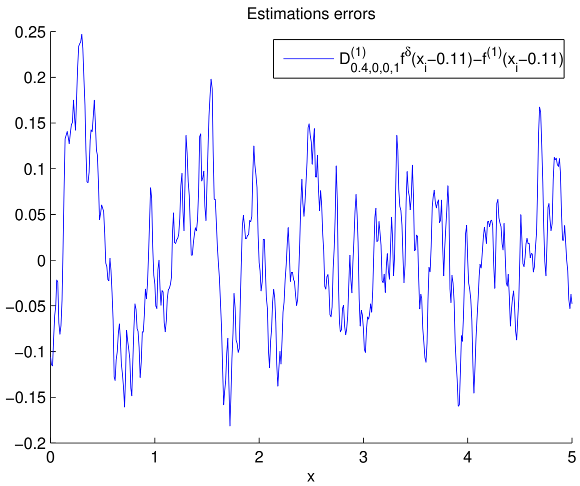

Fig. 5. $D^{(1)}_{h,\alpha,\beta,q}f^{\delta}(x_i-\theta_2 h)-f^{(1)}(x_i-\theta_2 h)$ with $h=0.4, \alpha=\beta=0, q=1, \theta_2=0.276$.

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