# Quantitative Languages Defined by Functional Automata 

Emmanuel Filiot, Raffaella Gentilini, Jean-François Raskin

## To cite this version:

Emmanuel Filiot, Raffaella Gentilini, Jean-François Raskin. Quantitative Languages Defined by Functional Automata. [Technical Report] 2011, pp.27. inria-00626216v1

## HAL Id: inria-00626216 https://inria.hal.science/inria-00626216v1

Submitted on 23 Sep 2011 (v1), last revised 29 Nov 2011 (v2)

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire HAL, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# Quantitative Languages Defined by Functional Automata 

Emmanuel Filiot ${ }^{1}$, Raffaella Gentilini ${ }^{2}$, and Jean-François Raskin ${ }^{1}$<br>1 Université Libre de Bruxelles<br>2 Università degli Studi di Perugia


#### Abstract

In this paper, we study several decision problems for functional weighted automata. To associate values with runs, we consider four different measure functions: the sum, the mean, the discounted sum of weights along edges and the ratio between rewards and costs. On the positive side, we show that the existential and universal threshold problems, the language inclusion problem and the equivalence problem are all decidable for the class of functional weighted automata and the four measure functions that we consider. On the negative side, we also study the quantitative extension of the realizability problem and show that it is undecidable for sum, mean and ratio. Finally, we show how to decide if the quantitative language defined by a functional weighted discounted sum automaton can be defined with a deterministic automata (it was already known for sum and mean).


1998 ACM Subject Classification Algorithms, Theory, Verification

Keywords and phrases Weighted automata, quantitative languages, functionality, synthesis, computer-aided verification

## 1 Introduction

Recently, there have been several efforts made to lift the foundations of computer aided verification and synthesis from the basic Boolean case to the richer quantitative case, e.g. [10, 8, 1]. This paper belongs to this line of research and contributes to the study of quantitative languages over finite words.

Our paper proposes a systematic study of the algorithmic properties of several classes of functional weighted automata (defining quantitative languages). A functional weighted automaton is a nondeterministic weighted automaton such that any two accepting runs $\rho_{1}$, $\rho_{2}$ on a word $w$ associate with this word a unique value $\mathrm{V}\left(\rho_{1}\right)=\mathrm{V}\left(\rho_{2}\right)$. As we show in this paper, several important verification problems are decidable for nondeterministic functional weighted automata while they are undecidable (or unknown to be decidable) for the full class of nondeterministic weighted automata. As functional weighted automata are a natural generalization of unambiguous weighted automata, and as unambiguity captures most of the nondeterminism that is useful in practice, our results are both theoretically and practically important. Also, the notion of functionality leads to useful insight into the relation between deterministic and nondeterministic weighted automata and into algorithmic idea for testing equivalence for example.

In this paper, we study automata in which an integer weight, or a pair of integer weights, is associated with each of their transitions. From those weights, an (accepting) run $\rho$ on a word $w$ associates a sequence of weights with the word, and this sequence is mapped to a rational value by a measure function. We consider four different measure functions ${ }^{1}$ : (i)

[^0]Sum computes the sum of the weights along the sequence, (ii) Avg returns the mean value of the weights, (iii) Dsum $\lambda_{\lambda}$ computes the discounted sum of the weights for a given discount factor $\lambda \in \mathbb{Q} \cap] 0,1[$, and $(i v)$ Ratio is applied to a sequence of pairs of weights, and it returns the ratio between the sum of weights appearing as the first component (rewards) and the sum of the weights appearing as the second component (costs). The value associated with a word $w$ accepted by $A$ is denoted by $L_{A}(w)$. While Sum and, to some extent, Avg are known because they can be seen as operations over a semiring of values [20, the case of Dsum ${ }_{\lambda}$ and Ratio are less studied. Those two measures are motivated by applications in computer aided verification and synthesis, see for example [12, 7].
Contributions Functionality is a semantical property. We show that it can be decided for the four classes of measure functions that we consider (either in polynomial time or polynomial space). Then we solve the following decision problems, along the line of [10]. First, we consider threshold problems. The existential (universal, respectively) threshold problem asks, given a weighted automaton $A$ and a threshold $\nu \in \mathbb{Q}$, if there exists a word (if for all words, respectively) $w$ accepted by $A: L_{A}(w) \geq \nu$. Those problems can be seen as generalizations of the emptiness and universality problems for finite state automata. Second, we consider the quantitative language inclusion problem that asks, given two weighted automata $A$ and $B$, if all words accepted by $A$ are also accepted by $B$, and for all accepted words $w$ of $A$, we have $L_{A}(w) \leq L_{B}(w)$. We show that all those problems are decidable for the four classes of measure functions that we consider in this paper when the automaton is functional. We show that the inclusion problem is PSpaceC for Sum, Avg and Dsum ${ }_{\lambda}$. For Ratio, we show decidability of the problem using a recent algorithm to solve quadratic diophantine equations [14], this is a new deep result in mathematics and the complexity of the algorithm is not yet known. Note that those decidability results are in sharp contrast with the corresponding results for the full class of nondeterministic weighted automata: for that class, only the existential threshold problem is known to be decidable, the language inclusion problem is undecidable for Sum, Avg, and Ratio while the problem is open for Dsum $_{\lambda}$. We also show that the equivalence problem can be decided in polynomial space for Ratio via an easy reduction to functionality.

Then, we consider a quantitative variant of the realizability problem introduced by Church, which is part of the foundations of game theory played on graphs [23] and synthesis of reactive systems [21]. It can be formalized as a game in which two players alternates in choosing letters in their respective alphabet. By doing so, they form a word which is obtained by concatenating the successive choices of the players. The realizability problem asks, given a weighted automaton $A$, alphabet $\Sigma=\Sigma_{1} \times \Sigma_{2}$, if there exists a strategy for choosing the letters in $\Sigma_{1}$ in the word forming game such that no matter how the adversary chooses his letters in $\Sigma_{2}$, the word $w$ that is obtained belongs to the language of $A$ and $A(w) \geq 0$. We show that this problem is undecidable for Sum, Avg, and Ratio even when considering unambiguous automata (the case Dsum $\lambda_{\lambda}$ is left open). However, we show that the realizability problem is decidable for the deterministic versions of the automata studied in this paper. This motivates the study of the determinizability problem.

The determinizability problem asks, given a functional weighted automaton $A$, if the quantitative language defined by $A$ is also definable by a deterministic automaton. This problem has been solved for Sum, Avg in [16]. It is known that Dsum $\lambda_{\lambda}$-automata are not
maximal value that appear in the sequence as the nondeterministic automata that use those measure functions can be made deterministic and all the decision problems for them have known and simple solutions.
determinizable in general 10. We give here a decidable necessary and sufficient condition for the determinizability of functional Dsum $\lambda_{\lambda}$ automata, and we show how to construct a deterministic automaton from the functional one when this is possible.

Related Works Motivated by computer-aided verification issues, our work follows the same line as [10]. However [10] is mainly concerned with weighted automata on infinite words, either non-deterministic, for which some important problems are undecidable (e.g. inclusion of Avg-automata), or deterministic ones, which are strictly less expressive than functional automata. The Ratio measure is not considered either. Their domains of quantitative languages are assumed to be total (as all states are accepting and their transition relation is total) while we can define partial quantitave languages thanks to an acceptance condition.

Weighted automata over semirings have been extensively studied (see [20] for a survey), and more generally rational series [4]. For instance, the functionality problem for weighted automata over the tropical semiring, i.e. Sum-automata, is known to be in PTime [16]. Moreover, it is known that determinizability of functional Sum-automata is decidable in PTime [16, as well as for the strictly more expressive class of polynomially ambiguous Sum-automata [15], for which the termination of Mohri's determinization algorithm [20] is decidable. However, the Dsum $\lambda_{\lambda}$ and Ratio-automata are not automata over any semiring, and therefore results on automata over semirings cannot be directly applied to those measures. The technics we use for deciding functionality and determinization are inspired by technics from word transducers [22, 6, 3, 11, 24].

The functionality problem has been studied for finite state (word) transducers. It was proved to be decidable in [22], and later in [6]. Based on a notion of delay between runs, an efficient PTime procedure for testing functionality has been given in 3]. The functionality problem for Sum,Avg and Dsum $\lambda_{\lambda}$-automata is also based on a notion of delay. Based on the twinning property [11 and the notion of delay, efficient procedures for deciding determinizability can be devised [3, 24]. This also inspired our determinization procedure for functional Dsum $_{\lambda}$-automata. In [9], Boker et. al. show that Dsum $\lambda_{\lambda}$-automata on infinite words with a trivial accepting condition (all states are accepting), but not necessarily functional, are determinizable for any discount factor of the form $1 / n$ for some $n \in \mathbb{N}_{\geq 2}$. Their proof is based on a notion of recoverable gap, similar to that of delays. In our paper, we provide a sufficient and necessary condition to check whether a functional Dsum $\lambda_{\lambda}$-automaton (over finite words) is determinizable. Finally in 13, the relation between discounted weighted automata over a semiring and weighted logics is studied.

To the best of our knowledge, our results on Dsum $\lambda_{\lambda}$ and Ratio-automata, as well as on the realizability problem, are new. Our main and most technical results are functionality of Dsum $\lambda_{\lambda}$ and Ratio-automata, inclusion problems, determinizability of functional Dsum $\lambda^{-}$automata, undecidability of the realizability of unambiguous Sum-automata, and solvability of the deterministic versions of the realizability problem. The latter reduce to games on graphs that are to the best of our knowledge new, namely finite Sum, Avg, Dsum ${ }_{\lambda}$, Ratiogames on weighted graphs with a combination of a reachability objective and a quantitative objective.

Omitted proofs can be found in the Appendix section.

## 2 Quantitative Languages and Functionality

Let $\Sigma$ be a finite alphabet. We denote by $\Sigma^{+}$the set of non-empty finite words over $\Sigma$. A quantitative language $L$ over $\Sigma$ is a mapping $L: \Sigma^{+} \rightarrow \mathbb{Q} \cup\{\perp\}^{2}$. For all $w \in \Sigma^{+}, L(w)$ is called the value of $w . L(w)=\perp$ means that the value of $w$ is undefined. For all $x \in \mathbb{Q}$, we let $\max (x, \perp)=x, \max (\perp, x)=x$ and $\max (\perp, \perp)=\perp$.

Let $n \geq 0$. Given a finite sequence $v=v_{0} \ldots v_{n}$ of integers (resp. a finite sequence $v^{\prime}=\left(r_{0}, c_{0}\right) \ldots\left(r_{n}, c_{n}\right)$ of pairs of natural numbers, $c_{i}>0$ for all $\left.i\right)$ and $\lambda \in \mathbb{Q}$ such that $0<\lambda<1$, we define the following functions:

$$
\operatorname{Sum}(v)=\sum_{i=0}^{n} v_{i} \quad \operatorname{Avg}(v)=\frac{\operatorname{Sum}(v)}{n} \quad \operatorname{Dsum}_{\lambda}(v)=\sum_{i=0}^{n} \lambda^{i} v_{i} \quad \operatorname{Ratio}\left(v^{\prime}\right)=\frac{\sum_{i=0}^{n} r_{i}}{\sum_{i=0}^{n} c_{i}}
$$

For empty sequences $\epsilon$, we also set $\operatorname{Sum}(\epsilon)=\operatorname{Avg}(\epsilon)=\operatorname{Dsum}_{\lambda}(\epsilon)=\operatorname{Ratio}(\epsilon)=0$.
Weighted Automata Let $V \in\{$ Sum, Avg, Dsum $\lambda$, Ratio $\}$. A weighted $V$-automaton over $\Sigma$ is a tuple $A=\left(Q, q_{I}, F, \delta, \gamma\right)$ where $Q$ is a finite set of states, $F$ is a set of final states, $\delta \subseteq Q \times \Sigma \times Q$ is the transition relation, and $\gamma: \delta \rightarrow \mathbb{Z}($ resp. $\gamma: \delta \rightarrow \mathbb{N} \times(\mathbb{N}-0)$ if $V=$ Ratio) is a weight function. The size of $A$ is defined by $|A|=|Q|+|\delta|+\sum_{t \in \delta} \log _{2}(\gamma(t))$. Note that $\left(Q, q_{I}, F, \delta\right)$ is a classical finite state automaton. We say that $A$ is deterministic (resp. unambiguous) if ( $Q, q_{I}, F, \delta$ ) is.

A run $\rho$ of $A$ over a word $w=\sigma_{1} \ldots \sigma_{n} \in \Sigma^{*}$ is a sequence $\rho=q_{0} \sigma_{1} q_{1} \sigma_{2} \ldots \sigma_{n} q_{n}$ such that $q_{0}=q_{I}$ and for all $i \in\{0, \ldots, n-1\},\left(q_{i}, \sigma_{i+1}, q_{i+1}\right) \in \delta$. It is accepting if $q_{n} \in F$. We write $\rho: q_{0} \xrightarrow{w} q_{n}$ to denote that $\rho$ is a run on $w$ starting at $q_{0}$ and ending in $q_{n}$. The domain of $A$, denoted by $\operatorname{dom}(A)$, is defined as the set of words $w \in \Sigma^{+}$on which there exists some accepting run of $A$.

The function $V$ is naturally extended to runs as follows:

$$
V(\rho)= \begin{cases}V\left(\gamma\left(q_{0}, \sigma_{1}, q_{1}\right) \ldots \gamma\left(q_{n-1}, \sigma_{n}, q_{n}\right)\right) & \text { if } \rho \text { is accepting } \\ \perp & \text { otherwise }\end{cases}
$$

The relation induced by $A$ is defined by $R_{A}^{V}=\left\{(w, V(\rho)) \mid w \in \Sigma^{+}, \rho\right.$ is a accepting run of $A$ on $\left.w\right\}$. It is functional if for all words $w \in \Sigma^{+}$, we have $\left|\left\{v \mid(w, v) \in R_{A}^{V}, v \neq \perp\right\}\right| \leq 1$. In that case we say that $A$ is functional. The quantitative language $L_{A}: \Sigma^{+} \rightarrow \mathbb{Q} \cup\{\perp\}$ defined by $A$ is defined by $L_{A}: w \mapsto \max \left\{v \mid(w, v) \in R_{A}^{V}\right\}$.

- Example 1. Fig. 1 illustrates two Sum-automata over the alphabet $\{a, b\}$. The first automaton (on the left) defines the quantitative language $w \in \Sigma^{+} \mapsto \max \left(\#_{a}(w), \#_{b}(w)\right)$, where $\#_{\alpha}(w)$ denotes the number of occurences of the letter $\alpha$ in $w$. Its induced relation is $\left\{\left(w, \#_{a}(w)\right) \mid w \in \Sigma^{+}\right\} \cup\left\{\left(w, \#_{b}(w)\right) \mid w \in \Sigma^{+}\right\}$. The second automaton (on the right) defines the quantitative language that maps any word of length at least 2 to the number of occurences of its last letter.

We say that a state $q$ is co-accessible (resp. accessible) by some word $w \in \Sigma^{*}$ if there exists some run $\rho: q \xrightarrow{w} q_{f}$ for some $q_{f} \in F$ (resp. some run $\rho: q_{I} \xrightarrow{w} q$ ). If such a word exists, we say that $q$ is co-accessible (resp. accessible). A pair of states ( $q, q^{\prime}$ ) is co-accessible if there exists a word $w$ such that $q$ and $q^{\prime}$ are co-accessible by $w$. In the sequel, we use the term $V$-automata to denote either Sum, Dsum ${ }_{\lambda}$, Avg or Ratio-automata.

[^1]

Figure 1 Examples of Sum-automata
Functional Weighted Automata The Sum-automaton on the left of Fig. 1 is not functional (e.g. the word $a b b$ maps to the values 1 and 2), while the one of the right is functional (and even unambiguous).

Concerning the expressiveness of functional automata, we can show that deterministic automata are strictly less expressive than functional automata which are again strictly less expressive than non-deterministic automata.

Lemma 2. Let $V \in\left\{\right.$ Sum, Avg, Dsum $_{\lambda}$, Ratio $\}$. The following hold:
Deterministic < Functional There exists a functional $V$-automaton that cannot be defined by any deterministic $V$-automaton;
Functional < Non-deterministic There exists a non-deterministic $V$-automaton that cannot be defined by any functional $V$-automaton.

Proof. Let $V \in\left\{\right.$ Sum, $\operatorname{Avg}$, Dsum $_{\lambda}$, Ratio $\}$. The automata of Fig. 1 can be seen as $V$ automata (with a constant cost 1 if $V=$ Ratio). The right $V$-automaton cannot be expressed by any deterministic $V$-automaton because the value of a word depends on its last letter. The left $V$-automaton cannot be expressed by any functional $V$-automaton.

As proved in Appendix, functional $V$-automata are equally expressive as unambiguous $V$-automata (i.e. at most one accepting run per input word). However we inherit the succinctness property of non-deterministic finite state automata wrt unambiguous finite state automata, as a direct consequence functional $V$-automata are exponentially more succinct than unambiguous $V$-automata. Moreover, considering unambiguous $V$-automata does not simplify the proofs of our results neither lower the computational complexity of the decision problems. Finally, testing functionality often relies on a notion of delay that gives strong insights that are useful for determinization procedures, and will allow us to test equivalence of functional (and even unambiguous) Ratio-automata with a better complexity than using our results on inclusion.

## 3 Functionality

In this section, we show that it is decidable whether a $V$-automaton $A=\left(Q, q_{I}, F, \delta, \gamma\right)$ is functional for all $V \in\left\{\right.$ Sum, Avg, Dsum ${ }_{\lambda}$, Ratio $\}$.

### 3.1 Functionality of Sum and Avg-Automata

It is clear that a Sum-automaton $A$ is functional iff the Avg-automaton $A$ is functional. Indeed, let $w \in \operatorname{dom}(A)$ and $v_{1}, v_{2} \in \mathbb{Z}$. We have $\left(w, v_{1}\right),\left(w, v_{2}\right) \in R_{A}^{\text {Sum }}$ iff $\left(w, \frac{v_{1}}{|w|}\right),\left(w, \frac{v_{2}}{|w|}\right) \in$ $R_{A}^{\text {Avg }}$. The result follows as $v_{1} \neq v_{2}$ iff $\frac{v_{1}}{|w|} \neq \frac{v_{2}}{|w|}$. So we can rephrase the following result of [16]:

- Theorem 3 ([16]). Functionality is decidable in PTime for Sum and Avg-automata.

The algorithm of [16] for checking functionality of Sum-automata is based on the notion of delay between two runs. This notion has been first introduced for deciding functionality of finite state (word) transducers 3]. Let $w \in \Sigma^{+}$and $\rho, \rho^{\prime}$ be two runs of a Sum-automaton $A$ on $w$. The delay between $\rho$ and $\rho^{\prime}$ is defined as delay $\left(\rho, \rho^{\prime}\right)=\operatorname{Sum}(\rho)-\operatorname{Sum}\left(\rho^{\prime}\right)$. For all pairs $(p, q)$, we define delay $(p, q)$ as the set of delays delay $(p, q)=\left\{\operatorname{delay}\left(\rho, \rho^{\prime}\right) \mid \exists w \in \Sigma^{*} \cdot \rho\right.$ : $\left.q_{I} \xrightarrow{w} p, \rho^{\prime}: q_{I} \xrightarrow{w} q\right\}$.

It is proved in [16] that a Sum-automaton $A$ is functional iff for all co-accessible pairs of states $(p, q)$, $|\operatorname{delay}(p, q)| \leq 1$. Intuitively, if $A$ is functional, then any delay delay $(p, q)$ associated with a pair $(p, q)$ co-accessible with the (same) word $w$ has to be recovered when reading $w$. If there are at least two different delays associated with $(p, q)$, one of them cannot be recovered when reading the same word $w$, therefore $A$ is not functional. The algorithm then consists first in computing all co-accessible pairs of states, and then all the triples $(p, q, k)$ in a forward manner, where $k$ represents some delay of $(p, q)$. If two triples $(p, q, k)$ and ( $p, q, k^{\prime}$ ) with $k \neq k^{\prime}$ are reached, then $A$ is not functional. Termination is obtained by a small witness property for non-functionality, which ensures that the triples need to be visited at most twice. A similar algorithm with another notion of delay is used for deciding functionality of Dsum $\lambda_{\lambda}$-automata.

### 3.2 Functionality of Dsum $_{\lambda}$-automata

- Definition 4 (Dsum $\lambda_{\lambda}$ Delay). Let $p, q \in Q$ and $d \in \mathbb{Q}$. The rational $d$ is a delay for $(p, q)$ if $A$ admits two runs $\rho: q_{I} \xrightarrow{w} p, \rho^{\prime}: q_{I} \xrightarrow{w} q$ on $w \in \Sigma^{*}$ such that

$$
\operatorname{delay}\left(\rho, \rho^{\prime}\right)={ }_{d e f} \frac{\operatorname{Dsum}_{\lambda}(\rho)-\operatorname{Dsum}_{\lambda}\left(\rho^{\prime}\right)}{\lambda^{|w|}}=d
$$

As for Sum-automata, at most one delay can be associated with co-accessible pairs of states of functional Dsum ${ }_{\lambda}$ automata.

- Lemma 5 (One Delay). Let $A=\left(Q, q_{I}, F, \delta, \gamma\right)$ be a functional Dsum ${ }_{\lambda}$-automaton. For all pairs of states $(p, q)$ : If $(p, q)$ is co-accessible, then $(p, q)$ admits at most one delay.

We now define an algorithm that checks whether a Dsum $\lambda_{\lambda}$-automaton is functional. In a first step, it computes all co-accessible pairs of states. Then it explores the set of accessible pairs of states in a forward manner and computes the delays associated with those pairs. If two different delays are associated with the same pair, or if a pair of final states with a nonzero delay is reached, it stops and returns that the automaton is not functional, otherwise it goes on until all co-accessible (and accessible) pairs have been visited and concludes that the automaton is functional.

- Lemma 6. Let $A=\left(Q, q_{I}, F, \delta, \gamma\right)$ be a Dsum $\lambda_{\lambda}$-automaton. If $A$ is not functional, there exists a word $w=\sigma_{0} \ldots \sigma_{n}$ and two accepting runs $\rho=q_{0} \sigma_{0} \ldots q_{n}, \rho^{\prime}=q_{0}^{\prime} \sigma_{0} \ldots q_{n}^{\prime}$ on it such that $\operatorname{Dsum}_{\lambda}(\rho) \neq \operatorname{Dsum}_{\lambda}\left(\rho^{\prime}\right)$ and for all positions $i<j$ in $w$, either $(i)\left(p_{i}, q_{i}\right) \neq\left(p_{j}, q_{j}\right)$ or (ii) $\operatorname{delay}\left(\rho_{i}, \rho_{i}^{\prime}\right) \neq \operatorname{delay}\left(\rho_{j}, \rho_{j}^{\prime}\right)$, where $\rho_{i}$ and $\rho_{i}^{\prime}$ (resp. $\rho_{j}$ and $\rho_{j}^{\prime}$ ) denote the prefixes of the runs $\rho$ and $\rho^{\prime}$ until position $i$ (resp. position $j$ ).

We can now prove the correctness of Algorithm DSumFunTest.

- Theorem 7. Given a Dsum $_{\lambda}$-automaton A, Algorithm DSumFunTest applied to $A$ returns Yes iff $A$ is functional and terminates within $O\left(|A|^{2}\right)$ steps.

```
Algorithm 1: Functionality test for Dsum \(\lambda_{\lambda}\)-automata. (DSumFunTest)
    Data: Dsum \(\lambda_{\lambda}\)-automaton \(A=\left(Q, q_{I}, F, \delta, \gamma\right)\).
    Result: Boolean certifying whether \(A\) is functional.
    begin
        CoAcc \(\leftarrow\) all co-accessible pairs of states;
        visited \(\leftarrow \emptyset ; \operatorname{delay}\left(q_{I}, q_{I}\right) \leftarrow 0 ; \operatorname{PUSH}\left(S,\left(\left(q_{I}, q_{I}\right), 0\right)\right)\);
        while \(S \neq \emptyset\) do
            \(((p, q), d) \leftarrow \operatorname{POP}(S) ;\)
        if \((p, q) \in F^{2} \wedge d \neq 0\) then returns No;
        if \((p, q) \in\) visited then
            if \(\operatorname{delay}(p, q) \neq d\) then returns No
                else
                    visited \(\leftarrow\) visited \(\cup\{(p, q)\}\);
                    delay \((p, q) \leftarrow d\);
                    foreach \(\left(p^{\prime}, q^{\prime}\right) \in \operatorname{CoAcc}\) s.t. \(\exists a \in \Sigma \cdot\left(p, a, p^{\prime}\right) \in \delta \wedge\left(q, a, q^{\prime}\right) \in \delta\) do
                    \(\operatorname{PUSH}\left(S,\left(\left(p^{\prime}, q^{\prime}\right), \gamma\left(p, a, p^{\prime}\right)-\gamma\left(q, a, q^{\prime}\right)+d\right)\right)\);
        returns Yes
```

Sketch, full proof in Appendix. If $\operatorname{DSumFunTest}(A)$ returns No, it is either because a pair of accepting states with non-null delay has been reached, which gives a counter-example to functionality, or it finds a pair of states with two different delays, so $A$ is not functional by Lemma 5

Conversely, if $A$ is non-functional, by Lemma there exists a word $w$ with two accepting runs having different values such that either no pair of states is repeated twice, in which case the algorithm can find a pair of final states with a non-null delay, or there is a pair of states that repeat twice (take the first that repeat) and has necessarily two different delays, in which case the algorithm will return No at line 6] if not before.

### 3.3 Functionality of Ratio-automata

Unlike Sum,Avg or Dsum $\lambda_{\lambda}$-automata, it is still open whether there exists a good notion of delay for Ratio-automata that would allow us to design an efficient algorithm to test functionality. However deciding functionality can be done by using a short witness property of non-functionality.
Lemma 8 (Pumping). Let $A$ be a Ratio-automaton with $n$ states. $A$ is not functional iff there exist a word $w$ such that $|w|<4 n^{2}$ and two accepting runs $\rho, \rho^{\prime}$ on $w$ such that $\operatorname{Ratio}(\rho) \neq \operatorname{Ratio}\left(\rho^{\prime}\right)$.

Proof. We prove the existence of a short witness for non-functionality. The other direction is obvious. Let $w$ be a word such that $|w| \geq 4 n^{2}$ and there exists two accepting runs $\rho_{1}, \rho_{2}$ on $w$ such that Ratio $(\rho) \neq \operatorname{Ratio}\left(\rho^{\prime}\right)$. Since $|w| \geq 4 n^{2}$, there exist states $p, q \in Q, p_{f}, q_{f} \in F$ and words $w_{0}, w_{1}, w_{2}, w_{3}, w_{4}$ such that $w=w_{0} w_{1} w_{2} w_{3} w_{4}$ and $\rho, \rho^{\prime}$ can be decomposed as follows:

$$
\left.\begin{array}{l}
\rho:
\end{array} q_{I} \xrightarrow{w_{0} \mid\left(r_{0}, c_{0}\right)} p p \xrightarrow{w_{1} \mid\left(r_{1}, c_{1}\right)} \quad p \xrightarrow{w_{2} \mid\left(r_{2}, c_{2}\right)} p p_{l}^{w_{3} \mid\left(r_{3}, c_{3}\right)} \quad p \xrightarrow{w_{4} \mid\left(r_{4}, c_{4}\right)} p_{f}\right)
$$

where $r_{i}, c_{i}$ denotes the sum of the rewards and the costs respectively on the subruns of $\rho$ on $w_{i}$, and similarly for $r_{i}^{\prime}, c_{i}^{\prime}$.

By hypothesis we know that $\left(\sum_{i=0}^{4} r_{i}\right) \cdot\left(\sum_{i=0}^{4} c_{i}^{\prime}\right) \neq\left(\sum_{i=0}^{4} c_{i}\right) \cdot\left(\sum_{i=0}^{4} r_{i}^{\prime}\right)$. For all subsets $X \subseteq\{1,2,3\}$, we denote by $w_{X}$ the word $w_{0} w_{i_{1}} \ldots w_{i_{k}} w_{4}$ if $X=\left\{i_{1}<\cdots<i_{k}\right\}$. For instance, $w_{\{1,2,3\}}=w, w_{\{1\}}=w_{0} w_{1} w_{4}$ and $w_{\{ \}}=w_{0} w_{4}$. Similarly, we denote by $\rho_{X}, \rho_{X}^{\prime}$ the corresponding runs on $w_{X}$. We will show that there exists $X \subsetneq\{1,2,3\}$ such that $\operatorname{Ratio}\left(\rho_{X}\right) \neq \operatorname{Ratio}\left(\rho_{X}^{\prime}\right)$. Suppose that for all $X \subsetneq\{1,2,3\}$, we have Ratio $\left(\rho_{X}\right)=\operatorname{Ratio}\left(\rho_{X}^{\prime}\right)$. We now show that it implies that $\operatorname{Ratio}(\rho)=\operatorname{Ratio}\left(\rho^{\prime}\right)$, which contradicts the hypothesis. For all $X \subseteq\{1,2,3\}$, we let:

$$
L_{X}=\left(\sum_{i \in X \cup\{0,4\}} r_{i}\right) \cdot\left(\sum_{i \in X \cup\{0,4\}} c_{i}^{\prime}\right) \quad R_{X}=\left(\sum_{i \in X \cup\{0,4\}} c_{i}\right) \cdot\left(\sum_{i \in X \cup\{0,4\}} r_{i}^{\prime}\right)
$$

By hypothesis, $L_{\{1,2,3\}} \neq R_{\{1,2,3\}}$ and for all $X \subsetneq\{1,2,3\}, L_{X}=R_{X}$. We now prove the following equalities:

$$
\begin{aligned}
& L_{\{ \}}+L_{\{1,2\}}+L_{\{1,3\}}+L_{\{2,3\}}-L_{\{1\}}-L_{\{2\}}-L_{\{3\}}=L_{\{1,2,3\}} \\
& R_{\{ \}}+R_{\{1,2\}}+R_{\{1,3\}}+R_{\{2,3\}}-R_{\{1\}}-R_{\{2\}}-R_{\{3\}}=R_{\{1,2,3\}}
\end{aligned}
$$

We only prove the equality with the $L$ values as it is symmetric for the $R$ values. For all $i, j \in\{0,4\}$, the subterm $r_{i} c_{j}^{\prime}$ occurs once in all expressions $L_{X}$, and $1+1+1+1-1-1-1=1$. For all $i, j \in\{1,2,3\}$ such that $i \neq j$, the subterm $r_{i} c_{j}^{\prime}$ appears once in $L_{\{i, j\}}$ and once in $L_{\{1,2,3\}}$. For all $i \in\{1,2,3\}$, the subterm $r_{i} c_{i}^{\prime}$ appears once in all $L_{X}$ such that $i \in X$, and there are exactly two such $L_{X}$ that are added to the left of the equation, one that is substracted to the left, and one added to the right. For instance, the subterm $r_{1} c_{1}^{\prime}$ appears in $L_{\{1,2,3\}}, L_{\{1,2\}}, L_{\{1,3\}}$ and $L_{\{1\}}$. It can be checked similarly that on the left of the equation, the coefficients for all other subterms are 1.

Therefore, since by hypothesis we have $L_{X}=R_{X}$ for all $X \subsetneq\{1,2,3\}$, we get $L_{\{1,2,3\}}=$ $R_{\{1,2,3\}}$, which is a contradiction. Thus there exists $X \subsetneq\{1,2,3\}$ such that $L_{X} \neq R_{X}$. In other words, there exists $X \subsetneq\{1,2,3\}$ such that $\operatorname{Ratio}\left(\rho_{X}\right) \neq \operatorname{Ratio}\left(\rho_{X}^{\prime}\right)$. This shows that when a witness of non-functionality has length at least $4 n^{2}$, we can find a strictly smaller witness of functionality. This achieves to prove the lemma.

As a consequence, we can design a non-deterministic PSpace procedure that will check non-functionality by guessing runs of length at most $4 n^{2}$, where $n$ is the number of states:

- Theorem 9. Functionality is decidable in PSpace for Ratio-automata, and in NLogSpace if the weights are encoded in unary.
- Remark. The pumping lemma states that if some Ratio-automaton with $n$ states is not functional, there exists a witness of non-functionality whose length is bounded by $4 n^{2}$, where $n$ is the number of states. Such a property also holds for Dsum $\lambda_{\lambda}$-automata (and is well-known for Sum and Avg-automata), but with the smaller bound $3 n^{2}$. Those bounds are used to state the existence of two runs on the same word such that the same pair of states is repeated 3 or 4 times along the two runs. Then it is proved that one can remove some part in between two repetitions and get a smaller word with two different output values. However for Ratio-automata, three repetitions are not enough to be able to shorten non-functionality witnesses. For instance, consider the following two runs on the alphabet $\{a, b, c, d\}$ and states $\left\{q_{I}, p, q, p_{f}, q_{f}\right\}$ where $p_{f}, q_{f}$ are final (those two runs can easily be realized by some Ratio-automaton):

$$
\begin{aligned}
& \rho: \quad q_{I} \xrightarrow{a \mid(2,2)} p \xrightarrow{\text { b|(2,1) }} p \xrightarrow{c \mid(2,2)} p \xrightarrow{d \mid(1,1)} p_{f} \\
& \rho^{\prime}: q_{I} \xrightarrow{a \mid(1,2)} q \xrightarrow{b \mid(2,1)} q \xrightarrow{c \mid(1,1)} q \xrightarrow{d \mid(2,1)} q_{f}
\end{aligned}
$$

It is easy to verify that the word $a b c d$ has two outputs given by $\rho$ and $\rho^{\prime}$ while the words $a d, a b d$ and $a c d$ has one output. For instance, the two runs $q_{I} \xrightarrow{a \mid(2,2)} p \xrightarrow{d \mid(1,1)} p_{f}$ and $q_{I} \xrightarrow{a \mid(1,2)} q \xrightarrow{d \mid(2,1)} q_{f}$ on $a d$ have both value 1 .

## 4 Decision Problems

In this section we investigate several decision problems for functional $V$-automata as defined in [10], $V \in\left\{\right.$ Sum, Avg, Dsum ${ }_{\lambda}$, Ratio $\}$. Given two $V$-automata $A, B$ over $\Sigma$ (and with the same discount factor when $V=\operatorname{Dsum}_{\lambda}$ ) and a threshold $\nu \in \mathbb{Q}$, we define the following decision problems:

```
Inclusion }\quad\mp@subsup{L}{A}{}\leq\mp@subsup{L}{B}{}\quad\mathrm{ holds if for all }w\in\mp@subsup{\Sigma}{}{+},\mp@subsup{L}{A}{}(w)\leq\mp@subsup{L}{B}{}(w
Equivalence }\quad\mp@subsup{L}{A}{}=\mp@subsup{L}{B}{}\quad\mathrm{ holds if for all }w\in\mp@subsup{\Sigma}{}{+},\mp@subsup{L}{A}{}(w)=\mp@subsup{L}{B}{}(w
~\nu-Emptiness }\quad\mp@subsup{L}{A}{~}\not=\varnothing holds if there exists w\in\mp@subsup{\Sigma}{}{+}\mathrm{ such that }\mp@subsup{L}{A}{}(w)~\nu,~\in{>,\geq
~\nu
```

It is known that inclusion is undecidable for non-deterministic Sum-automata [18], and therefore is also undecidable for Avg and Ratio-automata. To the best of our knowledge, it is open whether it is decidable for Dsum ${ }_{\lambda}$-automata.

- Theorem 10. Let $V \in\left\{\right.$ Sum, Avg, Dsum ${ }_{\lambda}$, Ratio $\}$ and let $A, B$ be two $V$-automata such that $B$ is functional. The inclusion problem $L_{A} \leq L_{B}$ is decidable. If $V \in\left\{\right.$ Sum, Avg, Dsum $\left.{ }_{\lambda}\right\}$ then it is PSpace-c and if additionnaly $B$ is deterministic, it is in PTime.

Proof. Let $V \in\left\{\operatorname{Sum}, \operatorname{Avg}, \operatorname{Dsum}_{\lambda}\right\}$. In a first step, we test the inclusion of the domains $\operatorname{dom}(A) \subseteq \operatorname{dom}(B)$ (it is PSpace-c and in PTime if $B$ is deterministic). Then we construct the product $A \times B$ as follows: $(p, q) \xrightarrow{a \mid n_{A}-n_{B}}\left(p^{\prime}, q^{\prime}\right) \in \delta_{A \times B}$ iff $p \xrightarrow{a \mid n_{A}} p^{\prime} \in \delta_{A}$ and $q \xrightarrow{a \mid n_{B}} q^{\prime} \in \delta_{B}$. Then $L_{A} \not \leq L_{B}$ iff there exists a path in $A \times B$ from a pair of initial states to a pair of accepting states with strictly positive sum if $V \in\{$ Sum, Avg $\}$, and with strictly positive discounted sum if $V=\operatorname{Dsum}_{\lambda}$. This can be checked in PTime for all those three measures, with shortest path algorithms for Sum and Avg, and as a consequence of a result of [2] about single player discounted games, for Dsum ${ }_{\lambda}$.

Let $V=$ Ratio. As for the other measures we first check inclusion of the domains. Let $\delta_{A}=\left\{x_{1}, \ldots, x_{n}\right\}$ and $\delta_{B}=\left\{y_{1}, \ldots, y_{m}\right\}$. Let $r_{A}=\left(r_{1}, \ldots, r_{n}\right)$ be the rewards associated with the transitions $x_{1}, \ldots, x_{n}$ respectively. Similarly, let $c_{A}=\left(c_{1}, \ldots, c_{n}\right)$ be the costs associated with $x_{1}, \ldots, x_{n}$. The vectors $r_{B}$ and $c_{B}$ are defined similarly.

Given a vector $a=\left(a_{1}, \ldots, a_{n}\right) \in \mathbb{N}^{n}$, we say that $a$ fits $A$ if there exists $w \in \Sigma^{+}$and an accepting run $\rho$ on $A$ such that $\rho$ passes by the transition $x_{i}$ exactly $a_{i}$ times, for all $1 \leq i \leq n$. We let $F(A)$ be the set of vectors fitting $A$ (i.e. the Parikh's image of the transitions of $A$ ). It is well-known by Parikh's theorem that $F(A)$ is a semi-linear set and can be effectively represented as the solution of a system of linear equations over natural numbers. $F(B)$ is defined similarly. We now define the set of pairs of vectors $\Gamma$ as follows:

$$
\Gamma=\left\{(a, b) \mid a \in F(A), b \in F(B), a \cdot r_{A} \cdot\left(b \cdot c_{B}\right)^{T}>a \cdot c_{A} \cdot\left(b \cdot r_{B}\right)^{T}\right\}
$$

where . denotes the pairwise multiplication, • the matrix multiplication, and.$^{T}$ the transposite. It is easy to check that $\Gamma \neq \varnothing$ iff $L_{A} \not \leq L_{B}$. The set $\Gamma$ can be defined as the solutions over natural numbers of a system of equations in linear and quadratic forms (i.e. in which products of two variables are permitted). It is decidable whether such a system has a solution [25, 14].

There is no known complexity bound for solving quadratic equations, so the proof above does not give us a complexity bound for the inclusion problem of functional Ratio-automata. However, thanks to the functionality test, which is in PSpace for Ratio-automata, we can test equivalence of two functional Ratio-automata $A_{1}$ and $A_{2}$ in PSpace: first check in PSpace that $\operatorname{dom}\left(A_{1}\right)=\operatorname{dom}\left(A_{2}\right)$ and check that the union of $A_{1}$ and $A_{2}$ is functional. This algorithm can also be used for the other measures:

- Theorem 11. Let $V \in\left\{\right.$ Sum, Avg, Dsum ${ }_{\lambda}$, Ratio $\}$. Equivalence of functional $V$-automata is PSpace-c.
- Theorem 12. Let $\nu \in \mathbb{Q}$. The $>\nu$-emptiness (resp. $\geq \nu$-emptiness) problem is in PTime for Sum-, Avg-, Ratio-, and Dsum $\lambda_{\lambda}$ automata (resp. Sum-, Avg-, and Ratio-automata).

It is open how to decide $\geq \nu$ for Dsum $_{\lambda}$-automata. Dually:

- Theorem 13. Let $\nu \in \mathbb{Q}$. The $\geq \nu$-universality (resp. $>\nu$-universality) problem is PS pacec for Sum-, Avg-, Ratio-, and Dsum $_{\lambda}$-automata (resp. Sum-, Avg-, and Ratio-automata).


## 5 Realizability

In this section, we consider the problem of quantitative language realizability. The realizability problem is better understood as a game between two players: the 'Player input' (the environment, also called Player $I$ ) and the 'Player output' (the controller, also called Player $O$ ). Player $I$ (resp. Player $O$ ) controls the letters of a finite alphabet $\Sigma_{I}$ (resp. $\Sigma_{O}$ ). We assume that $\Sigma_{O} \cap \Sigma_{I}=\varnothing$ and that $\Sigma_{O}$ contains a special symbol \# whose role is to stop the game. We let $\Sigma=\Sigma_{O} \cup \Sigma_{I}$.

Formally, the realizability game is a turn-based game played on an arena defined by a weighted automaton $A=\left(Q=Q_{O} \uplus Q_{I}, q_{0}, F, \delta=\delta_{I} \cup \delta_{O}, \gamma\right)$, whose set of states is partitioned into two sets, $\delta_{O} \subseteq Q_{O} \times \Sigma_{O} \times Q_{I}, \delta_{I} \subseteq Q_{I} \times \Sigma_{I} \times Q_{O}$, and such that $\operatorname{dom}(A) \subseteq(\Sigma \backslash\{\#\})^{*} \#$. Player $O$ starts by giving an initial letter $o_{0} \in \Sigma_{O}$, Player $I$ responds providing a letter $i_{0} \in \Sigma_{I}$, then Player $O$ gives $o_{1}$ and Player $I$ responds $i_{1}$, and so on. Player $O$ has also the power to stop the game at any turn with the distinguishing symbol \#. In this case, the game results in a finite word $\left(o_{0} i_{0}\right)\left(o_{1} i_{1}\right) \ldots\left(o_{j} i_{j}\right) \# \in \Sigma^{*}$, otherwise the outcome of the game is an infinite word $\left(o_{0} i_{0}\right)\left(o_{1} i_{1}\right) \cdots \in \Sigma^{\omega}$.

The players play according to strategies. A strategy for Player $O$ (resp. Player $I$ ) is a mapping $\lambda_{O}:\left(\Sigma_{O} \Sigma_{I}\right)^{*} \rightarrow \Sigma_{O}$ (resp. $\left.\lambda_{I}: \Sigma_{O}\left(\Sigma_{I} \Sigma_{O}\right)^{*} \rightarrow \Sigma_{I}\right)$. The outcome of the strategies $\lambda_{O}, \lambda_{I}$ is the word $w=o_{0} i_{0} o_{1} i_{1} \ldots$ denoted by outcome $\left(\lambda_{O}, \lambda_{I}\right)$ such that for all $0 \leq j \leq|w|$ (where $|w|=+\infty$ if $w$ is infinite), $o_{j}=\lambda_{O}\left(o_{0} i_{0} \ldots i_{j-1}\right)$ and $i_{j}=\lambda\left(o_{0} i_{0} \ldots o_{j}\right)$, and such that if $\#=o_{j}$ for some $j$, then $w=o_{0} i_{0} \ldots o_{j}$. We denote by $\Lambda_{O}$ (resp. $\Lambda_{I}$ ) the set of strategies for Player $O$ (resp. Player $I$ ).

A strategy $\lambda_{O} \in \Lambda_{O}$ is winning for Player $O$ if for all strategies $\lambda_{I} \in \Lambda_{I}$, outcome $\left(\lambda_{O}, \lambda_{I}\right)$ is finite and $L_{A}\left(\operatorname{outcome}\left(\lambda_{O}, \lambda_{I}\right)\right)>0$. The quantitative language realizability problem for the weighted automaton $A$ asks whether Player $O$ has a winning strategy and in that case, we say that $A$ is realizable.

Our first result on realizability is negative: we show that it is undecidable for weighted functional Sum-, Avg-automata, and Ratio-automata. In particular, we show that the halting problem for deterministic 2-counter Minsky machines 19] can be reduced to the quantitative language realizability problem for (functional) Sum-automata (resp. Avg-automata).

- Theorem 14. Let $V \in\{$ Sum, Avg, Ratio $\}$. The realizability problem for functional weighted $V$-automata is undecidable.

The proof of Theorem 14 (in Appendix) relies on the use of a nondeterministic weighted automaton. Indeed, as stated in the next theorem, the quantitative language realizability problem is decidable for the four measures when the automaton is deterministic, in NP $\cap$ coNP (see Appendix), though memoryfull strategies are necessary for winning those games.

- Theorem 15. The quantitative language realizability problem for deterministic weighted $V$-automata, $V \in\left\{\right.$ Sum, Avg, Dsum $_{\lambda}$, Ratio $\}$, is in NP $\cap$ coNP.


## 6 Determinization

A $V$-automaton $A=\left(Q, q_{I}, F, \delta, \gamma\right)$ is determinizable if it is effectively equivalent to a deterministic $V$-automaton ${ }^{3}$. $V$-automata are not determinizable in general. For example, consider the right automaton on Fig. 1. Seen as a Sum, Avg or Dsum $\lambda_{\lambda}$-automaton for any $\lambda$, it cannot be determinized, because there are infinitely many delays associated with the pair of states $(p, q)$. Those delays can for instance be obtained by the family of words of the form $a^{n}$.

We show that it can be decided whether a functional $V$-automaton is determinizable for $V \in\left\{\operatorname{Sum}, \operatorname{Avg}\right.$, Dsum $\left._{\lambda}\right\}$. However, it is still open for Ratio-automata, for which we do not have an adequate notion of delay.

To ease notations, for all $V$-automaton $A$ over an alphabet $\Sigma$, we assume that there exists a special ending symbol $\# \in \Sigma$ such that any word $w \in \operatorname{dom}(A)$ is of the form $w^{\prime} \#$ with $w^{\prime} \in(\Sigma-\#)^{*}$.

Determinizability is already known to be decidable in PTime for functional Sum-automata $[16]^{4}$. Determinizable functional Sum-automata are characterized by the so called twinning property, that has been introduced for finite word transducers [11]. Two states $p, q$ are twinned if both $p$ and $q$ are co-accessible and for all words $w_{1}, w_{2} \in \Sigma^{*}$, for all $n_{1}, n_{2}, m_{1}, m_{2} \in$ $\mathbb{Z}$, if $q_{I} \xrightarrow{w_{1} \mid n_{1}} p \xrightarrow{w_{2} \mid n_{2}} p$ and $q_{I} \xrightarrow{w_{1} \mid m_{1}} q \xrightarrow{w_{2} \mid m_{2}} q$, then $n_{2}=m_{2}$. In other words, the delays between the two runs cannot increase on the loop. If all pairs of states are twinned, then it is proved that the number of different accumulated delays on parallel runs is finite. The determinization for Sum-automata extends the classical determinization procedure of finite automata with delays. States are (partial) functions from states to delays. Clearly, a Sum-automaton $A$ is determinizable iff the Avg-automaton $A$ is determinizable. We can even use exactly the same determinization procedure as for Sum-automata.

- Theorem 16 ([16]). It is decidable in PTime whether a functional Sum or Avg-automaton is determinizable.

We now explain the determinization procedure for Dsum $\lambda_{\lambda}$-automata.

- Definition 17. We say that two states $p, q$ are twinned if both $p$ and $q$ are co-accessible and for all words $w_{1}, w_{2} \in \Sigma^{*}$, for all runs $\rho_{1}: q_{I} \xrightarrow{w_{1}} p, \rho_{2}: p \xrightarrow{w_{2}} p, \rho_{1}^{\prime}: q_{I} \xrightarrow{w_{1}} q$, $\rho_{2}^{\prime}: q \xrightarrow{w_{2}} q$, we have delay $\left(\rho_{1}, \rho_{1}^{\prime}\right)=\operatorname{delay}\left(\rho_{1} \rho_{2}, \rho_{1}^{\prime} \rho_{2}^{\prime}\right)$.

A Dsum $\lambda_{\lambda}$-automaton $A$ satisfies the twinning property if all pairs of states are twinned. We show that the twinning property is a decidable characterization of determinizable functional Dsum $\lambda_{\lambda}$-automata. First, we prove that it is decidable in PSpace:

[^2]- Lemma 18. Is it decidable in PSpace whether a Dsum $_{\lambda}$-automaton satisfies the twinning property.

We denote by $\mathcal{D}$ the set of possible delays between two runs of $A$, i.e. $\mathcal{D}$ is the set of delays delay $\left(\rho, \rho^{\prime}\right)$ for all runs $\rho, \rho^{\prime}$ on the same input word, such that the last states of $\rho$ and $\rho^{\prime}$ are both co-accessible.

- Lemma 19. If the twinning property holds, then $\mathcal{D}$ is finite of size at most $|\Sigma|^{|Q|^{2}}$.

Proof. As delays must be identical on parallel loops, any delay can be obtained with some pair of runs of length $|Q|^{2}$ at most (on longer pairs of runs, there must exist a parallel loop with identical delays that can be removed without affecting the value of the global delay of both runs, see Lemma 24 of the Appendix).

Determinization Assume that the twinning property holds. We define a determinization procedure that constructs from a functional Dsum $\lambda_{\lambda}$-automaton $A=\left(Q, q_{I}, F, \delta, \gamma\right)$ a deterministic Dsum $\lambda_{\lambda}$-automaton $A_{d}=\left(Q_{d}, f_{d}, F_{d}, \delta_{d}, \gamma_{d}\right)$. Wlog we assume that all states are co-accessible (otherwise we can remove non co-accessible states in linear time). We define $Q^{\prime}=\mathcal{D}^{Q}$ (which is finite by Lemma [19), the set of partial functions from states $Q$ to delays.We let $f_{I}^{\prime}: q_{I} \mapsto 0$ and $F^{\prime}$ is defined as $\left\{f \in Q^{\prime} \mid \operatorname{dom}(f) \cap F \neq \varnothing\right\}$. Then, given partial functions $f, f^{\prime} \in Q^{\prime}$ and a symbol $a \in \Sigma$, we let:

$$
\begin{aligned}
& \gamma^{\prime}\left(f, a, f^{\prime}\right)= \min \left\{\left.\frac{f(q)}{\lambda}+\gamma\left(q, a, q^{\prime}\right) \right\rvert\, q \in \operatorname{dom}(f) \wedge\left(q, a, q^{\prime}\right) \in \delta\right\} \\
&\left(f, a, f^{\prime}\right) \in \delta^{\prime} \quad \text { iff } \quad \text { for all } q^{\prime} \in \operatorname{dom}\left(f^{\prime}\right) \text { there exists } q \in \operatorname{dom}(f) \text { such that }\left(q, a, q^{\prime}\right) \in \delta \text { and } \\
& f^{\prime}\left(q^{\prime}\right)=\frac{f(q)}{\lambda}+\gamma\left(q, a, q^{\prime}\right)-\gamma^{\prime}\left(f, a, f^{\prime}\right)
\end{aligned}
$$

Let $Q_{d} \subseteq Q^{\prime}$ be the accessible states of $A^{\prime}:=\left(Q^{\prime}, f_{I}^{\prime}, F^{\prime}, \delta^{\prime}, \gamma^{\prime}\right)$. We define $A_{d}=\left(Q_{d}, f_{d}, \delta_{d}, \gamma_{d}\right)$ as the restriction of $A^{\prime}$ to the accessible states.

- Lemma 20. If the twinning property holds, $A_{d}$ and $A$ are equivalent, $A_{d}$ is deterministic and has $O\left(|\Sigma|^{|Q|^{3}}\right)$ states.

The proof is based on the following lemma:

- Lemma 21. Let $f \in Q_{d}$ be state of $A_{d}$ accessible by a run $\rho_{d}$ on some word $w \in \Sigma^{*}$. Then $\operatorname{dom}(f)$ is the set of states $q$ such that there exists a run on $w$ reaching $q$. Moreover, if $q \in$ $\operatorname{dom}(f)$ and $\rho$ is a run on $w$ reaching $q$, then $f(q)=\max \left\{\operatorname{delay}\left(\rho, \rho^{\prime}\right) \mid \rho^{\prime}\right.$ is a run of $A$ on $\left.w\right\}=$ $\frac{\operatorname{Dsum}_{\lambda}(\rho)-\operatorname{Dsum}_{\lambda}\left(\rho_{d}\right)}{\lambda^{|w|}}$ and $\operatorname{Dsum}_{\lambda}\left(\rho_{d}\right)=\min \left\{\operatorname{Dsum}_{\lambda}(\xi) \mid \xi\right.$ is a run of $A$ on $\left.w\right\}$.

If the twinning property does not hold, we show that $\mathcal{D}$ is infinite and that $A$ cannot be determinized. Therefore we get the following theorem:

- Theorem 22. A functional Dsum $_{\lambda}$-automaton is determinizable iff it satisfies the twinning property. Therefore determinizability is decidable in PSpace for functional Dsum $_{\lambda}$-automata.


## References

1 Benjamin Aminof, Orna Kupferman, and Robby Lampert. Rigorous approximated determinization of weighted automata. In LICS, pages 345-354, 2011.
2 D Andersson. An improved algorithm for discounted payoff games. In ESSLLI Student Session, pages 91-98, 2006.

3 Beal, Carton, Prieur, and Sakarovitch. Squaring transducers: An efficient procedure for deciding functionality and sequentiality. TCS: Theoretical Computer Science, 292, 2003.
4 Jean Berstel and Christophe Reutenauer. Rational Series and Their Languages. Number 12 in EATCS Monographs on Theoretical Computer Science. Springer-Verlag, 1988.
5 Henrik Björklund and Sergei Vorobyov. A combinatorial strongly subexponential strategy improvement algorithm for mean payoff games. Discrete Appl. Math., 155:210-229, 2007.
6 Meera Blattner and Tom Head. Single-valued $a$-transducers. JCSS, 15(3):310-327, 1977.
7 Roderick Bloem, Karin Greimel, Thomas A. Henzinger, and Barbara Jobstmann. Synthesizing robust systems. In FMCAD, pages 85-92. IEEE, 2009.
8 Udi Boker, Krishnendu Chatterjee, Thomas A. Henzinger, and Orna Kupferman. Temporal specifications with accumulative values. In LICS, pages 43-52, 2011.
9 Udi Boker and Thomas A. Henzinger. Determinizing discounted-sum automata. In CSL, pages 82-96, 2011.
10 Krishnendu Chatterjee, Laurent Doyen, and Thomas A. Henzinger. Quantitative languages. ACM Trans. Comput. Log, 11(4), 2010.
11 Christian Choffrut. Une caractérisation des fonctions séquentielles et des fonctions sousséquentielles en tant que relations rationnelles. Theor. Comput. Sci., 5(3):325-337, 1977.
12 Luca de Alfaro, Marco Faella, Thomas A. Henzinger, Rupak Majumdar, and Mariëlle Stoelinga. Model checking discounted temporal properties. Theor. Comput. Sci., 345(1):139-170, 2005.
13 Manfred Droste and George Rahonis. Weighted automata and weighted logics with discounting. In CIAA, pages 73-84, 2007.
14 Fritz Grunewald and Dan Segal. On the integer solutions of quadratic equations. Journal für die reine und angewandte Mathematik, 569:13-45, 2004.
15 Daniel Kirsten. A burnside approach to the termination of mohri's algorithm for polynomially ambiguous min-plus-automata. ITA, 42(3):553-581, 2008.
16 Daniel Kirsten and Ina Mäurer. On the determinization of weighted automata. Journal of Automata, Languages and Combinatorics, 10(2/3):287-312, 2005.
17 Ines Klimann, Sylvain Lombardy, Jean Mairesse, and Christophe Prieur. Deciding unambiguity and sequentiality from a finitely ambiguous max-plus automaton. Theor. Comput. Sci., 327(3):349-373, 2004.
18 Daniel Krob. The equality problem for rational series with multiplicities in the tropical semiring is undecidable. Int. Journal of Algebra and Computation, 4(3):405-425, 1994.
19 Marvin L. Minsky. Computation: Finite and Infinite Machines. Prentice-Hall, 1967.
20 Mehryar Mohri. Weighted automata algorithms. Handbook of Weighted Automata, pages 213-254, 2009.
21 A. Pnueli and R. Rosner. On the synthesis of a reactive module. In ACM Symposium on Principles of Programming Languages (POPL). ACM, 1989.
22 Marcel Paul Schützenberger. Sur les relations rationnelles. In Automata Theory and Formal Languages, volume 33 of LNCS, pages 209-213, 1975.
23 Wolfgang Thomas. Church's problem and a tour through automata theory. In Pillars of Computer Science, pages 635-655, 2008.
24 Andreas Weber and Reinhard Klemm. Economy of description for single-valued transducers. Inf. Comput., 118(2):327-340, 1995.
25 Karianto Wong, Aloys Krieg, and Wolfgang Thomas. On intersection problems for polynomially generated sets. In $I C A L P$, volume 4052 of $L N C S$, pages 516-527. Springer, 2006.

## A Quantitative Languages and Functionality

## A. 1 Functionality and Unambiguity

- Lemma 23. Let $V \in\left\{\right.$ Sum, Avg, Dsum ${ }_{\lambda}$, Ratio $\}$. For all functional $V$-automaton with $n$ states we can construct an equivalent unambiguous $V$-automaton with $O\left(n .2^{n}\right)$ states.

Proof. Our proof is independent on the measure. Let $A=\left(Q, q_{I}, F, \delta, \gamma\right)$ be a functional $V$-automaton. We order the transitions of $\delta$ by a total order denoted by $<_{\delta}$. We construct an equivalent unambiguous automaton $A^{\prime}=\left(Q^{\prime}, q_{I}^{\prime}, F^{\prime}, \delta^{\prime}, \gamma^{\prime}\right)$, where:

- $Q^{\prime}=Q \times 2^{Q}$;
- $q_{i}^{\prime}=\left(q_{I}, \varnothing\right)$;
- $F^{\prime}=F \times\{P \subseteq Q \mid F \cap P=\varnothing\}$;
- $\gamma^{\prime}:\left((p, P), a,\left(p^{\prime}, P^{\prime}\right)\right) \mapsto \gamma\left(p, a, p^{\prime}\right)$;

Before defining $\delta^{\prime}$ formally, let us explain intuitively the semantics of the states in $Q^{\prime}$. The automaton $A^{\prime}$ will guess a run of $A$ on first state component (called the current run). A pair $(p, P)$ represents the state $p$ of current run in the original automaton $A$ while $P$ represents the states reached by all the runs that are greater than the current run (for the order $<_{\delta}$ lexigraphically extended to runs). At the end of the word, the run is accepting iff $p$ is accepting and there is no accepting state in $P$. In other words, a run of $A^{\prime}$ on a word $w$ is accepting iff the run it defines on the first component is the smallest accepting run of $A$ on $w$.

When a new letter $a \in \Sigma$ is read, $A^{\prime}$ guesses a transition from $p$ to some state $p^{\prime}$, and goes to the state ( $p^{\prime}, S_{P} \cup S_{p, a, p^{\prime}}$ ), where $S_{P}$ are the successor states of of $P$ by $\delta$ on the input $a$, and $S_{p, a, p^{\prime}}$ are all the states reached from $p$ by a transition on $a$ bigger than $\left(p, q, p^{\prime}\right)$.

Formally, $\left((p, P), a,\left(p^{\prime}, P^{\prime}\right)\right) \in \delta^{\prime}$ iff

- $\left(p, a, p^{\prime}\right) \in \delta$;
- $P^{\prime}=\left\{q^{\prime} \mid \exists q \in P,\left(q, a, q^{\prime}\right) \in \delta\right\} \cup\left\{p^{\prime \prime} \mid\left(p, a, p^{\prime \prime}\right) \in \delta \wedge\left(p, a, p^{\prime}\right)<_{\delta}\left(p, a, p^{\prime \prime}\right)\right\}$.

It is clear by construction that $A$ and $A^{\prime}$ defines the same domain. As $A$ is functional, they also define the same function, because the value of a word is equal to the value of any run on it, and in particular to the value of the smallest run.

## B Functionality

## B. 1 Proof of Lemma 5

Proof. Consider a co-accessible pair of states $(p, q)$. Assume that $(p, q)$ admits two delays $d_{1}, d_{2}$. We show that if $A$ is functional, then $d_{1}=d_{2}$. Let $\rho_{1}: q_{0} \stackrel{w_{1}}{\sim} p, \rho_{1}^{\prime}: q_{0} \stackrel{w_{1}}{\sim} q$ (resp. $\rho_{2}: q_{0} \stackrel{w_{2}}{\rightsquigarrow} p, \rho_{2}^{\prime}: q_{0} \stackrel{w_{2}}{\rightsquigarrow} q$ ) be two runs witnessing the delay $d_{1}$ (resp. $d_{2}$ ), i.e.:

$$
\frac{\operatorname{Dsum}_{\lambda}\left(\rho_{1}\right)-\operatorname{Dsum}_{\lambda}\left(\rho_{1}^{\prime}\right)}{\lambda^{\left|w_{1}\right|}}=d_{1}, \frac{\operatorname{Dsum}_{\lambda}\left(\rho_{2}\right)-\operatorname{Dsum}_{\lambda}\left(\rho_{2}^{\prime}\right)}{\lambda^{\left|w_{2}\right|}}=d_{2}
$$

Since $(p, q)$ is co-accessible, there exists a word $u$ and two runs $\rho_{3}: p \stackrel{u}{\rightsquigarrow} f \in F, \rho_{3}^{\prime}: q \stackrel{u}{\rightsquigarrow}$ $f^{\prime} \in F$. Moreover, the hypothesis of functionality on $A$ implies:

$$
\begin{align*}
& \operatorname{Dsum}_{\lambda}\left(\rho_{1} \rho_{3}\right)-\operatorname{Dsum}_{\lambda}\left(\rho_{1}^{\prime} \rho_{3}^{\prime}\right)=0  \tag{1}\\
& \operatorname{Dsum}_{\lambda}\left(\rho_{2} \rho_{3}\right)-\operatorname{Dsum}_{\lambda}\left(\rho_{2}^{\prime} \rho_{3}^{\prime}\right)=0 \tag{2}
\end{align*}
$$

Let $v_{1,0} \ldots v_{1, m}$ (resp. $v_{1,0}^{\prime} \ldots v_{1, m}^{\prime}, v_{3,0} \ldots v_{3, l}, v_{3,0}^{\prime} \ldots v_{3, l}^{\prime}$ ) the sequence of weights occuring along $\rho_{1}$ (resp. $\rho_{1}^{\prime}, \rho_{3}, \rho_{3}^{\prime}$ ), where $m=\left|w_{1}\right|-1$ and $l=|u|-1$.

Then, Equation 1 implies:

$$
\begin{equation*}
\sum_{i=0}^{m} v_{1, i} \lambda^{i}+\lambda^{m+1} \sum_{i=0}^{l} v_{3, i} \lambda^{i}=\sum_{i=0}^{m} v_{1, i}^{\prime} \lambda^{i}+\lambda^{m+1} \sum_{i=0}^{l} v_{3, i}^{\prime} \lambda^{i} \tag{3}
\end{equation*}
$$

Let $v_{2,0} \ldots v_{2, n}$ (resp. $v_{2,0}^{\prime} \ldots v_{2, n}^{\prime}$ ) be the sequence of weights occuring along $\rho_{2}$ (resp. $\rho_{2}^{\prime}$ ), where $n=\left|w_{2}\right|-1$.

Then, Equation 2 implies:

$$
\begin{equation*}
\sum_{i=0}^{n} v_{2, i} \lambda^{i}+\lambda^{n+1} \sum_{i=0}^{l} v_{3, i} \lambda^{i}=\sum_{i=0}^{n} v_{2, i}^{\prime} \lambda^{i}+\lambda^{n+1} \sum_{i=0}^{l} v_{3, i}^{\prime} \lambda^{i} \tag{4}
\end{equation*}
$$

Equations 3 and 4 yield:

$$
\begin{align*}
\left(\sum_{i=0}^{m} v_{1, i} \lambda^{i}-\sum_{i=0}^{m} v_{1, i}^{\prime} \lambda^{i}\right) & =\lambda^{m+1}\left(\sum_{i=0}^{l} v_{3, i}^{\prime} \lambda^{i}-\sum_{i=0}^{l} v_{3, i} \lambda^{i}\right)  \tag{5}\\
\left(\sum_{i=0}^{n} v_{2, i} \lambda^{i}-\sum_{i=0}^{n} v_{2, i}^{\prime} \lambda^{i}\right) & =\lambda^{n+1}\left(\sum_{i=0}^{l} v_{3, i}^{\prime} \lambda^{i}-\sum_{i=0}^{l} v_{3, i} \lambda^{i}\right) \tag{6}
\end{align*}
$$

Dividing both the members of Equation 5 by $\lambda^{m+1}$ and both the members of Equation 6 by $\lambda^{n+1}$, and finally subtracting the obtained results, we get our thesis i.e.

$$
\frac{\operatorname{Dsum}_{\lambda}\left(\rho_{1}\right)-\operatorname{Dsum}_{\lambda}\left(\rho_{1}^{\prime}\right)}{\lambda^{\left|w_{1}\right|}}=\frac{\operatorname{Dsum}_{\lambda}\left(\rho_{2}\right)-\operatorname{Dsum}_{\lambda}\left(\rho_{2}^{\prime}\right)}{\lambda^{\left|w_{2}\right|}}
$$

## B. 2 Proof of Lemma 6

We first prove the following key result:

- Lemma 24. Let $A=\left(Q, q_{I}, F, \delta, \gamma\right)$ be a Dsum $\lambda_{\lambda}$-automaton. Let $w_{1}, w_{2}, w_{3} \in \Sigma^{*}$ such that there exist $p, p^{\prime}, q, q^{\prime} \in Q$ and the following runs:

$$
\begin{array}{lll}
\rho_{1}: q_{I} \xrightarrow{w_{1}} p & \rho_{2}: p \xrightarrow{w_{2}} p & \rho_{3}: p \xrightarrow{w_{3}} q \\
\rho_{1}^{\prime}: q_{I} \xrightarrow{w_{1}} p^{\prime} & \rho_{2}: p^{\prime} \xrightarrow{w_{2}} p^{\prime} & \rho_{3}: p^{\prime} \xrightarrow{w_{3}} q^{\prime}
\end{array}
$$

and such that $\operatorname{delay}\left(\rho_{1}, \rho_{1}^{\prime}\right)=\operatorname{delay}\left(\rho_{1} \rho_{2}, \rho_{1}^{\prime} \rho_{2}^{\prime}\right)$. Then $\operatorname{delay}\left(\rho_{1} \rho_{2} \rho_{3}, \rho_{1}^{\prime} \rho_{2}^{\prime} \rho_{3}^{\prime}\right)=\operatorname{delay}\left(\rho_{1} \rho_{3}, \rho_{1}^{\prime} \rho_{3}^{\prime}\right)$.
Proof. By hypothesis, we have the following equality:

$$
\begin{equation*}
\frac{\operatorname{Dsum}_{\lambda}\left(\rho_{1}\right)+\lambda^{\left|w_{1}\right|} \operatorname{Dsum}_{\lambda}\left(\rho_{2}\right)-\operatorname{Dsum}_{\lambda}\left(\rho_{1}^{\prime}\right)-\lambda^{\left|w_{1}\right|} \operatorname{Dsum}_{\lambda}\left(\rho_{2}^{\prime}\right)}{\lambda^{\left|w_{1}\right|+\left|w_{2}\right|}}=\frac{\operatorname{Dsum}_{\lambda}\left(\rho_{1}\right)-\operatorname{Dsum}_{\lambda}\left(\rho_{2}^{\prime}\right)}{\lambda^{\left|w_{1}\right|}} \tag{7}
\end{equation*}
$$

which implies:
$\operatorname{Dsum}_{\lambda}\left(\rho_{1}\right)+\lambda^{\left|w_{1}\right|} \operatorname{Dsum}_{\lambda}\left(\rho_{2}\right)-\operatorname{Dsum}_{\lambda}\left(\rho_{1}^{\prime}\right)-\lambda^{\left|w_{1}\right|} \operatorname{Dsum}_{\lambda}\left(\rho_{2}^{\prime}\right)=\lambda^{\left|w_{2}\right|}\left(\operatorname{Dsum}_{\lambda}\left(\rho_{1}\right)-\operatorname{Dsum}_{\lambda}\left(\rho_{2}^{\prime}\right)\right)$

By definition of the delays, we have:
$\operatorname{delay}\left(\rho_{1} \rho_{2} \rho_{3}, \rho_{1}^{\prime} \rho_{2}^{\prime} \rho_{3}^{\prime}\right)=$
$\frac{\operatorname{Dsum}_{\lambda}\left(\rho_{1}\right)+\lambda^{\mid w_{1}} \operatorname{Dsum}_{\lambda}\left(\rho_{2}\right)+\lambda^{\left|w_{1} w_{2}\right|} \operatorname{Dsum}_{\lambda}\left(\rho_{3}\right)-\operatorname{Dsum}_{\lambda}\left(\rho_{1}^{\prime}\right)-\lambda^{\mid w_{1}} \operatorname{Dsum}_{\lambda}\left(\rho_{2}^{\prime}\right)-\lambda^{\left|w_{1} w_{2}\right|} \operatorname{Dsum}_{\lambda}\left(\rho_{3}^{\prime}\right)}{\lambda^{\left|w_{1} w_{2} w_{3}\right|}}$

Thanks to Equation 8, it can be simplified into:
$\operatorname{delay}\left(\rho_{1} \rho_{2} \rho_{3}, \rho_{1}^{\prime} \rho_{2}^{\prime} \rho_{3}^{\prime}\right)=\frac{\lambda^{\left|w_{2}\right|}\left(\operatorname{Dsum}_{\lambda}\left(\rho_{1}\right)-\operatorname{Dsum}_{\lambda}\left(\rho_{1}^{\prime}\right)\right)+\lambda^{\left|w_{1} w_{2}\right|} \operatorname{Dsum}_{\lambda}\left(\rho_{3}\right)-\lambda^{\left|w_{1} w_{2}\right|} \operatorname{Dsum}_{\lambda}\left(\rho_{3}^{\prime}\right)}{\lambda^{\left|w_{1} w_{2} w_{3}\right|}}$

We can now simplify this expression by $\lambda^{\left|w_{2}\right|}$ and we get:
$\operatorname{delay}\left(\rho_{1} \rho_{2} \rho_{3}, \rho_{1}^{\prime} \rho_{2}^{\prime} \rho_{3}^{\prime}\right)=\frac{\operatorname{Dsum}_{\lambda}\left(\rho_{1}\right)-\operatorname{Dsum}_{\lambda}\left(\rho_{1}^{\prime}\right)+\lambda^{\left|w_{1}\right|} \operatorname{Dsum}_{\lambda}\left(\rho_{3}\right)-\lambda^{\left|w_{1}\right|} \operatorname{Dsum}_{\lambda}\left(\rho_{3}^{\prime}\right)}{\lambda^{\left|w_{1} w_{3}\right|}}$
which exactly means: $\operatorname{delay}\left(\rho_{1} \rho_{2} \rho_{3}, \rho_{1}^{\prime} \rho_{2}^{\prime} \rho_{3}^{\prime}\right)=\operatorname{delay}\left(\rho_{1} \rho_{3}, \rho_{1}^{\prime} \rho_{3}^{\prime}\right)$.

Lemma 6. Let $w \in \operatorname{dom}(A)$ such that $\left|R_{A}(w)\right|>1$. Clearly, there exist two runs $\rho, \rho^{\prime}$ on $w$ such that $\operatorname{Dsum}_{\lambda}(\rho) \neq \operatorname{Dsum}_{\lambda}\left(\rho^{\prime}\right)$. Moreover if $\rho$ and $\rho^{\prime}$ can be decomposed so that the premises of Lemma 24 are satisfied, then we can find a strictly shorter word with two runs on it which have different delays. We can repeat this operation until the goals of Lemma 6 are satisfied.

## B. 3 Proof of Theorem 7

Proof. We start to prove that $A$ is not functional iff $\operatorname{DSumFunTest}(A)$ returns No.
$(\Leftarrow)$ The following invariant holds overall the execution of the algorithm: If the stack $S$ contains the pair $((p, q), d)$, then $(p, q)$ is co-accessible and $A$ admits two runs $\rho: q_{I} \xrightarrow{w}$ $p, \rho^{\prime}: q_{I} \xrightarrow{w} q$ such that $\frac{\operatorname{Dsum}_{\lambda}(\rho)-\operatorname{Dsum}_{\lambda}(\rho)}{\lambda^{|w|}}=d$. This can be proved by a simple inductive argument on the number of iterations of the while loop at Line 3. Suppose that DSumFunTest $(A)$ returns No. There are two cases to consider. If the pair $\left((p, q) \in F^{2}, d \neq 0\right)$ is popped from the stack, then it witnesses the existence of two accepting runs $\rho: q_{I} \xrightarrow{w}$ $p, \rho^{\prime}: q_{I} \xrightarrow{w} q$ in $A$ for which $\operatorname{Dsum}_{\lambda}(\rho)-\operatorname{Dsum}_{\lambda}\left(\rho^{\prime}\right) \neq 0$. Thus, $A$ is not functional. In the second case, the pair $((p, q), d)$ popped from the stack witnesses that $A$ admits two delays for the co-accessible pair of states $(p, q)$. By Lemma 5is implies that $A$ is not functional.
$(\Rightarrow)$ Let $A$ be a non functional Dsum $\lambda_{\lambda}$-automaton. By Lemma 6, there exists a word $w$ such that $A$ admits two runs $\rho: q_{I} \xrightarrow{w} q_{f} \in F, \rho^{\prime}: q_{I} \xrightarrow{w} q_{f}^{\prime} \in F$ on $w$ such that $\operatorname{Dsum}_{\lambda}(\rho) \neq \operatorname{Dsum}_{\lambda}\left(\rho^{\prime}\right)$. Suppose that for all positions $i<j,\left(p_{i}, q_{i}\right) \neq\left(p_{j}, q_{j}\right)$. In that case DSumFunTest $(A)$ will output No at Line 5 (if not before). Otherwise let $i_{2}$ be the least position on the two runs $\rho, \rho^{\prime}$ such that there exists $i_{1}<i_{2}$ such that $\rho$ (resp. $\rho^{\prime}$ ) reach the state $p$ (resp. q) at positions $i_{1}$ and $i_{2}$. By condition (ii) of Lemma 6, the delays are different at $i_{1}$ and $i_{2}$ and therefore $\operatorname{DSumFunTest}(A)$ will determine that $A$ is not functional at Line 6, after processing $\rho, \rho^{\prime}$ upon position $i_{2}$ (if not before).

This algorithm terminates as any pair of states is visited at most twice. It is well-known that co-accessible states of a finite automaton can be computed in linear time. We can apply this procedure on the product of $A$ with itself to compute the co-accessible pairs of states in quadratic time.

## B. 4 Proof of Theorem 9

Proof. We give an non-deterministic CoPSpace (and hence PSpace) algorithm to test nonfunctionality. By the pumping lemma, if a ratio-automaton with $n$ states is not functional, there exists a word of length at most $4 n^{2}$ with two different values. We use the following nondeterministic iterative algorithm: non-deterministically choose two transitions in parallel on the same input letter and count the current length (up to $4 n^{2}$ ), and compute the current respective rewards and costs of the two current runs. Non-deterministically choose to stop before the length exceeds $4 n^{2}$ and check that the reached states are accepting and that the respective ratio of the two chosen runs are different. We therefore need to store the two current states, reward sum and cost sum, which are bounded by $4 n^{2} M_{r}$ and $4 n^{2} M_{c}$, where $M_{r}$ and $M_{c}$ are the maximal reward and cost (in absolute value). Those sums are represented in pspace, and in nlogspace if the weights are unary encoded.

## C Decision Problems

## C. 1 Proof of Theorem 12

Proof. For Dsum $\lambda_{\lambda}$ automata, we show that $L_{A}^{>\nu} \neq \emptyset$ iff Player 0 has a strategy to ensure a play from $v_{0}$ with discounted sum greater than $\nu$ in the one player (infinite) Dsum $\lambda_{\lambda}$ game $\Gamma=\left(V, E, w,\left\langle V_{0}, V_{1}\right\rangle\right)$, where:

- $V=\left\{p \mid p \in Q \wedge \exists w \in \Sigma^{*}(p \stackrel{w}{\rightsquigarrow} f \in F)\right\}$
- $V_{0}=V, V_{1}=\emptyset$
- $E=(V \times(\Sigma \cup \zeta) \times V) \cap\left(\left\{\left(p, a, p^{\prime}\right) \mid\left(p, a, p^{\prime}\right) \in \delta\right\} \cup\{(p, \zeta, p) \mid p \in F\}\right)$, where $\zeta \notin \Sigma$ is a fresh symbol
- For each $e=\left(p, a, p^{\prime}\right) \in E$ : If $\left(p, a, p^{\prime}\right) \in \delta$, then $w(e)=\gamma\left(p, a, p^{\prime}\right)$, else $w(e)=0$.

Once proved the above equivalence, our complexity bound follows easily, since checking wether $L_{A}^{>\nu} \neq \emptyset$ reduces to solving a 1 player Dsum $\lambda_{\lambda}$ game (that is in PTime [2]).
$(\Rightarrow)$ If $L_{A}^{>\nu} \neq \emptyset$, then $A$ admits an accepting run $r: q_{0}^{1}=r_{0} \xrightarrow[\rightsquigarrow]{w} r_{n} \in F$ such that $\operatorname{Dsum}_{\lambda}(\gamma(r)>\nu$. By construction, $\Gamma$ admits an (infinite) path $p$ with a positive discounted sum, i.e. Player 0 has a (memoryless) strategy to win the one-player discounted sum game $\Gamma$.
$(\Leftarrow)$ S uppose that Player 0 has a strategy to win the one-player discounted sum game $\Gamma$. Let $p$ be an infinite path on $\Gamma$ consistent with a winning strategy for player 0 . Then $\operatorname{Dsum}_{\lambda}(r)>0$. Let $W$ be the maximum absolute weight in $\Gamma$. For each prefix $r_{i}$ of length $i$ of $r$ we have:

$$
\begin{align*}
& \operatorname{Dsum}_{\lambda}\left(r_{i}\right)+\frac{W}{1-\lambda} \geq \operatorname{Dsum}_{\lambda}(r) \Rightarrow \\
& \operatorname{Dsum}_{\lambda}\left(r_{i}\right) \geq \operatorname{Dsum}_{\lambda}(r)-\frac{W \lambda^{i}}{1-\lambda} \tag{12}
\end{align*}
$$

Since $\operatorname{Dsum}_{\lambda}(r)>\nu$, there exists $i^{*}$ such that $\operatorname{Dsum}_{\lambda}(r)-\frac{W \lambda^{i^{*}}}{1-\lambda}>\nu$ that implies $\operatorname{Dsum}_{\lambda}\left(r_{i^{*}}\right)>$ $\nu$. By construction, each path in $\Gamma$ can be extended to reach a node in $F$. Let $r_{i}^{\prime}=$ $r_{0}^{\prime} \ldots r_{m}^{\prime} \in F$ be such a continuation of $r^{i^{*}}$. By Equation 12, our choice of $i^{*}$ guarantees
that $\operatorname{Dsum}_{\lambda}\left(r_{i}^{\prime}\right)>\nu$. Since $A$ is functional, $r^{\prime}$ witnesses the existence of a word $w$ such that $L_{A}(w)>\nu$.

For Sum automata, let $A$ be a Sum-automaton. $L_{A}^{\sim_{A}^{\nu}} \neq \emptyset$ iff $A$ admits a path to a final state whose sum of the weights is $\sim \nu$. This can be easily checked in PTime, using e.g. a shortest path algorithm (once the edges have been reversed).

For Avg-automata, let $A$ be an Avg-automaton. We can assume $\nu=0$ since the $\sim \nu$-emptiness problem for Avg-automata reduces to the $\sim 0$-emptiness problem for Avgautomata, by simply reweighting the input automaton [5]. $L_{A}^{\sim^{0}} \neq \emptyset$ iff $A$ admits a path to a final state whose sum of the weights is $\sim 0$, that can be easily checked in PTime.

Finally, let $A$ be a Ratio-automaton, let $\nu=\frac{m}{n}$. We consider the Sum automaton $A^{\prime}$, where each edge of $A$ having reward $r$ and cost $c$ is replaced by an edge of weight $r n-c m$. It can be easily proved that $L_{A}^{\sim^{\nu}} \neq \emptyset$ iff $L_{A^{\prime}}^{\sim} \neq \emptyset$.

## C. 2 Proof of Theorem 13

Proof. Let $A$ be a $V$-automaton, $V \in\left\{\operatorname{Sum}, \operatorname{Avg}\right.$, Dsum $\left._{\lambda}\right\}$ and consider the $\geq \nu$-universality (resp. $>\nu$-universality) problem for $V$-automata. We check wether $A$ admits an accepting run with $V(\gamma(r))<\nu$. This can be done in PTime for $V \in\left\{\right.$ Sum-, Avg-, Ratio, Dsum $\left.{ }_{\lambda}-\right\}$ (resp. $V \in\{$ Sum - , Avg - , Ratio $\}$ ), with a procedure similar to the one applied in the proof of Theorem 12 ,

## D Realizability

## D. 1 Proof of Theorem 14

Proof. A 2-counter machine $M$ consists of a finite set of control states $S$, an initial state $s_{I} \in S$, a final state $s_{F} \in Q$, a set $C$ of counters $(|C|=2)$ and a finite set $\delta_{M}$ of instructions manipulating two integer-valued counters. Instructions are of the form

## $c:=c+1$ goto $s^{\prime}$

$s$ : if $c=0$ then goto $s^{\prime}$ else $c:=c-1$ goto $s^{\prime \prime}$.
Formally, instructions are tuples $\left(s, \alpha, c, s^{\prime}\right)$ where $s, s^{\prime} \in S$ are source and target states respectively, the action $\alpha \in\{i n c, d e c, 0 ?\}$ applies to the counter $c \in C$. We assume that $M$ is deterministic: for every state $s \in S$, either there is exactly one instruction $(s, \alpha, \cdot, \cdot) \in \delta_{M}$ and $\alpha=i n c$, or there are two instructions $(s, d e c, c, \cdot),(s, 0 ?, c, \cdot) \in \delta_{M}$.

A configuration of $M$ is a pair $(s, v)$ where $s \in S$ and $v: C \rightarrow \mathbb{N}$ is a valuation of the counters. An accepting run of $M$ is a finite sequence $\pi=\left(s_{0}, v_{0}\right) \delta_{0}\left(s_{1}, v_{1}\right) \delta_{1} \ldots$ $\delta_{n-1}\left(s_{n}, v_{n}\right)$ where $\delta_{i}=\left(s_{i}, \alpha_{i}, c_{i}, s_{i+1}\right) \in \delta_{M}$ are instructions and ( $s_{i}, v_{i}$ ) are configurations of $M$ such that $s_{0}=s_{I}, v_{0}(c)=0$ for all $c \in C, s_{n}=s_{F}$, and for all $0 \leq i<n$, we have $v_{i+1}(c)=v_{i}(c)$ for $c \neq c_{i}$, and (a) if $\alpha=\operatorname{inc}$, then $v_{i+1}\left(c_{i}\right)=v_{i}\left(c_{i}\right)+1(b)$ if $\alpha=d e c$, then $v_{i}\left(c_{i}\right) \neq 0$ and $v_{i+1}\left(c_{i}\right)=v_{i}\left(c_{i}\right)-1$, and $(c)$ if $\alpha=0$ ?, then $v_{i+1}\left(c_{i}\right)=v_{i}\left(c_{i}\right)=0$. The corresponding run trace of $\pi$ is the sequence of instructions $\bar{\pi}=\delta_{0} \delta_{1} \ldots \delta_{n-1}$. The halting problem is to decide, given a 2-counter machine $M$, whether $M$ has an accepting run. This problem is undecidable [19].

Given a 2 -counters (deterministic) machine $M$, we construct a functional weighted functional Sum-automaton $A=\left(Q, q_{0}, \delta, \gamma\right)$ (resp. Avg-automata), where $Q=Q_{O} \cup Q_{I}, \Sigma=$ $\Sigma_{O} \cup \Sigma_{I}$ and $\delta \subseteq Q \times \Sigma \times Q$ such that $M$ halts if and only if $L(A)$ is realizable. In particular, $\Sigma_{O}=\delta_{M}$ and a strategy $\pi \in \Lambda_{O}$ for Player $O$ is winning if and only if for each $\lambda_{I} \in \Lambda_{I}$, the projection of outcome $\left(\pi, \gamma_{2}\right)$ onto $\Sigma_{O}$ is an accepting run of $M$. The alphabet $\Sigma_{I}$ for Player $I$


Figure 2 Gadget to check positive cheats


Figure 3 Gadget to check negative cheats
is the set of letters $\Sigma_{I}=\{g o\} \cup\left(\bigcup_{i=1,2}\{\right.$ cheatCi+, cheatCi- $\left.\}\right) \cup\left(\bigcup_{0 \leq j<|S|}\{\right.$ cheatR:sj\}). The role of Player $I$ is that of observing the play of Player $O$ and detecting whether he faithfully simulates $M$, or he cheats. In details, if Player $O$ cheats by declaring the $i$-th counter equal to 0 when it is not (positive cheat), then Player $I$ can use the action cheatCi+, $i \in\{1,2\}$, to force all the runs but one (with weight $\leq 0$ ) to die. Similarly, if Player $O$ cheats by decrementing a counter with value zero (negative cheat) or on the structural properties of a run of $M$, then Player $I$ can win by playing the corresponding observing action : cheatCi-, for negative cheats on counter $i \in\{1,2\}$, or cheatR:sj for a cheat on the run through $M$ detected at state $s j$.

The automaton $A$ consists of a nondeterministic initial choice between different gadgets, described below. Each gadget checks one of the properties of the sequence of actions provided by Player $O$, and verify whether Player $O$ simulates faithfully $M$ or he eventually cheats. Due to the initial nondeterministic choice, each final state (in one of the gadget) is accessible throughout the evolution of the play and Player $O$ has to ensure that all the properties checked in the gadgets are fulfilled. Otherwise, Player $I$ will have the ability to kill all the runs but one, and to ensure that the only surviving run (in the appropriate gadget) reaches the final state with weight $\leq 0$.

In particular, Figure 2 represents the gadget to check a positive cheat on counter $i$, $i \in\{1,2\}$. Player $I$ observes the inverted value of the counter $i$ throughout the path on $M$ simulated by Player $O$. Whenever Player $O$ declares that counter $i$ is equal to 0 , Player $I$ can use the action cheatCi+ to kill all the runs in $A$ but the one within the observing gadget. The evolution of such a run up to cheatCi+ will have a negative value (corresponding to


Figure 4 Neutral gadget.


Figure 5 Gadget to check that Player 1 plays $\left(s_{0}, \cdot, \cdot, \cdot\right)$ at the beginning.
the inverted value of the observed counter) if Player $O$ was cheating. Hence, as soon as Player $O$ playes $\sharp$ it will end in a final state with weight $\leq 0$. Symmetrically, the gadget for checking negative cheats (represented in Figure 3) uses the weights on the edges to store the value of the observed counter. If Player $O$ cheats decrementing counter $i$ when its value is 0 , Player $I$ can use the action cheatCi- to kill all the runs but the one (with negative value) in the gadget observing negative cheats.

Finally, Player $I$ can use the gadgets in Figures 56 to detect any structural cheat committed by Player $O$. If Player $O$ initially provides an action different from ( $s_{0}, \ldots, \ldots, \quad$ ), Player $I$ can punish him by playing action cheatR:s0. Similarly, if Player $O$ provides two actions that do not induce a (sub)-path in $M$, Player $I$ can punish him within the gadget in Figure 6 .

The automaton $A$ will contain a gadget to observe positive/negative cheats for each counter $i \in\{1,2\}$, a gadget to observe a structural cheat for each state $s \in S$ that can be traversed by a path in $M$, and a neutral gadget (represented in Figure 4), where Player $I$ simply observes the run provided by Player $O$ and let such a run to reach a final state as soon as Player $O$ provides an action simulating a step toward the halting state of $M$. The proof is concluded by showing that $M$ halts iff Player $O$ has a strategy to win the realizability game on $A$. Namely, we show that Player $O$ wins the realizability game iff he provides a


Figure 6 Gadget to check cheats along the run.
word $\pi$ which corresponds to an accepting run of $M$ (and then stop the game).
$(\Rightarrow)$ Suppose that $M$ halts. Let $\pi$ be the run of $M$ leading to the halting state, and consider $\lambda_{O}(\pi) \in \Lambda_{O}$, where $\lambda_{O}(\pi)$ denotes the strategy for Player $O$ induced by $\pi$, in which Player $O$ provides the word $\pi$ and then stop the game. Let $\lambda_{I} \in \Lambda_{I}$. There are two cases to consider. 1. In the first case, $\lambda_{I}$ does not provide any action in:

$$
\left(\bigcup _ { i = 1 , 2 } \{ \text { cheatCi+, cheatCi-\}} ) \cup \left(\bigcup_{0 \leq j<|S|}\{\text { cheatR:sj\} })\right.\right.
$$

Then, the only run to a final state in $A$ is the one within the neutral gadget, having weight 1.
2. In the second case, $\gamma_{I}$ contains an action in $\left(\bigcup_{i=1,2}\{\right.$ cheatCi+, cheatCi- $\left.\}\right)$. Let $\alpha$ be the first action in $\left(\bigcup_{i=1,2}\left\{\right.\right.$ cheatCi+, cheatCi-\}) on $\lambda_{I}$. There is only one gadget allowing a run containing $\alpha$. Since $\pi$ is faithfully simulating $M$, such a run leads to a final state in the corresponding gadget with value $>0$.
Note that $\lambda_{I}$ can not contain an action $\alpha \in\left(\bigcup_{0 \leq j<|S|}\{\right.$ cheatR:sj\}). In fact, Player $I$ can never play cheatR:sj, since Player $O$ does not commit any structural cheat on the run $\pi$. Hence, we conclude that $\forall \lambda_{I} \in \Lambda_{I}\left(L_{A}\right.$ (outcome $\left.\left(\lambda_{O}(\pi), \lambda_{I}\right)>0\right)$.
$(\Leftarrow)$ Suppose that the strategy $\lambda_{O} \in \Lambda_{O}$ is such that $\forall \lambda_{I} \in \Lambda_{I}\left(L_{A}\right.$ (outcome $\left.\left(\lambda_{O}, \lambda_{I}\right) \geq 0\right)$. By construction of $A, \lambda_{O}$ allows Player 1 to survive in the gadgets for detecting positive, negative or structural cheats if and only if the projection of the outcome onto $\Sigma_{O}$ is a faithful simulation of a run in $M$. If Player $I$ can not use an action in $\left(\bigcup_{i=1,2}\{\right.$ cheatCit, cheatCi- $\left.\}\right) \cup$ $\left(\bigcup_{0 \leq j<|S|}\{c h e a t R: s j\}\right)$ to win (using the gadget targeted to check the corresponding cheat), the only remaining strategy for Player $I$ is playing indefinitely $\neg$ cheat. In that case, Player $O$ wins only if he eventually provides an action simulating a step leading to an halting state in $M$ (and then stop the game). Thus, our hypothesis entail that $\lambda_{O}$ consists in providing a run for $M$ that leads to a final state, witnessing that $M$ halts.

## D. 2 Proof of Theorem 15

Proof. We first consider the case of deterministic Sum-automata. Let $A=\left(Q=Q_{O} \uplus\right.$ $\left.Q_{I}, q_{I}, F, \delta=\delta_{I} \cup \delta_{O}, \gamma\right)$ be a deterministic Sum-automaton. Without lost of generality, we assume that $A$ contains only one accepting state denoted by $f$ which is absorbing. Then we consider $A$ as a finite state game arena and compute the set of states $S \subseteq Q$ from which player 1 can force a visit to the accepting state $f$. Note that from any state $s$ in $S$, player 1 has a strategy to force a visit to $f$ within $n$ steps, where $n=|Q|$. Note also that by determinacy, the complement of this set is the set of states of $A$ from which player 2 has a strategy to prevent a visit to $f$. Clearly, player 1 has to avoid the states in $Q \backslash S$ at all cost and so they can be removed from $A$. Let $A^{\prime}$ be $A$ where we have kept only the states in $S$.

Now, we construct from $A^{\prime}$ a finite tree as follows. We unfold $A^{\prime}$ and stop a branch at a node when:

- it is labeled with $f$ and the sum of the weights on the branch up to the node is equal to $c>0$,
- it is labeled by a state $q$ that already appears on the branch from the root to the node. We call the node where $q$ already appears the ancestor of the leaf.
Let us note $L$ the set of leafs of this finite tree. We then partition the leafs of this tree into $L_{1}$, the set of leafs that are good for player 1 and $L_{2}$, the set of leafs that are good for player 2. $L_{1}$ contains:
- $\left(C_{1}\right)$ the leafs that are annotated with $f$ and for which the sum of weights is strictly positive and
- $\left(C_{2}\right)$ the leafs labeled with a repeating state and for which the sum of weights from the root to the leaf is strictly larger than the sum of weights from the root to the ancestor.
$L_{2}=Q \backslash L_{1}$ are the leafs that are good for player 2. Now, consider the game played on this finite tree where player 1 wants to reach $L_{1}$ and player 2 wants to reach $L_{2}$. The winner in this game can be determined by backward induction. We claim (and prove) below that player 1 win in this finite game tree iff he wins the original game.

Assume that player 1 wins the finite game tree. We show how to construct a winning strategy in the original game. The strategy is built as follows. In the original game, player 1 plays as in the final tree up to the point where he reaches a leaf (in $L_{1}$ ). If the leaf is of sort defined in $C_{1}$ above then we know that player 1 has won the original game. Otherwise, we know that the sum now is strictly greater than the sum up to the ancestor of the leaf that we have reached. Then player 1 continues to play as prescribed by its winning strategy in the tree from the ancestor. Continuing like that, each time that the game arrives at a leaf, the sum of weights has strictly increased from the last visit to that leaf. As a consequence, after a finite amount of time, the sum will be strictly larger than $n \cdot|-W|$ where $-W$ is the smallest negative weight in $A^{\prime}$. From that point, player 1 can use his strategy that forces the state $f$ and reach it with a sum that is strictly positive (this is because he can force $f$ within $n$ steps).

Now assume that player 2 wins the finite game tree. We show how to construct a winning strategy in the original game. The strategy simply follows the strategy of player 2 in the finite tree by applying the strategy from the ancestor when reaching a leaf. As only leaf in $L_{2}$ are reached when playing that way, we know that the sum on successive visits to repeating states is non-increasing. As a consequence, as player 1 cannot force a visit to a node labeled with $f$ and a strictly positive sum in the finite game tree, we know that this will not append in the original game neither when player 2 plays its strategy.

This proof clearly establishes that the problem belongs to NP $\cap$ coNP as we can guess for the winning player one edge per states and verifies in polynomial time that this leads to a winning strategy in the original game. Nevertheless, note that player 1 needs memory to win in the original game as he has to verify that he has reached a sufficiently high sum before applying the strategy that forces the visit to $f$.

As for the previous results Avg games can be reduced easily to Sum games, and as for the questions about thresholds, Ratio games can be reduced to Avg games.

We now turn to the case of $\operatorname{Dsum}_{\lambda}$. The solution for Dsum ${ }_{\lambda}$ is obtained by first removing from $A$ all states from which player 1 cannot force a visit to $f$. As above, we note the game where those states have been removed by $A^{\prime}$. Then, we consider $A^{\prime}$ as an (infinite) discounted sum game where player 1 tries to maximize the value of the discounted sum while player 2 tries to minimize this value. Let $v$ denotes the value of the initial state $q_{I}$ in that game. We claim that player 1 wins the initial game iff the value $v$ in $q_{I}$ is strictly positive. Indeed, if player 1 has a winning strategy in the original game, i.e. a strategy to force the game into $f$ with strictly positive discounted sum, then by playing this strategy in the discounted sum game, the infinite discounted sum will be equal to the discounted sum up to $f$ as from there only the self loop on $f$ is crossed and its weight is equal to 0 . Now assume that player 1 has a strategy that force a value $v>0$ in the discounted sum game. Then by playing that strategy for $i$ steps in the original game with $i$ large enough to make sure that $\lambda^{i} W+\cdots+\lambda^{i+n} W$ is small enough, he will be able to switch to its strategy that forces $f$ after at most $n$ steps and ensure to reach $f$ with a strictly positive discounted sum.

As infinite discounted sum games are in NP $\cap$ coNP 2] and since our reduction is polynomial, we also get that finite reachability discounted sum games are in NP $\cap$ coNP.

## E Determinization

## E. 1 Proof of Lemma 18

We prove the following short witness property for the twinning property:

- Lemma 25. Let $A$ be $a$ Dsum $_{\lambda}$-automaton. If $A$ does not satisfy the twinning property, there exist two words $w_{1}, w_{2} \in \Sigma^{*}$ such that $\left|w_{1}\right| \leq 2|Q|^{2}$ and $\left|w_{2}\right| \leq 2|Q|^{2}$, two states $p, q \in Q$ such that $p$ and $q$ are both co-accessible, and runs $\rho_{1}: q_{I} \xrightarrow{\overline{w_{1}}} p, \rho_{2}: p \xrightarrow{w_{2}} p$, $\rho_{1}^{\prime}: q_{I} \xrightarrow{w_{1}} q, \rho_{2}^{\prime}: q \xrightarrow{w_{2}} q$, such that delay $\left(\rho_{1}, \rho_{1}^{\prime}\right) \neq \operatorname{delay}\left(\rho_{1} \rho_{2}, \rho_{1}^{\prime} \rho_{2}^{\prime}\right)$.

Proof. Suppose that $\left|w_{2}\right|>2|Q|^{2}$ (the case $\left|w_{1}\right|>2|Q|^{2}$ is proved exactly the same way) and that $w_{1} w_{2}$ witnesses that the twinning property does not hold by the decomposition into runs $\rho_{1}, \rho_{2}, \rho_{1}^{\prime}, \rho_{2}^{\prime}$ as in the premisses of the lemma. We will show that we can shorten the runs $\rho_{1}, \rho_{1}^{\prime}$ and still get a witness that the twinning property does not hold.

Since $\left|w_{2}\right|>2|Q|^{2}$, there is a pair of states $\left(p^{\prime}, q^{\prime}\right)$ that repeat three times along the two parallel runs $\rho_{2}$ and $\rho_{2}^{\prime}$, i.e. $w_{2}$ can be decomposed as $w_{1}^{\prime} w_{2}^{\prime} w_{3}^{\prime} w_{4}^{\prime}$ and $\rho_{2}$ and $\rho_{2}^{\prime}$ can be decomposed as $r_{1} r_{2} r_{3} r_{4}$ and $r_{1}^{\prime} r_{2}^{\prime} r_{3}^{\prime} r_{4}^{\prime}$ respectively, where:

$$
\begin{array}{llll}
r_{1}: p \xrightarrow{w_{1}^{\prime}} p^{\prime} & r_{2}: p^{\prime} \xrightarrow{w_{2}^{\prime}} p^{\prime} & r_{3}: p^{\prime} \xrightarrow{w_{3}^{\prime}} p^{\prime} & r_{4}: p^{\prime} \xrightarrow{w_{4}^{\prime}} p \\
r_{1}^{\prime}: q \xrightarrow{w_{1}^{\prime}} q^{\prime} & r_{2}^{\prime}: q^{\prime} \xrightarrow{w_{2}^{\prime}} q^{\prime} & r_{3}^{\prime}: q^{\prime} \xrightarrow{w_{3}^{\prime}} q^{\prime} & r_{4}^{\prime}: p^{\prime} \xrightarrow{w_{4}^{\prime}} q
\end{array}
$$

Note that $r_{1}, r_{1}^{\prime}$ and $r_{4}, r_{4}^{\prime}$ may be empty (in this case $p=p^{\prime}$ and $q=q^{\prime}$ ), but $r_{2}, r_{3}, r_{2}^{\prime}, r_{3}^{\prime}$ are assumed to be non-empty.

Now, there are two cases: delay $\left(\rho_{1} r_{1}, \rho_{1}^{\prime} r_{1}^{\prime}\right) \neq \operatorname{delay}\left(\rho_{1} r_{1} r_{2}, \rho_{1}^{\prime} r_{1}^{\prime} r_{2}^{\prime}\right)$ and in that case the word $w_{1} w_{1}^{\prime} w_{2}^{\prime}$ is a witness that the twinning property does not hold, and $\left|w_{1} w_{1}^{\prime} w_{2}^{\prime}\right|<$ $\left|w_{1} w_{2}\right|$. In the second case, we have delay $\left(\rho_{1} r_{1}, \rho_{1}^{\prime} r_{1}^{\prime}\right)=\operatorname{delay}\left(\rho_{1} r_{1} r_{2}, \rho_{1}^{\prime} r_{1}^{\prime} r_{2}^{\prime}\right)$, but in that case, we can apply Lemma 24 and we $\operatorname{get} \operatorname{delay}\left(\rho_{1} r_{1} r_{3} r_{4}, \rho_{1}^{\prime} r_{1}^{\prime} r_{3}^{\prime} r_{4}^{\prime}\right)=\operatorname{delay}\left(\rho_{1} \rho_{2}, \rho_{1}^{\prime} \rho_{2}^{\prime}\right)$. Therefore delay $\left(\rho_{1}, \rho_{1}^{\prime}\right) \neq \operatorname{delay}\left(\rho_{1} r_{1} r_{3} r_{4}, \rho_{1}^{\prime} r_{1}^{\prime} r_{3}^{\prime} r_{3}^{\prime}\right)$ and $w_{1} w_{1}^{\prime} w_{3}^{\prime} w_{4}^{\prime}$ is a shorter witness that the twinning property does not hold.

We can iterate this reasoning until we find a witness whose size satisfy the premisses of the lemma.

We are now ready to prove Lemma 18 ,
Proof of Lemma 18. We define a non-deterministic PSpace algorithm to check whether a Dsum $_{\lambda}$-automaton does not satisfy the twinning property. The idea is to guess two runs on the same input word of size at most $4|Q|^{2}$ and two positions in those runs, and check the pair of states at the two positions are equal and that the respective delays are different. This algorithm uses a polynomial space (denomitors of the form $\lambda^{|w|}$ are stored in polynomial space) and thanks to Lemma 25, is correct.

## E. 2 Proofs of Lemma 21 and Lemma 20

We prove a stronger version of Lemma 21:

- Lemma 26. Let $f$ be an accessible state of $A_{d}$ by a run $\rho_{d}$ on some word $w \in \Sigma^{*}$. Then the following hold:

1. $\operatorname{dom}(f)$ is the set of states $q$ such that there exists a run on $w$ reaching $q$;
2. $\operatorname{Dsum}_{\lambda}\left(\rho_{d} T\right)=\min \left\{\operatorname{Dsum}_{\lambda}(\xi) \mid \xi\right.$ is a run of $A$ on $\left.w\right\}$.
3. If $q \in \operatorname{dom}(f)$ and $\rho$ is a run on $w$ reaching $q$, then

$$
f(q)=\max \left\{\operatorname{delay}\left(\rho, \rho^{\prime}\right) \mid \rho^{\prime} \text { is a run of } A \text { on } w\right\}=\frac{\operatorname{Dsum}_{\lambda}(\rho)-\operatorname{Dsum}_{\lambda}\left(\rho_{d} T\right)}{\lambda^{|w|}}
$$

4. For all $f^{\prime}: Q \rightarrow \mathbb{Q}$ and all $a \in \Sigma$ such that $\operatorname{dom}\left(f^{\prime}\right)=\left\{q^{\prime} \mid \exists q \in \operatorname{dom}(f),\left(q, a, q^{\prime}\right) \in \delta\right\}$ and for all $q^{\prime} \in \operatorname{dom}\left(f^{\prime}\right)$ :

$$
f^{\prime}\left(q^{\prime}\right)=\frac{f(q)}{\lambda}+\gamma\left(q, a, q^{\prime}\right)-\gamma_{d}\left(f, a, f^{\prime}\right) \text { for some } q \in \operatorname{dom}(f) \text { such that }\left(q, a, q^{\prime}\right) \in \delta
$$

we have $f^{\prime} \in Q_{d}$ and $\left(f, a, f^{\prime}\right) \in \delta_{d}$.
Proof. The five statements are proved by induction on $|w|$. It is clear when $|w|=0$.
Suppose that $|w|>0$ and $w=w^{\prime} a$ for some $a \in \Sigma$. Let $\rho_{d}: f_{I} \xrightarrow{w^{\prime}} f$ be a run of $A_{d}$ on $w^{\prime}$ and let $f^{\prime}$ such that $\left(f, a, f^{\prime}\right) \in \delta_{d}$, and $t_{d}=\left(f, a, f^{\prime}\right)$.

- The first statement is obvious by induction hypothesis and by definition of $\delta_{d}$.
- The second statement is proved as follows:

$$
\begin{aligned}
& \operatorname{Dsum}_{\lambda}\left(\rho_{d} T\right) \\
= & \operatorname{Dsum}_{\lambda}\left(\rho_{d}\right)+\lambda^{|w|} \gamma_{d}(T) \\
= & \operatorname{Dsum}_{\lambda}\left(\rho_{d}\right)+\lambda^{|w|} \min \left\{\left.\frac{f(q)}{\lambda}+\gamma\left(q, a, q^{\prime}\right) \right\rvert\, q \in \operatorname{dom}(f) \wedge\left(q, a, q^{\prime}\right) \in \delta\right\} \\
= & \operatorname{Dsum}_{\lambda}\left(\rho_{d}\right)+\lambda^{|w|} \min \left\{\left.\frac{\operatorname{Dsum}_{\lambda}(\rho)-\operatorname{Dsum}_{\lambda}\left(\rho_{d}\right)}{\lambda^{|w|}}+\gamma\left(q, a, q^{\prime}\right) \right\rvert\, q \in \operatorname{dom}(f) \wedge\left(q, a, q^{\prime}\right) \in \delta\right\}
\end{aligned}
$$ by induction hypothesis and for some run $\rho: q_{I} \xrightarrow{w^{\prime}} q$. This is independ on the choice of $\rho$ as any run $\rho^{\prime}$ reaching $q$ on $w^{\prime}$ satisfies $\operatorname{Dsum}_{\lambda}\left(\rho^{\prime}\right)=\operatorname{Dsum}_{\lambda}(\rho)$, as $A$ is functional and $q$ is co-accessible.

$=\operatorname{Dsum}_{\lambda}\left(\rho_{d}\right)+\min \left\{\operatorname{Dsum}_{\lambda}(\rho)-\operatorname{Dsum}_{\lambda}\left(\rho_{d}\right)+\lambda^{|w|} \gamma\left(q, a, q^{\prime}\right) \mid q \in \operatorname{dom}(f) \wedge\left(q, a, q^{\prime}\right) \in \delta\right\}$
$=\min \left\{\operatorname{Dsum}_{\lambda}(\rho)+\lambda^{|w|} \gamma\left(q, a, q^{\prime}\right) \mid q \in \operatorname{dom}(f) \wedge\left(q, a, q^{\prime}\right) \in \delta\right\}$
$=\min \left\{\operatorname{Dsum}_{\lambda}\left(\rho\left(q, a, q^{\prime}\right)\right) \mid q \in \operatorname{dom}(f) \wedge\left(q, a, q^{\prime}\right) \in \delta\right\}$
$=\min \left\{\operatorname{Dsum}_{\lambda}(\xi) \mid \xi\right.$ is a run on $\left.w\right\}$
(as it is independ on the choice of $\rho$ )

- The third statement is proved as follows: Let $q^{\prime} \in \operatorname{dom}\left(f^{\prime}\right)$.

$$
\begin{aligned}
f^{\prime}\left(q^{\prime}\right)= & \frac{f(q)}{\lambda}+\gamma\left(q, a, q^{\prime}\right)-\gamma_{d}\left(f, a, f^{\prime}\right) \\
= & \frac{\left(\text { for some } q \in \operatorname{dom}(f) \text { such that }\left(q, a, q^{\prime}\right) \in \delta\right)}{\operatorname{Dsum}_{\lambda}(\rho)-\operatorname{Dsum}_{\lambda}\left(\rho_{d}\right)}+\gamma\left(q, a, q^{\prime}\right)-\gamma_{d}\left(f, a, f^{\prime}\right) \\
& \left(\operatorname{byy}^{|w|} \text { induction hypothesis and for some } \rho: q_{I} \xrightarrow{w^{\prime}} q\right) \\
= & \frac{\operatorname{Dsum}_{\lambda}(\rho)+\lambda^{|w|} \gamma\left(q, a, q^{\prime}\right)-\operatorname{Dsum}_{\lambda}\left(\rho_{d}\right)-\lambda^{|w|} \gamma_{d}\left(f, a, f^{\prime}\right)}{\lambda^{|w|}} \\
= & \frac{\operatorname{Dsum}_{\lambda}\left(\rho\left(q, a, q^{\prime}\right)\right)-\operatorname{Dsum}_{\lambda}\left(\rho_{d} T\right)}{\lambda^{|w|}}
\end{aligned}
$$

This value does not depend on the choice $q$. Indeed, any run $\rho^{\prime}$ that reaches $q^{\prime}$ on $w$ satisfies $\operatorname{Dsum}_{\lambda}\left(\rho^{\prime}\right)=\operatorname{Dsum}_{\lambda}\left(\rho\left(q, a, q^{\prime}\right)\right)$, as $q^{\prime}$ is co-accessible and $A$ is functional.

Then, we prove the second part of the third statement:

$$
\begin{aligned}
f^{\prime}\left(q^{\prime}\right) & =\frac{\operatorname{Dsum}_{\lambda}\left(\rho\left(q, a, q^{\prime}\right)\right)-\operatorname{Dsum}_{\lambda}\left(\rho_{d} T\right)}{\lambda^{|w|}} \\
& =\frac{\operatorname{Dsum}_{\lambda}\left(\rho\left(q, a, q^{\prime}\right)\right)-\min \left\{\operatorname{Dsum}_{\lambda}(\xi) \mid \xi \text { is a run on } w\right\}}{\lambda^{|w|}} \\
& \left.\left.=\max \frac{\operatorname{Dsum}_{\lambda}\left(\rho\left(q, a, q^{\prime}\right)\right)-\operatorname{Dsum}_{\lambda}(\xi)}{\lambda^{|w|}} \right\rvert\, \xi \text { is a run on } w\right\} \lambda^{|w|} \\
& =\max \left\{\operatorname{delay}\left(\rho\left(q, a, q^{\prime}\right), \xi\right) \mid \xi \text { is a run on } w\right\}
\end{aligned}
$$

which achieves to prove the lemma, as again, this value does not depend on the choice of the run $\rho\left(q, a, q^{\prime}\right)$.

- We prove the fifth statement. Let $f^{\prime \prime}$ be a function as defined in the fifth statement. We have seen that the value $f^{\prime}\left(q^{\prime}\right)$ does not depend on the choice of $q$. We can therefore use exactly the same proof as $f^{\prime}$ to prove that for all $q^{\prime \prime} \in \operatorname{dom}\left(f^{\prime \prime}\right)$ :

$$
f^{\prime \prime}\left(q^{\prime \prime}\right)=\max \left\{\operatorname{delay}\left(\rho, \rho^{\prime}\right) \mid \rho, \rho^{\prime} \text { are runs of } A \text { on } w a \text { s.t. } \rho \text { reaches } q^{\prime \prime}\right\}
$$

By definition of $Q_{d}$, we get $f^{\prime \prime} \in Q_{d}$ and by definition of $\delta_{d},\left(f^{\prime}, a, f^{\prime \prime}\right) \in \delta_{d}$.
Proof of Lemma 20. First note that $A_{d}$ is complete. Indeed, for all $f \in Q_{d}$ and all $a \in \Sigma$, there exists $f^{\prime} \in Q_{d}$ such that $\left(f, a, f^{\prime}\right) \in \delta_{d}$. It suffices to define $f^{\prime}$ as follows: for all $q^{\prime} \in Q$ :

$$
f^{\prime}\left(q^{\prime}\right)=\frac{f(q)}{\lambda}+\gamma\left(q, a, q^{\prime}\right)-\gamma_{d}\left(f, a, f^{\prime}\right) \text { for some } q \in \operatorname{dom}(f) \text { such that }\left(q, a, q^{\prime}\right) \in \delta
$$

By Lemma 26 (statement 5), we get $\left(f, a, f^{\prime}\right) \in \delta_{d}$.
We show that $A_{d}$ is deterministic. Suppose that there exists $f, f^{\prime}, f^{\prime \prime} \in Q_{d}$ and $a \in \Sigma$ such that $\left(f, a, f^{\prime}\right) \in \delta_{d}$ and $\left(f, a, f^{\prime \prime}\right) \in \delta_{d}$. Clearly, by definition of $\delta_{d}, \operatorname{dom}\left(f^{\prime}\right)=\operatorname{dom}\left(f^{\prime \prime}\right)$. Since $f^{\prime}$ and $f^{\prime \prime}$ are accessible by definition of $A_{d}$, we can apply Lemma 20 and we clearly get that $f^{\prime}(q)=f^{\prime \prime}(q)$ for all $q \in \operatorname{dom}\left(f^{\prime}\right)=\operatorname{dom}\left(f^{\prime \prime}\right)$. Therefore $A_{d}$ is deterministic.

Let us prove that $L_{A_{d}}=L_{A}$. Let $w \in \Sigma^{+}$. We show that $w \in \operatorname{dom}\left(A_{d}\right)$ iff $w \in \operatorname{dom}(A)$. If $w \in \operatorname{dom}\left(A_{d}\right)$, then there exists an accepting run $\rho_{d}: f_{I} \xrightarrow{w} f$ of $A_{d}$ such that $f \in F_{d}$. Therefore there exists $q \in \operatorname{dom}(f)$ such that $q \in F$. By Lemma [26, there exists a run of $A$ on $w$ reaching $q \in F$, so that $w \in \operatorname{dom}(A)$.

Conversely, if $w \in \operatorname{dom}(A)$, then there exists an accepting run $\rho: q_{I} \xrightarrow{w} q$ with $q \in F$. Since $A_{d}$ is complete, there exists an accepting run of $A_{d}$ on $w$ reaching some $f \in Q_{d}$. Again by Lemma [26] we get $q \in \operatorname{dom}(f)$ and therefore, since $q \in F$, we have $f \in F_{d}$. Hence $w \in \operatorname{dom}\left(A_{d}\right)$.

Let $w \in \operatorname{dom}(A)$, we show that $L_{A_{d}}(w)=L_{A}(w)$. Since $\operatorname{dom}(A)=\operatorname{dom}\left(A_{d}\right), w \in$ $\operatorname{dom}\left(A_{d}\right)$ and therefore there exists an accepting run of $A_{d}$ on $w$ that we denote by $\rho_{d}$ : $f_{I} \xrightarrow{w} f \in F_{d}$. By Lemma 26, $\operatorname{Dsum}_{\lambda}\left(\rho_{d}\right)=\min \left\{\operatorname{Dsum}_{\lambda}(\xi) \mid \xi\right.$ is a run of $A$ on $\left.w\right\}$. Since $w \in \operatorname{dom}(A)$ and $\operatorname{dom}(A) \subseteq(\Sigma-\#)^{*} \#, w$ has necessarily the form $w^{\prime} \#$ and since all states of $A$ are assumed to be co-accessible, all the runs of $A$ on $w$ are necessarily accepting. Therefore $\operatorname{Dsum}_{\lambda}\left(\rho_{d}\right)=\operatorname{Dsum}_{\lambda}(\xi)$ for some accepting run $\xi$ of $A$ on $w$ (the choice of $\xi$ is not important as $A$ is functional). In other words, $L_{A_{d}}(w)=L_{A}(w)$.

## E. 3 Proof of Theorem 22

Proof. The forth direction has been already proved (Lemma 20) . We prove the back direction (i.e. the twinning property is a necessary condition). Suppose that the twinning property does not hold. There exist states $p, q$ such that $p$ and $q$ are co-accessible and there exists words $w_{1}, w_{2} \in \Sigma^{*}$, and runs $\rho_{1}: q_{I} \xrightarrow{w_{1}} p, \rho_{2}: p \xrightarrow{w_{2}} p, \rho_{1}^{\prime}: q_{I} \xrightarrow{w_{1}} q, \rho_{2}^{\prime}: q \xrightarrow{w_{2}} q$, such that delay $\left(\rho_{1}, \rho_{1}^{\prime}\right) \neq \operatorname{delay}\left(\rho_{1} \rho_{2}, \rho_{1}^{\prime} \rho_{2}^{\prime}\right)$.

We first show that there are infinitely many different delays. For all $i \geq 0$, let $\Delta(i)=$ $\operatorname{delay}\left(\rho_{1}\left(\rho_{2}\right)^{i}, \rho_{1}^{\prime}\left(\rho_{2}^{\prime}\right)^{i}\right)$. We show that for all $j \geq i \geq 0$, we have:

$$
\begin{equation*}
\lambda^{i\left|w_{2}\right|}(\Delta(j)-\Delta(i))=\Delta(j-i)-\Delta(0) \tag{13}
\end{equation*}
$$

Let us first develop the expression $\Delta(j)-\Delta(i)$ :

$$
\begin{aligned}
& \Delta(j)-\Delta(i) \\
= & \frac{\operatorname{Dsum}_{\lambda}\left(\rho_{1}\left(\rho_{2}\right)^{j}\right)-\operatorname{Dsum}_{\lambda}\left(\rho_{1}^{\prime}\left(\rho_{2}^{\prime}\right)^{j}\right)}{\lambda^{\left|w_{1}\right|+j\left|w_{2}\right|}}-\frac{\operatorname{Dsum}_{\lambda}\left(\rho_{1}\left(\rho_{2}\right)^{i}\right)-\operatorname{Dsum}_{\lambda}\left(\rho_{1}^{\prime}\left(\rho_{2}^{\prime}\right)^{i}\right)}{\lambda^{\left|w_{1}\right|+i\left|w_{2}\right|}} \\
= & \frac{\operatorname{Dsum}_{\lambda}\left(\rho_{1}\left(\rho_{2}\right)^{j}\right)-\operatorname{Dsum}_{\lambda}\left(\rho_{1}^{\prime}\left(\rho_{2}^{\prime}\right)^{j}\right)-\lambda^{(j-i)\left|w_{2}\right|} \operatorname{Dsum}_{\lambda}\left(\rho_{1}\left(\rho_{2}\right)^{i}\right)+\lambda^{(j-i)\left|w_{2}\right|} \operatorname{Dsum}_{\lambda}\left(\rho_{1}^{\prime}\left(\rho_{2}^{\prime}\right)^{i}\right)}{\lambda^{\left|w_{1}\right|+j\left|w_{2}\right|}}
\end{aligned}
$$

We can rewrite $\operatorname{Dsum}_{\lambda}\left(\rho_{1}\left(\rho_{2}\right)^{j}\right)-\lambda^{(j-i)\left|w_{2}\right|} \operatorname{Dsum}_{\lambda}\left(\rho_{1}\left(\rho_{2}\right)^{i}\right)$ into $\operatorname{Dsum}_{\lambda}\left(\rho_{1}\right)-\lambda^{(j-i)\left|w_{2}\right|} \operatorname{Dsum}_{\lambda}\left(\rho_{1}\right)+$ $\lambda^{\left|w_{1}\right|} \operatorname{Dsum}_{\lambda}\left(\left(\rho_{2}\right)^{j-i}\right)$, i.e. $\operatorname{Dsum}_{\lambda}\left(\rho_{1}\left(\rho_{2}\right)^{j-i}\right)-\lambda^{(j-i)\left|w_{2}\right|} \operatorname{Dsum}_{\lambda}\left(\rho_{1}\right)$. A similar rewriting can be obtained for $\operatorname{Dsum}_{\lambda}\left(\rho_{1}^{\prime}\left(\rho_{2}^{\prime}\right)^{j}\right)-\lambda^{(j-i)\left|w_{2}\right|} \operatorname{Dsum}_{\lambda}\left(\rho_{1}^{\prime}\left(\rho_{2}^{\prime}\right)^{i}\right)$. Therefore we get:

$$
\begin{aligned}
& \Delta(j)-\Delta(i) \\
= & \frac{\operatorname{Dsum}_{\lambda}\left(\rho_{1}\left(\rho_{2}\right)^{j-i}\right)-\operatorname{Dsum}_{\lambda}\left(\rho_{1}^{\prime}\left(\rho_{2}^{\prime}\right)^{j-i}\right)}{\lambda^{\left|w_{1}\right|+j\left|w_{2}\right|}}-\frac{\lambda^{(j-i)\left|w_{2}\right|}\left(\operatorname{Dsum}_{\lambda}\left(\rho_{1}\right)-\operatorname{Dsum}_{\lambda}\left(\rho_{1}^{\prime}\right)\right)}{\lambda^{\left|w_{1}\right|+j\left|w_{2}\right|}} \\
= & \frac{1}{\lambda^{i\left|w_{2}\right|}}\left(\frac{\operatorname{Dsum}_{\lambda}\left(\rho_{1}\left(\rho_{2}\right)^{j-i}\right)-\operatorname{Dsum}_{\lambda}\left(\rho_{1}^{\prime}\left(\rho_{2}^{\prime}\right)^{j-i}\right)}{\lambda^{\left|w_{1}\right|+(j-i)\left|w_{2}\right|}}-\frac{\lambda^{(j-i)\left|w_{2}\right|}\left(\operatorname{Dsum}_{\lambda}\left(\rho_{1}\right)-\operatorname{Dsum}_{\lambda}\left(\rho_{1}^{\prime}\right)\right)}{\lambda^{\left|w_{1}\right|+(j-i)\left|w_{2}\right|}}\right) \\
= & \frac{1}{\lambda^{i\left|w_{2}\right|}}\left(\frac{\operatorname{Dsum}_{\lambda}\left(\rho_{1}\left(\rho_{2}\right)^{j-i}\right)-\operatorname{Dsum}_{\lambda}\left(\rho_{1}^{\prime}\left(\rho_{2}^{\prime}\right)^{j-i}\right)}{\lambda^{\left|w_{1}\right|+(j-i)\left|w_{2}\right|}}-\frac{\operatorname{Dsum}_{\lambda}\left(\rho_{1}\right)-\operatorname{Dsum}_{\lambda}\left(\rho_{1}^{\prime}\right)}{\lambda^{\left|w_{1}\right|}}\right) \\
= & \frac{1}{\lambda^{i\left|w_{2}\right|}}(\Delta(j-i)-\Delta(0))
\end{aligned}
$$

By Equation [13, for all $i \geq 0$, we have $\lambda^{i\left|w_{2}\right|}(\Delta(i+1)-\Delta(i))=\Delta(1)-\Delta(0)$. Since by hypothesis, $\Delta(1) \neq \Delta(0)$, the sequence $(\Delta(i))_{i \geq 0}$ is either strictly increasing or strictly decreasing. Hence we get:

$$
\begin{equation*}
\forall i, j \geq 0, \quad(i \neq j) \Longrightarrow \Delta(i) \neq \Delta(j) \tag{14}
\end{equation*}
$$

We use Equation 14 to show that $A$ cannot be determinized. We suppose that there exists a deterministic automaton $A_{d}=\left(Q_{d}, f_{I}, F_{d}, \delta_{d}, \gamma_{d}\right)$ equivalent to $A$ and get a contradiction. We consider a word of the form $w_{1}\left(w_{2}\right)^{i}$, for $i$ taken large enough to satisfy the existence of a run of $A_{d}$ of the following form:

$$
f_{I} \xrightarrow{w_{1}\left(w_{2}\right)^{k_{1}}} f \xrightarrow{\left(w_{2}\right)^{k_{2}}} f \xrightarrow{\left(w_{2}\right)^{i-k_{2}-k_{1}}} f^{\prime}
$$

for some $k_{1}, k_{2}$ such that $k_{2}>0$, and some $f, f^{\prime} \in Q_{d}$.
Moreover, since $p$ and $q$ are both co-accessible, there exist two words $w_{3}, w_{3}^{\prime}$ and two runs of $A_{d}$ of the form:

$$
\begin{aligned}
& \rho_{d}: f_{I} \xrightarrow{w_{1}\left(w_{2}\right)^{k_{1}}} f \xrightarrow{\left(w_{2}\right)^{k_{2}}} f \xrightarrow{\left(w_{2}\right)^{i-k_{2}-k_{1}} w_{3}} g \\
& \rho_{d}^{\prime}: f_{I} \xrightarrow{w_{1}\left(w_{2}\right)^{k_{1}}} f \xrightarrow{\left(w_{2}\right)^{k_{2}}} f \xrightarrow{\left(w_{2}\right)^{i-k_{2}-k_{1}} w_{3}^{\prime}} g^{\prime}
\end{aligned}
$$

for some accepting states $g, g^{\prime} \in F_{d}$. Let $\rho_{d}=\rho_{d, 1} \rho_{d_{2}} \rho_{d, 3}$ and $\rho_{d}^{\prime}=\rho_{d, 1} \rho_{d_{2}} \rho_{d, 3}^{\prime}$ for some subruns $\rho_{d, 1}, \rho_{d_{2}}$ that correspond to $w_{1}\left(w_{2}\right)^{k_{1}}$ and $\left(w_{2}\right)^{k_{2}}$ respectively, and some subruns $\rho_{d, 3}$ and $\rho_{d, 3}^{\prime}$ that correspond to $\left(w_{2}\right)^{i-k_{1}-k_{2}} w_{3}$ and $\left(w_{2}\right)^{i-k_{1}-k_{2}} w_{3}^{\prime}$ respectively. By equation 14, we know that $\Delta\left(k_{1}\right) \neq \Delta\left(k_{1}+k_{2}\right)$. We show that this leads to a contradiction. Let also $\rho_{3}: p \xrightarrow{w_{3}} p_{f}$ and $\rho_{3}^{\prime}: q \xrightarrow{w_{3}^{\prime}} q_{f}$ be two runs in $A$, for some $p_{f}, q_{f} \in F$. Then we have:

$$
\begin{equation*}
\operatorname{Dsum}_{\lambda}\left(\rho_{d, 1} \rho_{d, 2} \rho_{d, 3}\right)=\operatorname{Dsum}_{\lambda}\left(\rho_{1}\left(\rho_{2}\right)^{i} \rho_{3}\right) \tag{15}
\end{equation*}
$$

```
\(\operatorname{Dsum}_{\lambda}\left(\rho_{d, 1} \rho_{d, 2} \rho_{d, 3}^{\prime}\right)=\operatorname{Dsum}_{\lambda}\left(\rho_{1}^{\prime}\left(\rho_{2}^{\prime}\right)^{i} \rho_{3}^{\prime}\right)\)
\(\operatorname{Dsum}_{\lambda}\left(\rho_{d, 1} \rho_{d, 3}\right)=\operatorname{Dsum}_{\lambda}\left(\rho_{1}\left(\rho_{2}\right)^{i-k_{2}} \rho_{3}\right)\)
\(\operatorname{Dsum}_{\lambda}\left(\rho_{d, 1} \rho_{d, 3}^{\prime}\right)=\operatorname{Dsum}_{\lambda}\left(\rho_{1}^{\prime}\left(\rho_{2}^{\prime}\right)^{i-k_{2}} \rho_{3}^{\prime}\right)\)
```

From which we get:
$\left.\left.\operatorname{Dsum}_{\lambda}\left(\rho_{d, 1} \rho_{d, 2} \rho_{d, 3}\right)\right)-\operatorname{Dsum}_{\lambda}\left(\rho_{d, 1} \rho_{d, 2} \rho_{d, 3}^{\prime}\right)\right)=\operatorname{Dsum}_{\lambda}\left(\rho_{1}\left(\rho_{2}\right)^{i} \rho_{3}\right)-\operatorname{Dsum}_{\lambda}\left(\rho_{1}^{\prime}\left(\rho_{2}^{\prime}\right)^{i} \rho_{3}^{\prime}\right)$
which is equivalent to:

$$
\begin{equation*}
\lambda^{\left|w_{1}\right|+\left(k_{1}+k_{2}\right)\left|w_{2}\right|}\left(\operatorname{Dsum}_{\lambda}\left(\rho_{d, 3}\right)-\operatorname{Dsum}_{\lambda}\left(\rho_{d, 3}^{\prime}\right)\right)=\operatorname{Dsum}_{\lambda}\left(\rho_{1}\left(\rho_{2}\right)^{i} \rho_{3}\right)-\operatorname{Dsum}_{\lambda}\left(\rho_{1}^{\prime}\left(\rho_{2}^{\prime}\right)^{i} \rho_{3}^{\prime}\right) \tag{20}
\end{equation*}
$$

Similarly:

$$
\begin{equation*}
\lambda^{\left|w_{1}\right|+k_{1}\left|w_{2}\right|}\left(\operatorname{Dsum}_{\lambda}\left(\rho_{d, 3}\right)-\operatorname{Dsum}_{\lambda}\left(\rho_{d, 3}^{\prime}\right)\right)=\operatorname{Dsum}_{\lambda}\left(\rho_{1}\left(\rho_{2}\right)^{i-k_{2}} \rho_{3}\right)-\operatorname{Dsum}_{\lambda}\left(\rho_{1}^{\prime}\left(\rho_{2}^{\prime}\right)^{i-k_{2}} \rho_{3}^{\prime}\right) \tag{21}
\end{equation*}
$$

Dividing Equation 20 by $\lambda^{k_{2}\left|w_{2}\right|}$ and combining it with Equation 21 we get:

$$
\begin{equation*}
\frac{\operatorname{Dsum}_{\lambda}\left(\rho_{1}\left(\rho_{2}\right)^{i} \rho_{3}\right)-\operatorname{Dsum}_{\lambda}\left(\rho_{1}^{\prime}\left(\rho_{2}^{\prime}\right)^{i} \rho_{3}^{\prime}\right)}{\lambda^{k_{2}\left|w_{2}\right|}}=\operatorname{Dsum}_{\lambda}\left(\rho_{1}\left(\rho_{2}\right)^{i-k_{2}} \rho_{3}\right)-\operatorname{Dsum}_{\lambda}\left(\rho_{1}^{\prime}\left(\rho_{2}^{\prime}\right)^{i-k_{2}} \rho_{3}^{\prime}\right) \tag{22}
\end{equation*}
$$

Let us rewrite the left hand side of Equation 22,

$$
\begin{aligned}
& \frac{\operatorname{Dsum}_{\lambda}\left(\rho_{1}\left(\rho_{2}\right)^{i} \rho_{3}\right)-\operatorname{Dsum}_{\lambda}\left(\rho_{1}^{\prime}\left(\rho_{2}^{\prime}\right)^{i} \rho_{3}^{\prime}\right)}{\lambda^{k_{2}\left|w_{2}\right|}} \\
= & \frac{\operatorname{Dsum}_{\lambda}\left(\rho_{1}\left(\rho_{2}\right)^{k_{1}+k_{2}}\right)-\operatorname{Dsum}_{\lambda}\left(\rho_{1}^{\prime}\left(\rho_{2}^{\prime}\right)^{k_{1}+k_{2}}\right)}{\lambda^{k_{2}\left|w_{2}\right|}}+ \\
& +\frac{\lambda^{\left|w_{1}\right|+\left(k_{1}+k_{2}\right)\left|w_{2}\right|}\left(\operatorname{Dsum}_{\lambda}\left(\left(\rho_{2}\right)^{i-k_{2}-k_{1}} \rho_{3}\right)-\operatorname{Dsum}_{\lambda}\left(\left(\rho_{2}^{\prime}\right)^{i-k_{2}-k_{1}} \rho_{3}^{\prime}\right)\right)}{\lambda^{k_{2}\left|w_{2}\right|}} \\
= & \lambda^{\left|w_{1}\right|+k_{1}\left|w_{2}\right|}\left(\Delta\left(k_{1}+k_{2}\right)+\operatorname{Dsum}_{\lambda}\left(\left(\rho_{2}\right)^{i-k_{2}-k_{1}} \rho_{3}\right)-\operatorname{Dsum}_{\lambda}\left(\left(\rho_{2}^{\prime}\right)^{i-k_{2}-k_{1}} \rho_{3}^{\prime}\right)\right)
\end{aligned}
$$

Similarly, we rewrite the right hand side of Equation 22.

$$
\begin{aligned}
& \operatorname{Dsum}_{\lambda}\left(\rho_{1}\left(\rho_{2}\right)^{i-k_{2}} \rho_{3}\right)-\operatorname{Dsum}_{\lambda}\left(\rho_{1}^{\prime}\left(\rho_{2}^{\prime}\right)^{i-k_{2}} \rho_{3}^{\prime}\right) \\
= & \operatorname{Dsum}_{\lambda}\left(\rho_{1}\left(\rho_{2}\right)^{k_{1}}\right)-\operatorname{Dsum}_{\lambda}\left(\rho_{1}^{\prime}\left(\rho_{2}^{\prime}\right)^{k_{1}}\right)+ \\
& +\lambda^{\left|w_{1}\right|+k_{1}\left|w_{2}\right|}\left(\operatorname{Dsum}_{\lambda}\left(\left(\rho_{2}\right)^{i-k_{2}-k_{1}} \rho_{3}\right)-\operatorname{Dsum}_{\lambda}\left(\left(\rho_{2}^{\prime}\right)^{i-k_{2}-k_{1}} \rho_{3}^{\prime}\right)\right) \\
= & \lambda^{\left|w_{1}\right|+k_{1}\left|w_{2}\right|}\left(\Delta\left(k_{1}\right)+\operatorname{Dsum}_{\lambda}\left(\left(\rho_{2}\right)^{i-k_{2}-k_{1}} \rho_{3}\right)-\operatorname{Dsum}_{\lambda}\left(\left(\rho_{2}^{\prime}\right)^{i-k_{2}-k_{1}} \rho_{3}^{\prime}\right)\right)
\end{aligned}
$$

Therefore Equation 22 rewrites into:

$$
\begin{gathered}
\lambda^{\left|w_{1}\right|+k_{1}\left|w_{2}\right|}\left(\Delta\left(k_{1}+k_{2}\right)+\operatorname{Dsum}_{\lambda}\left(\left(\rho_{2}\right)^{i-k_{2}-k_{1}} \rho_{3}\right)-\operatorname{Dsum}_{\lambda}\left(\left(\rho_{2}^{\prime}\right)^{i-k_{2}-k_{1}} \rho_{3}^{\prime}\right)\right) \\
= \\
\lambda^{\left|w_{1}\right|+k_{1}\left|w_{2}\right|}\left(\Delta\left(k_{1}\right)+\operatorname{Dsum}_{\lambda}\left(\left(\rho_{2}\right)^{i-k_{2}-k_{1}} \rho_{3}\right)-\operatorname{Dsum}_{\lambda}\left(\left(\rho_{2}^{\prime}\right)^{i-k_{2}-k_{1}} \rho_{3}^{\prime}\right)\right)
\end{gathered}
$$

And therefore $\Delta\left(k_{1}\right)=\Delta\left(k_{1}+k_{2}\right)$, which is a contradiction.


[^0]:    ${ }^{1}$ We do not consider the measure functions Min and Max that map a sequence to the minimal and the

[^1]:    2 As in [10, we do not consider the empty word as our weighted automata do not have initial and final weight functions. This eases our presentation but all our results carry over to the more general setting with initial and final weight function 20.

[^2]:    3 With the existence of an ending symbol, the notion of determinizability corresponds to the notion of subsequentializability 11.
    ${ }^{4}$ See [17, 15] for determinizability results on more general classes of Sum-automata.

