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A note on online and recursively arbitrarily vertex-partitionable balloons

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Abstract

In this note, we study arbitrarily vertex-partitionable balloons. In particular, we show that an online arbitrarily vertex-partitionable balloon has maximum degree at most 5. Moreover, we exhibit some infinite families of online and recursively vertex-partitionable balloons with four and five branches.

1 Preliminaries

In this note, the specific class of *balloons* is studied:

Definition 1 We denote by $B(b_1, \dots, \dots, b_k)$ the graph of order $2 + \sum_{1 \leq i \leq k} b_i$ consisting of two vertices r_1, r_2 , with k vertex-disjoint paths P_1, \dots, P_k of order b_1, \dots, b_k joining r_1 to r_2 . The graph $B(b_1, \dots, \dots, b_k)$ is called a k -balloon with roots r_1 and r_2 and branches P_1, \dots, P_k .

We will also need the following definition.

Definition 2 We denote by $PB(b_1, \dots, \overline{b_i}, \dots, \underline{b_j}, \dots, b_k)$ the partial k -balloon with branches of sizes b_1, \dots, b_k and roots r_1, r_2 where the branches denoted $\overline{b_i}$ are not adjacent to r_1 and the ones denoted $\underline{b_j}$ are not adjacent to r_2 .

Let n, τ_1, \dots, τ_k be positive integers such that $\sum_{1 \leq i \leq k} \tau_i = n$. Then $\tau = (\tau_1, \dots, \tau_k)$ is called a *decomposition* of n .

We study the problem of partitioning the vertex set of graphs according to certain conditions. The following definitions have been introduced in [1], [4] and [3], respectively.

Definition 3 Let G be a connected graph on n vertices.

- G is said to be arbitrarily vertex-partitionable (*AP for short*) if for every decomposition $\tau = (\tau_1, \dots, \tau_k)$ of n , there exists a partition V_1, \dots, V_k of $V(G)$ such that each part V_i has order τ_i and induces a connected subgraph of G .

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- G is said to be online arbitrarily vertex-partitionable (OLAP for short) if for every decomposition $\tau = (\tau_1, \dots, \tau_k)$ of n , there exists a partition $V_1 \dots, V_k$ of $V(G)$ such that each part V_i has order τ_i , induces a connected subgraph of G , and $V(G) \setminus V_i$ induces an OLAP subgraph of G .
- G is said to be recursively arbitrarily vertex-partitionable (RAP for short) if for every decomposition $\tau = (\tau_1, \dots, \tau_k)$ of n , there exists a partition $V_1 \dots, V_k$ of $V(G)$ such that each part V_i has order τ_i , and both V_i and $V(G) \setminus V_i$ induce a RAP subgraph of G .

It is worth noting that every graph with a hamiltonian path is RAP, every RAP graph is OLAP, every OLAP graph is AP, and every AP graph contains a (quasi)-perfect matching.

Observation 4 ([3]) Let $B = B(b_1, \dots, b_k)$ be an AP balloon with $b_1 \leq \dots \leq b_k$. If n is odd, B contains at most three branches of odd size. If n is even, B contains at most two branches of odd size.

Lemma 5 ([2]) Let $B(b_1, \dots, b_k)$ be an AP balloon with $b_1 \leq \dots \leq b_k$. Then for each $i \leq k$ we have $2b_i \geq \sum_{j < i} b_j$.

2 OLAP balloons have maximum degree at most 5

It is shown in [3] that every RAP balloon has maximum degree at most 5. We show that this holds as well for OLAP balloons.

Theorem 6 Let G be an OLAP k -balloon. Then $k \leq 5$.

Proof The proof is by contradiction. Let G be a minimum counterexample to the theorem: G is an OLAP k -balloon ($k \geq 6$) with least possible order.

For each $\lambda < n$, it is possible to find a subset of vertices V_λ with $|V_\lambda| = \lambda$, $G[V_\lambda]$ is connected, and $G[V \setminus V_\lambda]$ is OLAP. Since by the minimality of G , $G' = G[V \setminus V_\lambda]$ cannot be a partial k -balloon (otherwise adding an edge to G' would provide an OLAP k -balloon with order strictly less than G), for each λ , one of the following conditions must be fulfilled:

1. $G[V \setminus V_\lambda]$ is a k' -balloon with $k' = k - 1$ (i.e. V_λ is a branch of G), or
2. $G[V \setminus V_\lambda]$ is a path (i.e. V_λ consists of a root of G together with $k - 2$ branches), or
3. $G[V \setminus V_\lambda]$ is an OLAP caterpillar (i.e. V_λ consists of a root of G together with $k - 2$ branches but one vertex), or
4. $G[V \setminus V_\lambda]$ is the tripod $T_3(2, 4, 6)$ (i.e. V_λ consists of a root of G together with $k - 2$ branches but two vertices of the same branch)

We claim that G has branches of sizes respectively 1, 2, 3, 4, 5 and 6. We show this using contradiction. Let $\lambda \in \{1, 2, 3, 4, 5, 6\}$ and suppose there is no branch of size λ in G .

- $\lambda \in \{1, 2, 3\}$: for any choice of V_λ , $G[V \setminus V_\lambda]$ is either a partial k -balloon or a tree with a vertex of degree 4 (since at most two branches can be completely included in V_λ), a contradiction.
- $\lambda = 4$: similarly as in the previous case, for any choice of V_λ , $G[V \setminus V_\lambda]$ is either a partial k -balloon or a tree with a vertex of degree 3. By Observation 4, $G[V \setminus V_\lambda]$ cannot be a caterpillar, thus it is $T_3(2, 4, 6)$ and $G = B_6(1, 1, 1, 2, 4, 6)$, a contradiction since G has a branch of size 4.

- $\lambda = 5$: by the previous cases we know that G has branches of sizes 1, 2, 3 and 4. $G[V \setminus V_\lambda]$ cannot be a path since by Observation 4, five vertices cannot cover four branches and a root. If $G[V \setminus V_\lambda]$ is an OLAP caterpillar, then necessarily $k = 6$, the caterpillar is $Cat(4, 5)$ and G contains three branches of size 1: $G = B_6(1, 1, 1, 2, 3, 4)$. But then following Observation 4, G is not OLAP. Finally, if $G[V \setminus V_\lambda]$ is $T_3(2, 4, 6)$, V_λ must either totally cover the branch of size 3, or the branch of size 2 plus a vertex of the one of size 3. But then it is only possible to totally cover two branches, and the remaining part has a vertex of degree at least 4 and is not $T_3(2, 4, 6)$.
- $\lambda = 6$: by the previous cases we know that G has branches of sizes 1, 2, 3, 4 and 5. Now suppose there is one additional branch b_i ($i \leq 6$) such that $b_i \leq 5$. By Lemma 5 there can be at most one such additional branch, and then $b_i = 1$. Thus $G = B_k(1, 1, 2, 3, 4, 5, \dots)$. But now the only potential choice for V_λ would be the three smallest branches, a root and one additional vertex. If $k > 6$, $G[V \setminus V_\lambda]$ is a tree with a vertex of degree at least 4, which is not OLAP. Otherwise it is $T_3(2, 4, 5)$, which is not OLAP either. Thus, $G = B_k(1, 2, 3, 4, 5, b_6, \dots)$ with $b_6 \geq 7$. But again for $\lambda = 6$ it is not possible to decompose it. Thus G is not OLAP, a contradiction.

So, $G = B_k(1, 2, 3, 4, 5, 6, \dots)$. But using Lemma 5, G is not AP, a contradiction. \square

3 Families of OLAP balloons

In this section, we show that there are infinitely many OLAP k -balloons, for each $k \leq 5$. The case where $k \leq 3$ is trivial since then any k -balloon has a hamiltonian path and is therefore RAP.

3.1 OLAP 4-balloons

Proposition 7 $B(1, 1, n, m)$ is OLAP iff at most one of n and m is odd.

Proof Easily partitionable for $\lambda = 1, 3$. For $\lambda = 2$, one can prove by induction that $B(1, 1, \overline{2k}, m)$ and $B(1, 1, n, \overline{2k})$ is OLAP. \square

Proposition 8 Let $a \leq b \leq c$ be such that $Cat(b + 1, a + c + 2)$ or $Cat(c + 1, a + b + 2)$ or $Cat(a + 1, b + c + 2)$ is OLAP. Then $B(1, a, b, c)$ is OLAP.

Proof These caterpillars are spanning subgraphs of $B(1, a, b, c)$. \square

Proposition 9 $B(1, 2, 2, k)$, $k \geq 3$, is OLAP iff $k \equiv 0, 1 \pmod{3}$.

Proof Easily partitionable for $\lambda \neq 3$. If $k \equiv 0, 1 \pmod{3}$, also OK for $\lambda = 3$ because of $Cat(3, k + 1)$ which is OLAP.

Now suppose $k \equiv 2 \pmod{3}$. For $\lambda = 3$, for the balloon to be partitionable we should have either:

- $PB(1, 2, 2, \overline{k - 3})$ OLAP.
- $PB(1, 2, 2, \overline{a, b})$ ($a + b = k - 3$) OLAP

But in both cases these partial balloons are not $(3, \dots, 3)$ -partitionable. \square

Proposition 10 $B(1, 2, 3, k)$, $k \geq 3$, is OLAP.

Proof Straightforward. □

Proposition 11 $PB(2, 2, 2, \bar{k})$, is OLAP iff $k \equiv 0 \pmod 3$.

Proof By induction on k . □

Observation 12 $B(2, 2, 2, 3)$, $B(2, 2, 3, 4)$, $B(2, 2, 3, 6)$, $B(2, 2, 3, 18)$, $B(2, 2, 4, 6)$, $B(2, 3, 4, 6)$, are OLAP.

3.2 OLAP 5-balloons

The proof of the following result is omitted.

Proposition 13 Let G be a graph of Table 1. Then G is OLAP.

$B(1, 1, 1, 2, k)$ with $k \equiv 0, 4 \pmod 6$ $B(1, 1, 1, 4, k)$ with $k \in \{6, 8, 10, 18\}$ $B(1, 1, 1, 6, k)$ with $k \in \{8, 10, 12, 14\}$ $B(1, 1, 1, 8, 10)$
$B(1, 1, 2, 2, 3)$ $B(1, 1, 2, 2, 6)$ $B(1, 1, 2, 3, k)$ with $k \equiv 0 \pmod 2$ $B(1, 1, 2, 4, 6)$ $B(1, 1, 2, 5, k)$ with $k \equiv 0, 4 \pmod 6$ $B(1, 1, 3, 4, 6)$ $B(1, 1, 4, 4, 6)$
$B(1, 2, 2, 3, 4)$ $B(1, 2, 2, 4, 6)$ $B(1, 2, 3, 4, 5)$ $B(1, 2, 3, 4, 6)$ $B(1, 2, 3, 4, 7)$
$B(2, 2, 3, 4, 6)$

Table 1: Some OLAP 5-balloons

4 Families of RAP balloons

In this section, we show that there are infinitely many RAP k -balloons, for each $k \leq 5$.

4.1 RAP 4-balloons

Proposition 14 $PB(1, 1, 1, \bar{k})$, $B(1, 1, 1, k)$, $PB(1, 1, 3, \bar{k})$ and $B(1, 1, 3, k)$ are RAP iff k is even.

Proof If k is odd, they are not AP by Observation 4. On the other hand, $Cat(2, k + 3)$ is a spanning subgraph of $PB(1, 1, 1, \bar{k})/B(1, 1, 1, k)$ and $Cat(2, k + 5)$ is a spanning subgraph of $PB(1, 1, 3, \bar{k})/B(1, 1, 3, k)$. These caterpillars are RAP when k is even. □

Proposition 15 $PB(1, 1, 2, \bar{k})$ is RAP for any $k \geq 1$.

Proof Observe that it is true for $k \leq 3$ since $Cat(3, 5)$ and $Cat(4, 5)$ are spanning subgraphs of $PB(1, 1, 2, \bar{1})$, $PB(1, 1, 2, \bar{2})$ and $PB(1, 1, 2, \bar{3})$, respectively.

Now, suppose the proposition is true for all $i \leq k - 1$ and consider $PB(1, 1, 2, \bar{k})$. For $\lambda \leq 4$, $PB(1, 1, 2, \bar{k})$ can be partitioned into P_λ and $P_{n-\lambda}$. For $\lambda = 5$, it can be partitioned into $Cat(2, 3)$ and P_{k+1} . For $\lambda = 6$, it can be partitioned into $B(1, 1, 2)$ and P_k . Finally, for $\lambda \geq 7$, it can be partitioned into $PB(1, 1, 2, \overline{\lambda-6})$ and $P_{k-\lambda+6}$, the first being RAP by induction hypothesis. \square

Proposition 16 $PB(1, 2, 2, \bar{k})$ and $B(1, 2, 2, k)$ are RAP iff $k \equiv 1, 2 \pmod{3}$.

Proof On the one hand, observe that $Cat(3, k+4)$ is a spanning subgraph of $PB(1, 2, 2, \bar{k})$ and $B(1, 2, 2, k)$, which is RAP when $k \equiv 1, 2 \pmod{3}$. On the other hand, if $k \equiv 0 \pmod{3}$, they are not $(3, \dots, 3)$ -partitionable. \square

Proposition 17 $PB(1, 2, 3, \bar{k})$ is RAP for any $k \geq 3$.

Proof For any $\lambda \leq \frac{n}{2}$, $PB(1, 2, 3, \bar{k})$ must be partitionable into two vertex-disjoint RAP subgraphs of orders λ and $n - \lambda$.

- $\lambda = 1$: $P_1 + P_{n-1}$
- $\lambda = 2$: $P_2 + P_{n-2}$
- $\lambda = 3$: $P_3 + P_{n-3}$
- $\lambda = 4$: $P_4 + P_{k+4}$
- $\lambda = 5$: $P_5 + P_{k+3}$
- $\lambda = 6$: $P_6 + P_{k+2}$
- $\lambda = 7$: $Cat(3, 4) + P_{k+1}$
- $\lambda = 8$: $B(1, 2, 3) + P_k$

Note that by the previous cases, the proposition is true for $n \leq 16$ (i.e. $k \leq 8$).

Now, let $\lambda \geq 9$, suppose it is true for any $i \leq k - 1$, and consider $PB(1, 1, 2, 3, \bar{k})$. Now one can partition the graph into $P_{n-\lambda}$ and $PB(1, 2, 3, \overline{\lambda-8})$, which is RAP by induction hypothesis. \square

4.2 RAP 5-balloons

Proposition 18 $PB(1, 1, 2, 3, \overline{2k})$ is RAP for any $k \geq 0$.

Proof For any $\lambda \leq \frac{n}{2}$, $PB(1, 1, 2, 3, \overline{2k})$ must be partitionable into two vertex-disjoint RAP subgraphs of orders λ and $n - \lambda$.

- $\lambda = 1$: $P_1 + B(1, 2, 3, \overline{2k})$ (RAP by Proposition 17)
- $\lambda = 2$: $P_2 + B(1, 1, 3, \overline{2k})$ (RAP by Proposition 14)
- $\lambda = 3$: $P_3 + B(1, 1, 2, \overline{2k})$ (RAP by Proposition 15)
- $\lambda = 4$: $P_4 + Cat(4, 2k+1)$
- $\lambda = 5$: $Cat(2, 3) + P_{2k+4}$

- $\lambda = 6$: $P_6 + \text{Cat}(2, 2k + 1)$
- $\lambda = 7$: $\text{Cat}(3, 4) + P_{2k+2}$

Note that by the previous cases, the proposition is true for $n \leq 14$ (i.e. $k \leq 2$).

Now, let $\lambda \geq 8$, suppose it is true for any $i \leq k - 1$, and consider $PB(1, 1, 2, 3, \overline{2k})$.

- λ even:
 - if $\lambda \leq 2k$, one can partition the graph into P_λ and $PB(1, 1, 2, 3, \overline{2k - \lambda})$ by placing P_λ at the end of the last branch, and use the induction hypothesis.
 - if $\lambda > 2k$, then $n - \lambda \leq 7$ (since both $2k$ and λ are even), thus one can use the cases where $\lambda \leq 7$.
- λ odd. Then partition the graph into $PB(1, 1, 2, 3, \overline{\lambda - 9})$ (note that it is RAP by induction hypothesis since $\lambda - 9$ is even and positive) and $P_{2k - \lambda + 9}$.

□

This proves the existence of an infinite family of RAP 5-balloons:

Corollary 19 $B(1, 1, 2, 3, 2k)$ is RAP for any $k \geq 1$.

In [3], it is shown that $B(1, 1, 1, 2, 4)$ and $B(1, 1, 2, 2, 3)$ are RAP.

Proposition 20 Other RAP 5-balloons are $B(1, 2, 2, 3, 4)$ and $B(2, 2, 3, 4, 6)$.

References

- [1] O. Baudon, D. Barth and J. Puech. Decomposable trees: a polynomial algorithm for tripodes. *Discrete applied mathematics*, 119(3):205–216, 2002.
- [2] O. Baudon, F. Foucaud, J. Przybyło and M. Woźniak. Structure of k -connected Arbitrarily Partitionable Graphs. Manuscript, 2011.
- [3] O. Baudon, F. Gilbert and M. Woźniak. Recursively arbitrarily vertex-decomposable graphs. Submitted, 2010.
- [4] M. Hornák, Z. Tuza and M. Woźniak. On-line arbitrarily vertex decomposable trees. *Discrete Applied Mathematics* 155(11):1420–1429, 2007.