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ABSTRACT. Let D be a dendrite and $f : D \to D$ a continuous map. Denote by $E(D)$ and $B(D)$ the sets of endpoints and branch points of D respectively. We show that if $E(D)$ is countable (resp. $B(D)$ is discrete) then f is pointwise-recurrent if and only if f is pointwise periodic homeomorphism (resp. every point in $D \backslash E(D)$ is periodic).

1. Introduction

Recurrence and periodicity play an important role in studying dynamical systems. It is interesting to study maps $f : X \to X$ from a topological space X to itself that are pointwise-periodic (i.e. all points in X are periodic), or pointwise-recurrent (i.e. all points in X are recurrent). Montgomery [15] showed that if X is a connected topological manifold, a pointwise-periodic homeomorphism $f : X \to X$ must be periodic. Weaver [21] showed that if X is a continuum embedded in an orientable 2-manifold, an orientation-preserving C^1-homeomorphism $f : X \to X$ has this property. Gottschalk [9] proved that if X is a continuum then relatively recurrent homeomorphism $f : X \to X$(i.e. the closure of recurrent point set is dense in X) has every recurrent cut point periodic. In [20], Oversteegen and Tymchatyn showed that recurrent homeomorphisms of the plane are periodic. Kolev and Pérouème [11] proved that recurrent homeomorphisms of a compact surface with negative Euler characteristic are still periodic. Recently, Mai [13] showed that a graph map $f : G \to G$ is pointwise-recurrent if and only if one of the following statements holds:

(1) G is a circle, and f is a homeomorphism topologically conjugate to an irrational rotation

(2) f is a periodic homemorphism.

In this paper we will study pointwise-recurrent dendrite maps, their dynamical behaviors are both important and interesting in the study of dynamical systems and continuum theory. Recent interest in dynamics on dendrites is motivated by the fact that dendrites have often appear as Julia sets in complex dynamics (see [4]). In ([1], [3], [7], [14], [18] and [19]) several results concerning dendrites were obtained. In [19], we proved that every relatively recurrent monotone dendrite map have all its cut points periodic.

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Before stating our main results, we recall some basic properties of dendrites and dendrite maps.

A continuum is a compact connected metric space. A topological space is arcwise connected if any two of its points can be joined by an arc. We use the terminologies from Nadler [17]. An arc is any space homeomorphic to the compact interval $[0, 1]$. By a *dendrite* D, we mean a locally connected continuum which contains no homeomorphic copy to a circle. Every sub-continuum of a dendrite is a dendrite ([17], Theorem 10.10) and every connected subset of D is arcwise connected ([17], Proposition 10.9). In addition, any two distinct points x, y of a dendrite D can be joined by a unique arc with endpoints x and y, denote this arc by $[x, y]$ and let denote by $[x, y) = [x, y] \setminus \{y\}$ (resp. $(x, y] = [x, y] \setminus \{x\}$ and $(x, y) = [x, y] \setminus \{x, y\}$). A point $x \in D$ is called an *endpoint* if $D \setminus \{x\}$ is connected. It is called a *branch point* if $D\setminus\{x\}$ has more than two connected components. Denote by $E(D)$ and $B(D)$ the sets of endpoints, and branch points of D respectively. A point $x \in D \setminus E(D)$ is called a *cut point*. The set of cut points of D is dense in D. A tree is a dendrite with finite set of endpoints.

Let $\mathbb{Z}_+$ and $\mathbb{N}$ be the sets of non-negative integers and positive integers respectively. Let X be a compact metric space with metric d and $f : X \longrightarrow X$ be a continuous map. Denote by f^n the n-th iterate of f; that is, $f^0 = \mathrm{id}_X$: the Identity and $f^n = f \circ f^{n-1}$ if $n \geq 1$. For any $x \in X$ the subset $O_f(x) = \{f^n(x) : n \in \mathbb{Z}_+\}$ is called the f-orbit of x. A point $x \in X$ is called periodic of prime period $n \in \mathbb{N}$ if $f^n(x) = x$ and $f^i(x) \neq x$ for $1 \leq i \leq n-1$. A subset A of X is called *f-invariant* if $f(A) \subset A$. It is called *a minimal set of f* if it is non-empty, closed, f-invariant and minimal (in the sense of inclusion) for these properties. For a subset A of X, denote by $\overline{A}$ the closure of A and by $\mathrm{diam}(A)$ the diameter of A. We define the ω-limit set of a point x to be the set

$$\begin{aligned}\omega_f(x) &= \{y \in X : \exists\ n_i \in \mathbb{N}, n_i \to \infty, \lim_{i\to\infty} d(f^{n_i}(x), y) = 0\}\\ &= \bigcap_{n\in\mathbb{N}} \overline{\{f^k(x) : k \geq n\}}.\end{aligned}$$

The set $\omega_f(x)$ is a non-empty, closed and strongly invariant set, i.e. $f(\omega_f(x)) = \omega_f(x)$. A point $x \in X$ is said to be:

- *recurrent* for f if $x \in \omega_f(x)$.

- *almost periodic* if for any neighborhood U of x there exists $N \in \mathbb{N}$ such that $\{f^{n+i}(x) : i = 0, 1, \dots, N\} \cap U \neq \emptyset$ for all $n \in \mathbb{N}$.

- *regularly recurrent* if for any $\varepsilon > 0$, there is $N \in \mathbb{N}$ such that $d(x, f^{kN}(x)) < \varepsilon$ for all $k \in \mathbb{N}$.

It is easy to see that if x is regularly recurrent then it is almost periodic, hence $\omega_f(x)$ is a minimal set (see [6], Proposition 5, Chapter V).

Let $\mathrm{Fix}(f)$, $\mathrm{P}(f)$, $\mathrm{AP}(f)$ and $\mathrm{R}(f)$ denote the set of fixed points, periodic points, almost periodic and recurrent points respectively. Then we have the following inclusion relation $\mathrm{Fix(f)} \subset \mathrm{P(f)} \subset \mathrm{AP(f)} \subset \mathrm{R(f)}$. We say that f is

- *pointwise-periodic* if $\mathrm{P}(f) = X$.
- *pointwise-recurrent* if $\mathrm{R}(f) = X$.

- *relatively recurrent* if $\overline{\mathrm{R(f)}} = X$.

A continuous map from a dendrite into itself is called a *dendrite map.* Our main results are the following:

Theorem 1.1. *Let $f : D \to D$ be a dendrite map. If B(D) is discrete then f is pointwise-recurrent if and only if f is a homeomorphism and every cut point is periodic.*

Following ([2], Corollary 3.6), for any dendrite D, we have $\mathrm{B}(D)$ is discrete whenever $\mathrm{E}(D)$ is closed. Therefore:

Corollary 1.2. *Let $f : D \to D$ be a dendrite map. If E(D) is closed then Theorem 1.1 holds.*

Corollary 1.3. *If B(D) is discrete and $f : D \to D$ is pointwise-recurrent dendrite map then every endpoint of D is regularly recurrent.*

Remark 1.4. If $E(D)$ is closed and $f : D \to D$ is a pointwise-recurrent dendrite map, then an endpoint of D may not be in general periodic (see an example by Efremova and Makhrova in [7] on Gehman dendrite).

Theorem 1.5. *Let $f : D \to D$ be a dendrite map. If E(D) is countable then f is pointwise-recurrent if and only if f is pointwise-periodic homeomorphism.*

Corollary 1.6. [13] *Let T be a tree and $f : T \to T$ a continuous map. If f is pointwise-recurrent then f is periodic.*

Recall that the map f is periodic if $f^n = \mathrm{id}_T$ for some $n \in \mathbb{N}$.

2. **Preliminaries**

Lemma 2.1. ([14], Lemma 2.3) Let $(C_i)_{i\in\mathbb{N}}$ be a sequence of connected subsets of a dendrite (D, d). If $C_i \cap C_j = \emptyset$ for all $i \neq j$, then

$$\lim_{n\to+\infty} \mathrm{diam}(C_n) = 0.$$

Lemma 2.2. *Let D be a dendrite and $(p_n)_{n\in\mathbb{N}}$ be a sequence of D such that $p_{n+1} \in (p_n, p_{n+2})$ for all $n \in \mathbb{N}$, and $\lim\limits_{n\to+\infty} p_n = p_\infty$. Let U_n be the connected component of $D \setminus \{p_n, p_\infty\}$ that contains the open arc (p_n, p_∞). Then $\lim\limits_{n\to+\infty} \mathrm{diam}(U_n) = 0$*

Proof. It is easy to see that $U_{n+1} \subset U_n$ for all $n \in \mathbb{N}$. Suppose that Lemma 2.2 is not true, then there is $\delta > 0$ such that for all $n \in \mathbb{N}$, $\mathrm{diam}(U_n) > \delta$. We will construct an infinite sequence $(I_{n_i})_{i\in\mathbb{N}}$ of pairwise disjoint arcs such that $diam(I_{n_i}) \geq \frac{\delta}{3}$ for all $i \in \mathbb{N}$ which contradicts Lemma 2.1: Take $n_0 \in \mathbb{N}$ such that for all $n \geq n_0$, $diam([p_n, p_\infty]) < \frac{\delta}{3}$. For an integer $n \geq n_0$, let $a_n, b_n \in U_n$ be such that $d(a_n, b_n) > \delta$. There exist $c_n, d_n \in (p_n, p_\infty)$ such that $[a_n, c_n] \cap [p_n, p_\infty] = \{c_n\}$ and $[b_n, d_n] \cap [p_n, p_\infty] = \{d_n\}$. As $d(p_n, p_\infty) < \frac{\delta}{3}$, we have either $d(c_n, a_n) > \frac{\delta}{3}$ or $d(d_n, b_n) > \frac{\delta}{3}$. So we let $I_n = [c_n, a_n]$ if $d(c_n, a_n) > \frac{\delta}{3}$ and $I_n = [d_n, b_n]$ if $d(d_n, b_n) > \frac{\delta}{3}$. Choose an integer $m > n$ such that $p_m \in (c_n, p_\infty)$ if $I_n = [a_n, c_n]$ and $p_m \in (d_n, p_\infty)$ if $I_n = [b_n, d_n]$. Then $I_n \cap U_m = \emptyset$: Indeed, otherwise there exists $z \in I_n \cap U_m$. Take $I_n = [a_n, c_n]$. As $c_n \notin U_m$ and $z \in U$, then $[z, c_n]$ contains p_n or p_∞. In either cases, $[z, c_n] \supset [c_n, p_n] \neq \{c_n\}$, a contradiction. By the same way as I_n, we obtain an arc $I_m \subset U_m$ such that I_m intersect the arc $[p_m, p_\infty]$ in a single point and has diameter greater than $\frac{\delta}{3}$. So by repeating this process infinitely many times beginning from n_0, we obtain an infinite sequence of arcs $(I_{n_i})_{i\in\mathbb{N}}$ with diameter greater than $\frac{\delta}{3}$ and satisfying the following property: for all $i \in \mathbb{N}$, $I_{n_i} \subset U_{n_i}$ and $I_{n_i} \cap U_{n_{i+1}} = \emptyset$. This implies that $(I_{n_i})_{i\in\mathbb{N}}$ are pairwise disjoint, which is our claim. $\square$

Theorem 2.3. ([14], *Theorem 2.13*) *Let $f : D \to D$ be a dendrite map, $[x, y]$ be an arc in D, and U be the connected component of $D \setminus \{x, y\}$ containing the open arc (x, y). If there exist $k, m \in \mathbb{N}$ such that $\{f^k(x), f^m(y)\} \subset U$, then $U \cap P(f) \neq \emptyset$.*

We say that a dendrite map $f : D \to D$ is *monotone* if the preimage of any point by f is connected. Notice that if f is monotone so is f^n for any $n \in \mathbb{N}$.

Theorem 2.4. ([19], *Theorem 1.6*) *Let $f : D \to D$ be a dendrite map. If f is monotone then the following statements are equivalent:*

(i) *f is pointwise-recurrent.*
(ii) *f is relatively recurrent.*
(iii) *every cut point is a periodic point.*

Lemma 2.5. *Let $f : D \to D$ be a dendrite map, then the following properties are equivalent:*

(i) *any nondegenerate arc $I \subset f(D)$ contains a point with unique preimage by f.*
(ii) *the map f is monotone.*

Proof. $(i) \implies (ii)$: If f is not monotone, then there exists $z \in D$ such that $f^{-1}(z)$ is not connected. So one can find $a, b \in D$ with $a \neq b$ and $w_{-1} \in (a, b)$ such that $f(a) = f(b) = z$ and $w := f(w_{-1}) \neq z$. By continuity of f, we have $[z, w] \subset f([a, w_{-1}]) \cap f([b, w_{-1}]) \subset f(D)$. Hence each point in (z, w) has at least two preimages by f, a contradiction.

$(ii) \Longrightarrow (i)$: Let $I \subset f(D)$ be a nondegenerate arc. Then $(f^{-1}(\{x\}))_{x\in I}$ is a family of uncountably many pairwise disjoint connected non-empty subsets of D. Suppose that for every $x \in D$, $f^{-1}(\{x\})$ is not reduced to a point, then there is a non degenerate arc $I_x \subset f^{-1}(\{x\})$ containing no endpoints. By ([17], Corollary 10.28), D can be written as follow: $D = \cup_{n\in\mathbb{N}} A_n \cup E(D)$ where $(A_n)_{n\in\mathbb{N}}$ is a family of arcs with pairwise disjoint interiors. Hence, for each arc I_x, there is $n(x) \in \mathbb{N}$ such that $I_x \cap A_{n(x)}$ is a non degenerate arc. So necessarily, there is an arc A_{n_0} containing an uncountably many pairwise disjoint nondegenerate arcs of $(I_x)_{x\in I}$, which is a contradiction. □

Lemma 2.6. *Let X be a compact metric space and $f : X \to X$ is a pointwise-recurrent continuous map. Then every periodic point of f has a unique pre-image by f^n for all $n \in \mathbb{N}$.*

Proof. As $R(f) = R(f^n)$ for all $n \in \mathbb{N}$ (see [6]), it suffices to prove the Lemma for f. Suppose that for some periodic point p of period $n \in \mathbb{N}$, $f^{-1}(\{p\})$ contains more than one point. So there is $q \neq f^{n-1}(p)$ such that $f(q) = p$. Since q is recurrent and $\omega_f(q) = O_f(p)$, it follows that $q \in O_f(p)$, so there is $k \in \mathbb{N}$ such that $q = f^k(p)$. Thus $f^{n-1}(p) = f^n(q) = f^k(f^n(p)) = f^k(p) = q$, a contradiction. □

Lemma 2.7. *Let $f : D \to D$ be a dendrite map. Assume that f is a pointwise-recurrent. Let $(p_n)_{n\in\mathbb{N}} \subset D$ be a sequence of periodic points of f such that $p_{n+1} \in (p_n, p_{n+2})$ for all $n \in \mathbb{N}$, and $\lim\limits_{n\to+\infty} p_n = p_\infty$. Then p_∞ is a regularly recurrent point.*

Proof. For $n \in \mathbb{N}$, let U_n be defined as in Lemma 2.2, let $V_n := U_n \cup \{p_n, p_\infty\}$ and let denote by N_n the period of the point p_n. It is easy to see that if there is a sub-sequence of $(p_n)_{n\in\mathbb{N}}$ with bounded periods then by the continuity of f (and hence the continuity of its iterated maps), the point p_∞ is periodic, in particular, it is regularly recurrent point. Otherwise, we have to prove the Lemma 2.6 in the case of p_∞ is not periodic. Then without loss of generality, the sequence $(p_n)_{n\in\mathbb{N}}$ can be assumed such that

$$(p_n, p_\infty] \cap \mathrm{Fix}(f^{N_n}) = \emptyset. \tag{2.1}$$

We will prove that for all $n \in \mathbb{N}$, the orbit of the point p_∞ under the map $f^{N_{n+1}}$ is included into the set V_n: Indeed, otherwise for some $n \in \mathbb{N}$, there is $k \in \mathbb{N}$ such that $f^{kN_{n+1}}(p_\infty) \notin V_n$, so we have two possibilities both of them lead to a contradiction: $p_\infty \in (p_{n+1}, f^{kN_{n+1}}(p_\infty))$ or $p_n \in (p_{n+1}, f^{kN_{n+1}}(p_\infty))$. Let $m := kN_{n+1}$.

Suppose that $p_\infty \in (p_{n+1}, f^m(p_\infty))$. As $p_{n+1} \in \mathrm{Fix(f^m)}$, by the continuity of f^m, we have $f^m([p_{n+1}, p_\infty]) \supset [p_{n+1}, f^m(p_\infty)] \ni p_\infty$. Hence there is a point $p_{\infty,-1} \in (p_{n+1}, p_\infty)$ such that $f^m(p_{\infty,-1}) = p_\infty$. Similarly, there is a point $p_{\infty,-2} \in (p_{n+1}, p_{\infty,-1})$ such that $f^m(p_{\infty,-2}) = p_{\infty,-1}$. Thus, by induction we construct a sequence $(p_{\infty,-k})_{k\in\mathbb{N}}$ in (p_{n+1}, p_∞) such that for all $k \in \mathbb{N}$, $p_{\infty,-(k+1)} \in [p_{n+1}, p_{\infty,-k}]$ and

$$f^{km}(p_{\infty,-k}) = p_\infty. \tag{2.2}$$

As $p_{\infty,-1} \neq p_\infty$, there is $p_s \in (p_{\infty,-1}, p_\infty)$ for some $s \in \mathbb{N}$. Let $r \in \mathbb{N}$ be such that $f^r(p_{n+1}) = p_{n+1}$ and $f^r(p_s) = p_s$. Hence, $f^{rm}(p_{n+1}) = p_{n+1}$ and $f^{rm}(p_s) = p_s$. By (2.2), $f^{rm}(p_{\infty,-r}) = p_\infty$ and by the continuity of f^{rm}, $f^{rm}([p_{n+1}, p_{\infty,-r}]) \supset [p_{n+1}, p_\infty] \ni p_s$, hence p_s has a pre-image q by the map f^{rm} in the arc $[p_{n+1}, p_{\infty,-r}]$ and as $p_s \notin [p_{n+1}, p_{\infty,-r}]$, $q \neq p_s$, this contradicts Lemma 2.6 since p_s is a fixed point of f^{rm}.

Suppose now the second case, $p_n \in (p_{n+1}, f^m(p_\infty))$. By the continuity of f^m, $f^m([p_{n+1}, p_\infty]) \supset [p_{n+1}, f^m(p_\infty)] \ni p_n$. So p_n has a preimage q by the map f^m in the arc $[p_{n+1}, p_\infty]$. By Lemma 2.6, q is a periodic point that belongs to the orbit of p_n hence q has the same period as p_n. Hence, $q \in (p_n, p_\infty] \cap Fix(f^{N_n})$, this contradict (2.1).

It follows that for any $n \in \mathbb{N}$, the orbit of the point p_∞ under the map $f^{N_{n+1}}$ is included into the set V_n and as $\operatorname{diam}(V_n) = \operatorname{diam}(U_n)$, $\lim_{n\to+\infty} \operatorname{diam}(V_n) = 0$, by Lemma 2.5. This implies that p_∞ is regularly recurrent point. □

3. **Proof of Theorem 1.1 and Corollary 1.3**

Proof of Theorem 1.1: The "if" part of the Theorem results clearly from Theorem 2.4 since f is in particular monotone. Lets prove the "only if" part:

Assume that f is pointwise-recurrent, then f is surjective. We will use Lemma 2.5: Let $I \subset D$ be a nondegenerate arc. Since $B(D)$ is discrete, there exists a non-degenerate open arc $J \subset I$ containing no branch points, hence J is an open subset in D. So let $x \in J$. Since f is pointwise-recurrent, one can find $n, m \in \mathbb{N}$ such that $x, f^m(x) \in J$ and $f^{n+m}(x) \in (x, f^m(x))$. Thus $(x, f^m(x)) \subset I$ is the connected component of $D \setminus \{x, f^m(x)\}$ containing the open arc $(x, f^m(x))$. By Theorem 2.3, $(x, f^m(x))$ contains a periodic point p and by Lemma 2.6, p has a unique preimage by f. We conclude by Lemma 2.5 that f is monotone. Therefore, by ([19], Corollary 1.7), f is a homeomorphism and by Theorem 2.4, every cut point is periodic. The proof is complete. □

Proof of Corollary 1.3: Let $e \in E(D)$. By Theorem 1.1, one can find a sequence of periodic points $(p_n)_{n\in\mathbb{N}} \subset D$ such that $p_{n+1} \in (p_n, p_{n+2})$ for all $n \in \mathbb{N}$ and $\lim_{n\to+\infty} p_n = e$. Hence by Lemma 2.7, e is regularly recurrent. □

4. **Proof of Theorem 1.5**

We need the following result.

Theorem 4.1. [14] *Let $f : D \to D$ be a dendrite map. If $E(D)$ is countable then $\overline{R(f)} = \overline{P(f)}$.*

Lemma 4.2. *Let $f : D \to D$ be a dendrite map. If $E(D)$ is countable and f is pointwise-recurrent then $\omega_f(x) \cap P(f) \neq \emptyset$ for all $x \in D$.*

Proof. If $\omega_f(x) \subset E(D)$ then $\omega_f(x)$ is compact countable. If $\omega_f(x)$ is infinite then it is perfect. Since for any $y \in \omega_f(x)$, $y = \lim_{k\to+\infty} f^{n_k}(x)$ where $(n_k)_{k\in\mathbb{N}}$ is an infinite sequence of positif integers. As $x \in \omega_f(x)$, then $O_f(x) \subset \omega_f(x)$, hence y is an accumulation point of $\omega_f(x)$. So $\omega_f(x)$ is uncountable, a contradiction. Therefore, $\omega_f(x)$ is finite, that is $\omega_f(x)$ is a periodic orbit and so x is a periodic point. One can then assume that $\omega_f(x)\backslash E(D) \neq \emptyset$, so let $y \in \omega_f(x)\backslash E(D)$. Since $R(f) = D$, it follows by Theorem 4.1 that $\overline{P(f)} = D$, so one can find $p, q \in P(f)$ such that $y \in [p, q]$. Let $N \in \mathbb{N}$ be such that $f^N(p) = p$ and $f^N(q) = q$. We already have $y \in [p, f^N(y)] \cup [q, f^n(y)]$. One can suppose that $y \in [p, f^N(y)]$, the same proof being true with p replaced by q. In this case, we have $y \in f^N([p, y])$, so there is $y_{-1} \in [p, y]$ such that $f^N(y_{-1}) = y$. Again, $y_{-1} \in [p, y] = [p, f^N(y_{-1})] \subset f^N([p, y_{-1}])$, so there is $y_{-2} \in [p, y_{-1}]$ such that $f^N(y_{-2}) = y_{-1}$. Thus, we construct by induction a sequence $(y_{-n})_{n\in\mathbb{N}} \subset [p, y]$ such that $y_{-(k+1)} \in (y_{-k}, y_{-(k+2)})$, $f^N(y_{-(k+1)}) = y_{-k}$ for every $k \in \mathbb{N}$, and $\lim_{n\to+\infty} y_{-n} = y_\infty \in [p, y]$. Therefore one has $f^N(y_\infty) = y_\infty$ (by the continuity of f^N) and so $y_\infty \in P(f)$. Since $R(f) = D$ and for any $k \in \mathbb{N}$, $f^{kN}(y_{-k}) = y$, it follows that $y_{-k} \in \omega_f(y_{-k}) = \omega_f(y)$. Therefore $y_\infty \in \omega_f(y)$ and as $\omega_f(y) \subset \omega_f(x)$, we conclude that $y_\infty \in \omega_f(x) \cap P(f)$. □

Lemma 4.3. *Let $f : D \to D$ be a dendrite map and let p_∞ as in Lemma 2.7. If $E(D)$ is countable and f is pointwise-recurrent, then p_∞ is a periodic point.*

Proof. By Lemma 2.7, p_∞ is regularly recurrent, hence $\omega_f(p_\infty)$ is a minimal set. It follows, by Lemma 4.2, that $\omega_f(p_\infty)$ is a periodic orbit, and so p_∞ is a periodic point since $p_\infty \in \omega_f(p_\infty)$. □

Lemma 4.4. *Let D be a dendrite with countable set of endpoints. Then every sub-dendrite of D has countable set of endpoints.*

Proof. Let Y be a sub-dendrite of D. By ([8], page 157, (see also [17])), Y is a monotone retraction of D by $r : D \to Y$. Then we have $r(E(D)) = E(Y)$. Indeed, $r(E(D)) \subset E(Y)$ follows from the fact that $r(x)$ lies in any arc joining x to any point of Y. Let $a \in E(Y)$ and let Z the connected component of $D \setminus Y$ such that $\overline{Z}$ contains a. Then $r(\overline{Z}) = \{a\}$. As $\overline{Z} \cap E(D) \neq \emptyset$, there is $b \in \overline{Z} \cap E(D)$ with $r(b) = a$. It follows that $Card(E(Y)) \leq Card(E(D))$ and hence $E(Y)$ is countable. □

Proof of Theorem 1.5: The "if" part of the Theorem is clear. Lets prove the "only if" part: By Theorem 4.1, one has $\overline{P(f)} = D$.

Claim .1. If $x \in D$ is not periodic then for any $z \in D$ with $z \neq x$, (x, z) contains a non periodic branch point.
Indeed, $[x, z]$ contains an infinite sequence of branch points that converges to x. Since otherwise, there is $y \in (x, z]$ such that $(x, y) \cap B(D) \neq \emptyset$. So (x, y) is an open subset of D, hence as $\overline{P(f)} = D$, one can find a sequence $(p_n)_{n\in\mathbb{N}}$ of periodic points in (x, y) such that $p_{n+1} \in (p_n, p_{n+2})$ for all $n \in \mathbb{N}$, and $\lim\limits_{n\to+\infty} p_n = x$. By Lemma 4.3, x is periodic, a contradiction. Now, Suppose that every branch point in (x, z) is periodic, then again by Lemma 4.3, x is periodic. Therefore, necessarily one branch point in (x, z) must be not periodic.

Claim.2. If there exists $x \in D$ not periodic then D contains a sub-dendrite with uncountably set of endpoints.
Suppose that $x \in D$ is not periodic. By Claim.1, there is $a \in B(D)$ not periodic. Now, as a is a branch point, there exist two nondegenerate arcs I_0, I_1 with one endpoint is x and form with $[x, a]$ a family of three arcs having pairwise disjoint interior. As a is not periodic then by Claim.1, there is $a_0 \in I_0$ (resp. $a_1 \in I_1$) a non periodic branch point distinct from a. Denote by J_0 (resp. J_1) the arc $[a, a_0]$ (resp. $[a, a_1]$). In the second step, as a_0 and a_1 are branch points, there exist two nondegenerate arcs I_{00} and I_{01} (resp. I_{10} and I_{11}) with one endpoint is a_0 (resp. a_1) and form with $[a, a_0]$ (resp. $[a, a_1]$) a family of three arcs having pairwise disjoint interior. Similarly, by Claim.1, we find, for any $i, j \in \{0, 1\}$, a non periodic branch point $a_{ij} \in I_{ij}$ distinct from a_i, let denote by $J_{ij} := [a_i, a_{ij}]$. Thus by induction, we construct for all $n \in \mathbb{N}$ a sequence of arcs J_{α_n} and a sequence of non periodic branch points a_{α_n} where $\alpha_n \in \{0, 1\}^n$ satisfying the following properties:
(i) $J_{\alpha_n 0} = [a_{\alpha_n}, a_{\alpha_n 0}]$ and $J_{\alpha_n 1} = [a_{\alpha_n}, a_{\alpha_n 1}]$,
(ii) $J_{\alpha_n} \cap J_{\alpha_n 0} = J_{\alpha_n} \cap J_{\alpha_n 1} = J_{\alpha_n 0} \cap J_{\alpha_n 1} = \{a_{\alpha_n}\}$. any $n \in \mathbb{N}$, $D_n = \cup_{\alpha_n \in \{0,1\}^n} J_{\alpha_n}$, then clearly D_n is a tree. Also we have $D_{n+1} \subset D_n$ for all $n \in \mathbb{N}$. Hence $D_\infty = \overline{\cup_{n\in\mathbb{N}} D_n}$ is a sub-dendrite. Take $\beta \in \{0, 1\}^{\mathbb{N}}$. For each $n \in \mathbb{N}$, we have $[a, a_{\beta(n)}] \subset [a, a_{\beta(n+1)}]$ (where for $i \in \mathbb{N}$, we denote by $\beta(i)$ the first word of length i in the sequence β). Then $\overline{\cup_{n\in\mathbb{N}}[a, a_{\beta(n)}]}$ is an arc where the sequence $(a_{\beta(n)})_{n\in\mathbb{N}}$ is monotone in that arc so it converges to a point namely a_β that belongs to the dendrite D_∞. The points a_β for

$\beta \in \{0,1\}^{\mathbb{N}}$ are the endpoints of the sub-dendrite D_∞. Hence, D_∞ has an uncountable set of endpoints .

We deduce then Theorem 1.5 from Claim.2 and Lemma 4.4. □

Proof of Corollary 1.6: By Theorem 1.5, f is pointwise-periodic homeomorphism. As $E(T)$ is finite, there is $N \in \mathbb{N}$ such that every endpoint of T is fixed by f^N. Let $a, b \in T$ be two distinct endpoints of T. As f is a homeomorphism (and so is f^N), the arc $[a, b]$ is f^N-invariant, hence every point in $[a, b]$ is fixed by f^N. Since $T = \cup_{a,b\in E(T)}[a, b]$, every point in the tree T is fixed by f^N. So f is periodic. □

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