La renormalisation constructive pour la théorie quantique des champs non commutative

Zhituo Wang

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La renormalisation constructive pour la théorie quantique des champs noncommutative

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devant le jury composé de :

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Christoph	Kopper	examinateur

Resumé

Dans cette these on considere la renormalisation constructive pour la theorie quantique des champs noncommutative. La théorie constructive (ou la renormalisation constructive) propose l'étude mathématiquement rigoureuse de l'existence et des propriétés non perturbatives de la théorie quantique des champs.

Les méthodes traditionnelles de la théorie constructive sont les développements en amas et le groupe de renormalisation de Wilson.

Mais il y a aussi des défauts de ces deux méthodes : premièrement, les techniques du développement en amas et de Mayer sont compliquées, donc sont difficiles à utiliser. Deuxièmement, ces méthodes ne peuvent pas s'appliquer pour les théories quantiques des champs noncommutatives, où il n'y a pas de localité sur l'espace et l'interaction est non-locale.

Récemment une nouvelle méthode a été trouvée qui s'appelle loop vertex expansion (LVE), ou développement de vertex à boucle, qui est une combinaison de la technique des champs intermédiaires et de la formule des forêt (la formule de BKAR), qui peut résoudre ces deux problèmes avec succès.

Avec cette méthode, on n'a pas besoin du développement de Mayer et le développement en amas est aussi simplifié. Au lieu d'un ensemble de polymères, on a qu'un seul polymère ou arbre, donc on obtient la fonction de Schwinger connexe automatiquement ; et comme le terme d'interaction devient non-local aussi, cette méthode s'applique bien pour les théories quantique des champs noncommutatives, par exemple, le modèle de Grosse-Wulkenhaar, qui est un modèle $\lambda\phi^4$ avec un potentiel harmonique dans l'espace de Moyal. C'est le premier modèle de la théorie quantique des champs noncommutative qui est renormalisable. De plus, la fonction β est nulle quand on attend le point fixe ultraviolet de cette théorie. Donc c'est aussi un modèle naturel qu'on peut construire non-perturbativement.

Dans cette thèse nous allons construire le modèle de Grosse-Wulkenhaar à 2-dimensions avec la LVE.

Abstract

The main subject of this thesis is about a new method of constructive renormalization theory, called the Loop vertex expansion (LVE). Constructive renormalization theory is to study the nonperturbative properties of Euclidean quantum field theory. The traditional methods are cluster/Mayer expansions and the renormalization group analysis. But these methods are not suitable for the construction of quantum field theories defined on noncommutative manifolds. Since in the noncommutative quantum fields theories the interactions are nonlocal, the Cluster and Mayer expansions fail to work. This problem could be solved by the loop vertex expansion method, which is a combination of the intermediate fields technique with the BKAR tree formula. The reason is that in the intermediate field representation of the partition function the interactions are also nonlocal.

The Grosse-Wulkenhaar model is a is a quantum theory of scalar fields defined in the noncommutative Moyal space with harmonic potential. This model is not only renormalisable to all orders but also the beta function is zero at the fixed point of this theory. So this model is a candidate to be fully constructed.

As a first step, we constructed the 2-dimensional Grosse-Wulkenhaar model, with the method of loop vertex expansions.

In this thesis we studied also the construction of other noncommutative manifolds, namely the noncommutative type 1 Cartan domain, with the method of coherent states quantization. We studied also the graph polynomials for commutative and noncommutative quantum field theories.

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1

Introduction

Dans la partie principale de cette these on considere la theorie euclidienne constructive des champs. La théorie constructive (ou la renormalisation constructive) propose l'étude mathématiquement rigoureuse de l'existence et des propriétés non perturbatives de la théorie quantique des champs.

Les méthodes traditionnelles de la théorie constructive sont les développements en amas et le groupe de renormalisation de Wilson.

Pour utiliser le développement en amas, on décompose tout d'abord l'espace euclidien en cubes unitaires et on teste quels cubes sont vraiment occupés par les champs. La clé pour la convergence du développement en amas est la décroissance de la covariance et le fait que la constante de couplage soit petite.

Après avoir effectué le développement en amas, on obtient une forêt ou un emsemble d'arbres qui ne se superposent pas, et on les appelle les polymères avec l'interaction de coeur dur. Ensuite, on effectue le développement de Mayer pour enlever les coeurs durs et obtenir les fonctions connexes. Ce développement de Mayer est lui aussi un développement en amas mais entre les polymères crées par le premier développement.

Mais il y a aussi des défauts de ces deux méthodes :

- Premièrement, les techniques du développement en amas et de Mayer sont compliquées, donc sont difficiles à utiliser.
- Deuxièmement, ces méthodes ne peuvent pas s'appliquer pour les théories quantiques des champs noncommutatives, où il n'y a pas de localité sur l'espace et l'interaction est non-locale.

Récemment une nouvelle méthode a été trouvée qui s'appelle loop vertex expansion (LVE), ou développement de vertex à boucle, qui est une combinaison de la technique des champs intermédiaires et de la formule des forêt (la formule de BKAR), qui peut résoudre ces deux problèmes avec succès.

Avec cette méthode, on n'a pas besoin du développement de Mayer et le développement en amas est aussi simplifié. Au lieu d'un ensemble de polymères, on a qu'un seul polymère ou arbre, donc on obtient la fonction de Schwinger connexe automatiquement ; et comme le terme d'interaction devient non-local aussi, cette méthode s'applique bien pour les théories quantique des champs noncommutatives, par exemple, le modèle de Grosse-Wulkenhaar, qui est un modèle $\lambda\phi^4$ avec un potentiel harmonique dans l'espace de Moyal. C'est le premier modèle de la théorie quantique des champs noncommutative qui est renormalis-

able. De plus, la fonction β est nulle quand on attend le point fixe "ultraviolet" [1] de cette théorie. Donc c'est aussi un modèle naturel qu'on peut construire non-perturbativement.

Dans cette thèse nous allons introduire la méthode de LVE et construire le modèle de Grosse-Wulkenhaar à 2-dimensions.

Dans le premier chapitre nous allons introduire brièvement et rappeler les techniques traditionnelles de la théorie constructive. Le deuxième chapitre est dédie à l'introduction de la méthode de LVE. On va prendre le modèle ϕ_2^4 comme exemple.

Dans le chapitre 3, nous allons étudier la construction du modèle ϕ^{2k}, et ensuite au chapitre 4, nous allons construire le modèle de Grosse-Wulkenhaar à deux dimensions, avec la méthode LVE.

Nous allons construire un autre modèle de l'espace noncommutatif avec la méthode des états cohérents.

1. Nous mettons ultraviolet entre parenthèses parce que cette théorie mélange l'infrarouge et l'ultraviolet traditionnels.

2

La renormalisation constructive

2.1 Introduction

La théorie quantique des champs (TQC) est une extension logique et physiquement nécessaire du formalisme de la mécanique quantique non relativiste, pour expliquer les expériences de la physique des hautes énergies où les particules sont couplées et leur nombre n'est pas conservé. Par conséquent on doit remplacer la fonction d'onde de chaque particule par un opérateur linéaire dans l'espace de Hilbert, qui s'appelle l'opérateur de champ, possédant une infinité de degrés de liberté, capable de créer ou détruire des particules ou des antiparticules. Cette théorie est en accord avec les résultats expérimentaux et elle est à la base de la description des particules élémentaires.

Bien qu'il y ait beaucoup de succès en TQC, les propriétés mathématiques n'étaient pas bien comprises jusqu'aux années 1960. Les premiers essais pour comprendre la TQC plus rigoureusement est la théorie axiomatique, due à A. Wightman, R. Haag, R.Jost, D. Kastler...[1] [2], dans laquelle des propriétés générales, comme l'invariance de Lorentz, la causalité et l'unitarité, sont exprimées. Grâce à leur théorème de reconstruction, on peut reconstruire l'espace hilbertien et les opérateurs des champs physiques.

2.2 Théorie quantique euclidienne des champs

Bien que la théorie axiomatique soit mathématiquement bien définie, on ne peut trouver aucun modèle non-trivial satisfaisant à ces axiomes.

On étudie premièrement la TQC euclidienne, car dans le cadre euclidien il y a plusieurs propriétés qui sont meilleures que dans le cadre minkowskien : par exemple, c'est seulement dans ce cadre que l'intégrale de chemin est mathématiquement bien définie comme mesure de probabilité. Ensuite, on reconstruit la TQC dans l'espace de Minkowski par la continuation analytique.

Osterwalder et Schrader ont prouvé que [3][4], si la théorie euclidienne respecte cer-

tains axiomes, appelés axiomes d'Osterwalder-Schrader, on peut reconstruire la théorie minkowskienne à partir de celle-ci.

- OS1 la propriété de régularité nécessaire pour la continuation analytique dans l'espace de Minkowski ;
- OS2 l'invariance euclidienne par rotations et translations ;
- OS3 la positivité par réflection ;
- OS4 la symétrie par rapport à la permutation des arguments externes pour les bosons et antisymémtrie pour les fermions ;
- OS5 la décomposition en clusters. Les fonctions de Schwinger doivent se factoriser asymptotiquement lorsque deux ensembles d'arguments sont éloignés.

2.3 Renormalisation perturbative et analyse multiéchelle

On rencontre toujours des divergences quand on calcule les corrections quantiques pour les fonctions de Schwinger dans des modèles physiques de dimension suffisante. Elles sont une conséquence inévitable du caractère ponctuel des particules et de la conservation des probabilités. Si on a rencontré des termes divergents, on va les compenser par des contre-termes qui sont de la même forme que l'action de départ. Cette procédure pour compenser les divergences s'appelle la renormalisation, et le premier théorème mathématique complet qui assure que on peut ainsi enlever toutes les divergences est le théorème de BPHZ (voir [7]).

Si dans un graphe de Feynman l'amplitude associée à un sous-graphe est divergente, nous devons pour la renormaliser comparer l'amplitude du sous-graphe a celle d'un contre-terme *local*, c'est à dire de la forme de l'action initiale. Ceci est souvent fait dans les livres classiques dans la représentation de moment. Mais en theorie constructive on préfère travailler dans la repésentation directe. Dans ce cas pour comparer l'amplitude du sous-graphe au contre-terme local nous détachons les lignes extérieures attachées aux différents vertex de ce sous-graphe et les rattachons à un même vertex, et calculons les termes de correction par un développement de Taylor. Nous pouvons annuler excatement le terme avec les lignes externes au meme vertex (qui est local) avec son contre-terme. Comme les sous-graphes divergents sont toujours non-locaux et les contre-termes sont locaux, il nous reste après cette compensation les termes de correction du developpement de Taylor. De cette façon les paramètres initiaux de la théorie, comme la constante de couplage, le terme de masse ou le coefficient de la fonction d'onde changent de valeur. On dit qu ils sont renormalises et ce procédé est la renormalisation BPHZ.

Un point de vue plus fondamentale que BPHZ sur la renormalisation est le groupe de renormalisation, introduit par K. Wilson. L'idée essentielle est que l'on doit exprimer la théorie á l'aide des couplages physiques à des échelles très différentes, c'est-à-dire qu'il y a une trajectoire qui relie les couplages nus et renormalisés, qu'on appelle le flot du groupe de renormalisation. On commence par une TQC définie à une énergie haute, qui est loin d'être accessible expérimentalement, et par la procédure de la renormalisation on peut obtenir progressivement la physique des basses énergies ou des grandes distances. De cette façon la renormalisation est vue comme l'évolution d'un système dynamique mais c'est l'échelle qui joue le rôle du temps.

Une représentation explicite du groupe de renormalisation est l'expansion et le dévelop-

-pement d'espace de phase, appelée aussi l'analyse multiéchelle, développée par Glimm, Jaffe, Feldman, Magnen, Rivasseau et Sénéor [9, 7]. Dans cette représentation du groupe de renormalisation, nous décomposons les covariances de la théorie $C_{\Lambda}^{\Lambda_0}(p)$ en ρ tranches :

$$C_{\Lambda}^{\Lambda_0}(p) = \sum_{q=1}^{\rho} C^q(p), \tag{1}$$

où $M^{q-1} \leq |p| \leq M^q$, $\Lambda = M^0$, $\Lambda^0 = M^\rho$.

L'amplitude $A(G)$ du graphe G étant le produit des propagateurs des lignes, elle peut s'écrire comme :

$$A(G) = \prod_{l \in G} C_{\Lambda}^{\Lambda_0}(p) = \sum_{\mu} \prod_{l \in G} C^{\mu(l)}(p_l) = \sum_{\mu} A^{\mu}(G), \tag{2}$$

où l'attribution μ est une collection d'indices de tranche pour chaque ligne $\mu = \{\mu(l) =: l \in G\}$. L'espace de phase est representé avec une direction horizontale pour l'espace direct, et une direction verticale qui représente les échelles. Tout graphe avec une attribution μ peut etre décomposee selon cette représentation. Les propagateurs, qui, à tranche fixée, joignent des points d'espace différents, sont des lignes horizontales dans cette repésentation, et les vertex qui joignent des propagateurs d'échelle différentes sont des lignes verticales. Les objets importants sont alors les sous-graphes connexes dont toutes les lignes internes l_i ont une attribution de tranche plus élevée que toutes les attributions externes l_e, c'est-à-dire, $\mu(l_i) > \mu(l_e)$. Ces sous-graphes s'appellent *sous-graphes quasi-locaux*, et on s'intéresse plus particulièrement à ceux qui sont divergents. Les sous-graphes quasi-locaux forment pour la relation d'inclusion un arbre appellé arbre de Gallavoti-Nicolò [5][6]. En choisissant un arbre du graphe complet qui est sous arbre dans chaque sous-graphe quasi-local (c'est toujours possible) on peut établir excatement le comptage de pussance et verifier que seuls les sous graphes quasi-locaux divergents sont responsables des divergences.

D'autre part c'est *seulement pour les graphes quasi-locaux* que l'opération de Taylor définie plus haut donne des corrections *plus petites que le terme local*.

Le développement d'espace de phase permet donc d'effectuer une analyse plus fine des propriétés de divergence que la thérie BPHZ. Dans BPHZ on renormlaise les sous-graphes divergents meme quand ils ne sont pas quais-locaux. Alors on introduit des restes de Taylor qui peuvent etre tres grands. Les amplitudes renormalisees sont bien finies mais peuvenet etre tres grandes (comme n ! pour un graphe d'ordre n, et cela crée des nouvelles divergences non-perturbatives appellées renormalons).

Au contraire d'après l'analyse multi-échelle, il ne faut insérer que des contre-termes correspondant aux sous-graphes quasi-locaux ; on exprime la théorie en termes d'une infinité de constantes effectives, une par étage conformément à la philosophie du groupe de renormalisation et on n'a pas de renormalons, donc on peut faire la théorie constructive !

2.4 Renormalisation constructive

Il y a deux sortes de divergences dans la théorie quantique des champs perturbative : les divergences des fonctions de corrélation à chaque ordre de perturbation qu'on a introduit

dans la section précédente, et la divergence de la série de perturbation elle-même, c'est-à-dire que, la série n'est pas sommable, et c'est dû à la combinatoire : le nombre des graphes de Feynman prolifère si on développe la théorie complètement [10]. Ce deuxieme type de divergence est résolu par la theorie constructive si la constante de couplage reste petite. En gros au lieu de développer tous les graphes, on développe seulement des arbres couvrants assurant la connexité des fonctions de Schwinger et on resomme le reste. Les arbres en effet proliferent bien moins vite que les graphes : Pour le modèle ϕ^4 le nombre de graphes d'ordre n est $(4n)!! \sim (n!)^2$, mais le nombre des arbres couvrants sur n sommets est seulement $n^{n-2} \sim n!$, beaucoup plus petit que $(n!)^2$. On va introduire de tels formulalisme des arbres dans les sections suivantes, par exemple le developpement de vertex a boucles. Mais tout le probleme consiste a montrer que la theorie constructive reste compatible avec la renormalisation. C'est le but de cette these de faire un nouveau pas dans cette direction avec la construction du modele GW2.

Si on considère la théorie quantique des champs sérieusement, on voudrait en effet construire des fonctions de corrélation uniquement reliees a la serie de perturbation. par resommation de la série perturbative, et la bonne notion dans le cas de ϕ^4 est la Borel sommabilite. Et de plus, les fonctions de corrélation que l'on a construites doivent être indépendante de l'échelle du cutoff ultraviolet et du volume de l'espace où la théorie des champs est définie.

C'est le but de la théorie de la constructive qui est introduite et développée par Arthur Wightman, E. Nelson, J. Glimm et A. Jaffe [8][7][9]. Jusqu'à maintenant les théories qui ont été construites sont le modèle de ϕ^4 dans l'espace euclidien à deux et trois dimensions, et à quatre dimensions dans la région de l'infrarouge, la théorie de Yang-Mills à 4 dimensions dans la région de l'ultraviolet, et les théories de fermions à deux dimensions. Par exemple, le modèle de Gross-Neveu [11].

Pour les théories fermioniques les méthodes qui ont été utilisées sont la formule des forêts et l'inégalité de Gram. Dû au principe de Pauli, les théories de fermions juste renormalisables sont construites avec beaucoup de succès, le problème étant presque aussi facile que leur renormalisation perturbative. Mais pour les théories bosoniques, comme les bosons peuvent se condenser mais pas s'exclure, les techniques sont plus compliquées.

Les méthodes que nous allons utiliser sont le développement en clusters, avec lequel on peut prendre la limite thermodynamique pour la fonction de partition et ensuite effectuer le développement de Mayer pour obtenir la fonction de corrélation connexe. Je vais introduire ces méthodes avec plus de détails dans les sections suivantes.

Dans ce chapitre nous allons introduire les théorèmes et techniques fondamentaux pour la théorie constructive et on va introduire la loop vertex expansion (LVE) ou développement de vertex à boucle, plus explicitement dans les chapitres suivants. Dans la section 2 on va introduire la formule de forêt ou la formule de BKAR (Brydges-Kennedy-Abdessalam-Rivasseau). Dans la section 3 nous allons introduire le développement en clusters et de Mayer. Ensuite, en section 4 nous allons introduire la sommabilité de Borel et montrer un théorème important de Sokal-Nevanlinna.

2.5 La formule de forêt

Comme nous l'avons dit dans la section précédente, pour éviter la prolifération de la combinatoire de la série perturbative, on doit effectuer le développement en arbres. Dans cette section nous allons introduire la formule de BKAR (Brydges-Kennedy Abdessalam-Rivasseau)[12][13], qui est la formule pour le développement de la fonction de Schwinger en forêts. Cette formule est un outil canonique pour calculer les poids des arbres couvrants dans un graphe de Feynman. Considérons le développement de la fonction connexe en graphes de Feynman

$$S = \sum_G \mathcal{A}_G \tag{1}$$

Cette formule n'est pas bien définie parce que

$$\sum_G |\mathcal{A}_G| = \infty. \tag{2}$$

Donc on doit utiliser un autre développement par rapport aux arbres et on obtient

$$S = \sum_G A_G = \sum_G \sum_{T \subset G} w(G,T)\mathcal{A}_G \tag{3}$$

où $w(G,T)$ est le poids relatif d'un arbre T dans un graphe G. Après échange de l'ordre des sommes sur G et T on a :

$$S = \sum_T \mathcal{A}_T, \text{ où } \mathcal{A}_T = \sum_{G \supset T} w(G,T)\mathcal{A}_G, \tag{4}$$

et nous avons alors $\sum_T |\mathcal{A}_T| < \infty$ pour les théories fermioniques et les théories bosoniques avec champs intermédiares.

2.5.1 Une formule combinatoire

Nous pouvons récrire cette formule d'une autre façon comme le poids explicite d'une forêt dans un graphe. Comme chaque forêt peut se diviser comme arbres couvrants disconnectés, on considère seulement la formule pour les arbres. On a le théorème suivant :

Théorème 2.5.1 *Le poids de chaque arbre couvrant dans un graphe est :*

$$w(G,T) = \prod_{\ell \in T} \int_0^1 \prod_{\ell \in T} dw_\ell \prod_{\ell \notin T} x_\ell^T(\{w\}) \tag{5}$$

où $\ell = (v, v')$, $x_\ell^T(\{w\})$ est le minimum parmi tous les paramètres $w_{\ell'}$ sur les lignes ℓ' qui forment le chemin unique dans T qui connecte les points v et v'. On a

$$\sum_{T \in G} w(G,T) = 1. \tag{6}$$

2.5.2 BKAR

Pour le cas plus compliqué comme dans la théorie quantique des champs, il y a une méthode canonique pour calculer le poids $w(T,G)$, c'est la formule de BKAR [12][13]. La formule de BKAR est le développement de Taylor avec le reste de l'intégrale. On considère un graphe avec N points, le nombre de lignes est donc $\frac{N(N-1)}{2}$. On attribue à chaque ligne une variable $w_l \in [0,1]$ et $l = (v, v')$. On considère une fonction $f(W) = f(x_1, \cdots, x_{N(N-1)/2})$ et on a :

Théorème 2.5.2 *La formule générale des forêt est :*

$$f(1,\cdots,1) = \sum_F \Big(\prod_{l\in F}\int_0^1 dw_l\Big)\Big(\prod_{l\in F}\Big(\frac{\partial}{\partial x_l}\Big)f\Big)(x_F^l(w)) \tag{7}$$

où

- $x_F(w) = \min w_l$ *s'il existe une trajectoire entre deux vertex* v *et* v' *et* $\min w_l$ *est le paramètre minimum des lignes du chemin connectant* v *et* v'.
- $x_F^l(w) = x_F^{vv'}(w) = x_F^{v'v}(w)$, *c'est-à-dire le paramètre* $x_F^l(w)$ *est symétrique. Et* $x_F^{vv}(w) = 1$.
- $x_F(w) = 0$, *si* v *et* v' *ne sont pas connectés.*

La raison pour attribuer les poids des arbres couvrants vient de la positivité de la mesure. on dit que cette formule est positive parce que la matrice $x_F^l(w)$ est positive. Et la combinaison convexe de matrices positives est encore positive. Cette propriété est fondamentale pour appliquer l'analyse constructive aux modèles bosoniques.

2.6 Développement en clusters et de Mayer

Le calcul de la fonction de Green connexe comprend généralement deux étapes : on commence par le développement en amas, pour prendre la limite thermodynamique de la fonction de partition Z. Dans cette étape on coupe l'espace de volume V en cubes et on définit l'ensemble I_n comme l'ensemble des cubes. Le cluster est donc le sous-ensemble des cubes. Après le développement en amas on obtient un ensemble de polymères avec interaction de coeur dur.

Ensuite, on effectue le développement de Mayer pour calculer la fonction connexe en enlevant l'interaction du coeur dur.

On va prendre la théorie de ϕ_d^4 définie sur l'espace R^d comme un exemple pour montrer le développement en amas [7][13].

L'action est définie comme :

$$S = \frac{1}{2}\int_{R^d} d^dx\big[(\partial_\mu\phi)^2 + m^2\phi^2\big] + \int_V \lambda\phi^4(x). \tag{1}$$

et la fonction de partition est définie comme

$$Z(V) = \int d\mu_C e^{-\lambda \int_V \phi^4(x)}. \tag{2}$$

où $d\mu_C$ est la mesure gaussienne libre, définie par la covariance

$$C(x,y) = \int_\epsilon^\infty \frac{d\alpha}{\alpha^{d/2}} e^{-\alpha m^2 - |x-y|^2/4\alpha}. \tag{3}$$

Ici ϵ est le régulateur ultraviolet (UV). Et la fonction de Schwinger dans le vide est définie comme

$$p = \lim_{V \to R^d} \frac{1}{|V|} log Z(V) \tag{4}$$

On écrit le volume comme $V = \cup_{i \in I_n} \Delta_i$ où Δ_i est un cube de volume unité. On définit la fonction caractéristique χ_Δ par

$$\chi_{\Delta_i}(x) = \begin{cases} 1, \text{ si } x \in \Delta_i \\ 0, \text{ sinon} \end{cases} \tag{5}$$

et on a $\chi_V = \sum_{i \in I_n} \chi \Delta_i$. Comme l'interaction est entièrement dans V, on peut écrire le propagateur comme $C_V = \chi_V(x) C(x,y) \chi_V(y)$ sans changement de Z.

On considère l'ensemble de cubes I_n et on définit l'ensemble des lignes $l = (i,j) \in P_n$ s'il y a de l'interaction entre Δ_i et Δ_j. On peut interpoler les propagateurs C_V comme :

$$\begin{aligned} C_V((w_l)_{l \in P_n})(x,y) &= \sum_{i=1}^n \chi_{\Delta_i}(x) C(x,y) \chi_{\Delta_i}(y) \\ + \sum_{(i,j) \in P_n} & w_{ij} \big[\chi_{\Delta_i}(x) C(x,y) \chi_{\Delta_j}(y) + \chi_{\Delta_j}(x) C(x,y) \chi_{\Delta_i}(y) \big]. \end{aligned} \tag{6}$$

Notons que $C_V(1, \cdots, 1) = C_V$. On va utiliser la formule en notant f la fonction de partition et en remplaçant la covariance C par $C_V((w_l)_{l \in P_n})$. Ici la positivité est essentielle parce que la mesure de l'interpolation doit être positive. Finalement on a pour la fonction partition :

$$\begin{aligned} Z(V) = f(I) = \sum_F \int d\mu_{C_V(W_l)}(\phi) \Big(\prod_{l \in F} \int dw_l \Big) \\ \Big[\prod_{l=(ij) \in F} \int dx dy \chi_{\Delta_i}(x) \chi_{\Delta_j}(y) C(x,y) \frac{\delta}{\delta \phi(x)} \frac{\delta}{\delta \phi(x)} \Big] e^{-\lambda \int_V \phi^4(x) d^d x}. \end{aligned} \tag{7}$$

Comme les termes de l'interaction et de covariance se factorisent sur les clusters de la forêt F, la fonction $Z(V)$ se factorise aussi comme le produit des contributions pour chaque cluster qui ne se superpose pas avec l'autre. On appelle aussi la forêt de clusters le gaz de polymères Y_i. Pour chaque cluster ou polymère qui est connecté on peut utiliser la formule de l'arbre. Et finalement on a obtenu :

$$Z(V) = \int d\mu_{C_V}(\phi) e^{-S_V(\phi)} = \sum_{\{Y_1,\cdots,Y_n\};\ Y_i\cap Y_j=\emptyset, \cup Y_i = V} \prod_{i=1}^{n} A(Y_i) \tag{8}$$

$$A(Y) = \sum_{T\in Y} \int d\mu_{C_V(w_l)}(\phi) \Big(\prod_{l\in T} \int dw_l \Big)$$

$$\Big[\prod_{l=(ij)\in T} \int dxdy \chi_{\Delta_i}(x)\chi_{\Delta_j}(y) C(x,y) \frac{\delta}{\delta\phi(x)} \frac{\delta}{\delta\phi(x)} \Big] e^{-\lambda \int_V \phi^4(x) d^d x}.$$

où Δ_i et Δ_j sont deux points de la ligne l, T est l'arbre couvrant le polymère Y et la mesure $d\mu_{C_V(w_l)}(\phi)$ est la mesure gaussienne avec la covariance

$$C_Y(w_l)(x,y) = w_{ij}\chi_Y(x) C(x,y) \chi_Y(y). \tag{9}$$

La définition pour les facteurs w_l est comme avant.

On dois effectuer le développement (A.4) dans la représentation multi-échelle des propagateurs et pour chaque polymère déchelle i on peut choisir un cube qui dépendente aussi léchelle, comme racine et l'appeler Δ_0, et on peut prouver que dans chaque tranche pour la constante de couplage λ petite et $\mathrm{Re}\lambda > 0$, il existe un facteur K positif tel que :

$$\sum_{Y,\Delta_0\in Y^i} |A(Y)| K^{|Y^i|} \leq 1. \tag{10}$$

On peut trouver la preuve dans le livre de V. Rivasseau [7].

Avec le développement en amas en utilisant la formule de l'arbre on peut calculer la fonction de partition qui se factorise comme l'ensemble de polymères avec l'interaction de coeur dur, qui est définie comme

$$V(X,Y) = \begin{cases} 0, \text{ si } X\cap Y = \emptyset \\ \infty, \text{ si } X\cap Y \neq \emptyset \end{cases} \tag{11}$$

Pour résoudre cette interaction et obtenir la fonction connexe, on doit utiliser le développement de Mayer, que l'on va introduire dans la section prochaine.

2.7 Le développement de Mayer

Le développement de Mayer est à résoudre l'interaction du coeur dur. Comme les amplitudes des polymères sont invariantes par la translation, et A_0, l'amplitude de l'amas avec un seul cube $Y = \Delta$, est indépendante de Δ_i, on peut redéfinir l'amplitude en amas et la fonction de partition comme $Z_r = Z(V)/A_0^{|V|}$.

Nous avons pour les amplitudes des polymères :

$$Z_r(V) = 1 + \sum_{n\geq 1} \sum_{\{Y_1,\cdots,Y_n\},\ |Y_i|\geq 2, Y_i\cap Y_j = 0} \prod_{i=1}^{n} A_r(Y_i), \tag{1}$$

et la fonction de partition devient :

$$Z_r(V) = 1 + \sum_{n\geq 1} \frac{1}{n!} \sum_{\{Y_1,\cdots,Y_n\},\ |Y_i|\geq 2} \prod_{i=1}^{n} A_r(Y_i) \prod_{1\leq i<j\leq n} e^{-V(Y_i,Y_j)}. \tag{2}$$

On peut écrire le dernier terme comme :

$$\prod_{1\leq i<j\leq n} e^{-V(Y_i,Y_j)} = \prod_{1\leq i<j\leq n} [(e^{-V(Y_i,Y_j)} - 1) + 1] = \prod_{i,j} [\epsilon_{ij}^{Y} + 1], \tag{3}$$

où $\epsilon_{ij}^{Y} = (e^{-V(Y_i,Y_j)} - 1)$ et $i \neq j$.

Définissons $P_n = \{l = (i,j)\}$ comme l'ensemble des paires de points (i,j), et pour une séquence de polymères $(Y_1, \cdots, Y_n)$ fixée, considérons la fonction

$$f((w_l)_{l\in P_n}) = \prod_{l\in P_n} (1 + w_l \epsilon_l^{Y}), \tag{4}$$

et $f(I) = f(1, \cdots, 1)$. À nouveau nous allons utiliser la formule de BKAR, et Z_r devient :

$$\begin{aligned} Z_r(V) &= 1 + \sum_{n\geq 1} \frac{1}{n!} \sum_{\{Y_1,\cdots,Y_n\},|Y_i|\geq 2} F(I) \prod_{i=1}^{n} A_r(Y_i) \qquad (5)\\ &= 1 + \sum_{n\geq 1} \frac{1}{n!} \sum_{\{Y_1,\cdots,Y_n\},|Y_i|\geq 2} f(I) \prod_{i=1}^{n} A_r(Y_i) \sum_{F} \Big(\prod_{l\in F} \int_0^1 dw_l \Big) \\ &\quad \Big(\prod_{l\in F} \epsilon_l^{Y} \Big) \Big(\prod_{l\notin F} (1 + w_l^{F}(w) \epsilon_l^{Y}) \Big) \\ &= \sum_{n\geq 0} \frac{1}{n!} \Big(\sum_{k\geq 0} \frac{1}{k!} \sum_{\{Y_1,\cdots,Y_k\},|Y_i|\geq 2} \Big(\prod_{i=1}^{n} A_r(Y_i) \Big) C^{T}(Y_1, \cdots, Y_k) \Big)^{n}. \end{aligned}$$

où

$$\begin{aligned} C^{T}(Y_1, \cdots, Y_k) &= \sum_{G^c} \prod_{l\in G^c} \epsilon_l^{Y} \\ &= \sum_{T} \prod_{l\in T} \int_0^1 dw_l \Big(\prod_{l\in P_k,\ l\in T} \epsilon_l^{Y} \Big) \Big(\prod_{l\in P_k,\ l\notin T} (1 + w_l^{T}(w) \epsilon_l^{Y} \Big). \end{aligned} \tag{6}$$

où G^c est l'ensemble des graphes connexes défini sur l'ensemble des polymères $\{Y_1, \cdots, Y_k\}$ et T est l'ensemble des arbres définis sur l'ensemble des polymères.

A cause de la factorisation on obtient facilement que :

$$\log Z_r(V) = n \log A_0 + \sum_{k\geq 1} \frac{1}{k!} \sum_{\{Y_1,\cdots,Y_k\},|Y_i|\geq 2} \Big(\prod_{i=1}^{n} A_r(Y_i) \Big) C^{T}(Y_1, \cdots, Y_k), \tag{7}$$

qui est la fonction connexe et on peut prendre la limite thermodynamique facilement :

$$p = \lim_{V\to R^d} \frac{1}{|V|} \log Z_r(V). \tag{8}$$

Il paraîtque, dans la formule (7), il existe la structure des arbres et des forêts sur deux objets différents : le lien entre les cubes qui fait l'amas Y_i et le lien de Mayer entre les amas différents. Si on effectue le calcul du groupe de renormalisation , c'est-à-dire, l'analyse multi-échelle pour la fonction connexe, le formalisme sera très compliqué. Pour éviter cette complication, on pouvait utiliser la formule de jungle [13]. Comme dans cette thèse nous allons utiliser le développement de loop vertex avec lequel on peut résoudre ce problème autrement et plus facilement, on ne va pas introduire les formules de jungle et on suggère aux lecteurs intéressés de consulter le papier original de Abdesselam et Rivasseau [13].

2.8 La somme de Borel

Naturellement les séries perturbatives en théorie quantique des champs comme ϕ^4 et QED, peuvent s'écrire comme $(-1)^n\lambda^n n!$ et ne sont pas naîvement sommables. Mais si les valeurs des constantes de couplage peuvent se continuer analytiquement à certains domaines complexes, les séries devienent sommables dans le sens de Borel [14, 15] [7][23][22]. Dans cette section nous allons introduire le théorème pour la sommabilité de Borel et la preuve d'après Sokal. De plus, on va aussi introduire la sommabilité de Borel-Leroy aux ordres plus élevés. C'est utile pour la construction de la théorie comme ϕ^{2k} où la série de perturbation est comme $(-1)^n\lambda^n(n!)^k$.

Théorème 2.8.1 *(Nevanlinna-Sokal)[23]*

Une série $\sum_{n=0}\frac{a_n}{n!}\lambda^n$ est sommable de Borel à la fonction $f(\lambda)$ si les conditions suivantes sont respectées :

- *pour un nombre $R>0$, $f(\lambda)$ est une fonction analytique dans le disque $C_R=\{\lambda\in C : \Re\lambda^{-1}>R^{-1}\}$.*
- *la fonction $f(\lambda)$ admet $\sum_{n=0}^{N}a_n\lambda^n$ comme un développement asymptotique avec le reste de Taylor d'ordre N R^Nf quand la variable $\lambda\to 0$ dans C_R, et le reste de Taylor d'ordre N est bornée dans C_R par :*

$$\left|R^Nf\right|\leqslant AB^NN!|\lambda|^{N+1}. \tag{1}$$

où A et B sont des constantes.

Et la transformation de Borel devient :

$$B_f(u)=\sum_{n=0}^{\infty}\frac{a_N}{N!}u^N. \tag{2}$$

Ici $B_f(u)$, transformée de Borel de f, est holomorphe pour $|u|<B^{-1}$. Il admet la continuation analytique dans la bande $\{u\in C : |\Im u|<R, \Re u>0\}$, et on obtient, pour $\lambda\in C_R$:

$$f(\lambda) = \int_0^\infty B_f(u) exp[-(u/\lambda)](u/\lambda) du. \tag{3}$$

Dans ce cas là on dit que la série perturbative est sommable de Borel et la somme est la fonction f. Nous avons aussi le théorème suivant pour la sommabilité de Borel à l'ordre plus élevé :

Théorème 2.8.2 *(Nevanlinna-Le Roy)[22]*

La série $\sum_{n=0} \frac{a_n}{n!}\lambda^n$ *est sommable de Borel à la fonction* $f(\lambda)$ *à l'ordre* k *si elle satisfait les conditions suivantes :*

- $f(\lambda)$ *est analytique dans le domaine* $C_R^k = \{\lambda \in C : \Re\lambda^{-1/k} > R^{-1}\}$, *où* $k > 0$ *est un nombre entier.* C_R^k *est un disque si* $k = 1$.
- *La fonction* $f(\lambda)$ *admet le développement asymptotique uniforme* $\sum_{n=0}^\infty a_n\lambda^n$ *pour tous les ordres* n *quand* $\lambda \to 0$ *dans* C_R^k *avec la borne du reste de Taylor :*

$$\left|R^N f\right| \leqslant AB^N\Gamma(kN+1)|\lambda|^{N+1}. \tag{4}$$

où A *et* B *sont des constantes.*

Et la transformation de Borel-Le Roy d'ordre k *est :*

$$B_f^{(k)}(u) = \sum_{n=0}^\infty \frac{a_n}{\Gamma(kn+1)} u^n, \tag{5}$$

qui est holomorphe quand $|u| < B^{-1}$, *et elle admet pour continuation analytique sur la bande :* $\{u \in C : |\Im u| < R, \Re u > 0\}$, *et pour* $0 \leqslant R$, *on obtient :*

$$f(\lambda) = \frac{1}{k\lambda}\int_0^\infty B_f^{(k)}(u) exp[-(u/\lambda)^{1/k}](u/\lambda)^{(1/k-1)} du. \tag{6}$$

2.9 Le développement de loop vertex (LVE)

Il existe aussi des défauts pour les développements en amas et de Mayer : premièrement, quand on coupe l'espace euclidien par amas, on brise l'invariance par rotation. Et comme ces méthodes assument la localité de l'espace et des interactions des théories, elles sont très difficiles à utiliser, voire impossible, pour les théories des champs définies sur l'espace non-commutatif.

Dans cette thèse nous allons introduire une nouvelle méthode pour la construction de la théorie bosonique qui s'appelle le développement de loop vertex. Cette méthode a été d'abord introduite par Vincent Rivasseau pour la construction de la théorie matricielle

[16], puis V. Rivasseau et J. Magnen ont trouvé qu'on peut appliquer cette méthode aussi à la théorie des champs commutative [17], le modèle ϕ_4^4 dans une tranche de moments, par exemple.

Et puis cette méthode est généralisée pour la théorie bosonique ϕ^{2k} à zéro dimension [18] et la théorie de ϕ^4 à deux dimensions, par V. Rivasseau et l'auteur [18]. En même temps cette méthode était utilisée pour la construction du modèle de Grosse-Wulkenhaar à deux dimensions [21], qui est le premier modèle non-commutatif qu'on peut construire rigoureusement.

Dans le chapitre suivant, nous allons introduire cette méthode pour construire le modèle ϕ_2^4 avec plus de détails. Et ensuite nous allons montrer comment cette méthode fonctionne pour le modèle ϕ^{2k} à zéro dimension et le modèle de Grosse-Wulkenhaar à deux dimensions.

3

Théorie constructive ϕ_2^4 avec LVE

3.1 Introduction

Nous avons introduit au chapitre précédent les méthodes traditionnelles de la théorie constructive des champs que sont les développements en amas et de Mayer, ainsi que leurs défauts respectifs. Il apparaît alors difficile d'utiliser ces méthodes pour construire les TQC sur des espaces non-commutatifs.

Récemment Vincent Rivasseau a développé une nouvelle méthode en théorie constructive, la *Loop Vertex expansion* (LVE) [16][17]. C'est une combinaison de la méthode des champs intermédiaires et de la formule de BKAR [12][13]. Grâce à la méthode LVE, nous pouvons obtenir directement la fonction de Green connexe ; il est donc inutile de recourir à un développement de Mayer. D'autre part, le développement en amas, qui permet d'atteindre la limite thermodynamique, est lui aussi simplifié. Enfin, et c'est le point le plus important pour nous, cette méthode permet de traiter les TQC non-commutatives telles que le modèle de Grosse-Wulkenhaar par exemple.

Dans ce chapitre, on illustre, sur l'exemple du modèle ϕ_2^4, les principales idées de la LVE et comment elle permet de construire ce modèle.

Dans le cas d'une théorie à zéro composantes la méthode du champ intermédiaire avait déjà été développée au niveau constructif par D. Iagolinitzer et J. Magnen [19], qui ont vérifié que l'on peut alors contrôler la théorie infrarouge en dimension 4 où elle est juste renormalisable. Cela nous rend confiants que l'on pourra bientôt construire rigoureusement le modèle de Grosse-Wulkenhaar en dimension 4 grâce à l'ensemble de ces techniques.

3.2 La construction pour ϕ_2^4

Dans cette section et le reste de ce chapitre, nous allons montrer comment construire le modèle ϕ_2^4 avec la méthode de LVE. On commence par redéfinir le modèle ϕ_2^4 par la fonction de partition ordonnée suivant l'ordre de Wick. Ceci permet d'écrire les contre-

termes explicitement :

$$Z(\lambda, \Lambda, \mathcal{V}) = \int d\mu_{C^\Lambda}(\phi) e^{-\frac{\lambda}{2}\int_{\mathcal{V}} d^2x :\phi^4(x): + j\phi}, \tag{1}$$

où $j(x)$ est le champ de source.

L'ordre de Wick pour l'interaction est : $\phi^4(x) :\equiv \phi^4 - 6T_\Lambda \phi^2 + 3T_\Lambda^2$, oú $T_\Lambda = 2C_\Lambda(x,x)$ est le contre-terme qui diverge comme $\log \Lambda$ quand $\Lambda \to \infty$.

Pour les fonctions à n points où $n = 2p$ est pair, la fonction de Schwinger connexe s'écrit comme :

$$S(x_1, \cdots, x_n) = \frac{1}{(2p)!} \frac{\partial^{2p}}{\partial j(x_1) \cdots \partial j(x_{2p})} \frac{1}{\mathcal{V}} \log Z(\Lambda, j(x))|_{j=0}. \tag{2}$$

On introduit alors le champ intermédiaire σ et la fonction de partition devient :

$$\begin{aligned} Z(\Lambda, j(x)) &= \int d\mu_{C^\Lambda}(\phi) e^{-\frac{\lambda}{2}\int_{\mathcal{V}} d^2x [(\phi^2 - 3T_\Lambda)^2 - 6T_\Lambda^2] + j(x)\phi(x)} \\ &= \int d\mu_{C^\Lambda}(\phi) \int d\sigma e^{-\frac{1}{2}\int d^2x\ \sigma(x)^2} e^{3\lambda|\mathcal{V}|T_\Lambda^2 + i\sqrt{\lambda}\int_{\mathcal{V}}\ d^2x\ \sigma\ (\phi^2 - 3T_\Lambda)} \end{aligned} \tag{3}$$

Effectuons l'integration sur les champs ϕ et la fonction de partition devient alors :

$$Z(\Lambda, j(x)) = \int d\nu(\sigma) e^{3\lambda|\mathcal{V}|T_\Lambda^2 + \mathrm{Tr}_{\mathcal{V}}\left(3\mathrm{i}\sqrt{\lambda}\mathrm{T}_\Lambda\sigma - \frac{1}{2}\log(1 + 2\mathrm{i}\sqrt{\lambda}\mathrm{C}^{1/2}\sigma\mathrm{C}^{1/2})\right) + \int \frac{1}{2}\mathrm{j(x)R(x,y)j(y)}}$$

où

$$C_R(x, y, \sigma) = [C^{1/2} \frac{1}{1 + 2i\sqrt{\lambda}C^{1/2}\sigma C^{1/2}} C^{1/2}](x, y). \tag{4}$$

est la résolvante.

Dans ce chapitre, on ne considère que le cas $p = 0$. Le terme d'interaction s'écrit dans ce cas

$$V(\lambda, \Lambda, \mathcal{V}, \sigma) = 3\lambda|\mathcal{V}|T_\Lambda^2 + \mathrm{Tr}_{\mathcal{V}}\left(-\frac{1}{2}\log(1 + 2\mathrm{i}\sqrt{\lambda}\mathrm{C}^{1/2}\sigma\mathrm{C}^{1/2}) + 3\mathrm{i}\sqrt{\lambda}\mathrm{T}_\Lambda\sigma\right). \tag{5}$$

Du point de vue de la localité, les rôles respectifs du propagateur et du vertex sont ainsi échangés : le propagateur dans ce modèle devient $\delta(x - y)$, qui est trivial ; tandis que l'interaction devient non-locale.

Développons le terme d'interaction comme suit :

$$\begin{aligned} &\mathrm{Tr}_{\mathcal{V}}[-\frac{1}{2}\log(1 + 2\sqrt{\lambda}\mathrm{C}^{1/2}\sigma\mathrm{C}^{1/2})] \\ =\ &\mathrm{Tr}_{\mathcal{V}}[-\mathrm{i}\sqrt{\lambda}\mathrm{T}_\Lambda\sigma - \frac{1}{2}\log_2(1 + 2\mathrm{i}\sqrt{\lambda}\mathrm{C}^{1/2}\sigma\mathrm{C}^{1/2})]. \end{aligned} \tag{6}$$

où

$$\log_p(1 + x) = \log(1 + x) - \sum_{q=1}^{p-1}(-1)^{q-1}\frac{x^q}{q}. \tag{7}$$

Le premier terme de (6) peut partiellement compenser le contre-terme.

Comme nous l'avons déjà indiqué, la méthode LVE est une combinaison de la méthode des champs intermédiaires et de la formule de BKAR. Il nous reste donc à appliquer la formule de BKAR ou, plus explicitement, la formule des forêts, pour obtenir la fonction de Schwinger connexe.

On tout d'abord développe l'exponentielle comme $\sum_n \frac{V^n}{n!}$ et on attribue à chaque vertex un indice v, $v = 1, \cdots, n$, de sorte que :

$$\begin{aligned} Z(\lambda, \Lambda, \mathcal{V}) &= \int d\nu(\sigma) \sum_{n=0}^{\infty} \frac{V^n}{n!} = \int d\nu(\sigma) \sum_{n=0}^{\infty} \prod_{v=1}^{n} \frac{V_v(\sigma_v)}{n!} \\ = & \sum_0^{\infty} \frac{1}{n!} \int d\nu(\{\sigma_v\}) \prod_{v=1}^{n} \Big[3\lambda |\mathcal{V}| T_\Lambda^2 \\ + & \ \mathrm{Tr}_{\mathcal{V}} \Big(-\frac{1}{2} \log_2(1 + 2\mathrm{i}\sqrt{\lambda} \mathrm{C}^{1/2} \sigma_{\mathrm{v}} \mathrm{C}^{1/2}) + 2\mathrm{i}\sqrt{\lambda} \mathrm{T}_\Lambda \sigma_{\mathrm{v}} \Big) \Big]. \end{aligned} \tag{8}$$

où la mesure gaussienne devient :

$$d\nu(\{\sigma_v\}) = d\nu(\sigma_v) e^{-\mathrm{Tr}\sigma_{\mathrm{v}}^2/2} \prod_{v' \neq v}^{n} \delta(\sigma_v - \sigma_{v'}) d\sigma_{v'} \ . \tag{9}$$

Nous calculons à présent la puissance $\frac{1}{|\mathcal{V}|} \ln Z(\lambda, \Lambda, \mathcal{V})$ avec la fomule en arbres :

Théorème 3.2.1 (LVE à multi-échelle)

$$\log Z(\lambda, \Lambda, \mathcal{V}) = \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{\mathcal{T} \text{ with } n \text{ vertex}} G_{\mathcal{T}} \tag{10}$$

$$G_{\mathcal{T}} = \Big\{ \prod_{\ell \in \mathcal{T}} \int d^2 x_\ell d^2 y_\ell \big[\int_0^1 dw_\ell \big] \Big\} \int d\nu_{\mathcal{T}}(\{\sigma^v\}, \{w\})$$

$$\Big\{ \prod_{\ell \in \mathcal{T}} \big[\frac{1}{2} \delta(x_\ell - y_\ell) \frac{\delta}{\delta \sigma^{v(\ell)}(x_\ell)} \frac{\delta}{\delta \sigma^{v'(\ell)}(y_\ell)} \big] \Big\} \prod_{v=1}^{n} V_v.$$

3.2.1 Les représentations graphiques

Il y a deux représentations graphiques pour la LVE : d'une part la représentation directe, constituée des structures en arbres des vertex décorées des contre-termes et, d'autre part, la représentation duale. La figure (A.5) fournit un exemple de ces deux représentations.

Dans la représentation duale, on remplace chaque ligne de propagateur par son dual. On n'a donc qu'une grande boucle décorée des contre-termes. Les vertex dans la représenta -tion directe correspondent à des régions différentes dans la représentation duale. Les graphes de la représentation directe étant planaires, la représentation duale est bien définie. La représentation duale est plus intéressante en raison de l'existence d'un ordre cyclique pour tous les objets. Cf. l'appendice de ce chapitre pour plus de détails.

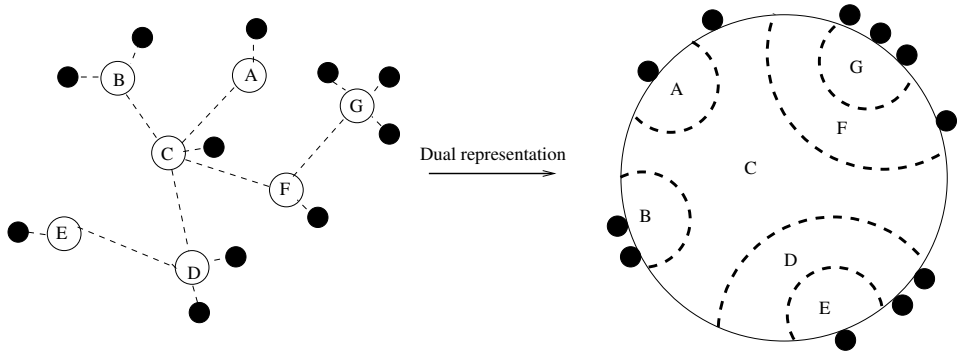

FIGURE 3.1: Le graphe direct de LVE et sa représentation duale. A gauche, dans le graphe direct, les disques noirs sont les contre-termes, les boucles sont les vertex à boucles et les lignes pointillées les propagateurs du champ σ. A droite, dans le graphe dual, les vertex à boucles sont remplacés par des lignes pointillées qui divise la grande boucle en différentes régions.

On peut définir l'amplitude pour chaque objet dans la représentation duale :

$$P_u = (-iT_\Lambda)D(x_{\bar{b}}) \text{ si } u = \bar{b} \in \bar{B}, \tag{11}$$
$$P_u = D(x_{\bar{\ell}(h)}) \text{ si } u = h \in H, \tag{12}$$
$$P_u = \hat{R}(\sigma_{\bar{v}_0(\bar{\ell})}) \text{ si } u = \bar{v}_0 \in \bar{S}, \tag{13}$$
$$P_u = R(\sigma_{\bar{v}_1(\bar{\ell})}) \text{ si } u = \bar{v}_1 \in \mathcal{L}. \tag{14}$$

où $\bar{B}$ est l'ensemble des contre-termes, $\mathcal{L}$ est la boucle duale, H est l'ensemble des demi-lignes, S est l'ensemble des résolvantes qui ne sont pas des feuilles et où chaque région correspond à un propagateur habillé.

Nous pouvons tout d'abord renormaliser les termes aux ordres 1 et 2 – cf. appendice – et l'amplitude pour un graphe dual peut s'écrire comme :

$$\log Z(\lambda, \Lambda, \mathcal{V}) = R_2 + \sum_{\bar{\mathcal{T}} \text{ cycles décorés avec } l \geq 3 \text{ lignes}} G_{\bar{\mathcal{T}}}$$
$$G_{\bar{\mathcal{T}}} = (-\lambda)^{n-1} \int_{\mathcal{V}} [\prod_{\bar{\ell}} d^2 x_{\bar{\ell}} \prod_{\bar{b} \in \bar{B}} d^2 x_{\bar{b}}] \ \mathrm{Tr}_{\mathcal{V}} \left\{ \tilde{\prod}_{\mathrm{u} \in \mathcal{O}} \mathrm{P_u} \right\} \tag{15}$$

L'amplitude est bornée par :

$$P_\Lambda \leq \sum_n \frac{n^{n-2}}{n!} 2^{n-1} \lambda^n \ln \Lambda^n \leq \sum (\lambda K)^n \log \Lambda^n. \tag{16}$$

qui est divergent quand $\Lambda \to \infty$.

La fonction de Schwinger connexe est donc divergente lorsque $\Lambda \to \infty$. Il est alors nécessaire de développer les résolvantes qui contiennent les champs σ, afin de compenser les divergences issues des contre-termes. On appellera ce développement, *développement de nettoyage*.

3.3 Le développement de nettoyage

L'idée du développement de nettoyage est que on developpe et contracte explicitement les champs σ qui sont cachées dans les resolventes pour compenser les tadpoles. Nous effectuons le développement de nettoyage dans la représentation dual, où la planarité est plus explicite, et dans la representation multi-échelle du propagateur. Nous coupons ensuite les propagateurs par les échelles :

$$\begin{aligned} C^{\Lambda} &= \sum_{j=0}^{j_{max}} C_j(x,y), \ o\grave{u} \\ C_j(x,y) &= \int_{M^{-2j}}^{M^{-2j+2}} e^{-\alpha m^2 - \frac{(x-y)^2}{4\alpha}} \ \frac{d\alpha}{\alpha} \leq K e^{-cM^j|x-y|}, \end{aligned} \tag{1}$$

où $\Lambda = M^{j_{max}}$, $M > 1$ est une constante et où K et c sont les noms génériques respectifs pour les constantes grandes et petites.

Nous décomposons également les contre-termes T_Λ comme :

$$T_\Lambda = \sum_{j \leq jmax} T_j, \ \ T_{\bar{j}} = \sum_{k \leq j} T_k, \tag{2}$$

où

$$T_k = C_k(x,x). \tag{3}$$

Pour la résolvante R, on a :

$$R_{jk}(\sigma x) \equiv \frac{1}{1 + 2i\sqrt{\lambda} C_j^{1/2}(.\ ,x)\sigma(x) C_k^{1/2}(x,\ .)}, \tag{4}$$

et

$$R_j = \sum_{k,l \ \ \sup(k,l)=j} R_{kl}. \tag{5}$$

Nous définissons aussi les termes :

$$D_{jk}(x) \equiv 2i\sqrt{\lambda} C_j^{1/2}(.\ ,x) C_k^{1/2}(x,\ .), \tag{6}$$

et

$$D_j = \sum_{k,l \ \ \sup(k,l)=j} D_{kl}. \tag{7}$$

Et

$$D'_{\bar{j}} = \sum_{k \leq j} D'_k, \ \ R'_{\bar{j}} = \sum_{k \leq j} R'_k, \tag{8}$$

et

$$D_k(x,\sigma) = D_k(x) \cdot \sigma(x). \tag{9}$$

La loi pour la multiplication de deux propagateurs $D'_{ij}(x)$ est :

$$\begin{aligned} \int dz dy' D'_{jk}(y,x,z) D'_{lm}(y',x',z') &= \int dz dy' (2i\sqrt{\lambda})^2 \delta_{kl} \delta(z-y') C_j^{1/2}(y,x) \\ & \quad C_k^{1/2}(x,z) C_l^{1/2}(y',x') C_m^{1/2}(x',z') \\ &= -4\lambda C_j^{1/2}(y,x) C_k(x,x') C_m^{1/2}(x',z'). \end{aligned} \tag{10}$$

Nous avons le formule suivant pour chaque résolvante :

$$\begin{aligned} R^j(\sigma) &= \frac{1}{1+\sum_{k\leq j} D^k(\sigma)} = \frac{1}{1+\sum_{k<j} D^k\sigma + D^j\sigma} \\ &= R^{j-1}(\sigma)\frac{1}{1+\sigma D^j R^{j-1}(\sigma)} \\ &= R^{j-1}(\sigma) - R^{j-1}(\sigma)\ \sigma\ D^j\ R^j(\sigma). \end{aligned} \tag{11}$$

C'est l'equation algebrique du groupe de renormalisation pour les resolventes. Voir que dans le deuxième terme il y a un champs σ explicite, et nous utilisons l'intégrale par parties pour contracter ce champ σ avec un autre qui est caché dans les resolventes. Nous commençons ce développement par la resolvente à échelle j_{max} et continuons vers $j_{max}-1, \cdots, j_{min}$.

Lorsqu'on effectue l'intégrale sur les champs σ, on génère soit un *tadpole* intérieur, avec lequel on peut compenser les contre-termes, soit une ligne de *crossing* ou de *nesting*, qui acquièrent un facteur convergent.

Plus explicitement, pour chaque *crossing* à l'échelle j, l'amplitude est bornée par :

$$\int d^2x C_j(x,y) = \int_{M^{-2j}}^{M^{-2j+2}} e^{-\alpha m^2 - \frac{(x-y)^2}{4\alpha}}\ \frac{d\alpha}{\alpha} \leq K e^{-cM^j|x-y|} \sim KM^{-2j}, \tag{12}$$

Et pour chaque ligne de *nesting* à l'échelle j, on a

$$\begin{aligned} |\int d^2x C_R^j(\sigma,x,y)| &= |\int d^2x R_j(\sigma) C_j(x,y)| \leqslant \int d^2x |R_j(\sigma)|\cdot|C_j(x,y)| \\ &\leqslant \int d^2x |C_j(x,y)| \sim KM^{-2j}, \end{aligned} \tag{13}$$

où on a utilisé $|R_j(\sigma)| < 1$. Ainsi, pour chaque résolvante à l'échelle j, on gagne un facteur KM^{-2j}, [7].

Cependant, ce développement ne pouvant être poursuivi indéfiniment pour les raisons mentionnées précédemment, nous devons imposer les *règles d'arrêt* suivantes : nous arrête -rons le développement dès lors que le nombre de *crossings* atteindra la valeur $N_j = a\cdot j$, où a est un nombre arbitraire à fixer.

3.3.1 Resommation des contre-termes restants

Après le développement de nettoyage, l'amplitude est écrite comme :

$$A^N_{\bar{\mathcal{T}}}|_{|\bar{\mathcal{T}}|=N+1} = \prod_{l,l'\in\bar{\mathcal{T}}} \int d\nu(w,\sigma) \int_0^1 dw_{l'}\lambda^N \prod \hat{R}_l(\sigma) \prod R_{l'}(\sigma) \prod_{m=1}^{N-n} (T_\Lambda)^m. \tag{14}$$

Il reste de nombreux contre-termes dans la partie non encore nettoyée. On se propose de les resommer par exponentiation puis de les borner. L'idée essentielle de la resommation est l'intégration par parties :

$$\int d\nu(w,\sigma) f(\sigma) g(\sigma) = e^{\frac{1}{2}\frac{\partial}{\partial\sigma(x)} C(x,x',w) \frac{\partial}{\partial\sigma(x')}} f(\sigma)g(\sigma))|_{\sigma=0}. \tag{15}$$

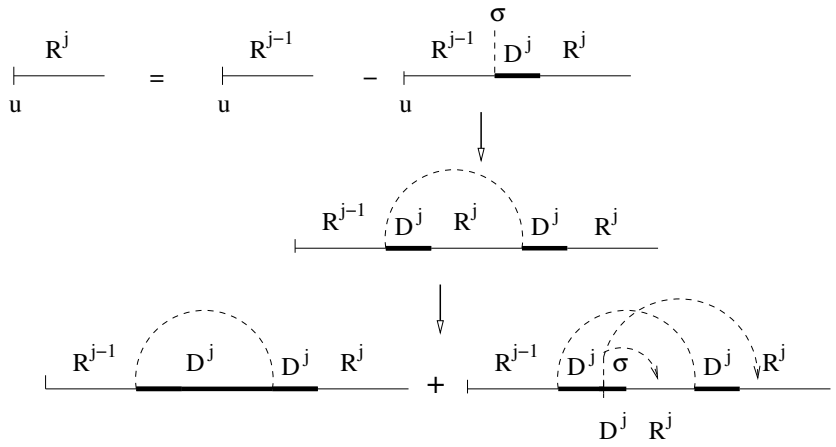

FIGURE 3.2: Le développement de nettoyage.

Considérons l'amplitude générale :

$$G = \int d\nu(\sigma, w) \prod_{l,l' \in \bar{\mathcal{T}}} \lambda^{|\bar{\mathcal{T}}|} [\hat{R}_l(\sigma)][R_{l'}(\sigma)] e^{\int 2i\sqrt{\lambda}\sigma T_\Lambda}. \tag{16}$$

On peut développer puis resommer les termes correspondant aux contre-termes pour obtenir

$$A_{\bar{\mathcal{T}}} = \int d\nu(\sigma, w) \prod_{l,l' \in \bar{\mathcal{T}}} \lambda^{|\bar{\mathcal{T}}|} \hat{R}_l(\sigma) R_{l'}(\sigma) e^{\int 2i\sqrt{\lambda}\sigma T_\Lambda} e^{2\lambda T_\Lambda^2}. \tag{17}$$

Après la resommation des contre-termes, on obtient un facteur divergent $e^{2\lambda T_\Lambda^2}$ qui doit être compensé par le facteur convergent provenant des *crossings* :

$$e^{-aj_{max}^2} \cdot j_{max}! \cdot e^{2\lambda T_\Lambda^2} \sim e^{-aj_{max}^2 + j_{max} \ln j_{max} + 2\lambda j_{max}^2} < 1 \tag{18}$$

comme $T_\Lambda \sim j_{max}$.

On peut choisir la constante a de telle sorte que : $a > 3\lambda$. La procédure de resommation est représentée en figure C.17.

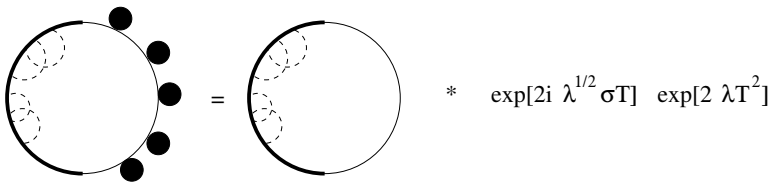

FIGURE 3.3: Représentation schématique de la resommation des contre-termes.

On obtient finalement la borne :

$$\begin{aligned} |G_{\bar{T}}|_{|T|=n-1} &< \int d\nu(\sigma, w) \prod_{l,l' \in \mathcal{T}} |\lambda|^{n-1} |\hat{R}_l(\sigma)| \; |R_{l'}(\sigma, w_{l'})| \; |e^{\int 2i\sigma T_\Lambda}| \\ &\times e^{2\lambda T_\Lambda^2} e^{-aj_{max}^2} \leqslant (K|\lambda|)^{n-1}, \end{aligned} \tag{19}$$

où K est une constante arbitraire.

Pour un volume fini $\mathcal{V}$, l'amplitude s'écrit comme

$$G_T = \int d\nu(\sigma, w) \prod_{l,l' \in \bar{\mathcal{T}}} \lambda^{|\bar{\mathcal{T}}|} \hat{R}_l(\sigma) R_{l'}(\sigma, w_{l'}) e^{\int 2i\sqrt{\lambda}\sigma T_\Lambda} e^{|\mathcal{V}|2\lambda T_\Lambda^2}, \tag{20}$$

qui est bornée par le terme divergent $e^{|\mathcal{V}|2\lambda T_\Lambda^2}$.

Pour compenser ce facteur, on utilise le développement en amas et, pour chaque cube unitaire, le développement de nettoyage permet d'obtenir le facteur $e^{-aT_\Lambda^2}$.

On a finalement la borne :

$$\begin{aligned} |G_n| &\leqslant \frac{(2K|\lambda|)^{n-1}}{(n-1)!} \sum_T \sum_\tau \prod_{i=1}^{k_v} \int d\nu(\sigma) \prod_{l \in T} |\hat{R}_l(\sigma)| |e^{\int d^2x 2i\sigma T_\Lambda}| \\ &\leq (2K|\lambda|)^{n-1} \frac{1}{(n-1)!} \sum_T (k-1)!, \end{aligned} \tag{21}$$

où $|\hat{R}(\sigma)| \leq 2$, K est un facteur arbitraire.

D'après le théorème de Cayley, le nombre d'arbres étiquetés à n nœuds présentant un nombre de coordination fixe égal à k vaut $\frac{n!}{(k-1)!}$.

On obtient donc : $|G_n| \leq (2K|\lambda|)^{n-1}$, et la fonction connexe

$$P(\lambda, \mathcal{V}) = \frac{1}{|\mathcal{V}|} \log Z(\lambda, \mathcal{V}) = \sum_n G_n, \tag{22}$$

est bornée tant que $|\lambda| < \frac{1}{2K}$.

3.4 La limite thermodynamique et le développement en amas

Pour obtenir la limite thermodynamique pour la fonction de Schwinger connexe, nous devons la développer en amas afin d'établir quelles régions de $\mathcal{R}^2$ sont occupées par les champs σ. Dans ce but, on découpe $\mathcal{R}^2$ en cubes unitaires. Un graphe en amas est représenté figure (A.6).

3.5 La sommation de Borel

Dans cette dernière section, nous étudions la sommabilité de Borel pour la série perturbative renormalisée [15, 14].

Considérons le développement de Taylor pour une fonction $f(\lambda)$ à l'ordre N avec reste intégral $R^N F$:

$$R^N f = f(\lambda) - \sum_{n=0}^{N} a_n \lambda^n = \lambda^{N+1} \int_0^1 \frac{(1-t)^N}{N!} f^{(N+1)}(t\lambda) dt. \tag{1}$$

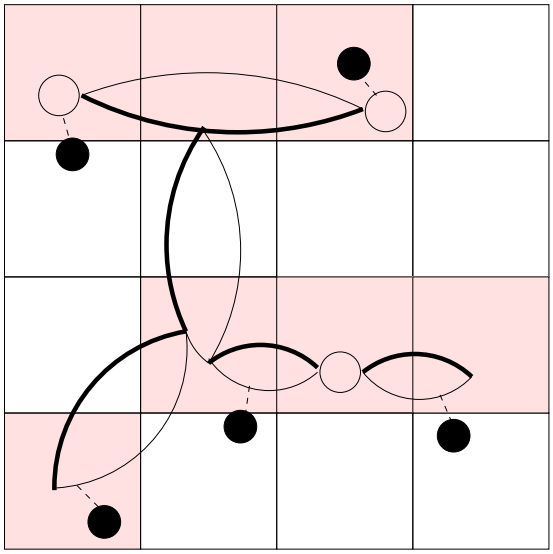

FIGURE 3.4: Graphe en amas de LVE s'étendant sur un ensemble Γ.

La sommabilité de Borel et, en particulier, le théorème de Nevanlinna ont déjà été introduits au chapitre précédent. Dans cette section, on montre que

Théorème 3.5.1 *La série perturbative pour la fonction de Schwinger connexe de ϕ_2^4 est Borel sommable.*

On sait que le domaine d'analyticité de la fonction de Schwinger connexe, c'est-à-dire le plan de Borel, est donné par $\mathrm{Re}\lambda > 0$. En d'autres termes,

$$-\frac{\pi}{4} \leq \mathrm{Arg}\sqrt{\lambda} \leq \frac{\pi}{4} \tag{2}$$

et $|\lambda| < \frac{1}{2K}$.

Comme les propagateurs C et les champs σ sont réels, on a, pour chaque résolvante,

$$|\mathcal{R}| = |\frac{1}{1 + 2i\sqrt{\lambda}C^{1/2}\sigma}C^{1/2}| \leq \sqrt{2}, \tag{3}$$

et, pour le terme de feuille,

$$|\hat{\mathcal{R}}| = |\frac{1}{1 + 2i\sqrt{\lambda}C^{1/2}\sigma C^{1/2}} - 1| \leq \sqrt{2} + 1. \tag{4}$$

Posons $\theta = \mathrm{Arg}\sqrt{\lambda}$ et soit $|\sqrt{\lambda}|$ la norme de $\sqrt{\lambda}$. Dans le plan de Borel, le terme $e^{2i\sqrt{\lambda}\sigma T_\Lambda}$ devient alors :

$$e^{2i\sqrt{\lambda}\sigma} = e^{2i|\sqrt{\lambda}|\sigma\cos\theta T_\Lambda}e^{-2|\sqrt{\lambda}|\sigma\sin\theta T_\Lambda}, \tag{5}$$

qui ne peut être borné par 1 en raison de la divergence du second terme pour les valeurs négatives de σ.

On réécrit donc ce terme sous la forme :

$$\int d\mu\sigma e^{-1/2\int d^2x\sigma^2} e^{-2\int d^2x|\sqrt{\lambda}|\sin\theta\sigma T_\Lambda} = \int d\mu\sigma e^{-1/2\int d^2x(\sigma+2|\sqrt{\lambda}|T_\Lambda)+2\mathcal{V}\sin^2\theta|\lambda|T_\Lambda^2}, \quad (6)$$

et on borne le terme en $\cos\theta$ par 1.

Le terme restant, $e^{2\mathcal{V}\sin^2\theta|\lambda|T_\Lambda^2}$, est dominé par $e^{\mathcal{V}|\lambda|T_\Lambda^2}$, où $\theta = \pm\pi/4$. Une telle divergence est sans danger dans la mesure où l'on peut choisir la valeur de a dans le facteur $e^{-a\mathcal{V}j_{max}^2} \sim e^{-a\mathcal{V}T_\Lambda^2}$ de manière à compenser le facteur divergent.

On a donc, une fois encore, une borne :

$$\begin{aligned} |G_N| &= \int d\nu(\sigma,w) \prod_{l,l'\in\mathcal{T}} |\lambda^N||\hat{\mathcal{R}}_l(\sigma)||\mathcal{R}_{l'}(\sigma,w_{l'})| \\ &\cdot \quad |e^{\int i\cos\theta\sigma T_\Lambda}|e^{|\mathcal{V}|(2+1)\lambda T_\Lambda^2} e^{-a\mathcal{V}T_\Lambda^2} \leq (K|\lambda|)^N. \end{aligned} \quad (7)$$

On développe à présent la fonction de Schwinger connexe jusqu'à l'ordre N en λ grâce à la formule de Taylor avec reste intégral (1) et on effectue explicitement les contractions de Wick entre les champs σ. Il vient alors

$$|\sum_{n=N+1}^{\infty} G_n| < |\lambda|^{N+1} K^N (2N)!! \leq |\lambda|^{N+1} K^N (N)!, \quad (8)$$

où le facteur K est une constante arbitraire multipliée soit par un facteur de $\sqrt{2}$ provenant de $\mathcal{R}$, soit par un facteur de $\sqrt{2}+1$, provenant de $\hat{\mathcal{R}}$.

Ceci achève la preuve du théorème.

4

LVE pour le modèle ϕ^{2k} à zéro dimension

4.1 Introduction

Nous allons construire dans ce chapitre le modèle $\lambda\phi^{2k}$ à zéro dimension avec la méthode de LVE. La procédure est similaire à celle du cas de ϕ_2^4 sauf que nous avons besoin d'introduire plusieurs champs intermédiaires.

Nous allons tout d'abord illustrer comment la méthode LVE fonctionne pour le modèle ϕ^6, et ensuite nous allons construire le modèle ϕ^{2k}.

4.2 La construction du modèle ϕ^6 à zéro dimension

Nous considérons dans cette section le modèle $\lambda\phi^6$ à zéro dimension, où le champ ϕ est un nombre réel et la constante de couplage λ est un nombre réel positif. La fonction de partition est définie comme :

$$Z(\lambda) = \int \frac{d\phi}{\sqrt{2\pi}} e^{-\frac{1}{2}\phi^2} e^{-\lambda\phi^6}, \tag{1}$$

où la covariance pour la mesure gaussienne est $< \phi^2 >= 1$.

Introduisons d'abord un champ intermédiaire réel σ, la fonction de partition devient :

$$Z(\lambda) = \int \frac{d\phi}{\sqrt{2\pi}} e^{-\frac{1}{2}\phi^2} e^{-\lambda\phi^6} = \int \frac{d\phi}{\sqrt{2\pi}} e^{-\frac{1}{2}\phi^2} \int \frac{d\sigma}{\sqrt{2\pi}} e^{-\frac{1}{2}\sigma^2} e^{i\sqrt{2\lambda}\phi^3\sigma}. \tag{2}$$

Ensuite, nous introduisons trois autres champs intermédiaires a, b et c et effectuons

l'intégrale sur les champs ϕ et σ ; la fonction de partition devient :

$$Z(\lambda) = \int \frac{da\sqrt{i}}{\sqrt{2\pi}} \frac{db}{\sqrt{2i\pi}} \frac{dc}{\sqrt{2i\pi}} e^{i(b^2+c^2-a^2)/2} e^{V(\lambda,a,b,c)} = \int d\mu(a,b,c) e^{(\lambda,a,b,c)} \tag{3}$$

où

$$V = -\frac{1}{2}\mathrm{Tr}\ln[\begin{pmatrix}1 & 0\\ 0 & 1\end{pmatrix} + \mathrm{i}(2\lambda)^{1/4}\begin{pmatrix}c-a & b-a\\ b-a & 0\end{pmatrix}] = -\frac{1}{2}\mathrm{Tr}\ln(\mathbf{I}+\mathrm{iH}), \tag{4}$$

Il paraît que l'opérateur H est hermitien.

Nous attribuons à chaque vertex un indice v, $v = 1, \cdots, n$, de sorte que :

$$e^V = \sum_n \frac{V^n}{n!} = \sum_n \frac{1}{n!} \prod_{v=1}^{n} V_v(a^v, b^v, c^v). \tag{5}$$

[][]La covariance pour les champs a, b, c.....

Pour obtenir la fonction de Schwinger connexe, nous allons utiliser la formule de forêt, comme dans la section précédente.

Comme il y a trois champs intermédiaires différents, graphiquement nous avons trois ensembles de lignes, correspondant aux propagateurs pour les champs a, b et c. Nous les appelons $\mathcal{T}_a$, $\mathcal{T}_b$ et $\mathcal{T}_c$.

La fonction de Schwinger connexe est :

Théorème 4.2.1

$$\begin{aligned}
\log Z(\lambda) &= \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{\mathcal{T} \text{ with } n \text{ vertices}} Y_{\mathcal{T}} && (6)\\
Y_{\mathcal{T}} &= \Big\{ \prod_{\ell\in\mathcal{T}} [\int_0^1 dw_\ell] \Big\} \int d\nu_{\mathcal{T}}(\{a^v, b^v, c^v\}, \{w\}) \\
&\times \Big\{ \prod_{\ell\in\mathcal{T}_a} [\delta_{v,v'} \frac{\delta}{\delta a^{v(\ell)}} \frac{\delta}{\delta a^{v'(\ell)}}] \Big\} \Big\{ \prod_{\ell\in\mathcal{T}_b} [\delta_{v,v'} \frac{\delta}{\delta b^{v(\ell)}} \frac{\delta}{\delta b^{v'(\ell)}}] \Big\} \\
&\times \Big\{ \prod_{\ell\in\mathcal{T}_c} [\delta_{v,v'} \frac{\delta}{\delta c^{v(\ell)}} \frac{\delta}{\delta c^{v'(\ell)}}] \Big\} \prod_{v=1}^{n} V_v && (7)
\end{aligned}$$

où les mesures gaussiennes $d\nu_T(\{a^v, b^v, c^v\}, \{w\})$ pour les champs a^v, b^v et c^v ont les covariances :

$$\begin{aligned}
< a^v, a^{v'} > &= -iw^T(v, v', \{w\}), && (8)\\
< b^v, b^{v'} > &= iw^T(v, v', \{w\}), && (9)\\
< c^v, c^{v'} > &= iw^T(v, v', \{w\}), && (10)\\
< a^v, b^v > &= < b^v, c^v > = < a^v, c^v > = 0, && (11)
\end{aligned}$$

Ici $w^T(v, v', \{w\})$ est 1 si $v = v'$, et a la valeur du minimum des w_ℓ, où ℓ parcourt sur la trajectoire unique de v à v' dans T, quand $v \neq v'$.

Cette formule n'est pas convergente absolue, comme les intégrales (7) sur a, b et c sont oscillantes. Donc nous avons besoin de déformer les contours pour les chemins d'intégrales.

Pour chaque champ a nous avons la décomposition comme :

$$a = \sum_{k=1}^{p+1} \sum_{v=1}^{n} a_{k,v}, \tag{12}$$

où la covariance pour chaque champ indépendant est :

$$< a_{k,v}, a_{k,v'} >= (w_{k-1} - w_k) X^{\mathcal{F},k}_{v,v'}. \tag{13}$$

Comme la covariance est dégénérée dans chaque bloc, pour chaque bloc on a besoin d'effectuer une seule déformation du contour de l'intégrale.

On obtient donc :

$$d\mu_{iI_q} = \frac{da_1}{\sqrt{2\pi}} e^{-ia_1^2/2} \prod_{i=2}^{q} \delta(a_1 - a_i) da_i. \tag{14}$$

Comme chaque intégrale dans la fonction de partition a la forme

$$\int_{-\infty}^{\infty} da f(a) e^{ia^2/2}, \tag{15}$$

où f indique le produit des résolvantes, qui sont analytiques et bornées dans la région $\mathcal{B} = \{\Im a \leq A^{-1}\}$, où A est un nombre réel et grand, on peut déformer le contour de l'intégrale comme :

$$a_1' \;=\; a_1 - i\frac{a_1}{A|a_1| + 1},\; a_1' \to a_1 - i\,\mathrm{sgn}\, a_1/A \text{ if } a_1 \to \pm\infty. \tag{16}$$

La borne pour l'intégrale sur a_1 devient

$$\begin{aligned}
&\left| \int da_1 f(a_1') e^{-ia_1^2/2(w_{k-1}-w_k) - \frac{2a_1^2}{2(w_{k-1}-w_k)(A|a_1|+1)} + i\frac{a_1^2}{2(w_{k-1}-w_k)(A|a_1|+1)^2}} \right| \\
&\leq \sup_{\mathcal{B}} |f| \int da_1 e^{-\frac{2a_1^2}{2(w_{k-1}-w_k)(A|a_1|+1)}} \\
&\leq 2(w_{k-1} - w_k) A \sup_{\mathcal{B}} |f|.
\end{aligned} \tag{17}$$

Donc chaque fois quand on effectue l'intégrale sur un champ intermédiaire on obtient $\sup_{\mathcal{B}} |f|$ fois un facteur $2(w_{k-1} - w_k)A$.

On obtient finalement la borne pour tous les champs a_k :

$$\begin{aligned}
&\prod_{k=1}^{p+1} \prod_{v=1}^{n} 2A(w_{k-1} - w_k) \leq \prod_{k=1}^{p+1} \prod_{v=1}^{n} e^{2A(w_{k-1}-w_k)} = \prod_{v=1}^{n} e^{\sum_{k=1}^{p+1} 2A(w_{k-1}-w_k)} \\
\leq\; &\prod_{v=1}^{n} e^{2A} \leq (e^{2A})^n,
\end{aligned} \tag{18}$$

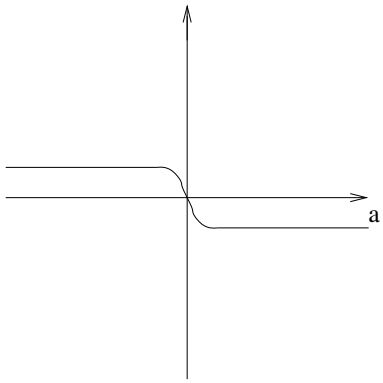

FIGURE 4.1: Le contour de chemin d'intégrale pour a.

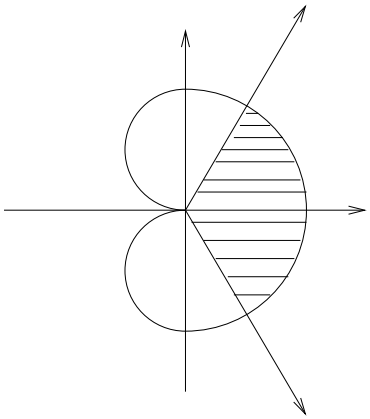

FIGURE 4.2: Le domaine de l'analyticité C_R^2

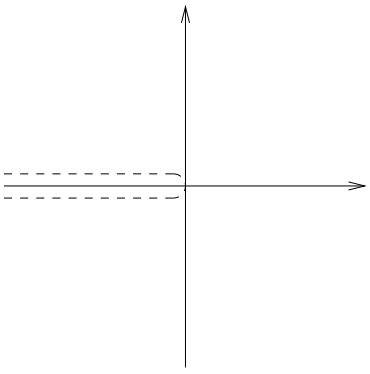

FIGURE 4.3: Le domaine de l'analyticité $\mathcal{D}^2$

où on a utilisé la formule

$$\sum_k (w_{k-1} - w_k) \leq 1. \tag{19}$$

Notons que le terme $1 + iH$ se change aussi quand on déforme le contour de l'intégrale,

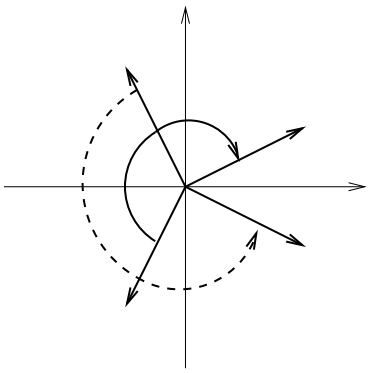

FIGURE 4.4: La continuation analytique

et le changement est comme :

$$1 + iH - (2\lambda)^{1/4} \begin{pmatrix} \epsilon(a) & \epsilon(a) \\ \epsilon(a) & 0 \end{pmatrix} \tag{20}$$

où $\epsilon = 1/A$ est un nombre petit.

Comme $\lambda << 1$, $\|\lambda^{1/4} \begin{pmatrix} \epsilon(a) & \epsilon(a) \\ \epsilon(a) & 0 \end{pmatrix} \| << 1$.

Donc après la déformation du contour de l'intégrale, la borne pour la résolvente devient $\sqrt{2} + O(1/A)$ si on choisit $-\pi < Arg\lambda < +\pi$, $-\pi/4 < Arg\lambda^{1/4} < +\pi/4$. Nous allons ignorer les facteurs $O(1/A)$ dans le reste de ce chapitre, comme ils ne changent pas la borne de la résolvante.

Le reste de cette section est dédié à la preuve du théorème suivant :

Théorème 4.2.2 *La fonction de Schwinger du vide connexe* $\log Z(\lambda)$ *avec le terme d'intéraction* $\lambda\phi^6$ *est sommable de Borel-Le Roy à l'ordre* 2.

Preuve : premièrement, nous allons prouver que la fonction de Schwinger connexe à l'ordre n est convergent uniformément dans le domaine $\mathcal{D}_\epsilon^2 = \{\lambda | -\pi < Arg\lambda < \pi \text{ et } |\lambda| < \epsilon\}$, ensuite, nous allons montrer que le reste intégral de Taylor est aussi borné , et nous avons des lemmes comme le suivant :

Lemme 4.2.1 *Chaque terme* $Y_\mathcal{T}(\lambda)$*, défini dans le domaine* $\mathcal{D}_\epsilon^2$*, est borné par* $\epsilon^{(n-1)/2}K^n$.

Preuve :

Considérons la résolvente et diagonalisons le terme dans le dénominateur, on obtient :

$$\frac{1}{1+iH} = \frac{1}{\begin{pmatrix} 1 + i(2\lambda)^{1/4}\omega_+ & 0 \\ 0 & 1 + i(2\lambda)^{1/4}\omega_- \end{pmatrix}}, \tag{21}$$

où

$$\begin{aligned}\omega_+ &= \big(c-a+\sqrt{(c-a)^2+4(b-a)^2}\,\big)/2>0\\ \omega_- &= \big(c-a-\sqrt{(c-a)^2+4(b-a)^2}\,\big)/2<0.\end{aligned} \tag{22}$$

Comme le domaine d'analyticité contient au moins $\mathcal{D}^2$, nous avons :

$$-\pi/4 < \mathrm{Arg}(\lambda^{1/4}) < \pi/4, \tag{23}$$

c'est-à-dire

$$|(1+i\lambda^{1/4}\omega_+)^{-1}| < \sqrt{2}\ ,\ \ |(1+i\lambda^{1/4}\omega_-)^{-1}| < \sqrt{2}. \tag{24}$$

Donc chaque résolvante est bornée par :

$$\|\frac{1}{1+iH}\| \le \sqrt{2}(1+O(1/A)), \tag{25}$$

et pour un arbre avec n sommets, le produit de toutes les résolventes est borné par $\sqrt{2}^{2(n-1)}$. Finalement, comme le facteur global de la trace fait 2, on obtient :

$$\left|\mathrm{Tr}\prod_{\text{around }\mathcal{T}}\frac{1}{1+\mathrm{iH}}\right| \le 2\cdot\sqrt{2}^{2(n-1)} = 2^n. \tag{26}$$

Pour chaque arbre $\mathcal{T}$ de l'ordre n on a un facteur $\lambda^{(n-1)/2}$ dans $Y_{\mathcal{T}}$. Comme pour chaque produit des résolventes on a 3 possibilités de choisir un champ intermédiaire parmi a, b ou c, on a aussi un facteur 3^{n-1}.

Enfin, considérons le facteur $(e^{2A})^n$ qui vient de la déformation du contour de chemin, nous avons

$$|Y_{\mathcal{T}}(\lambda)| \le 2^n 3^{n-1}|\lambda|^{(n-1)/2}(e^{2A})^n \le \epsilon^{(n-1)/2}K^n \tag{27}$$

où $K = 6e^{2A}$.

D'après le théorème de Cayley, le nombre d'arbres étiquetés à l'ordre n est n^{n-2}. Donc avec le lemme précédent, on obtient : $\mathcal{D}^2_\epsilon$:

$$\begin{aligned}\sum_{n=1}^{\infty}\frac{1}{n!}\sum_{\mathcal{T}\text{ avec }n\text{ vertex}}|Y_{\mathcal{T}}(\lambda)| &\le \sum_{n=1}^{\infty}\frac{n^{n-2}}{n!}\epsilon^{(n-1)/2}K^n\\ &\le \sum_{n=1}^{\infty}\epsilon^{(n-1)/2}(eK)^n,\end{aligned} \tag{28}$$

Et le reste intégral de Taylor est borné par :

$$R^N\log Z(\lambda) = \sum_{n=1}^{\infty}\frac{1}{n!}\sum_{\mathcal{T}\text{ avec }n\text{ vertex}}R^N Y_{\mathcal{T}}(\lambda). \tag{29}$$

Nous avons, pour les arbres $\mathcal{T}$ avec les sommets $n \ge 2N+3$, $R^N[Y_{\mathcal{T}}(\lambda)] = Y_{\mathcal{T}}$.

Comme nous avons :

$$\begin{aligned}|\sum_{n=2N+3}^{\infty}\frac{1}{n!}\sum_{\mathcal{T}\text{ avec }n\text{ vertex}}R^N Y_{\mathcal{T}}(\lambda)| &\le |\lambda|^{N+1}\sum_{n=2N+3}^{\infty}\epsilon^{(n-1)/2-(N+1)}(eK)^n\\ &\le |\lambda|^{N+1}K^N\end{aligned} \tag{30}$$

pour $\lambda \in \mathcal{D}_\epsilon^2$. nous n'avons pas besoin de considérer le cas $n \leq 2N+2$.

Avec la définition $\bar{Y}$ par $Y_\mathcal{T}(\lambda) = \lambda^{(n-1)/2}\bar{Y}_\mathcal{T}(\lambda)$, nous avons

$$R^N Y_\mathcal{T} = \lambda^{(n-1)/2} R^{N-(n-1)/2} \bar{Y}_\mathcal{T}, \tag{31}$$

On considère les contractions de Wick explicitement, on a le lemme suivant :

Lemme 4.2.2 *Pour les arbres avec les vertex $n \leq 2N+2$ définis dans le domaine $\mathcal{D}_\epsilon^2$ $\mathcal{T}$, nous avons*

$$|\lambda^{(n-1)/2} R^{N-(n-1)/2} \bar{Y}_\mathcal{T}| \leq |\lambda|^{N+1} K^N \Gamma(2N-n+1). \tag{32}$$

Avec le lemme précédent on a montré que le reste de Taylor à l'ordre N est borné par $|\lambda|^{N+1}K^N\Gamma(2N+1)$. Donc on a montré le théorème B.3.4.

4.3 Le modèle ϕ^{2k}

Nous allons construire dans cette section le modèle scalaire $\lambda\phi^{2k}$, pour lequel nous avons besoin d'introduire plusieurs champs intermédiaires.

$$e^{-\lambda\phi^{2k}} = \int d\sigma_1 e^{-\sigma_1^2/2 + i\sqrt{\lambda}\phi^k\sigma_1} \tag{1}$$

et

$$2\sqrt{\lambda}\phi^k\sigma_1 = [(\lambda^{\frac{1}{2k}}\phi\sigma_1 + \lambda^{\frac{k-1}{2k}}\phi^{k-1})^2 - \lambda^{\frac{1}{k}}\phi^2\sigma_1^2 - \lambda^{\frac{k-1}{k}}\phi^{2k-2}]. \tag{2}$$

pour le premier terme on va introduire un autre champ σ_2 :

$$e^{i(\lambda^{\frac{1}{2k}}\phi\sigma_1 + \lambda^{\frac{k-1}{2k}}\phi^{k-1})^2} = \int d\sigma_2 e^{i(\lambda^{\frac{1}{2k}}\phi\sigma_1 + \lambda^{\frac{k-1}{2k}}\phi^{k-1})\sigma_2} e^{-i\sigma_2^2/2}. \tag{3}$$

Et nous avons, pour le second terme :

$$e^{-i\lambda^{\frac{1}{k}}\phi^2\sigma_1^2} = \int d\sigma_3 e^{-i\lambda^{\frac{1}{2k}}\sigma_3\phi\sigma_1} e^{i\sigma_3^2/2}. \tag{4}$$

Le troisième terme est un polynôme ϕ^{2k-2} qui a le même type que l'interaction initiale mais avec l'ordre abaissé par 2, et la constante de couplage abaissé par $\lambda^{\frac{1}{k}}$.

On doit répéter cette procédure afin de réduire l'ordre du polynôme de l'interaction jusqu'à $\phi^2\sigma_i$. La fonction de partition devient alors :

$$Z(\lambda) = \int \prod_r da_r \prod_s db_s \prod dc \; e^{i(a_1^2 - a_2^2 - a_3^2 + b_1^2 - b_2^2 - b_3^2 \pm c^2 \cdots)/2} e^V \tag{5}$$

où

$$V = -\frac{1}{2}\mathrm{Tr}\ln[\mathrm{A} + \mathrm{iH}(\{\mathrm{a}\},\{\mathrm{b}\},\{\mathrm{c}\}...)]. = -\frac{1}{2}\mathrm{Tr}\ln[\mathrm{G}]., \tag{6}$$

et $A = diag(1, 1, i, -i...)$. Si k est un nombre pair, le nombre de 1 dans A est 1 et d'autre elements de A sont $\pm i$; si k est impair, il y a deux 1 dans A et les autre élement sont $\pm i$.

La matrice H est symmétrique et a la forme :

$$H = \lambda^{\frac{1}{2k}} \begin{pmatrix} \lambda^{\frac{1}{2k}} g_1(a_i, b_i, c_i...) & g_2(a_i, b_i, c_i...) & g_3(a_i, b_i, c_i...) & ... \\ g_2(a_i, b_i, c_i...) & 0 & 0 & ... \\ g_3(a_i, b_i, c_i...) & 0 & 0 & ... \\ ... & ... & ... & ... \end{pmatrix}, \tag{7}$$

où $g_j(a_i)$ est la somme des termes linéairs en champs intermédiaires qui appraîtent dans le déterminant.

Nous nous avons le lemme suivant pour la matrice $G = A + iH$:

Lemme 4.3.1 *La matrice d'inverse de G existe, et elle est bornée par $\sqrt{2}$.*

4.3.1 Le domaine de l'analycité et la déformation du contour

Dans la théorie $\lambda\phi^{2k}$, le domaine de l'analyticité est :

$$-\frac{(k-1)\pi}{2} \leqslant \text{Arg } \lambda \leqslant -\frac{(k-1)\pi}{2}. \tag{8}$$

Comme pour chaque champs ϕ on a une constant de couplage $\lambda^{\frac{1}{2k}}$ associé , on a la relation pour chauqe élement a_i :

$$-\frac{(k-1)\pi}{4k} \leqslant \text{Arg } a_i \leqslant \frac{(k-1)\pi}{4k}. \tag{9}$$

De la même façns nous pouvons montrer que l'inverse de la matrice G est borné si la constante de couplage λ est dans son domaine analytique .

Nous avons :

$$|1 + i\omega_{\pm}| \geq c \sin \frac{\pi}{2k} \tag{10}$$

où k est un nombre entier arbitarire et positif.

De nouveau nous avons besoin de deformer le countour de l'intégralee pour chaque champs intermédiaire. La procédure pour la déformation est sililaire que le cas ϕ^6 et nous n'allons pas la répéter ici.

Similairement nous pouvons obtenir la borne pour la fonction de Schwinger connex pour chaque ordre N, et la preuve de la sommabilité de Borel pour la théorie ϕ^{2k} est similaire que pour la théorie de ϕ^6.

Nous avons finalement le théorème suivant :

Théorème 4.3.1 *La fonction de Schwinger connexe* $\log Z(\lambda)$ *pour du modèle $\lambda\phi^{2k}$ est sommable de Borel-Le Roy de l'ordre $k - 1$.*

5

Théorie constructive $GW2$ avec LVE

5.1 Introduction

La théorie quantique des champs définie dans l'espace non-commutatif (NCTQC)[24] a été considérée comme un moyen d'étudier la gravité quantique. Le modèle NCTQC le plus simple est une théorie des champs scalaire ϕ^4 définie dans l'espace Moyal à 4 dimensions. Mais il y a un problème avec ce modèle, c'est le mélange UV/IR. C'est-à-dire, si on effectue l'intégrale sur les champs de hautes échelles pour les graphes nonplanaires, l'amplitude est divergente quand les moments des champs extérieurs tendent vers zéro [25].

La percée vient du travail de Grosse et Wulkenhaar [26][28][27]. H.Grosse et R.Wulkenhaar ont montré, dans une série de papiers, que le modèle de ϕ_4^4 avec un potentiel harmonique dans l'espace Moyal à 4 dimensions est perturbativement renormalisable à tous les ordres. Avec le potentiel harmonique l'action a une symétrie entre les coordonées et les moments découverts par E. Langman et R. Szabo, appelée la dualité de Langman-Szabo [141].

La preuve de Grosse et Wulkenhaar est initialement dans l'espace de matrice [28, 29], ensuite on a montré la renormalisabilité de ce modèle dans l'espace direct [41] et dans la représentation paramétrique [68].

De la même façon, on a construit les autres modèles qui sont aussi renormalisables, par exemple, le modèle de Gross-Neveu à 2 dimensions [31], le modèle de ϕ_6^3 [32], le modèle ϕ_3^6 dans le plan de Moyal avec une dimension commutative [35] et le modèle de ϕ^4 dans l'espace de Moyal dégénéré [36].

Il existe un nouveau point fixé dans l'espace de paramètre dans le modèle de Grosse-Wulkenhaar (GW_4), pour qui la fonction β pour la constante de couplage est nulle [37][38][39]. Donc on peut construire ce modèle non-perturbativement. Comme une première étape on va construire le modèle de Grosse-Wulkenhaar dans l'espace de Moyal à 2 dimensions (GW_2) avec la méthode LVE [21].

Nous allons introduire dans la section 2 les propriétés fondamentales sur l'espace de Moyal M^d. Ensuite, nous allons définir dans la section 3 le modèle de GW_2 et la représentation des champs intermédiaires. Et dans la section 4, nous allons étudier les représentations graphiques de LVE. Dans la section 5 on va adapter l'argument de Nelson

et finalement on va montrer la sommabilité de Borel pour la série perturbative.

5.2 L'espace Moyal

5.2.1 L'espace de Moyal

L'espace de Moyal $\mathbb{R}^D_\theta$ à D dimensions, où D un nombre pair est engendré par les opérateurs x^μ qui obéissent aux relations : $[x^\mu, x^\nu] = i\Theta^{\mu\nu}$, où

$$\Theta = \begin{pmatrix} 0 & \theta_1 & & & \\ -\theta_1 & 0 & & (0) & \\ & & \ddots & & \\ & & & 0 & \theta_{D/2} \\ & (0) & & -\theta_{D/2} & 0 \end{pmatrix}, . \tag{5.2.1}$$

$\Theta^{\mu\nu} = -\Theta^{\nu\mu}$, est une matrice asymétrique. C'est le modèle le plus simple et le plus étudié en géométrie non-commutative (voir [42, 24] pour plus de détails).

Pour les fonctions dans l'espace Schwartz $\mathcal{S}(\mathbb{R}^D)$, définies sur $\mathbb{R}^D_\theta$, le produit de * est définie comme :

$$(f \star_\Theta g)(x) = \int_{\mathbb{R}^D} \frac{d^D k}{(2\pi)^D} d^D y\, f(x + \tfrac{1}{2}\Theta \cdot k) g(x+y) e^{\imath k \cdot y} \tag{5.2.2}$$

$$= \frac{1}{\pi^D |\det \Theta|} \int_{\mathbb{R}^D} d^D y d^D z\, f(x+y) g(x+z) e^{-2\imath y \Theta^{-1} z} . \tag{5.2.3}$$

Nous allons mettre $D = 2$ pour ce chapitre. Avec les coordornées x_μ on peut définir les opérateurs de création et d'anihilation comme : [26, 28, 42] :

$$a = \frac{1}{\sqrt{2}}(x_1 + \mathrm{i}x_2) , \qquad \bar{a} = \frac{1}{\sqrt{2}}(x_1 - \mathrm{i}x_2) ,$$
$$\frac{\partial}{\partial a} = \frac{1}{\sqrt{2}}(\partial_1 - \mathrm{i}\partial_2) , \qquad \frac{\partial}{\partial \bar{a}} = \frac{1}{\sqrt{2}}(\partial_1 + \mathrm{i}\partial_2) . \tag{5.2.4}$$

et pour toutes les fonctions $f \in \mathcal{S}_D$ on a la relation :

$$(a \star f)(x) = a(x)f(x) + \frac{\theta_1}{2}\frac{\partial f}{\partial \bar{a}}(x) , \qquad (f \star a)(x) = a(x)f(x) - \frac{\theta_1}{2}\frac{\partial f}{\partial \bar{a}}(x) ,$$
$$(\bar{a} \star f)(x) = \bar{a}(x)f(x) - \frac{\theta_1}{2}\frac{\partial f}{\partial a}(x) , \qquad (f \star \bar{a})(x) = \bar{a}(x)f(x) + \frac{\theta_1}{2}\frac{\partial f}{\partial a}(x) . \tag{5.2.5}$$

On peut définir les bases $f_{mn}(x)$ en $\mathbb{R}^2$ pour les fonctions ϕ comme :

$$f_{mn}(x) := \frac{1}{\sqrt{n!m!\,\theta_1^{m+n}}}\, \bar{a}^{\star m} \star f_0 \star a^{\star n} \tag{5.2.6}$$

$$= \frac{1}{\sqrt{n!m!\,\theta_1^{m+n}}} \sum_{k=0}^{\min(m,n)} (-1)^k \binom{m}{k}\binom{n}{k} k!\, 2^{m+n-2k}\, \theta^k\, \bar{a}^{m-k}\, a^{n-k} f_0 ,$$

où $f_0(x) = 2\mathrm{e}^{-\frac{1}{\theta}(x_1^2+x_2^2)}$. Il y a des relations suivantes pour les fonctions de base :

$$(f_{mn} \star f_{kl})(x) = \delta_{nk} f_{ml}(x) \ . \tag{5.2.7}$$

et

$$\int d^2x f_{mn}(x) = 2\pi\theta\delta_{mn}. \tag{5.2.8}$$

Nous pouvons développer toutes les fonctions $\phi(x) \in \mathcal{S}(\mathbb{R}^2)$ comme :

$$\phi(x) = \sum_{mn} \phi_{mn} f_{mn}(x). \tag{5.2.9}$$

5.2.2 Le modèle de Grosse-Wulkenhaar à 2 dimensions

Le modèle de Grosse-Wulkenhaar à 2 dimensions, GW_2 est défini comme

$$\begin{aligned} S &= \int d^2x \big[\frac{1}{2}\partial_\mu\phi \star \partial^\mu\phi + \frac{\Omega^2}{2}(\tilde{x}_\mu\phi) \star (\tilde{x}^\mu\phi) + \frac{\mu^2}{2}\phi\star\phi \\ &+ \ : \frac{\lambda}{4}\phi\star\phi\star\phi\star\phi : \big]. \end{aligned} \tag{5.2.10}$$

où $\phi(x)$ est un champ réel, $\tilde{x}_\mu = 2(\Theta^{-1})_{\mu\nu}x^\nu$ et nous avons utilisé la métrique euclidienne.

L'action (9) possède une symétrie entre les coordonnées et les moments comme :

$$S[\phi;\mu,\lambda,\Omega] \mapsto \Omega^2 S[\phi;\frac{\mu}{\Omega},\frac{\lambda}{\Omega},\frac{1}{\Omega}]. \tag{5.2.11}$$

quand on effectue la transformée de Fourier :

$$p_\mu \leftrightarrow \tilde{x}_\mu, \quad \hat{\phi}(p) \leftrightarrow \pi\sqrt{|\det\theta|}\phi(x), \tag{5.2.12}$$

oú

$$\hat{\phi}(p) := \int d^2x e^{-ip\Theta x}\phi(x), \quad p\Theta x := p_\mu\Theta^{\mu\nu}x_\nu. \tag{5.2.13}$$

Cette dualité, appelée aussi la dualité de Langman-Szabo [141], est essentielle pour la renormalisation de ce modèle.

Dans ce chapitre nous allons étudier ce modèle dans la base de la matrice, et l'action de GW_2 devient :

$$\begin{aligned} S[\phi] &= 2\pi\theta \mathrm{Tr}\Big[\frac{1}{2}\phi\Delta\phi + : \frac{\lambda}{4}\phi_\star^4 : \Big] \\ &= 2\pi\theta \sum_{m,n,k,l} \Big[\frac{1}{2}\phi_{mn}\Delta_{mn;kl}\phi_{kl} + \frac{\lambda}{4} : \phi_{mn}\phi_{nk}\phi_{kl}\phi_{lm} : \Big]. \end{aligned} \tag{5.2.14}$$

où ϕ_{mn} est une matrice réele et symmetrique.

Pour $\Omega = 1$, qui est le point fixé de cette théorie, l'opérateur laplacien devient alors :

$$\Delta_{mn} = \Big[\mu^2 + \frac{2(1+\Omega^2)}{\theta}(m+n+1)\Big] = \Big[\mu^2 + \frac{4}{\theta}(m+n+1)\Big], \tag{5.2.15}$$

et la covariance s'écrit comme :

$$C_{mn} = \frac{1}{\mu^2 + \frac{4}{\theta}(m+n+1)}. \tag{5.2.16}$$

Comme $\Omega = 1$ est un point fixé pour ce modèle, nous allons mettre $\Omega = 1$ dans le reste de ce chapitre.

L'ordre de Wick pour le terme d'interaction devient alors :

$$:\phi_{mn}\phi_{nk}\phi_{kl}\phi_{lm}: := \phi_{mn}\phi_{nk}\phi_{kl}\phi_{lm} - 8\phi_{mp}\phi_{pm}T^{\Lambda}_{m} + 6\mathrm{Tr_m}(\mathrm{T}^{\Lambda})^2_{\mathrm{m}}. \tag{5.2.17}$$

où Λ est la coupure ultraviolette,

$$T^{\Lambda}_{m} = \sum_{q=0}^{\Lambda} \frac{1}{q+m} = \log\frac{\Lambda+m}{m} \sim \log\Lambda, \text{for } 1 \leqslant m \ll \Lambda, \tag{5.2.18}$$

et

$$T^2_{\Lambda} = \mathrm{Tr_m}(\mathrm{T}^{\Lambda}_{\mathrm{m}})^2 = \sum_{\mathrm{m}}(\sum_{\mathrm{p}} \frac{1}{\mathrm{m}+\mathrm{p}})(\sum_{\mathrm{q}} \frac{1}{\mathrm{m}+\mathrm{q}}) \sim (2\ln^2 2 + \frac{\pi^2}{6})\Lambda. \tag{5.2.19}$$

Notons que les graphes non-planaires à l'ordre 1 sont ignorés dans (5.2.17), parce qu'ils sont finis.

Les facteurs combinatoires pour les contre-termes sont présentés sur les figures C.2 et C.3.

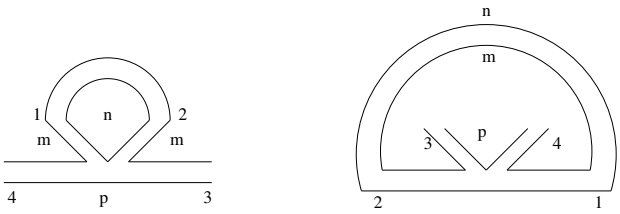

FIGURE 5.1: Les contre-termes $\mathrm{Tr}(\phi^2\mathrm{T})$. Il y a 4 possibilités pour contracter deux champs voisins, par exemple : soit 1 avec 2, soit 2 avec 3 , soit 3 avec 4 ou soit 4 avec 1. Et pour chaque contraction il y a deux configurations différentes comme montré sur cette figure. Donc au total nous avons un facteur 8.

5.3 Le développement de vertex à boucle

5.3.1 La représentation des champs intermédiaires

Dans la théorie quantique des champs la quantité la plus intéressante est la fonction de Schwinger connexe ou la fonction de Green euclidienne, qui est définie comme :

$$S(l_1r_1, \cdots, l_{2p}r_{2p}) = \frac{1}{(2p)} \frac{\partial^{2p}}{\partial j_{l_1r_1} \cdots \partial j_{l_{2p}r_{2p}}} \log Z(\Lambda, j)|_{j=0}, \tag{5.3.1}$$

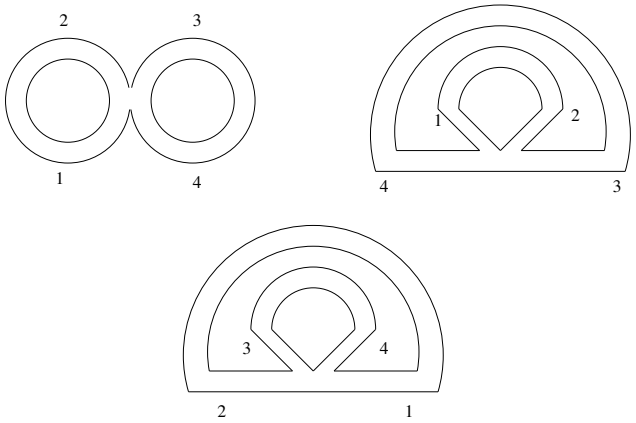

FIGURE 5.2: Les contre-termes T^2_Λ. Il y a 2 possibilités pour contracter les champs afin de former les graphes du vide. Par exemple, 1 se contracte avec 2 et 3 avec 4, ou 1 avec 4 et 2 avec 3. Pour chaque cas il y a 3 configurations différentes. Donc au total cela fait 6.

où

$$Z(\Lambda, j) = \int d\mu(\phi) e^{-\frac{\lambda}{4}\mathrm{Tr}[(\phi^2 - 4\mathrm{T}_\Lambda)^2 + 10\mathrm{T}^2_\Lambda] + \mathrm{Trj}\phi} \tag{5.3.2}$$

est la fonction de partition avec les champs de source j_{mn} et

$$d\mu(\phi_{mn}) = \pi^{-N(N-1)/2} e^{-1/2[\mathrm{Tr}\phi_{\mathrm{mn}}\Delta_{\mathrm{mn}}\phi_{\mathrm{nm}}]} \prod_m d\phi_{mm} \prod_{n<m} d\phi_{nm}, \tag{5.3.3}$$

est la mesure gaussienne normalisée avec la covariance C.

Nous introduisons les champs matriciels symétriques σ_{mn} et effectuons l'intégrale sur les champs ϕ_{mn} ; nous obtenons :

$$\begin{aligned} Z(\Lambda, j) &= \int d\mu(\phi) e^{-\frac{\lambda}{4}\mathrm{Tr}[(\phi^2 - 4\mathrm{T}_\Lambda)^2 + 10\mathrm{T}^2_\Lambda] + \mathrm{Trj}\phi} \qquad (5.3.4) \\ &= \int d\nu(\sigma) e^{\frac{5}{2}\lambda T^2_\Lambda + \mathrm{Tr}\left(2\mathrm{i}\sqrt{2\lambda}\mathrm{T}_\Lambda\sigma - \frac{1}{2}\log(1 + \mathrm{i}\sqrt{2\lambda}\mathrm{C}^{1/2}\sigma\mathrm{C}^{1/2})\right) + \frac{1}{2}\mathrm{Tr}[\mathrm{jC}^{1/2}\mathrm{RC}^{1/2}\mathrm{j}]} \end{aligned}$$

où

$$R = \frac{1}{1 + [i\sqrt{2\lambda}C^{1/2}\sigma C^{1/2}]}, \; R_{mn}(\sigma) = [C^{1/2}\frac{1}{1 + [i\sqrt{2\lambda}C^{1/2}\sigma C^{1/2}]}C^{1/2}]_{mn}. \tag{5.3.5}$$

Nous allons développer le terme d'exponentiel comme $\sum_n \frac{V(\sigma)^n}{n!}$, et nous attribuons à chaque vertex $V(\sigma)$ un indice v, $v = 1, \cdots, n$.

La mesure gaussienne pour les champs σ_v devient alors

$$d\nu(\{\sigma_v\}) = d\nu(\sigma_{v_0}) \prod_{v' \neq v_0}^{n} \delta(\sigma'_v - \sigma_{v_0}) d\sigma_{v'} \; , \tag{5.3.6}$$

où v_0 est un vertex arbitraire marqué. Pour obtenir la fonction de Schwinger connexe, nous allons utiliser la formule des arbres comme :

Théorème 5.3.1

$$S^c = \sum_{\pi} \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{\mathcal{T} \text{ avec } n \text{ vertex}} \Big\{ \prod_{\ell \in \mathcal{T}} [\int_0^1 dw_\ell] \Big\}$$
$$\int d\nu_{\mathcal{T}}(\{\sigma^v\}, \{w\}) \Big\{ \prod_{\ell \in \mathcal{T}} [\frac{1}{2}[\delta_{m_\ell p_\ell} \delta_{n_\ell q_\ell} + \delta_{m_\ell q_\ell} \delta_{n_\ell p_\ell}] \frac{\delta}{\delta \sigma_{mn}^{v(\ell)}} \frac{\delta}{\delta \sigma_{pq}^{v'(\ell)}}] \Big\}$$
$$\prod_{v=1}^{n} V_v \prod_{k=1}^{2p} R_{l_{1\pi(k)}, r_{1,\pi(k)}}(\sigma), \tag{5.3.7}$$

où

$$d\nu(\sigma) = \pi^{-N(N-1)/2} e^{-\mathrm{Tr}\sigma^2} \prod_{ij} d\sigma_{ij}, \tag{5.3.8}$$

la covariance devient :

$$< \sigma_{mn}, \sigma_{kl} >= \int d\mu(\sigma) \sigma_{mn} \sigma_{kl} = \delta_{nk} \delta_{ml}. \tag{5.3.9}$$

et le produit sur k parcourt les 2p termes extérieurs $R_{l_{1\pi(k)}, r_{1,\pi(k)}}(\sigma)$.

Nous considérons dans le reste de ce chapitre que le cas $p = 0$, c'est-à-dire, la fonction de Schwinger du vide (SV), parce qu'on peut obtenir toutes les autres fonctions de Schwinger facilement par les dérivations par rapport aux champs j.

La fonction de partition sans source s'ecrit comme :

$$Z(\lambda) = \int d\mu(\sigma) e^{2i\sqrt{2\lambda} \mathrm{Tr}_{\mathrm{m}} \mathrm{T}^{\Lambda}_{\mathrm{m}} \sigma_{\mathrm{mm}} - \frac{1}{2} \mathrm{Tr} \log[1 + \mathrm{i}\sqrt{2\lambda} (\mathrm{C}^{1/2})_{\mathrm{mn}} \sigma_{\mathrm{nk}} (\mathrm{C}^{1/2})_{\mathrm{km}}] + \frac{5}{2} \lambda \mathrm{T}^2_{\Lambda}}, \tag{5.3.10}$$

où

$$\mathrm{Tr} \log\big(1 \otimes 1 + \mathrm{i}\sqrt{2\lambda} \, [\, (\mathrm{C}^{1/2} \sigma \mathrm{C}^{1/2}) \otimes 1 + 1 \otimes (\mathrm{C}^{1/2} \sigma \mathrm{C}^{1/2})]\big). \tag{5.3.11}$$

La formule de multiplication pour les covariances est défini comme :

$$(C^{1/2})_{mn} (C^{1/2})_{nl} = \delta_{ml} C_{mn}, \quad (C^{1/2})_{mn} (C^{1/2})_{kl} = \delta_{nk} \delta_{ml} C_{mn}. \tag{5.3.12}$$

Nous avons tout d'abord le lemme suivant :

Lemme 5.3.1 *Les graphes engendré par* (5.3.10) *sont planaires..*

Développons le terme d'interaction comme :

$$\begin{aligned} & -\frac{1}{2} \mathrm{Tr} \log[1 + \mathrm{i}\sqrt{2\lambda} \mathrm{C}^{1/2} \sigma \mathrm{C}^{1/2}] \\ = \; & -2 \cdot \frac{i}{2} \sqrt{2\lambda} \mathrm{Tr}_{\mathrm{m}} \mathrm{T}^{\Lambda}_{\mathrm{m}} \sigma_{\mathrm{mm}} - \frac{1}{2} \mathrm{Tr} \log_2[1 + \mathrm{i}\sqrt{2\lambda} \mathrm{C}^{1/2} \sigma \mathrm{C}^{1/2}], \end{aligned} \tag{5.3.13}$$

où le facteur 2 est dû au fait qu'il y a deux tadpoles différents (voir la figure 14) et $\log_n(x)$ est défini comme :

$$\log_n(x) = \log(x) - [x - x^2/2 + x^3/3 \cdots + (-1)^{n+1} x^n/n]. \tag{5.3.14}$$

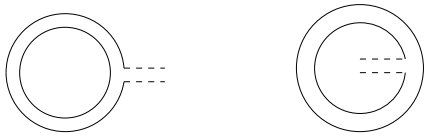

FIGURE 5.3: Les tadpoles.

On voit que dans l'équation (5.3.10), le tadpole de feuille peut annuler partiellement le contre-terme. Donc on peut réécrire le vertex d'interaction comme :

$$\begin{aligned} V(\sigma) &= -\frac{1}{2}\mathrm{Tr}\log_2[1+\mathrm{i}\sqrt{2\lambda}(\mathrm{C}^{1/2})_{\mathrm{mn}}\sigma_{\mathrm{nk}}(\mathrm{C}^{1/2})_{\mathrm{km}}] \\ &+ i\sqrt{2\lambda}\mathrm{Tr}_{\mathrm{m}}\mathrm{T}^{\Lambda}_{\mathrm{m}}\sigma_{\mathrm{mm}} + \frac{5}{2}\lambda\mathrm{T}^2_{\Lambda}, \end{aligned} \tag{5.3.15}$$

et la fonction de partition devient alors :

$$Z(\lambda) = \int d\nu(\sigma) e^{i\sqrt{2\lambda}\mathrm{Tr}_{\mathrm{m}}\mathrm{T}^{\Lambda}_{\mathrm{m}}\sigma_{\mathrm{mm}} - \frac{1}{2}\mathrm{Tr}\log_2[1+\mathrm{i}\sqrt{2\lambda}(\mathrm{C}^{1/2})_{\mathrm{mn}}\sigma_{\mathrm{nk}}(\mathrm{C}^{1/2})_{\mathrm{km}}] + \frac{5}{2}\lambda\mathrm{T}^2_{\Lambda}}. \tag{5.3.16}$$

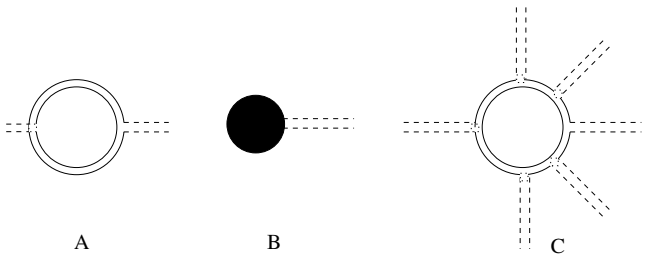

FIGURE 5.4: Les élements basic de LVE. Le graph A indique la feuille K, graph B le contre-term et graph C, le graphe de LVE plus générale.

5.4 Les représentations graphiques et leur amplitude pour la LVE

Il y a deux représentations graphiques pour la LVE : d'une part la représentation directe, constituée des structures en arbres des vertex décorés des contre-termes et, d'autre part, la représentation duale.

Dans la représentation duale, on remplace chaque ligne de propagateur par son dual. On a donc qu'une grande boucle décorée de contre-termes. Les vertex dans la représentation directe correspondent à des régions différentes dans la représentation duale. Les graphes de la représentation directe étant planaires, la représentation duale est bien définie. La représentation duale est plus intéressante en raison de l'existence d'un ordre cyclique pour tous les objets. Cf. l'appendice de ce chapitre pour plus de détails.

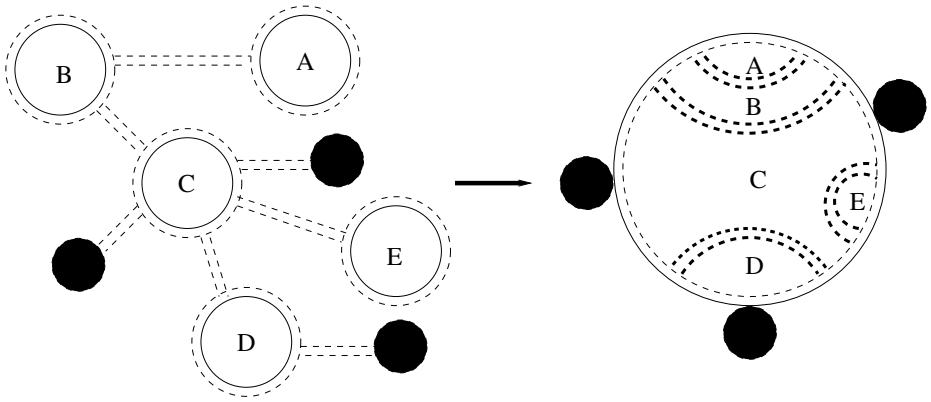

FIGURE 5.5: Le graphe direct de LVE et sa représentation duale. A gauche, dans le graphe direct, les disques noirs sont les contre-termes, les boucles sont les vertex à boucles et les lignes tiretées les propagateurs du champ σ. A droite, dans le graphe dual, les vertex à boucles sont remplacés par des lignes tiretées qui divisent la grande boucle en différentes régions.

Voila une figure (Figure C.12) pour un graphe dual.

On peut définir l'amplitude pour chaque objet dans la représentation duale

$$P_u = T_\Lambda D^{\bar{\ell}_c}_{mn} \text{ if } u = \bar{b} \in \bar{B}, \tag{5.4.1}$$

$$P_u = K^{\bar{\ell}}_{mn}(\sigma) \text{ if } u = \bar{v}_0 \in \bar{S}, \tag{5.4.2}$$

$$P_u = R^{\bar{\ell}_l,\bar{\ell}_r}_{mn}(\sigma) \text{ if } u = \bar{v}_1 \in \mathcal{L}, \tag{5.4.3}$$

où $\bar{B}$ est l'ensemble des contre-termes, $\mathcal{L}$ est la boucle duale, H est l'ensemble des demi-lignes, S est l'ensemble des résolvantes qui ne sont pas des feuilles et où chaque région correspond à un propagateur habillé.

Nous pouvons tout d'abord renormaliser les termes aux ordres 1 et 2 – cf. appendice – et l'amplitude pour un graphe dual peut s'écrire comme :

$$P_\Lambda = \log Z(\lambda, \Lambda, \mathcal{V}) = \sum_{\bar{\mathcal{T}} \text{ lines, } |\bar{\mathcal{T}}|=n-1} G_{\bar{\mathcal{T}}}$$

$$G_{\bar{\mathcal{T}}} = (-\lambda)^{n-1} \int d\nu(\sigma, w) \mathrm{Tr}\Big\{ \tilde{\prod}_{\mathrm{u}\in\mathcal{O}} \mathrm{P_u} \Big\}. \tag{5.4.4}$$

où l'opérateur Tr indique l'ordre cyclique pour les objets P_u.

Ici on doit redéfinir la mesure gausienne pour le champ σ_v : si on contracte deux champ σ correspondant à deux régions $\bar{v}$ et $\bar{v}'$, on va ajouter un facteur faible w.

5.5 Le développement de nettoyage

Dans cette section on va introduire le développement de nettoyage pour annuler (partialement) les contretermes et le facteur de Nelson. On effectue ce développement dans la représentation duale avec les conditions d'arrêt comme suivantes :

– On va commencer le développement de netroyage (ddn) d'un point marqué et fixé dans le graphe dual et on poursuit ce développement d'après un ordre cyclique fixé, par exemple l'ordre des aiguilles d'une montre .
– Si on engendre un tadpole, on va le compenser avec un contre-terme correspondant et on recommence ce développement à partir d'un point suivant.
– si on engendre une ligne de crossing ou nesting, on va continuer jusqu'à ce que le nombre de crossing $n_j = aj^2_{max}$, puis on va arrêter le développement.

Comme ce développement est similaire au cas de ϕ^4_2, on ne va pas répéter les détails ici.

5.5.1 La représentation multi-échelles pour les propagateurs et les résolvantes

On introduit le paramètre de Schwinger et on découpe le propagateur d'après son échelle comme :

$$C_{mn} = \sum_{j=0}^{\infty} C^j_{mn}, \tag{5.5.1}$$

où

$$C^j_{mn} = \int_{M^{-2i}}^{M^{-(2i-2)}} d\alpha e^{-\alpha(\mu^2+\frac{4}{\theta}(m+n+1))}, \tag{5.5.2}$$

et M est une constante arbitraire. On trouve facilement que

$$|C^j_{mn}| \leq KM^{-2j} e^{-M^{-2j}||\mu^2+\frac{4}{\theta}(m+n+1)||}. \tag{5.5.3}$$

Comme il y a de l'ordre cyclique pour les résolvantes, on peut l'écrire comme

$$R_{mn} = \frac{\partial}{\partial\sigma_{mn}}[-\frac{1}{2}\mathrm{Tr}\log(1+\mathrm{i}\sqrt{2\lambda}\mathrm{C}\sigma)] = -\mathrm{i}\frac{1}{2}\sqrt{2\lambda}\mathrm{C}_{\mathrm{mn}}\mathrm{Tr}\frac{1}{1+\mathrm{i}\sqrt{2\lambda}\mathrm{C}\sigma}. \tag{5.5.4}$$

Définissons les propagateurs par,

$$D_{mn} = i\sqrt{2\lambda}C_{mn}, \tag{5.5.5}$$

nous avons :

$$R(\sigma) = \mathrm{Tr}\frac{1}{1+\mathrm{D}_{\mathrm{mn}}\sigma_{\mathrm{nm}}}, \text{ et } \mathrm{R}(\sigma)_{\mathrm{mn}} = \mathrm{R}(\sigma)\mathrm{D}_{\mathrm{mn}}, \tag{5.5.6}$$

où les propagateurs D_{mn} et $R(\sigma)$ sont des opérateurs matrices.

Définisons la résolvante coupé et la résolvante habillée par :

$$R^j(\sigma) = \mathrm{Tr}\frac{1}{1+\sum_{\mathrm{k}\leqslant\mathrm{j}}\mathrm{i}\sqrt{2\lambda}\mathrm{C}^{\mathrm{k}}_{\mathrm{nm}}\sigma_{\mathrm{mn}}} = \mathrm{Tr}\frac{1}{1+\sum_{\mathrm{k}\leqslant\mathrm{j}}\mathrm{D}^{\mathrm{k}}_{\mathrm{nm}}\sigma_{\mathrm{mn}}}, \tag{5.5.7}$$

et

$$R^j_{mn} = D^j_{mn}R^j. \tag{5.5.8}$$

respectivement. Pour chaque résolvante on a l'équation algébrique :

$$\begin{aligned} R^j(\sigma) &= \frac{1}{1+\sum_{k\leq j} D^k\sigma} = \frac{1}{1+\sum_{k<j} D^k\sigma + D^j\sigma} \\ &= R^{j-1}(\sigma)\frac{1}{1+D^j\sigma R^{j-1}(\sigma)} \\ &= R^{j-1}(\sigma) - R^{j-1}(\sigma)\ D^j\ \sigma\ R^j(\sigma). \end{aligned} \tag{5.5.9}$$

C'est la formule fondamentale pour le développement de nettoyage.

Nous allons premièrment montrer un théorème pour la renormalisation, avant d'introduire le développement de nettoyage, comme :

Théorème 5.5.1 *Chaque tadpole intérieur doit être compensé par le contre-terme correspondant.*

Preuve 5.5.1 *Nous considérons un tadpole intérieur qui est engendré dans une résolvante $R(\sigma)$ arbitraire de l'échelle j, et nous effectuons le développement de Taylor pour cette résolvante :*

$$R^j(\sigma) = \frac{1}{1+i\sqrt{2\lambda}C^j\sigma} = 1 - i\sqrt{2\lambda}C^j\sigma + (i\sqrt{2\lambda}C^j\sigma)^2 + \text{Remainder}, \tag{5.5.10}$$

et l'amplitude pour ce tadpole intérieur devient :

$$A_{tadpole} = \int d\nu(\sigma) \sum_j (\sum_{mn}^{\Lambda} i\sqrt{2\lambda}C^j_{mn}\sigma_{nm})^2 = -2\lambda^2_\Lambda. \tag{5.5.11}$$

Comme l'amplitude pour un contre-terme attaché à cette résolvante est :

$$T_c = \int d\nu(\sigma)\frac{1}{2!} \times 2 \times (i\sqrt{2\lambda}\sum_{mn} C_{mn}\sigma_{nm})\mathrm{Tr_p}\sigma_{\mathrm{pp}}\mathrm{T_p} = 2\lambda \mathrm{T}^2_\Lambda, \tag{5.5.12}$$

nous avons donc

$$T_{tadpole} + T_c = 0. \tag{5.5.13}$$

Comme cette compensation est exacte, ce théorème est valide pour le tadpole et le contreterme à l'echelle j arbitraire.

On va commencer le développement de l'échelle j_{max} jusqu'à l'échelle d'arrêt. Pour chaque résolvante dans l'amplitude duale on va utiliser la formule (5.5.9). Pour chaque champ σ engendré par (5.5.9), on va utiliser la formule de l'intégration par partie. Par exemple, pour la deuxieme partie de (12) on a :

$$\begin{aligned} &\int d\mu(\sigma) D^{j_{max}}_{mn}\sigma_{nm} R^{j_{max}}(\sigma) R^{j_{max}-1}(\sigma) \times \text{ autre résolvantes d'échelle } \mathrm{j_{max}} \\ = \ & D^{j_{max}}_{mn} \int d\mu(\sigma) \frac{\partial}{\partial\sigma_{nm}} [R^{j_{max}}(\sigma) R^{j_{max}-1}(\sigma) \times \text{ autre résolvantes}]. \end{aligned} \tag{5.5.14}$$

Ici on néglige le cas où $\frac{\partial}{\partial\sigma}$ s'applique au terme $R^{j_{max}-1}$ parce que ce terme n'est pas très important, et on ne considère que le terme $\frac{\partial}{\partial\sigma}R^{j_{max}}$. Pour mieux montrer le développement de nettoyage, on va tout d'abord développer chaque résolvante par la formule de (5.5.9) pour voir si on engendre un propagateur pur ou une résolvante. Voir aussi la figure 5.6.

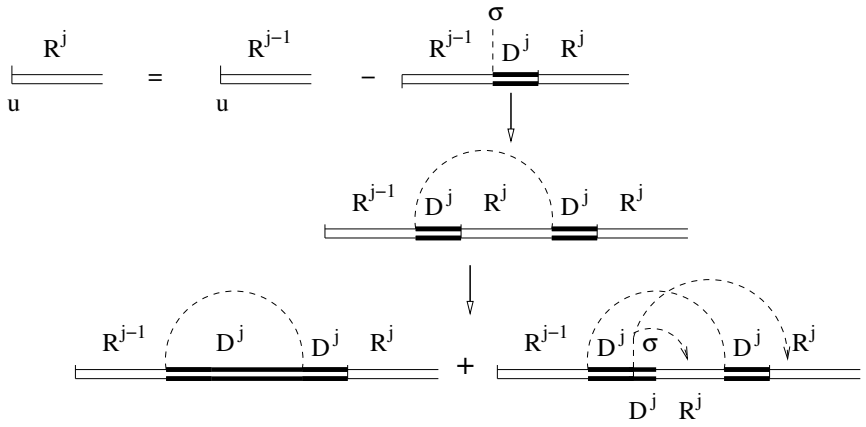

FIGURE 5.6: Le développement de nettoyage.

La seconde ligne sur la Figure 5.6 indique qu'on effectue la formule (5.5.9) quatre fois. Il y a plusieurs cas possibles quand on effectue l'intégrale pour les champs σ :

- Le champ σ_1 se contracte avec le champ σ_2 ; dans ce cas là on engendre un tadpole d'intérieur de l'échelle j_{max} qui doit-être annulé par le contreterme $T^{j_{max}}$.
- σ_1 est contracté avec σ_3 ; ici on engendre un crossing à l'échelle j_{max} Pour chaque crossing à l'échelle j, on peut borner son amplitude par :

$$||C^j_{mn}|| \leqslant |KM^{-2j}e^{-M^{-2j}||\mu^2+\frac{4}{\theta}(m+n+1)||}| < KM^{-2j}, \tag{5.5.15}$$

  Donc dans ce cas-là on a gagné un facteur convergent $M^{-2j_{max}}$ (voir formule (**??**)).jia gong shi
- σ_1 est contracté avec σ_4 ou autre champ qui est caché dans la résolvante et σ_2 est contracté avec σ_3 ; cela engendre aussi un tadpole intérieur et on doit l'annuler par le contre-terme.
- C'est bien possible que chaque champ σ ne se contracte que avec un champ caché dans la résolvante. Dans ce cas on ne peut que engendrer les lignes de nesting (voir la figure 5.7). Et pour chaque nesting de l'échelle j, l'ampliude est bornée par :

$$|R^{\bar\ell_l,\bar\ell_r\,j}_{mn}[\prod R^{\bar\ell'_l,\bar\ell'_r\,j}_{m'n'}]| < |R^{\bar\ell_l,\bar\ell_r\,j}_{mn}C^j_{mn}| \leqslant ||R^j|||C_{mn}|| < KM^{-2j}, \tag{5.5.16}$$

  Donc pour chaque ligne de nesting on peut gagner un facteur M^{-2j}.
- si le champ σ_1 se contracte avec les résolvantes dans l'autre region du graphe dual, on va engendrer quand même une ligne de crossing et gagner le bon facteur. Voir la Figure 5.8.

Dans la procédure du développement de nettoyage on considère aussi les conditions d'arrêt : on va arrêter ce développement quand le nombre de crossing et nesting atteindra $n(j_{max}) = aj^2_{max}$.

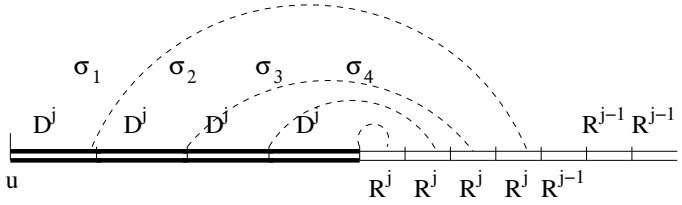

FIGURE 5.7: Graphe de LVE avec 4 lignes de nesting.

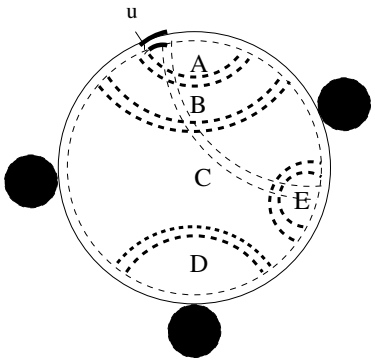

FIGURE 5.8: Lignes de crossings

Donc au total on peut gagner le facteur $M^{-aj_{max}^3}$ mais perdre un facteur divergent $(aj_{max})! \sim e^{2j_{max}^2 \ln j_{max}}$ dû à la combinatoire. Mais on peut compenser ce facteur facilement avec le facteur convergent.

Ce sont les idées essentielles du développement de nettoyage.

5.6 La resommation des contre-termes et l'argument de Nelson

Nous considérons la fonction G sans développer des countre-termes :

$$G = \int d\nu(\sigma, w) \prod_{l \in \mathcal{T}} \mathrm{Tr}[\mathrm{K_{mn}} \mathrm{R_{mn}}(\sigma)] \mathrm{e}^{\mathrm{Tri}\sqrt{2\lambda}\sigma \mathrm{T}_\Lambda}. \tag{5.6.1}$$

Utilisons la formule :

$$\int d\nu(w, \sigma) f(\sigma) g(\sigma) = e^{\frac{1}{2} \frac{\partial}{\partial \sigma} C(\sigma, \sigma', w) \frac{\partial}{\partial \sigma}} f(\sigma) g(\sigma))|_{\sigma=0}, \tag{5.6.2}$$

Nous obtenons alors :

$$G = \int d\nu(\sigma,w) \sum_{N=0}^{\infty} \frac{1}{N!} [\frac{1}{2} \frac{\partial}{\partial\sigma} \frac{\partial}{\partial\sigma'}]^N \{ \prod_{\bar{l}\in\bar{\mathcal{T}}} \int_0^1 dw_{l'} \mathrm{Tr}[\mathrm{K}^{\bar{\ell}}(\sigma) \mathrm{R}^{\bar{\ell}_1,\bar{\ell}_r}(\sigma)] \mathrm{e}^{\mathrm{Tri}\sqrt{2\lambda}\sigma \mathrm{T}_\Lambda}] \}$$

$$= \sum_{N_1=0}^{\infty} \sum_{N_2=0}^{\infty} \sum_{N_3=0}^{\infty} \prod_{l\in\mathcal{T}} \int_0^1 dw_l \frac{1}{N_1! N_2!} (\frac{1}{2})^{N_1+N_2} [(\frac{\partial}{\partial\sigma})^{N_2} \{\frac{\partial}{\partial\sigma} \frac{\partial}{\partial\sigma'}\}^{N_1}] [K(\sigma) R(\sigma)]$$

$$\{(\frac{\partial}{\partial\sigma'})^{N_2} \frac{1}{N_3!} [\frac{1}{2} \frac{\partial}{\partial\sigma} \frac{\partial}{\partial\sigma'}]^{N_3} e^{\sum_q i\sqrt{2\lambda}\sigma_{mm} T_m^\Lambda}\}|_{\sigma=0}. \tag{5.6.3}$$

où T_m^Λ est défini par la formule (5.2.18).

Les dérivations de N_1 et N_2 dans cette formule ont engendrés les termes connexes et les dérivations de N_3 ont engendrés les termes disconnectés (voir le graphe B dans la Figure 5.4).

Graphiquement la somme sur ces N_3 termes disconnectés indiquent que nous détachons et contractons les contre-termes du graphe à la représentation duale.

Ici il y a des points différents du cas commutatif : comme dans le cas GW_2 tous les vertex sont non-locaux, on ne peut pas effectuer la somme sur tous les contre-termes à toutes les échelles naïvement, sinon le résultat divergera. Plus précisément : comme chaque contreterme est attaché à un propagateur pur à la forme $\lambda C_{nm} T_m^\Lambda$, et C_{nm} est convergent comme M^{-2j}, où $2j \sim \ln\max(m,n)$, l'amplitude pour chaque contre-terme est borné par

$$C_{nm} T_m^\Lambda \sim \frac{1}{m} \ln(\Lambda/m) \leq \frac{\ln\Lambda}{m}. \tag{5.6.4}$$

Ce terme sera plus petit que 1 et n'est pas dangereux si $m > \ln\Lambda$.

C'est-à-dire, on a seulement besoin d'effectuer la somme sur les contre-termes quand l'indice m est plus petit que $\ln\Lambda$.

Nous avons donc :

$$- \quad 2\lambda \sum_{m=1}^{\ln\Lambda} \ln^2(\Lambda/m) < -2\lambda \sum_{m=1}^{\ln\Lambda} \ln^2\Lambda = -2\lambda(\ln\Lambda)^3, \tag{5.6.5}$$

et le résultat pour la somme devient :

$$A_T^N = \int d\nu(\sigma,w) \prod_{\bar{\ell}\in\bar{\mathcal{T}}} \mathrm{Tr} \overrightarrow{\prod} [\mathrm{K}_{\mathrm{mn}}^{\bar{\ell}}(\sigma) \mathrm{R}_{\mathrm{mn}}^{\bar{\ell}_1,\bar{\ell}_r}(\sigma)] \mathrm{e}^{\mathrm{Tri}\sqrt{2\lambda}\sigma \mathrm{T}_\Lambda} \mathrm{e}^{\lambda(\ln\Lambda)^3}. \tag{5.6.6}$$

La procédure pour la resommation est présenté sur la Figure 5.9.

Le facteur $e^{\lambda(\ln\Lambda)^3}$ n'est pas dangereux, comme on peut le borner par le facteur convergent qu'on a gagné par les lignes de crossing et nesting. Donc on obtient

$$e^{-aj_{max}^3} \cdot (j_{max}^2)! \cdot e^{\lambda \ln 32\Lambda} \sim e^{-aj_{max}^3 + 2j_{max}^2 \ln j_{max} + \lambda j_{max}^3} < 1, \tag{5.6.7}$$

dès que le coefficient a est bien choisi, par exemple $a > 2\lambda$. Ici nous avons utilisé la relation $\ln\Lambda \sim j_{max}$ et $(aj_{max}^2)! \sim e^{2j_{max}^2 \ln j_{max}}$.

Donc enfin nous avons pour l'amplitude :

$$|A_T|_{|T|=n} \quad < \quad \int d\nu(\sigma,w) \prod_{l,l'\in\mathcal{T}} \lambda^{n-1} |\mathrm{Tr}\{||\mathrm{K}_{\mathrm{mn}}^{\bar{\ell}}(\sigma)|| \; ||\mathrm{R}_{\mathrm{mn}}^{\bar{\ell}_1,\bar{\ell}_r}(\sigma)|| \; |\mathrm{e}^{\mathrm{Tri}\sqrt{2\lambda}\sigma \mathrm{T}_\Lambda}|\}|$$

$$\times \quad e^{\lambda \ln^3\Lambda} e^{-aj_{max}^3} \cdot (aj_{max}^2)! \leqslant (K\lambda)^{n-1}. \tag{5.6.8}$$

où K est une constante arbitraire. C'est l'argument de Nelson dans la langue de LVE.

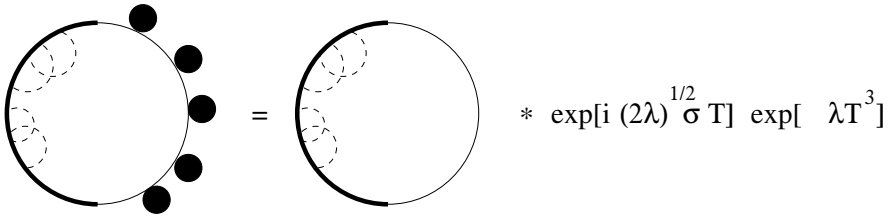

FIGURE 5.9: Représentation schématique de la resommation des contre-termes.

5.7 La somme de Borel

Dans cette section nous allons montrer que la série perturbative est sommable dans le sens de Borel :

Théorème 5.7.1 *La fonction Schwinger perturbative pour le modèle de GW_2 est Borel sommable.*

Preuve 5.7.1 *Pour que la série soit Borel sommable, le domaine d'analyticité doit être $|\lambda| < \frac{1}{K}$ et $\mathrm{Re}\lambda > 0$ [14, 15, 23], C'est-à-dire :*

$$-\frac{\pi}{4} \leq \mathrm{Arg}\sqrt{\lambda} \leq \frac{\pi}{4}. \tag{5.7.1}$$

On réécrit la résolvante comme :

$$R = \frac{1}{1 + i\sqrt{2\lambda}\mathrm{Tr}[\mathrm{C}^{1/2}\sigma\mathrm{C}^{1/2}]}. \tag{5.7.2}$$

Comme la matrice $C^{1/2}\sigma C^{1/2}$ est hermitienne, ses valeurs propres doivent être réelles. Donc la résolvante est bornée par :

$$\parallel R \parallel = |\frac{1}{1 + i\sqrt{2\lambda}\mathrm{Tr}[\mathrm{C}^{1/2}\sigma\mathrm{C}^{1/2}]}| \leqslant \sqrt{2}, \tag{5.7.3}$$

et la feuille est bornée par

$$\parallel K \parallel = \parallel R - 1 \parallel \leqslant 1 + \sqrt{2}. \tag{5.7.4}$$

Comme la constante de couplage devient complexe dans le domaine de l'analyticité, le contre-terme devient :

$$e^{\mathrm{Tri}\sqrt{2\lambda}\sigma\mathrm{T}_\Lambda} = e^{\mathrm{Tri}|\sqrt{2\lambda}|\cos\theta\sigma\mathrm{T}_\Lambda} e^{-\mathrm{Tr}|\sqrt{2\lambda}|\sin\theta\sigma\mathrm{T}_\Lambda}, \tag{5.7.5}$$

où $\theta = \mathrm{Arg}\sqrt{\lambda}$. *On peut borner le premier terme* (6) *par* 1, *mais le deuxième terme est divergent pour* σ *négatif.*

On reécrit ce terme sous la forme :

$$\begin{aligned} & \int d\mu(\sigma) e^{-1/2\mathrm{Tr}\sigma^2} e^{-\mathrm{Tr}|\sqrt{2\lambda}|\sin\theta\sigma \mathrm{T}_\Lambda} \\ = & \int d\mu(\sigma) e^{-1/2\mathrm{Tr}(\sigma+\sqrt{2\lambda}\sin\theta \mathrm{T}_\Lambda)^2} e^{\sin^2\theta T_\Lambda^2}, \end{aligned} \tag{5.7.6}$$

telle que ce terme est borné par $e^{\sin^2\theta T_\Lambda^2}$, *qui est dominé par* $e^{1/2T_\Lambda^2}$ *pour* $\theta = \pm\pi/4$. *Une telle divergence est sans danger, parce que nous pouvons la compenser par le terme* $e^{-aj_{max}^3} \sim e^{-aT_\Lambda^3}$ *qu'on a gagné.*

On effectue le développement de Taylor avec reste intégral comme :

$$G(t\lambda)|_{t=1} = \frac{G^{(n)}(\lambda)}{n!}|_{t=0} + \int_0^1 dt \frac{(1-t)^n}{n!} G^{(n+1)}(t\lambda), \tag{5.7.7}$$

et on effectue les contractions de Wick, on trouve finalement que ce terme est borné par :

$$||R^{n+1}|| < |\lambda|^{n+1} K^n (2n)!! \leq |\lambda|^{n+1} K'^n \cdot n!, \tag{5.7.8}$$

où K *et* K' *sont des constantes positives . Donc nous avons montré que la série perturbative est sommable dans le sens de Borel.*

6

La construction d'autres espaces non-commutatifs

Comme les TQC dans l'espace non-commutatif, le modèle de Grosse-Wulkenhaar par exemple, ont des propriétés meilleures que les modèles de TQC commutatives correspondantes, il est naturel de considérer d'autres espaces noncommutatifs et aussi le modèle de TQC sur l'espace non-commutatif correspondant.

Nous allons construire dans ce chapitre un modèle d'espace non-commutatif par la déformation de certains domaines de Cartan classiques du type 1, qui sont l'espace quotient comme $SU(m,n)/S(U(m)\times U(n))$. La méthode que nous allons utiliser est la quantification par états cohérents, développée par F. Berezin, H. Grosse et P. Presnajder [89].

6.1 Les états cohérents pour un groupe de Lie G arbitraire

Soit G un groupe de Lie non-abélien non-compacte, et T_g une représentation unitaire irréductible de G dans un espace hilbert : $\mathcal{H}$. Soit $|z_0\rangle =\in \mathcal{H}$ un état normalisé dans $\mathcal{H}$, et H le groupe de stabilité pour l'état $|z_0\rangle$, défini comme $T_h|z_0\rangle = e^{i\alpha(h)}|z_0\rangle$ [88].

On peut construire alors, pour chaque point $z = g_z z_0$ de l'espace de quotient $D = G/H$, un ensemble d' états cohérents comme : $|z\rangle = T(g_z)|z_0\rangle$. Et nous avons pour les $|z\rangle$

$$\int d\mu(z,\bar{z})|z\rangle\langle z| = 1, \tag{1}$$

où $d\mu(z,\bar{z})$ est la mesure invariante normalisée. Nous pouvons définir, pour chaque opérateur de la représentation T_g, un opérateur $\hat{F}$ dans $\mathcal{H}$ par $\hat{F} = \int dg\tilde{F}(g)T(g)$. Et pour chaque

$\hat{F}$ on peut construire une fonction biholomorphe $F(\bar{z}, z)$ par :

$$F(z, \bar{z}) = \langle z|\hat{F}|z\rangle = \int dg\tilde{F}(g)\omega(g, z),\ \omega(g, z) = \langle z|T_g|z\rangle, \tag{2}$$

où $\tilde{F}(g)$ est une distribution avec le support compact.

Comme le groupe G est non-abélien, les opérateurs T_g différents ne se commutent pas, nous avons obtenu donc une algèbre non-commutative des fonctions définies dans l'espace commutatif $D = G/H$. Nous espérons que la non-commutativité de l'algèbre peut induire une structure non-commutative pour l'espace D. C'est-à-dire, on peut construire un espace non-commutative $\hat{D}$ par l'algèbre non-commutative des fonctions définis en D.

Considérons une autre fonction $G(\bar{z}, z)$ définie de la même façon :

$$G(z, \bar{z}) = \langle z|\hat{G}|z\rangle = \int dg\tilde{G}(g)\omega(g, z), \tag{3}$$

Le produit d'étoile entre ces deux fonctions est défini par [89] :

$$(F \star G)(z, \bar{z}) = \langle z|\hat{F}\hat{G}|z\rangle = \int dg_1 dg_2 \tilde{F}(g_1)\tilde{F}(g_2)\omega(g_1 g_2, z). \tag{4}$$

Comme chaque distribution $\tilde{F}$ peut être exprimée comme les dérivations de la fonction δ du groupe G :

$$\tilde{F}_{A_1 \ldots A_n}(g) \ = \ (\mathcal{X}_{A_1} \ \ldots \ \mathcal{X}_{A_n}\delta)(g), \tag{5}$$

où $\mathcal{X}_A$ sont les vecteurs invariants à gauche par l'action du groupe G, nous pouvons finalement obtenir le produit d'étoile quand on effectue l'intégrale par partie pour la fonction δ. Plus explicitement, nous avons :

$$\begin{aligned} & (F \star G)(z) \\ = \ & (-1)^{n+m}(\mathcal{X}_{A_n} \ \ldots \ \mathcal{X}_{A_1}\, \mathcal{X}_{B_m} \ \ldots \ \mathcal{X}_{B_1}\omega)(g_1 g_2, z)|_{g=e}. \end{aligned} \tag{6}$$

D'après la formule $\omega(g, z) =< z|T_g|z >= \omega_0(g_z^{-1} g g_z)$, il suffit de calculer $\omega_0(g)$ pour obtenir le produit $*$ explicitement.

Nous allons appliquer cette méthode dans ce chapitre à la quantification pour l'espace quotient $D = SU(2, 1)/S(U(2)\times U(1))$, et nous considérons le cas $SU(2, 2)/S(U(2)\times U(2))$ dans l'appendice.

6.2 Le groupe $SU(2, 1)$ et sa représentation unitaire irréductible

Le groupe $G = SU(m, 1)$ est un sous-groupe de $SL(m + 1, C)$:

$$g = \begin{pmatrix} a_{m\times m} & b_{m\times 1} \\ c_{1\times m} & d \end{pmatrix} \in G \tag{1}$$

satisfaisant à l'équation :

$$g^\dagger \Gamma g = g\Gamma g^\dagger = \Gamma, \quad \Gamma = \begin{pmatrix} E_{m\times m} & 0 \\ 0 & -1 \end{pmatrix}. \tag{2}$$

où

$$\Gamma = \begin{pmatrix} E_{m\times m} & 0 \\ 0 & -1 \end{pmatrix}. \tag{3}$$

Nous avons donc deux restrictions équivalentes comme :

$$\begin{aligned} a^\dagger a &= E + c^\dagger c,\ d^\dagger d = E + b^\dagger b,\ a^\dagger b = c^\dagger d,\ or & (4)\\ aa^\dagger &= E + bb^\dagger,\ dd^\dagger = E + cc^\dagger,\ ac^\dagger = bd^\dagger. & (5)\end{aligned}$$

Le sous-groupe maximal du G est

$$K = S(U(m)\times U(1)) = \{\begin{pmatrix} K_1 & 0 \\ 0 & K_2 \end{pmatrix}\},\ \det(K_1K_2) = 1. \tag{6}$$

Pour chaque élément $g \in SU(2,1)$, nous avons la décomposition de Cartan comme :

$$g = \begin{pmatrix} N_1 & ZN_2 \\ Z^\dagger N_1 & N_2 \end{pmatrix}\begin{pmatrix} K_1 & 0 \\ 0 & K_2 \end{pmatrix}, \tag{7}$$

où

$$N_1 = (E - ZZ^\dagger)^{-1/2}, N_2 = (1 - Z^\dagger Z)^{-1/2}. \tag{8}$$

et

$$Z = bd^{-1} = \begin{pmatrix} z_1 \\ z_2 \\ \cdots \\ z_m \end{pmatrix},\ Z^\dagger = (z_1^\dagger, z_2^\dagger, \cdots z_m^\dagger) \tag{9}$$

sont les coordornées de l'espace quotient $D = SU(m,1)/S(U(m)\times U(1))$, qui est défini par l'équation :

$$D = \{Z \,|\, 1 - |Z|^2 > 0\} \ = \ \{Z \,|\, 1 - |z_1|^2 - |z_2| - \cdots - |z_m|^2 > 0\}. \tag{10}$$

Il y a les propriétés suivantes pour le domaine D :

– D est un domaine pseudo-convexe, donc nous pouvons construire l'espace hilbertien holomorphe avec le noyau de Bergman :

$$K(W^\dagger, Z) = (1 - W^\dagger Z)^{-(m+1)}. \tag{11}$$

– D est une variété kählerienne avec la métrique définie comme : $g_{i\bar{j}} = -\partial_{\bar{z}^i}\partial_{z^j}\ \log K(Z^\dagger, Z)$. Plus explicitement, nous avons

$$g_{i\bar{j}} = [\frac{\delta_{ij}}{1 - |Z|^2} + \frac{z_i\bar{z}_j}{(1 - |Z|^2)^2}],\quad g^{i\bar{j}} = (1 - |Z|^2)(\delta_{ij} - \bar{z}_i z_j). \tag{12}$$

– le tenseur de Ricci pour D est $R_{i\bar{j}} = -(m+1)g_{i\bar{j}}$, et la courbure scalaire est $R = g^{i\bar{j}}R_{i\bar{j}} = -(m+1)$.

– nous pouvons vérifier que $g_{i\bar{j}}$ est une solution de l'équation d'Einstein dans le vide

$$R_{i\bar{j}} - \frac{1}{2}g_{i\bar{j}}R + \Lambda g_{i\bar{j}} = 0 \tag{13}$$

avec une constante cosmologique $\Lambda = \frac{m+1}{2}$, qui est positive.

6.2.1 L'algèbre de Lie

L'algèbre de Lie correspondant à $\mathbf{g} = su(m,1)$ est définie par : $M = \{ \begin{pmatrix} A & B \\ B^\dagger & D \end{pmatrix} \} \in \mathbf{g}$, $M^\dagger \Gamma = -\Gamma M$. c'est-à-dire $A^\dagger = -A, D^\dagger = -D, \mathrm{tr}(A+D) = 0$.

Nous avons, pour un élément de l'algèbre de Lie $\mathbf{g}$, la décomposition de Cartan unique comme :

$$\mathbf{g} = \mathbf{l} + \mathbf{p}, \tag{14}$$

où

$$\mathbf{l} = \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} \ , \ \ et \ \mathbf{p} = \begin{pmatrix} 0 & B \\ B^\dagger & 0 \end{pmatrix}. \tag{15}$$

où A est une matrice anti-hermitienne $m \times m$ et D est un nombre imaginaire pur.

Soit $\mathbf{a} \subset \mathbf{p}$ est la sous-algèbre abélienne maximale, on peut choisir la forme de l'élement $H_t \in \mathbf{a}$ comme :

$$H_t = \begin{pmatrix} O_{(m-1)\times(m-1)} & O_{(m-1)\times 1} & O_{(m-1)\times 1} \\ O_{1\times(m-1)} & 0 & t \\ O_{1\times(m-1)} & t & 0 \end{pmatrix} \tag{16}$$

où t est un nombre réel.

Les racines de $(\mathbf{g}, \mathbf{a})$ sont $\pm\alpha, \ \pm 2\alpha$ avec les multiplicités $m_\alpha = 2$ et $m_{2\alpha} = 1$; et les racines positives sont α et 2α.

Le groupe abélien correspondant est

$$\delta := \{\delta_t |\ \delta_t = \exp H_t, \ H_t \in \mathbf{a}\}, \tag{17}$$

donc nous avons :

$$\delta_t = \begin{pmatrix} E_{(m-1)\times(m-1)} & O & 0 \\ O & \cosh t & \sinh t \\ 0 & \sinh t & \cosh t \end{pmatrix}. \tag{18}$$

6.2.2 Les représentations du groupe $SU(m,1)$

Il y a trois représentations unitaires irreductibles pour le groupe G [77][98] :
- la série principale ,
- la série discrète,
- et la série supplémentaire.

Nous considérons seulement la série discrète, qui est réalisée dans l'espace hilbertien $\mathcal{L}^2_N(D)$ de fonctions holomorphes, dont le produit scalaire est défini par :

$$(f,g)_N = \int d\mu_N(Z,\bar{Z}) \ \bar{f}(\bar{Z}) g(Z), \tag{19}$$

où $d\mu_N(Z,\bar{Z}) = c_N[\det(E - Z^\dagger Z)]^{N-(m+1)}|dZ|$, est une mesure normalisée et $c_N = \pi^{-2}(N-2)(N-1)$, pour que $\int d\mu_N(Z,\bar{Z}) = 1$.

La représentation de la série discrète est définie par l'opérateur de la représentation T^N_g :

$$T^N_g f(Z) = (cZ+d)^{-N} f(Z'), \tag{20}$$

où $N \geq 1$ est un nombre naturel qui caractérise la représentation discrète et $Z' = (aZ + b)(cZ + d)^{-1}$.

Nous allons réaliser cette représentation par les oscillateurs harmoniques comme suivant : introduisons une matrice de 3×1 des oscillateurs bosoniques satisfaisant aux relations de commutation suivantes :

$$\hat{Z} \;=\; \begin{pmatrix} \hat{a} \\ \hat{b}^\dagger \end{pmatrix} : \quad [\hat{a}_\alpha, \hat{a}^\dagger_\beta] \;=\; \delta_{\alpha\beta}, \alpha, \beta = 1, 2. \;\; [\hat{b}, \hat{b}^\dagger] \;=\; 1 \tag{21}$$

Les oscillateurs harmoniques sont définis dans l'espace de Fock par $\hat{a}_\alpha \,|0\rangle \;=\; \hat{b}\,|0\rangle \;=\; 0$, et l'état générale est engendré par :

$$|m_\alpha,\, n\rangle \;=\; \prod_\alpha \frac{(\hat{a}^\dagger_\alpha)^{m_\alpha}\,(\hat{b}^\dagger)^n}{\sqrt{m_\alpha!\, n!}}\,|0\rangle\,. \tag{22}$$

Nous pouvons vérifier que le terme quadratique $\hat{Z}^\dagger \Gamma \hat{Z}$ est invariant sous l'action du groupe :

$$\hat{Z} \;\mapsto\; g\,\hat{Z}, \quad \hat{Z}^\dagger \;\mapsto\; \hat{Z}^\dagger\, g^\dagger. \tag{23}$$

Nous pouvons construire, pour chaque base X de l'algèbre $su(2,1)$, où $X \;=\; \{X^A\}$, $A = 1, \cdots 8$, les opérateurs dans l'espace hilbertien comme :

$$\hat{X^A} \;=\; -\mathrm{Tr}\hat{Z}^\dagger \Gamma X^A \hat{Z} \;= -\hat{z}_i^\dagger\, \Gamma_{ij}\, X^A_{ij}\, \hat{z}_j\,, \tag{24}$$

$$\hat{X^A}^\dagger \;=\; -\mathrm{Tr}\,(\hat{Z}^\dagger X^{A\dagger} \Gamma \hat{Z}) \;=\; \mathrm{tr}\,(\hat{Z}^\dagger \Gamma X^A \hat{Z}) \;=\; -\,\hat{X}^A. \tag{25}$$

En utilisant les relations de commutation pour les opérateurs d'annihilation et de création, nous avons :

$$[\hat{X^A}, \hat{X^B}] \;=\; [\mathrm{Tr}(\hat{Z}^\dagger \Gamma X^A \hat{Z}),\; \mathrm{Tr}(\hat{Z}^\dagger \Gamma X^B \hat{Z})] \;= -\mathrm{Tr}(\hat{Z}^\dagger\, \Gamma\, [X^A,\; X^B]\, \hat{Z})\,. \tag{26}$$

Donc les opérateurs $\hat{X}^A$ satisfont les relations de commutation de l'algèbre $su(2,1)$, et nous réalisons la représentation unitaire pour le groupe $SU(2,1)$ dans l'espace de Fock, par l'application exponentielle de l'algèbre vers le groupe :

$$\hat{T}(g) \;=\; e^{\xi^A \hat{X}_A}. \tag{27}$$

Ensuite nous considérons l'expression explicite du produit $*$. Construisons tout d'abord l'état du vide $|z_0\rangle$ comme :

$$|z_0\rangle = \frac{(\hat{b}^\dagger)^N}{\sqrt{(N+1)!}}|0> = \frac{1}{\sqrt{N}}\,|0, 0; N\rangle. \tag{28}$$

Et nous avons :

$$\omega_0(g) = \langle z_0|\,\hat{T}(g)\,|z_0\rangle = \frac{1}{\cosh t}[(1 + \ln\cosh t) e^{i(\alpha(q) - \alpha(k))}]^N. \tag{29}$$

Nous pouvons définir les fonctions comme $\xi_A(z) = \frac{1}{N}\,\langle z|\hat{X}_A|z\rangle$. D'après un théorème de Harish-Chandra, nous pouvons toujours encastrer l'espace hermitien symétrique dans

la partie non-compacte de l'algèbre de Cartan. Donc nous pouvons identifier les fonctions $\xi_4, \xi_5, \xi_6, \xi_7$ comme les coordonnées de l'espace non-commutif. Et le produit des coordonées devient :

$$(\xi_A \star \xi_B)(z) \;=\; \frac{1}{2N^2}\,\langle z|\{\hat{X}_A, \hat{X}_B\}|z\rangle \;+\; \frac{1}{2N^2}\,\langle z|[\hat{X}_A, \hat{X}_B]|z\rangle, \tag{30}$$

où le premier terme est

$$\frac{1}{2N^2}\,\langle z|\{\hat{X}_A, \hat{X}_B\}|z\rangle \;=\; (1 + A_N)\,\xi_A(z)\,\xi_B(z) \;+\; B_N\,\delta_{AB}, \tag{31}$$

et le second terme est :

$$\frac{1}{2N^2}\,\langle z|[\hat{X}_A, \hat{X}_B]|z\rangle \;=\; \frac{1}{2N}\,f^C_{A,B}\,\xi_C(z). \tag{32}$$

où les constantes B_N et C_N sont dépendantes des constantes de Bernouilli et les parties dominantes des constantes B_N et C_N sont $1/N$.

6.3 Un modèle de TQC

Nous considérons dans cette section un modèle de champ scalaire défini sur le domaine non-commutatif $\hat{D}$.

Le laplacien invariant pour le domaine $\hat{D}$ est [102][103] :

$$\Delta_N = (1 - |z|^2)(\sum_{i=1}^{2} \frac{\partial^2}{\partial\bar{z}\partial z} - \bar{R}R - N\bar{R}) \tag{1}$$

où $R = \sum_1^2 z_i\partial/\partial z_i$ est un operateur différentielle de premier ordre et

$$T_g\Delta_N f(z) = \Delta_N T_g f(z) \tag{2}$$

Les fonctions propres du laplacien sont [102] :

$$\phi_\lambda(z) = (1 - |z|^2)^{(-N+2-i\lambda)/2} F(\frac{N+2-i\lambda}{2},\; \frac{-N+2-i\lambda}{2};\; d;\; |z|^2) \tag{3}$$

où $F(\frac{N+2-i\lambda}{2}\;\frac{-N+2-i\lambda}{2};\; m;\; |Z|^2)$ est la fonction hypergéométrique, et $\lambda = \alpha(\mathbf{g}_c)$.

Ses valeurs propres sont :

$$-\frac{1}{4}((N-2)^2 + \lambda^2). \tag{4}$$

Nons ne considérons que la série discrète, où les valeurs de λ sont entières imaginaires comme

$$\lambda = i(N - 2 - 2l) \tag{5}$$

où $l = 0, 1, \cdots, \left[\frac{N-2}{2}\right]$.

Donc le spectre du laplacien est discrèt et il contient un nombre fini de points.

L'action de la théorie quantique des champs scalaires est définie comme :

$$S = \int d\mu(z)\{-\frac{1}{2}\Phi \star \Delta_n\Phi(z) - \frac{1}{2}[\mu^2 + \eta R]\Phi^2(z) - \lambda\Phi^4_\star(z)\} \tag{6}$$

où le champ Φ est identifié comme la fonction $F(z,\bar{z}) =< z|\hat{F}|z >$ et il peut s'écrire comme un polynôme de coordonnées non-commutatves ξ_i.

On peut développer le champ Φ d'après les fonctions propres du laplacien :

$$\Phi = \sum_{N,l} C_{N,l}\phi(N,l,z,\bar{z}), \tag{7}$$

Utilisons la quantification par l'intégral de chemin, la valeur moyenne pour l'opérateur $F[\Phi]$ est : Φ :

$$\langle F[\Phi]\rangle \;=\; \frac{\int D\Phi\, e^{-S[\Phi]}\, F[\Phi]}{\int D\Phi\, e^{-S[\Phi]}}, \tag{8}$$

où

$$\int D\Phi e^{-\frac{1}{2}\int_D \Phi^2} = \int_D \prod_{z\bar{z}} d\Phi(z,\bar{z}) e^{-\frac{1}{2}\int_D \Phi(z,\bar{z})^2}, \tag{9}$$

Le propagateur dans l'espace de Fourier est :

$$< \Phi, \Phi >= \frac{1}{\mu^2 - (m+1)\eta + \frac{1}{4}((N-2)^2 + \lambda^2)} \tag{10}$$

Le propagateur pour la série discrète devient alors :

$$\frac{1}{l(N-2) - l^2 + m^2 - R\eta}. \tag{11}$$

Nous calculons la fonction de deux points à une boucle, et on obtient :

$$G_2 = \lambda \sum_{N=3}^{\Lambda} \sum_{l=0}^{[(N-2)/2]} \frac{1}{N} \frac{1}{l(N-2) - l^2 + m^2}. \tag{12}$$

Effectuons la somme de l'indice l, et la fonction à deux points devient :

$$G_2 = \lambda \sum_{N=3}^{\Lambda} \frac{1}{N} \frac{1}{N-2} [\ln(N-2) - \ln\epsilon]. \tag{13}$$

où $0 < \epsilon \sim \frac{1}{\Lambda} \ll 1$ est le régulateur infrarouge. Nous effectuons la somme de N et nous obtenons :

$$G_2 \sim \lambda(1 - \frac{2}{\Lambda}\ln\Lambda - \frac{2}{\Lambda}) < \infty, \tag{14}$$

qui n'est pas divergente.

7

Les représentations paramétriques et les polynômes des graphes

7.1 Introduction

Dans les sections précédentes on a introduit les renormalisations perturbatives et constructives pour les modelès TQC commutatives et non-commutatifs. Dans ce chapitre on va introduire les propriétés combinatoires dans la TQC perturbative. On va montrer que les polynômes de Symanzik pour la représentation paramétrique de TQC sont reliés aux polynômes de Tutte [106].

Nous allons aussi montrer que les polynômes de Symanzik pour les TQC non-commuta -tives sont reliés aux polynômes de Bollobás-Riordan pour les graphes de rubans.

7.2 Les représentations paramétriques

Parmi les représentations directes et la représentation de l'espace des moments, il existe une autre représentation qui est aussi beaucoup utilisée, c'est la représentation paramétrique.

Passons à la représentation paramétrique [115] des intégrales de Feynman, le point de départ est fourni par la représentation du propagateur libre :

$$\frac{1}{p^2+m^2} = \int_0^\infty d\alpha e^{-\alpha(p^2+m^2)}, \tag{1}$$

où α est appelé le paramètre de Schwinger. Pour un graphe de Feynman G, nous allons associer à chaque ligne e de G un paramètre α_e. Puis, nous effectuons la transformée de Fourier et effectuons l'intégrale sur les variables d'espace et de moment.

Donc finalement les fonctions de corrélations peuvent s'écrire comme la combinaison de polynômes dépendant seulement de ces paramètres de Schwinger et des moments extérieurs, qui s'appellent les polynômes de Symanzik [115].

Cette représentation est plus compacte et plus jolie que les deux autres, et dans cette représentation, c'est plus facile d'étudier les propriétés combinatoires de la TQC.

Considérons l'amplitude d'un graphe de Feynman dans la représentation de moment tronquée :

$$A_G^T(p_1,...,p_N) = \int \prod_{e=1}^{E} d^d k_e \frac{1}{k_e^2+m^2} \prod_{v=1}^{V-1} \delta(\epsilon_{fv} p_f + \epsilon_{ev} k_e). \tag{2}$$

Nous introduisons les paramètres de Schwinger pour les propagateurs, nous effectuons les transformées de Fourier, et nous introduisons les régulateurs q_v pour chaque vertex, l'amplitude devient :

$$A_G^T(p_1,...,p_N;\{q_v\}) = \int \prod_{e=1}^{E} \left[d\alpha_e e^{-\alpha_e m^2} d^d k_e\right] \prod_{v=1}^{V} d^d x_v e^{-\alpha_e k_e^2 - q_v x_v^2 + 2i(p_f \epsilon_{fv} x_v + k_e \epsilon_{ev} x_v)}. \tag{3}$$

Et nous avons :

$$\lim_{q_v \to 0} A_G^T(p_1,...,p_N;q_v) = \delta(\sum_{f=1}^{N} p_f) A_G^T(p_1,...,p_N). \tag{4}$$

Nous pouvons réécrire la formule (3) dans une forme plus compacte comme :

$$A_G(p_f,q_v) = \int \prod_e d\alpha_e e^{-\alpha_e m^2} d^d k_e \int \prod_v d^d x_v e^{-Y X_G Y^t} \tag{5}$$

où X_G est une matrice carrée de dimension $d(E+V+N)$, c'est-à-dire :

$$X_G = \begin{pmatrix} \alpha_e & -i\epsilon_{ev} & 0 \\ -i\epsilon_{ev} & q_v & -i\epsilon_{fv} \\ 0 & -i\epsilon_{fv} & 0 \end{pmatrix} \tag{6}$$

ici α_e et q_v indiquent les matrices diagonales $\alpha_e \delta_{e,e'}$ et $q_v \delta_{v,v'}$.

Décomposons X_G comme $X_G = \begin{pmatrix} Q_G & -iR_G^t \\ -iR_G & 0 \end{pmatrix}$, nous pouvons obtenir la représentation engendrée pour l'amplitude de Feynman comme :

$$\begin{aligned} A_G(p_f,q_v) &= \int \prod_e d\alpha_e e^{-\alpha_e m^2} \frac{1}{\det Q_G^{d/2}} e^{-P R_G Q_G^{-1} R_G^t P^t} \\ &= \int \prod_e d\alpha_e e^{-\alpha_e m^2} d^d k_e \frac{e^{-\mathcal{V}/\mathcal{U}}}{\mathcal{U}^{d/2}}, \end{aligned} \tag{7}$$

où $\mathcal{U}_G(\alpha_e,q_v) = \det Q_G$ et $\mathcal{V}_G(\alpha_e,q_v,p_f)/\mathcal{U}_G(\alpha_e,q_v) = P R_G Q_G^{-1} R_G^t P^t$. Les polynômes $\mathcal{U}_G$ et $\mathcal{V}_G$ sont appelés le premier et le deuxième polynôme de Symanzik généralisés, respectivement.

Nous avons le théorème suivant donnant la forme explicite des polynômes $\mathcal{U}_G$ et $\mathcal{V}_G$:

Théorème 7.2.1

$$\mathcal{U}_G(\alpha_e, q_v) = \sum_{\mathcal{F}} \prod_{e \notin \mathcal{F}} \alpha_e \prod_{\mathcal{C}} q_{\mathcal{C}}, \tag{8}$$

$$\mathcal{V}_G(\alpha_e, q_v, p_f) = \sum_{\mathcal{F}} \prod_{e \notin \mathcal{F}} \alpha_e \sum_{\mathcal{C}} p_{\mathcal{C}}^2 \prod_{\mathcal{C}' \neq \mathcal{C}} q_{\mathcal{C}'}, \tag{9}$$

où la somme sur $\mathcal{F}$ parcourt tout les forêts du graphe, les indices $\mathcal{C}$ et $\mathcal{C}'$ indiquent les parties connexes dans la forêt (les vertex isolés sont inclus), et les variables $p_{\mathcal{C}}$ sont $q_{\mathcal{C}}$ les sommes naturelles sur les variables associées aux composantes connexes.

Soient U_G et V_G les polynômes de Symanzik ordinaires, nous pouvons les obtenir facilement des polynômes de Symanzik généralisés $\mathcal{U}_G$ comme :

Théorème 7.2.2 *Pour chaque graphe G connexe et chaque vertex v, nous avons :*

$$U_G(\alpha_e) = \frac{\partial}{\partial q_v} \mathcal{U}_G(\alpha_e, q_v) \ \ |_{q_{v'}=0 \ \forall v'}. \tag{10}$$

Nous avons pour $p_G = 0$,

$$V_G(\alpha_e, p_f) = -\frac{1}{2} \sum_{v \neq v'} p_v \cdot p_{v'} \ \ \frac{\partial^2}{\partial q_v \partial q_{v'}} \mathcal{U}_G(\alpha_e, q_v) \ \ |_{q_{v''}=0 \ \forall v''}. \tag{11}$$

Preuve : c'est une conséquence du théorème E.3.2.

7.3 La théorie des graphes et le polynôme de Tutte

Un graphe G est défini comme un ensemble de vertex V et d'arêtes E avec les relations d'incidence entre eux [106].

Une matrice d'incidence ϵ_{ve} est une matrice rectangulaire de E fois V avec les élements définies comme :

- ϵ_{ve} égal à +1, si e commence par v,
- ϵ_{ve} égal à -1, si e termine au v,
- ϵ_{ve} égal à 0, pour les autres cas.

Un objet mathématique important pour décrire les graphes est le polynôme de Tutte, qui peut être défini soit par une relation de récurrence linéaire soit par une somme sur les sous-ensembles.

Définition- suppression et contraction

Soit $G = (V, E)$ un graphe, et e une ligne régulière, c'est-a-dire, que e n'est ni une ligne qui forme un tadpole ni une ligne de pont, pour chaque pont on associe une variable x et chaque tadpole une variable y, nous avons alors :

$$T_G(x, y) = T_{G/e}(x, y) + T_{G-e}(x, y). \tag{1}$$

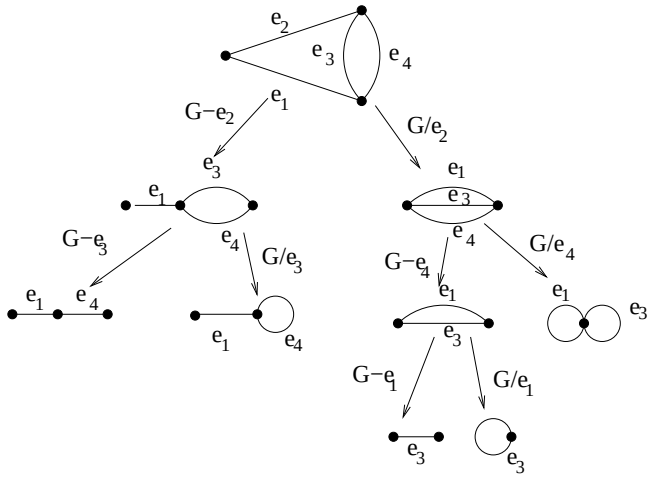

FIGURE 7.1: Les opérations de contraction-suppression pour un graphe

pour un graphe terminal avec m ponts et n tadpole, le polynôme correspondant est défini par :

$$T_G(x, y) = x^m y^n. \tag{2}$$

Définition-somme sur sous-ensembles

Soit $G = (V, E)$ est un graphe, le polynôme de Tutte correspondant $T_G(x, y)$ est défini par :

$$T_G(x, y) = \sum_{A \subset E} (x-1)^{r(E)-r(A)} (y-1)^{n(A)}, \tag{3}$$

où $r(A) = |V| - k(A)$ est le rang du sous-graphe A et $n(A) = |A| + k(A) - |V|$ est le nombre cyclomatique du sous-graphe.

Nous pouvons prouver que ces deux définitions sont équivalentes. Les lecteurs intéressés peuvent consulter l'appendice pour les détails.

Les polynômes de Tutte qui apparaît dans la TQC sont les polynômes de Tutte multi-variables [108], où on associe à chaque vertex une variable q et chaque ligne e une variable β_e.

Comme les polynômes de Tutte ordinaires, nous pouvons définir les polynôme de Tutte de multivariables $Z_G(q, \{\beta\})$ de deux façons, une façon par contraction-suppression et une façon globale par somme de sous-ensembles :

Définition- suppression et contraction

Pour chaque graghe G et chaque ligne $e \in G$, le polynôme de Tutte multivariable est défini comme :

$$Z_G(q, \{\beta\}) = \beta_e Z_{G/e}(q, \{\beta - \{\beta_e\}\}) + Z_{G-e}(q, \{\beta - \{\beta_e\}\}). \tag{4}$$

Le graphe terminal est un ensemble de points isolés, donc la formule pour chaque graphe terminal est : $Z_G(q, \beta) = q^v$.

Définition- la somme sur sous-ensembles

$$Z_G(q, \beta) = \sum_{A \subset E} q^{k(A)} \prod_{e \in A} \beta_e, \tag{5}$$

où $k(A)$ est le nombre de parties connexes dans le sous-graphe (V, A).

Nous pouvons prouver que les polynômes de Symanzik généralisés satisfont les relations de contraction-suppression pour les lignes de graphe, c'est-à-dire, les polynômes de Symanzik généralisés sont des polynômes de Tutte multivariables. Nous avons le théorème :

Théorème 7.3.1 *Pour chaque ligne $e \in G$ qui n'est pas une ligne de tadpole, nous avons :*

$$\mathcal{U}_G(\alpha_e, q_v) = \alpha_e \, \mathcal{U}_{G-e}(\alpha_e, q_v) + \mathcal{U}_{G/e}(\alpha_e, q_v), \tag{6}$$

et

$$\mathcal{V}_G(\alpha_e, q_v, p_f) = \alpha_e \mathcal{V}_{G-e}(\alpha_e, q_v, p_f) + \mathcal{V}_{G/e}(\alpha_e, q_v, p_f), \tag{7}$$

et le graphe terminal qui contient seulement les vertexs isolés et les lignes de tadpole, nous avons :

$$\mathcal{U}_G(\alpha_e, q_v) = \prod_e \alpha_e \prod_v q_v, \tag{8}$$

$$\mathcal{V}_G(\alpha_e, q_v, p_f) = \prod_e \alpha_e \sum_v p_v^2 \prod_{v' \neq v} q_v. \tag{9}$$

L'idée essentielle pour le prouver est que nous écrirons les déterminants comme l'intégrale sur les variables grassmaniennes. Pour plus de détails, voir l'appendice.

On peut aussi utiliser cette méthode pour étudier les relations entre les polynômes de Symanzik pour la théorie quantique des champs définis dans l'espace de Moyal noncommutatif, où les graphes de Feynman sont réelment les graphes à rubans, et les polynômes de graphe à ruban.

Dans [44] nous considerons la théorie de ϕ^4 scalaires en plan Moyal, et nous trouvons que les polynômes de Symanzik sont exactements les polynômes Bollobàs-Riordan. Et dans le papier [71], V.Rivasseau, T.Krajewski et F.Vigne-Tourneret ont considéré les polynômes de Symanzik pour le modèle de Grosse-Wulkenhaar. Ils trouvent que ces polynômes de Symanzik sont un nouveau genre de polynômes où les opérations graphiques ne sont seulement les suppressions et contractions, mais aussi les super-suppressions et anti-contractions.

A

Paper 1

Constructive Renormalization for Φ_2^4 Theory with Loop Vertex Expansion

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Abstract In this paper we construct the 2 dimensional Euclidean ϕ^4 quantum field theory using the method of loop vertex expansion. We reproduce the results of standard constructive theory, for example the Borel summability of the Schwinger functions in the coupling constant. Our method should be also suitable for the future construction of Grosse-Wulkenhaar models on non-commutative space-time.

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A.1 Introduction

In QFT theory the most important quantities are the Green functions, which are the vacuum expectation values of products of physical fields, and contain all the physical information. To compute these Green functions, perturbation theory expands the partition function in a power series of the coupling constant.

However when actually computing each term of the series we typically get infinities. Renormalization is a technique to get rid of these infinities. It allows to compute the mass, charge and magnetic moment of elementary particles or form factors in a way which agreed very well with experiments. The modern interpretation of renormalization is the renormalization group of Wilson. It starts with a bare action which is valid at a very high energy scale and after many renormalization-group steps leads to an effective theory accessible to today's experiments. In good cases the theory is *renormalizable*, which means that the effective action has the same form as the original bare action, but with modified constants.

However there is a second type of divergences in perturbative quantum field theory which is often overlooked. The power series itself usually has zero radius of convergence. It is often stated in textbooks that perturbation theory is therefore expected only to be asymptotic to the true functions. However since any formal power series is asymptotic to infinitely many smooth functions, perturbative field theory alone doesn't give any well defined recipe to compute to arbitrary accuracy the physical quantities. In a deep sense it is not a theory at all. Therefore we need to extend perturbative renormalization into constructive renormalization [7].

Constructive renormalization considers as equally important the finiteness of the renormalized arbitrary coefficients of the perturbation series and the *summability* of the series. For the Fermionic case, the summation of the series is easier thanks to the Pauli's principle [50, 52], which translates into compensations between graphs within a given order of perturbation theory. But the Bosonic case is much harder, as Bosons can condense. We need to combine amplitudes from infinitely many different orders, as the compensations occur only between these different orders. Also the best we can hope for is only Borel summability [23]. The standard techniques for proving such Borel summability are cluster and Mayer expansions [14, 15]. These methods require to divide the space into cubes. This decomposition seems odd since it breaks the rotation invariance of the theory. What's more, the cluster expansion seems unsuited for non-local theories, for example, noncommutative QFTs like the Grosse-Wulkenhaar model.

Recently a new constructive Bosonic method has been invented, namely the loop vertex expansion (LVE) [16, 17, 18, 57]. In this method the role of vertices and propagators is exchanged and one gets a natural explicit summation of the amplitudes of different orders. The method gives direct access to thermodynamic quantities such as connected functions, and requires only a single use of a forest formula [12, 13] to compute connected quantities. It is a definite improvement over cluster and Mayer expansions, which require to use *twice* such a formula.

In this paper we include for the first time renormalization into the loop vertex expansion to construct the Euclidean ϕ^4 theory in two dimensions. This is a first step on the road to a full construction of the noncommutative $\phi_2^{\star 4}$ [26] and ultimately the $\phi_4^{\star 4}$ Grosse-Wulkenhaar model [28, 40, 29, 39].

For details on the standard construction of commutative ϕ_2^4 theory let's refer to [59, 7]. Recall that it was proved in [14] that the Schwinger functions of this model are the Borel sum of their perturbative expansion.

This paper is organized as follows. In section 2 we recall the notion of intermediate fields and the BKAR formula [12, 13]. We rewrite the ϕ_2^4 theory in the language of the loop vertex expansion distinguishing more clearly the direct and the dual version of the LVE

representation. We perform already a fraction of the renormalization, canceling the so-called leaf-tadpoles. In section 3 we recall why, according to [16, 17], the LVE can compute the thermodynamic limit for the model with ultraviolet cutoff without any renormalization ; however this can be done only for a coupling constant smaller and smaller as the ultraviolet cutoff tends to infinity, and is therefore not enough. In section 4 we explain how to superimpose a standard cluster expansion on the LVE to decide exactly the volume occupied by an LVE term. The main section is Section 5, which implements a new expansion called the *cleaning expansion*. It uses the canonical cyclic ordering of the dual LVE representation to turn around any loop vertex term and "clean" it, canceling in this process any inner tadpoles met with their corresponding counter terms. This cleaning is continued until sufficiently many convergent factors have been gathered to pay for the *Nelson's bound* [7]. This must be done in each unit square actually occupied by the loop vertex term, and that's why we need the cluster expansion of Section 4. Note however than in contrast with the standard constructions of ϕ^4_2 we *never* use at any time any Mayer expansion, since the thermodynamic quantities are computed right at the start by the LVE. In the last sections we check more explicitly the combinatorics of tadpole versus countertem compensation and of Borel summability of the ordinary perturbation theory.

We think that the methods of this paper could also apply to the construction of the ϕ^4 theory on a $2d$ curved background. The method is particularly well suited to the case of the ϕ^4_2 theory on a 2 dimensional compact Riemannian manifold, since in that finite volume case the cluster expansion of section 4 is unnecessary.

A.2 The Model and its Loop Vertex Expansion

A.2.1 The ϕ^4_2 Model

The free Bosonic theory has variance $C^{-1} = -\nabla^2 + m^2$. In 2 dimensions the corresponding covariance or propagator is formally given by the kernel

$$C(x,y) = \frac{1}{4\pi}\int_0^\infty \frac{d\alpha}{\alpha} e^{-\alpha m^2 - \frac{(x-y)^2}{4\alpha}}. \tag{1}$$

It diverges at $x = y$ so we need ultraviolet regularization, hence we define the covariance with ultraviolet cutoff Λ as

$$C_\Lambda(x,y) = \frac{1}{4\pi}\int_{\Lambda^{-2}}^\infty \frac{d\alpha}{\alpha} e^{-\alpha m^2 - \frac{(x-y)^2}{4\alpha}}. \tag{2}$$

We define the tadpole term $T_\Lambda = 2C_\Lambda(x,x)$ as the value of the propagator at coinciding points. It diverges proportionally to $\log\Lambda$ as $\Lambda \to \infty$.

It is well known that only the tadpole graphs are divergent in ϕ^4_2 theory and the renormalization reduces to the Wick ordering. The model with ultraviolet cutoff Λ and infrared volume cutoff $\mathcal{V}$ is therefore defined as :

$$Z(\lambda,\Lambda,\mathcal{V}) = \int d\mu_{C^\Lambda}(\phi) e^{-\frac{\lambda}{2}\int_{\mathcal{V}} d^2x :\phi^4(x):} \tag{3}$$

where $d\mu_C$ is the normalized Gaussian measure with covariance C. The Wick ordering in $:\phi^4(x):\equiv \phi^4 - 6T_\Lambda\phi^2 + 3T_\Lambda^2$ is taken with respect to C^Λ [59].

Writing the Wick product explicitly the partition function becomes :

$$\begin{aligned} Z &= \int d\mu_{C^\Lambda} e^{-\frac{\lambda}{2}\int_\mathcal{V} d^2x[\phi^4 - 6T_\Lambda\phi^2 + 3T_\Lambda^2]} \\ &= \int d\mu_{C^\Lambda} e^{-\frac{\lambda}{2}\int_\mathcal{V} d^2x[(\phi^2 - 3T_\Lambda)^2 - 6T_\Lambda^2]}. \end{aligned} \tag{4}$$

A.2.2 The Intermediate Field Representation

Introducing the intermediate field σ and integrating the terms that are quadratic in $\phi(x)$ we get

$$Z(\lambda,\Lambda,\mathcal{V}) = \int d\nu(\sigma) e^{3\lambda|\mathcal{V}|T_\Lambda^2 + \mathrm{Tr}_\mathcal{V}\left(3\mathrm{i}\sqrt{\lambda}\mathrm{T}_\Lambda\sigma - \frac{1}{2}\log(1+2\mathrm{i}\sqrt{\lambda}\mathrm{C}^{1/2}\sigma\mathrm{C}^{1/2})\right)}, \tag{5}$$

where $\mathrm{Tr}_\mathcal{V}$ means integration over 2 dimensional volume $\mathcal{V}$ for each σ field argument and $d\nu(\sigma)$ is the ultralocal measure on σ with covariance $\delta(x-y)$.

Defining a new interaction vertex

$$V(\lambda,\Lambda,\mathcal{V},\sigma) = 3\lambda|\mathcal{V}|T_\Lambda^2 + \mathrm{Tr}_\mathcal{V}\left(-\frac{1}{2}\log(1+2\mathrm{i}\sqrt{\lambda}\mathrm{C}^{1/2}\sigma\mathrm{C}^{1/2}) + 3\mathrm{i}\sqrt{\lambda}\mathrm{T}_\Lambda\sigma\right), \tag{6}$$

the partition function can be written as :

$$Z(\lambda,\Lambda,\mathcal{V}) = \int d\nu(\sigma) e^{V(\lambda,\Lambda,\mathcal{V},\sigma)}. \tag{7}$$

There are three basic propagators in the loop vertex expansion, the propagators C, the resolvents R, whose definition will be given in section 2.4 and the ultralocal propagators for the σ fields. They are shown in Figure A.1.

FIGURE A.1: The basic propagators in LVE. A stands for a pure propagator, B for a a resolvent, C for the ultralocal propagator for the σ field. D is the subtracted resolvent introduced further below.

Remark that the standard renormalization of ϕ_2^4 involves canceling all tadpoles through Wick-ordering. But if we were to apply directly the LVE formalism at this stage, as in [17], tadpoles would appear in two different ways [57]. Leaves with a single propagator of type A are explicitly visible tadpoles, called leaf-tadpoles, see Figure A.2 ; but there is a second kind of tadpoles hidden in the LVE. Indeed *consecutive* σ fields hidden in any resolvent part of any given loop vertex can still contract together upon Gaussian integration of the σ field. The corresponding tadpoles are called *inner tadpoles*. They are not explicitly

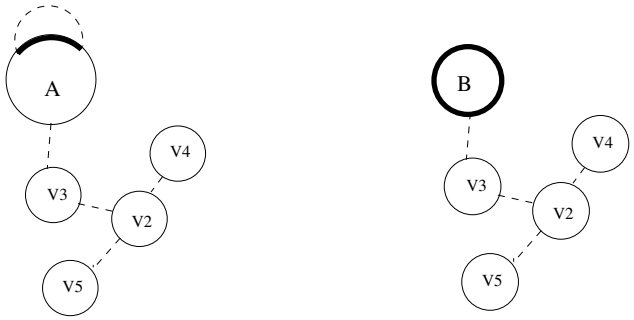

FIGURE A.2: Inner tadpole A and leaf tadpole B. The bold lines mean the pure propagators while the ordinary lines mean the resolvents.

visible after performing the LVE and are responsible for the non-trivial aspects of the ϕ_2^4 renormalization in the LVE formalism. An inner tadpole and a leaf tadpole are shown in Figure A.2.

A very simple procedure can eliminate completely all leaf-tadpoles. Expanding the log function as :

$$\log(1+2\sqrt{\lambda}C^{1/2}\sigma C^{1/2}) = 2i\sqrt{\lambda}C^{1/2}\sigma C^{1/2} + \log_2(1+2i\sqrt{\lambda}C^{1/2}\sigma C^{1/2}),$$

where

$$\log_p(1+x) = \log(1+x) - \sum_{q=1}^{p-1}(-1)^{q-1}\frac{x^q}{q}, \tag{9}$$

we simplify the interaction vertex as :

$$\begin{aligned} V(\lambda,\Lambda,\mathcal{V},\sigma) &= CC(\lambda,\Lambda,\mathcal{V}) + CT(\lambda,\Lambda,\mathcal{V},\sigma) + W(\lambda,\Lambda,\mathcal{V},\sigma) \\ CC(\lambda,\Lambda,\mathcal{V}) &= 3\lambda|\mathcal{V}|T_\Lambda^2 \\ CL(\lambda,\Lambda,\mathcal{V},\sigma) &= 2i\sqrt{\lambda}T_\Lambda \mathrm{Tr}_{\mathcal{V}}\sigma \\ W(\lambda,\Lambda,\mathcal{V},\sigma) &= -\frac{1}{2}\mathrm{Tr}_{\mathcal{V}}\log_2(1+2\mathrm{i}\sqrt{\lambda}\mathrm{C}^{1/2}\sigma\mathrm{C}^{1/2}). \end{aligned} \tag{10}$$

CC is called the constant counter term, pictured as a single big black dot in Figure A.3 part C ; CL is called the linear counter term, pictured as one of the two smaller black dots in part B of Figure A.3, and W is the (non-trivial) loop vertex.

This regrouping will forbid the formation of any leaf-tadpole in the LVE below. Remark that it also reduces the coefficient of the linear counter term which from an initial value of $3i\sqrt{\lambda}T_\Lambda\sigma$ becomes $2i\sqrt{\lambda}T_\Lambda\sigma$ [1]. This remaining linear counter term is there to compensate all the *inner* tadpoles, in particular the ones that will appear in the cleaning expansions in section (5.2). This compensation is not obvious to explicitly perform and this is the main source of difficulty of this paper.

Let us now expand the exponential as $\sum_n \frac{V^n}{n!}$. To compute the connected function while avoiding an additional factor $n!$, we give a kind of *fictitious* index v, $v = 1, \cdots, n$

1. Remark therefore that a third of the tadpoles are leaf-tadpoles and the majority, namely the remaining two-thirds, are inner-tadpoles.

to all the σ fields of the vertex V. This means we consider n different copies σ_v of σ with a degenerate Gaussian measure

$$d\nu(\{\sigma_v\}) = d\nu(\sigma_v) e^{-\mathrm{Tr}\sigma_{\mathrm{v}}^2/2} \prod_{v' \neq v}^{n} \delta(\sigma_v - \sigma_{v'}) d\sigma_{v'} \ , \tag{11}$$

whose covariance or ultralocal propagator

$$< \sigma_v, \sigma_{v'} >= \delta(x - y) \tag{12}$$

does not depend on the fictitious indices v and v'.

We obtain

$$\begin{aligned} Z(\lambda, \Lambda, \mathcal{V}) &= \int d\nu(\sigma) \sum_{n=0}^{\infty} \frac{V^n}{n!} = \int d\nu(\sigma) \sum_{n=0}^{\infty} \prod_{v=1}^{n} \frac{V_v(\sigma_v)}{n!} \\ &= \sum_{0}^{\infty} \frac{1}{n!} \int d\nu(\{\sigma_v\}) \prod_{v=1}^{n} \Big[3\lambda |\mathcal{V}| T_\Lambda^2 \\ &+ \mathrm{Tr}_{\mathcal{V}} \Big(-\frac{1}{2} \log_2(1 + 2\mathrm{i}\sqrt{\lambda} \mathrm{C}^{1/2} \sigma_{\mathrm{v}} \mathrm{C}^{1/2}) + 2\mathrm{i}\sqrt{\lambda} \mathrm{T}_\Lambda \sigma_{\mathrm{v}} \Big) \Big]. \end{aligned} \tag{13}$$

The real interesting quantities which do have limits as $\mathcal{V} \to \mathcal{R}^2$ are the connected functions, the simplest of all being the pressure $\frac{1}{|\mathcal{V}|} \ln Z(\lambda, \Lambda, \mathcal{V})$. Feynman graphs formally allow the computation of such quantities, but through a divergent expansion, as there are too many graphs ($(4n)!!$ at order n). As noticed in the introduction, the loop vertex expansion, hereafter called LVE, performs the same task of computing connected function but without introducing too many terms. Here it gives

Théorème A.2.1 (Loop Vertex Expansion)

$$\frac{1}{\mathcal{V}} \log Z(\lambda, \Lambda, \mathcal{V}) = \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{\mathcal{T} \text{ with } n \text{ vertices}} G_{\mathcal{T}} \tag{14}$$

$$G_{\mathcal{T}} = \mathrm{Tr}_{\mathcal{V}} \Big\{ \prod_{\ell \in \mathcal{T}} \int \mathrm{d}^2\mathrm{x}_\ell \mathrm{d}^2\mathrm{y}_\ell \big[\int_0^1 \mathrm{dw}_\ell \big] \Big\} \int \mathrm{d}\nu_{\mathcal{T}}(\{\sigma^{\mathrm{v}}\}, \{\mathrm{w}\})$$

$$\Big\{ \prod_{\ell \in \mathcal{T}} [\frac{1}{2} \delta(x_\ell - y_\ell) \frac{\delta}{\delta \sigma^{v(\ell)}(x_\ell)} \frac{\delta}{\delta \sigma^{v'(\ell)}(y_\ell)}] \Big\} \prod_{v=1}^{n} V_v,$$

where

- *each line ℓ of the tree joins two different loop vertices $V_{v(\ell)}$ and $V_{v'(\ell)}$ at points x_ℓ and y_ℓ. These points are in fact identified through a $\delta(x_\ell - y_\ell)$ ultralocal σ propagator.*
- *the sum is over trees joining n loop vertices. These trees have therefore $n-1$ lines, corresponding to σ propagators.*

– *the normalized Gaussian measure* $d\nu_{\mathcal{T}}(\{\sigma_v\},\{w\})$ *over the vector field* σ_v *has a covariance*

$$< \sigma^v(x), \sigma^{v'}(y) >= \delta(x-y) w^{\mathcal{T}}(v,v',\{w\}), \tag{15}$$

which depends on the "fictitious" indices. Here $w^{\mathcal{T}}(v,v',\{w\})$ *equals 1 if* $v = v'$*, and equals the infimum of the* w_ℓ *for* ℓ *running over the unique path from* v *to* v' *in* $\mathcal{T}$ *if* $v \neq v'$*.*

The proof is a completely standard consequence of the BKAR formula and is given in detail in [16, 17]. The BKAR formula rewrites Z as a sum over forests. The key observation is the factorization of the forests contributions according to their connected parts, which are the trees of the forest. This allows to compute $\log Z$ as the sum of the *same* contributions, but indexed by trees. Here like in the computation of connected functions through connected Feynman graphs, Joyal theory of species [43] is the mathematical explanation behind the scene. The $d\nu_{\mathcal{T}}$ measure is well-defined since the matrix $w^{\mathcal{T}}$ is positive.

The n-point Euclidean Green's functions, where $n = 2p$ is an even number, can also be written in the LVE representation [17]. They could be derived eg by introducing source fields. The partition function with source fields reads :

$$Z(\Lambda, j(x)) = \int d\mu_{C^\Lambda} e^{-\frac{\lambda}{2}\int_{\mathcal{V}} d^2x[(\phi^2 - 3T_\Lambda)^2 - 6T_\Lambda^2] + j(x)\phi(x)} \tag{16}$$

The connected n-point functions are given by :

$$S(x_1,\cdots,x_n) = \frac{1}{(2p)!}\frac{\partial^{2p}}{\partial j(x_1)\cdots\partial j(x_{2p})}\frac{1}{\mathcal{V}}\log Z(\Lambda, j(x))|_{j=0}, \tag{17}$$

and we have a similar following theorem for the connected Schwinger's function :

Théorème A.2.2

$$\begin{aligned} S^c = &\sum_\pi \sum_{n=0}^{\infty}\frac{1}{n!}\sum_{\mathcal{T}}\Big\{\prod_{\ell\in\mathcal{T}}\int d^2x_\ell d^2y_\ell\big[\int_0^1 dw_\ell\big]\Big\} \\ &\int d\nu_{\mathcal{T}}(\{\sigma^v\},\{w\})\Big\{\prod_{\ell\in\mathcal{T}}\big[\frac{1}{2}\delta(x_\ell - y_\ell)\frac{\delta}{\delta\sigma^{v(\ell)}(x_\ell)}\frac{\delta}{\delta\sigma^{v'(\ell)}(y_\ell)}\big]\Big\} \\ &\prod_{v=1}^{n} V_v \prod_{r=1}^{p} C_R(\sigma_r, x_{\pi(r,1)}, x_{\pi(r,2)}), \end{aligned} \tag{18}$$

where the sum over π *runs over the parings of the* $2p$ *external variables into pairs* $(x_{\pi(r,1)}, x_{\pi(r,2)})$*,* $r = 1,\cdots,p$*, the tree* $\mathcal{T}$ *now joins the* n *vertices and the* p *resolvents* C_R *(whose explicit form is given in formula (26) below).*

For simplicity we now treat only the vacuum connected function, as the more general case could be obtained easily by introducing external resolvents.

A.2.3 The first terms, $n = 1$ and $n = 2$

Remark that the constant part CC in (10) cannot bear any derivation. Hence it can appear only in the empty tree $\mathcal{T} = \emptyset$. The linear counter term can bear only one derivative hence it can appear only as a leaf of $\mathcal{T}$, see Figure A.3. All other terms have at least one regular resolvent loop.

We want to develop therefore further each $G_{\mathcal{T}}$ according to how many leaves of $\mathcal{T}$ are counter terms and how many are (nontrivial) loop vertices W. Let us first treat the two first terms of the expansion, corresponding to $\mathcal{T}$ the empty tree and the tree with a single line, since they are special.

We now explicit a bit better the first terms Z_1 and Z_2 of the expansion (17) to show the compensation of the special CC counter term.

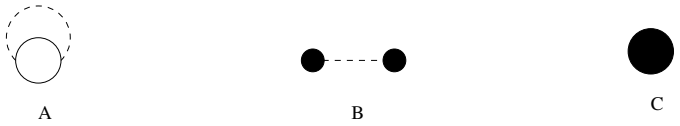

FIGURE A.3: The lowest order graphs. Graph C represents the counterterm $3\lambda T^2_\Lambda$.

We expand first the $n = 1$ Z_1 term in (17) as

$$\begin{aligned} Z_1 &= \int d\nu(\sigma)\Big(3\lambda|\mathcal{V}|T^2_\Lambda + \mathrm{Tr}_{\mathcal{V}}\big(-\frac{1}{2}\log_2(1 + 2\mathrm{i}\sqrt{\lambda}\mathrm{C}^{1/2}\sigma\mathrm{C}^{1/2})\big)\Big) \\ &= 2\lambda|\mathcal{V}|T^2_\Lambda + \int d\nu(\sigma)\mathrm{Tr}_{\mathcal{V}}\big(-\frac{1}{2}\log_4(1 + 2\mathrm{i}\sqrt{\lambda}\mathrm{C}^{1/2}\sigma\mathrm{C}^{1/2})\big). \end{aligned} \tag{19}$$

The $n = 2$ term, Z_2, corresponds to a single tree with two leaves that can be either counter terms CT or loop vertices W. In hopefully transparent notations :

$$Z_2 = Z_{2;CT,CT} + 2Z_{2;CT,W} + Z_{2;W,W}. \tag{20}$$

The contribution with two counterterms is

$$Z_{2,CT,CT} = \frac{1}{2}\int d\nu(\sigma) Tr_{\mathcal{V}}(2i\sqrt{\lambda}T_\Lambda\sigma)^2 = -2\lambda|\mathcal{V}|T^2_\Lambda, \tag{21}$$

hence cancels out exactly as expected with the $2\lambda|\mathcal{V}|T^2_\Lambda$ term in (19).

To write down in a compact notation the contributions $Z_{2;CT,W}$ and $Z_{2;W,W}$ which contain loop vertices it is convenient to introduce further notations. We define the resolvent R and the subtracted resolvent $\hat{R}$ through

$$R(\sigma) = \frac{1}{1 + 2i\sqrt{\lambda}\sigma C}, \quad \hat{R}(\sigma) = \frac{1}{1 + 2i\sqrt{\lambda}\sigma C} - 1 = -\frac{2i\sqrt{\lambda}\sigma C}{1 + 2i\sqrt{\lambda}\sigma C}.$$

We also define the dressed resolvents and dressed subtracted resolvent

$$C_R = CR, \quad \hat{C}_R = C\hat{R} \tag{22}$$

where C is the ϕ propagator. The dressed resolvent is pictured as a B line in Figure A.1 and the dressed subtracted resolvent is pictured as drawing D of the same figure.

We can also define more symmetric operators

$$\begin{aligned} D'(x) &= 2i\sqrt{\lambda}C^{1/2}(.,x)C^{1/2}(x,.), \\ R'(\sigma) &= \frac{1}{1+2i\sqrt{\lambda}C^{1/2}\sigma C^{1/2}}, \quad \hat{R}'(\sigma) = \frac{1}{1+2i\sqrt{\lambda}C^{1/2}\sigma C^{1/2}} - 1, \end{aligned} \tag{23}$$

and the full dressed resolvents and subtracted resolvents

$$D'_R = 2i\sqrt{\lambda}C^{1/2}R'C^{1/2}, \quad \hat{D}'_R = 2i\sqrt{\lambda}C^{1/2}\hat{R}'C^{1/2} \tag{24}$$

By the cyclic ordering, traces of C_R or $\hat{C}_R$ operators are equal to traces of D'_R or $\hat{D}'_R$ operators.

The result of deriving once with respect to σ a (non trivial) loop vertex W gives

$$\frac{\partial}{\partial\sigma(x)}[-\frac{1}{2}\mathrm{Tr}_{\mathcal{V}}\log_2(1+2\mathrm{i}\sqrt{\lambda}\mathrm{C}^{1/2}\sigma\mathrm{C}^{1/2})] = -\mathrm{i}\sqrt{\lambda}\hat{\mathrm{D}}_{\mathrm{R}}(\sigma,\mathrm{x},\mathrm{x}). \tag{25}$$

The result of deriving twice or more gives

$$\begin{aligned} &\frac{\partial}{\partial\sigma(x_1)}\cdots\frac{\partial}{\partial\sigma(x_p)}[-\frac{1}{2}\mathrm{Tr}_{\mathcal{V}}\log_2(1+2\mathrm{i}\sqrt{\lambda}\mathrm{C}^{1/2}\sigma\mathrm{C}^{1/2})] \\ = \; & -\frac{1}{2}(2i\sqrt{\lambda})^p\sum_{\tau} D_R(x_1,x_{\tau(2)},\sigma)\cdots D_R(x_{\tau(p)},x_1,\sigma) \end{aligned} \tag{26}$$

with $p \geq 2$ and the sum over τ is over the $(p-1)!$ permutations of $[2,...p]$. Hence only the first derivation can lead to a trace with a single $\hat{C}_R$ operator.

Therefore we get

$$\begin{aligned} Z_{2,CT,W} &= \lambda T_\Lambda \int d\nu(\sigma)\int_{\mathcal{V}} dx\hat{D}_R(\sigma)(x,x) \\ Z_{2,W,W} &= -\frac{\lambda}{2}\int_0^1 dw\int d\nu(\sigma_1,\sigma_2,w)\int_{\mathcal{V}} dx[\hat{D}_R(\sigma_1)(x,x)\hat{D}_R(\sigma_2)(x,x)] \end{aligned} \tag{27}$$

where $d\nu$ has covariance $\begin{pmatrix} C & wC \\ wC & C \end{pmatrix}$. Hence finally :

$$\begin{aligned} Z_1+Z_2 &= \int d\nu(\sigma)\mathrm{Tr}_{\mathcal{V}}\big(-\frac{1}{2}\log_4(1+2\mathrm{i}\sqrt{\lambda}\mathrm{C}^{1/2}\sigma\mathrm{C}^{1/2})\big) \\ &+ 2\lambda T_\Lambda\int d\nu(\sigma)\int_{\mathcal{V}} dx\hat{D}_R(\sigma)(x,x) \\ &- \frac{\lambda}{2}\int_0^1 dw\int d\nu(\sigma_1,\sigma_2,w)\int_{\mathcal{V}} dx[\hat{D}_R(\sigma_1)(x,x)\hat{D}_R(\sigma_2)(x,x)] \end{aligned} \tag{28}$$

We can expand further two σ fields hidden in

$$\int d\nu(\sigma)\mathrm{Tr}_{\mathcal{V}}\big(-\frac{1}{2}\log_4(1+2\mathrm{i}\sqrt{\lambda}\mathrm{C}^{1/2}\sigma\mathrm{C}^{1/2})\big) \tag{29}$$

term, by writing

$$\log_4(1+x) = \int_0^1 dwx[\frac{1}{1+wx} - 1 + \frac{x}{2} - \frac{x^2}{3}]. \tag{30}$$

Integrating by parts we get

$$\begin{aligned} & \int d\nu(\sigma)\mathrm{Tr}_{\mathcal{V}}\big(-\frac{1}{2}\log_4(1+2\mathrm{i}\sqrt{\lambda}\mathrm{C}^{1/2}\sigma\mathrm{C}^{1/2})\big) \\ = \;& \lambda\int_0^1 dw \int d\nu(\sigma)\mathrm{Tr}_{\mathcal{V}}[-\mathrm{wD(x)R(w},\sigma(\mathrm{x}))\mathrm{D(x)R(w},\sigma(\mathrm{x})) \\ + \;& \frac{1}{2}D(x)D(x)]. \end{aligned} \tag{31}$$

In this way we have checked how the first counterterm cancel, and written the remainder in a form which is similar to the more ordinary LVE terms at $n \geq 3$, so that they can be tretaed in the same way by the cleaning expansion below.

Direct Resolvent Representation

Let's index the leaves of a tree with a letter f. A decorated tree $\mathcal{T}$ is an ordinary tree plus an index c_f with values 0 or 1 for each leaf f telling whether the leaf is an ordinary loop vertex ($c_f = 1$) or a counter term ($c_f = 0$). If we call F the set of leaves we can rewrite the outcome of the action of the

$$\prod_{\ell\in\mathcal{T}}[\frac{1}{2}\delta(x_\ell - y_\ell)\frac{\delta}{\delta\sigma^{v(\ell)}(x_\ell)}\frac{\delta}{\delta\sigma^{v'(\ell)}(y_\ell)}] \tag{32}$$

operator in (17) as

$$\begin{aligned} & \log Z(\lambda,\Lambda,\mathcal{V}) = a_1 + a_2 + \sum_{\mathcal{T}\text{ decorated tree with } n\geq 3 \text{ vertices}} G_{\mathcal{T}} \\ & G_{\mathcal{T}} = \prod_{\ell\in\mathcal{T}}(-\lambda)\int_{\mathcal{V}} d^2x_\ell \prod_{f\in F, c_f=0}(2T_\Lambda) \\ & \prod_{f\in F, c_f=1}\mathrm{Tr}_{\mathcal{V}}\mathrm{D}(\mathrm{x}_{\ell_\mathrm{f}})\hat{\mathrm{R}}(\sigma)\prod_{\mathrm{v}\notin\mathrm{F}}\mathrm{Tr}_{\mathcal{V}}\tilde{\prod_{\ell\in\mathrm{v}}}\mathrm{D}(\mathrm{x}_{\ell_\mathrm{f}})\mathrm{R}(\sigma), \end{aligned} \tag{33}$$

where a_1 and a_2 are the finite remainder terms of order $n = 1$ and $n = 2$ in the expansion, respectively, the $\vec{\prod}$ means that one has to take the ordered product of the operators along the loop vertex.

Indeed a $\frac{\partial}{\partial\sigma(x)}$ derivation acting on $-\frac{1}{2}\log_2[1+2i\sqrt{\lambda}C^{1/2}\sigma C^{1/2}]$ gives an $-D(x)\cdot\hat{R}(\sigma)$ term but a $\frac{\partial}{\partial\sigma}(x)$ derivation further acting on a $\hat{R}(\sigma)$ gives a $R(\sigma)DR(\sigma)$ term.

A.2.4 Graphic Representations

Direct Representation

We introduce two equivalent graphic representations for the LVE. For the first representation we defined $\mathcal{T}$ as the spanning tree of a given term $G_{\mathcal{T}}$ in the LVE. We remark

that the loop vertex V_v is the sum of the nontrivial loop vertex $W_v = \mathrm{Tr}_{\mathcal{V}}\Big(-\frac{1}{2}\log_2(1 + 2\mathrm{i}\sqrt{\lambda}\mathrm{C}^{1/2}\sigma_{\mathrm{v}}\mathrm{C}^{1/2})\Big)$ the linear counter term $CT_v = 2i\sqrt{\lambda}T_\Lambda\sigma_v$ also called the counter term, and the constant counter term $CC = 3\lambda|\mathcal{V}|T_\Lambda^2$.

The direct representation pictures a term in the LVE as a decorated tree between loop vertices. This tree joins together the derived loop vertices $\prod_{\ell|v(\ell)=v}\frac{\delta}{\delta\sigma^{v(\ell)}(x_\ell)}V_v$ for $v \in W$, $|W| = n$. When the tree is non empty each such loop vertex bears at least one derivation hence is a cyclic product of ordinary ϕ^4 resolvent lines of the B type in Figure A.1 [17]. The lines of the tree are of type C in Figure A.1 hence should not be confused with the resolvent lines forming the loop vertices, that's why we picture them as dashed lines. They are labeled by an index $\ell \in \mathcal{T}$, and correspond to σ propagators that join the different loop vertices.

A *leaf* of the tree is an "extremal" vertex, i.e. a vertex of coordination 1. Leaves will play a particular role in the LVE above because of renormalization and the presence of counterterms, absent in [17]. In developing (17), two different types of leaves occur. The ordinary ones are loop vertices, bearing a single σ propagator and we label them by an index $v_0 \in S$ (as "simple loops"). But there are also counterterms leaves, which we picture as small black disks, labeled by an index $b \in CTL$. These counterterms correspond to the fact that the $2i\sqrt{\lambda}T_\Lambda\sigma_v$ term in (17), which is linear, can bear only one $\frac{\partial}{\partial\sigma}$ derivation. Hence their value is $2i\sqrt{\lambda}T_\Lambda$. The loop vertices which are not leaves have coordination at least 2 and are indexed by a different index, $v_1 \in W - S$, where W is the set of loop vertices (hence excluding the counterterms).

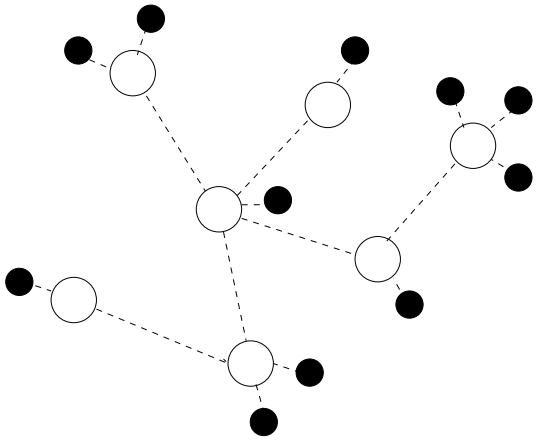

FIGURE A.4: A tree of loop vertices with $|V| = 7$, $|B| = 11$, $|n| = 18$. The dash lines are the propagators for the σ fields while the ordinary lines are the fully dressed resolvents which contain the resolvents and pure propagators.

Hence a term in the LVE direct representation is pictured as a cactus, decorated with an arbitrary number of black counterterms.

For $n \geq 2$ we have relations, such as $|\mathcal{T}| = n - 1$, $|B| + |V| = n$.

Dual Representation

In the dual representation, the σ propagators are replaced by their dual, still pictured as propagators. Since a LVE term is *planar*, this notion of duality is globally well-defined. We define

– $\mathcal{C}$ as the cycle in the dual representation of the LVE corresponding to G (see below).
– $\bar{\mathcal{T}}$ as the tree dual to $\mathcal{T}$ (see below).

A LVE dual term for a decorated tree $\{\mathcal{T}, B\}$ of the direct representation corresponds to a cycle $\mathcal{C}$ made of objects forming a set $\mathcal{O}$. There is a dual tree $\bar{\mathcal{T}}$ pairings half lines plus dual decorations $\bar{B}$. Hence it is a fairly complicated combinatoric object which we shall describe in loose terms first.

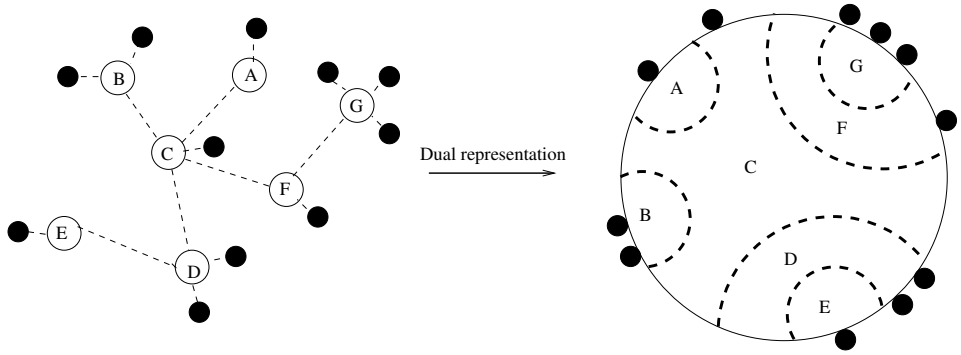

FIGURE A.5: The dual graph of the tree of loop vertices. The regions A, B, E and G are in the right figure are the corresponding leaves of the left one. The ordinary dash lines for the σ propagators on the LHS are drawn as bold dash line for the dual graph.

This dual object first consists of a single huge loop vertex, called the *cycle* $\mathcal{C}$ of the tree. This cycle contains all the ordinary resolvent lines of all the loop vertices of the direct representation, but read *in the cyclic order obtained by turning around the tree*. These full ϕ-resolvent lines should be again carefully distinguished from the dual tree lines $\bar{\ell}$ of $\bar{\mathcal{T}}$, each of which corresponds to a line ℓ of the direct picture. To better distinguish the two pictures, we picture the dual tree lines of the dual representation $\bar{\ell}$ as bold dash lines rather than dotted lines. The important fact reflecting the tree character of $\mathcal{T}$ is that these bold dash lines $\bar{\ell}$ when drawn inside the disk $\mathcal{D}$ bounded by the cycle $\mathcal{C}$ *cannot cross*. Hence they divide this disk $\mathcal{D}$ into different connected regions. Each connected region $\bar{v} \in \bar{V}$ correspond to a single loop vertex $v \in V$ of the direct expansion, namely the one made of the lines forming the boundary of the region. This explains how the two pictures, dual of each other, are equivalent and how anyone can be reconstructed from the other.

Counterterms decorations have also to be added to this basic dual picture. They are pictured as arbitrarily many "black dots" $\bar{b} \in \bar{B}$ decorating the ordinary full lines of the cycle.

Remark that since the lines joining a counterterm to a loop vertex are omitted in the dual representation, we have $|\bar{\mathcal{T}}| = |\mathcal{T}| - |B|$.

The simple loop vertices of S, namely the loop vertices which were leaves in $\mathcal{T}$, can still be identified in this dual representation : they are indeed the minimal connected regions

$\bar{s} \in \bar{S} \subset \bar{V}$ of the disk, namely those bounded by a single line $\bar{\ell}$.

This second representation is very interesting as it gives a canonical (up to an orientation choice) cyclic ordering of all ingredients occurring in a LVE term.

In this dual picture the measure $d\nu$ correspond to the following rule : the weakening factor between a $\sigma_{\bar{v}}$ and a $\sigma_{\bar{v}'}$ is the infimum of the w parameters of the lines $\bar{\ell}$ that have to be *crossed* to join the two regions $\bar{v}$ and $\bar{v}'$.

In the set $\mathcal{O}$ there are four types of different objects which can occur when we follow the cycle $\mathcal{C}$, which have different values and need to be carefully distinguished :

- Black vertices $\bar{b} \in \bar{B}$
- Half-dash lines. They are labeled by an index $h \in H$; obviously H has even cardinal, as $|H| = 2|\bar{\mathcal{T}}|$. The wavy lines $\bar{\ell} = (h_\ell, h_{\ell'})$ form a pairing on this set.
- Simple leaves i.e resolvent lines sandwiched between two consecutive objects in H which are paired. They form a set $\bar{S}$ and are indexed by an index $\bar{v}_0$.
- Non-leaf resolvent lines, ie resolvent lines not in $\bar{S}$. They form a set $\mathcal{L}$, indexed by an index $\bar{v}_1$.

Hence $\mathcal{O} = \bar{B} \cup H \cup \bar{S} \cup \mathcal{L}$.

We have $|\bar{B}| = |B|$, $|\bar{S}| = |S|$, $|\bar{V}| = |V|$ and we can check that a contribution with tree $\bar{\mathcal{T}}$ and decorations $\bar{B}$ of n vertices is of order $O(\lambda^{n-1})$, as we have $n-1 = |\mathcal{T}| = |\bar{B}| + |\bar{\mathcal{T}}|$.

We can also check

$$|\bar{S}| + |\bar{B}| + |\mathcal{L}| = 2(n-1); \quad |S| + |\mathcal{L}| = |\bar{B}| + |H|. \tag{34}$$

A.2.5 Resolvent Representations

It is convenient to introduce further notations to write down in more compact form the contribution, or amplitude of a tree in the LVE. or more explicitly,

$$\mathcal{R}(x_1, x, x_2) = D(x_1, y_1) R(y_1, x, x_2), \quad \hat{\mathcal{R}}(x_1, x, x_2) = D(x_1, y_1) \hat{R}(y_1, x, x_2), \tag{35}$$

where D should be thought of as an operator with kernel on $L^2(\mathcal{R}^2)$, but with a middle point distinguished, Similarly $R(\sigma)$ and $\hat{R}(\sigma)$ are operators on $L^2(\mathcal{R}^2)$, but they depend on σ. Their existence is ensured by the i factor, see eg [16, 17].

Dual Resolvent Representation

The beauty of the LVE representation is that all the various traces of the loop vertices in the direct representation give rise to a *single* trace in the dual representation. This is the fundamental observation which made the representation suited for constructive matrix and non commutative field theory [16].

Consider a dual tree $\{\mathcal{C}, \mathcal{O}, \bar{\mathcal{T}}\}$ made of the set of objects $\mathcal{O} = \bar{B} \cup H \cup \bar{S} \cup \mathcal{L}$ cyclically ordered according to $\mathcal{C}$, together with the pairing rules for the wavy lines in H encoded in $\bar{\mathcal{T}}$.

We need a label u to describe the various object met when turning around the cycle $\mathcal{C}$. To every counterterm in $\bar{b} \in \bar{B}$ is associated a position x_b and to every half wavy line $h \in H$ a position $x_{\bar{\ell}(h)}$, the same for the two ends of any $\bar{\ell} \in \bar{\mathcal{T}}$.

Then to each object $u \in \mathcal{O}$ is associated an operator P_u, with value

$$\begin{aligned} P_u &= (-iT_\Lambda) D(x_{\bar{b}}) \text{ if } u = \bar{b} \in \bar{B}, && (36)\\ P_u &= D(x_{\bar{\ell}(h)}) \text{ if } u = h \in H, && (37)\\ P_u &= \hat{R}(\sigma_{\bar{v}_0(\bar{\ell})}) \text{ if } u = \bar{v}_0 \in \bar{S}, && (38)\\ P_u &= R(\sigma_{\bar{v}_1(\bar{\ell})}) \text{ if } u = \bar{v}_1 \in \mathcal{L}. && (39) \end{aligned}$$

Then in the dual resolvent representation we have :

$$\begin{aligned} \log Z(\lambda, \Lambda, \mathcal{V}) &= R_2 + \sum_{\bar{\mathcal{T}} \text{ decorated cycle with } l \geq 3 \text{ lines}} G_{\bar{\mathcal{T}}} \\ G_{\bar{\mathcal{T}}} &= (-\lambda)^{n-1} \int_{\mathcal{V}} [\prod_{\bar{\ell}} d^2 x_{\bar{\ell}} \prod_{\bar{b} \in \bar{B}} d^2 x_{\bar{b}}] \ \mathrm{Tr}_{\mathcal{V}} \Big\{ \tilde{\prod}_{u \in \mathcal{O}} \mathrm{P_u} \Big\} \end{aligned} \qquad (40)$$

where R_2 is the finite remainder terms up to 2nd order expansion and $\vec{\prod}$ means that one has to take the ordered product of the operators along the cycle $\mathcal{C}$.

A.3 Thermodynamic Limit at fixed UV cutoff

To perform the thermodynamic limit one has to divide by the volume to quotient out the translation invariance. This is an almost trivial but subtle "global gauge fixing" step which consists in fixing a preferred root line ℓ_0 (hence a former ϕ^4 vertex) in the tree $\mathcal{T}$ at the origin. The pressure has then the expansion :

$$\begin{aligned} &\frac{1}{|\mathcal{V}|} \log Z(\lambda, \Lambda, \mathcal{V}) = \int d\nu(\sigma) \mathrm{Tr}_{\mathcal{V}} \big(-\frac{1}{2} \log_4 (1 + 2\mathrm{i}\sqrt{\lambda} \mathrm{C}^{1/2} \sigma \mathrm{C}^{1/2}) \big) \\ + &\int d\nu(\sigma_v) 2i\sqrt{\lambda} T_\Lambda \hat{R} + \int d\nu(\sigma_v \sigma_{v'}) \hat{R} \hat{R} \\ + &\sum_{n=3}^{\infty} \frac{1}{n!} \sum_{\mathcal{T} \text{ with } n \text{ vertices}} G_{\mathcal{T}, \ell_0}, \\ &G_{\mathcal{T}, \ell_0} = \Big\{ \prod_{\ell \in \mathcal{T}} \int d^2 x_\ell d^2 y_\ell \big[\int_0^1 dw_\ell \big] \Big\} \int d\nu_{\mathcal{T}}(\{\sigma^v\}, \{w\}) \\ &\Big\{ \prod_{\ell \in \mathcal{T}} [\frac{1}{2} \delta(x_\ell - y_\ell) \frac{\delta}{\delta \sigma^{v(\ell)}(x_\ell)} \frac{\delta}{\delta \sigma^{v'(\ell)}(y_\ell)}] \Big\} \prod_{v=1}^{n} V_v|_{x_{\ell_0} = 0} \end{aligned} \qquad (1)$$

Théorème A.3.1 *This expansion (1) for the pressure P_Λ is absolutely convergent and defines an analytic function in the half-disk $\mathcal{D}_\Lambda - \{\lambda \mid \Re(\lambda^{-1}) \geq K \log \Lambda$, where K is a large constant.*

Proof

This is just an application of the techniques of [16, 17] which we recall briefly.

– The number of trees is n^{n-2},
– each resolvent R is bounded by 1 in the disk,
– each resolvent $\hat{R}$ is bounded by 2 in the disk,
– each decorating counterterm is bounded by $\log \Lambda$.

So the connected function is bounded by

$$P_\Lambda \leq \sum_n \frac{n^{n-2}}{n!} 2^{n-1} \lambda^n \ln \Lambda^n \leq \sum (\lambda K)^n \log \Lambda^n. \tag{2}$$

which is a convergent geometric series as long as $\Re(\lambda^{-1}) \geq K \log \Lambda$

However the radius of convergence goes to zero as $\Lambda \to \infty$. This divergence comes from the linear counterterms that are unbounded. To improve the bounds we *must* combine several different terms in the LVE in order to compensate *many* of these counterterms with the inner tadpoles that can be generated when contracting the σ fields hidden in the resolvents. However we cannot compensate *all* these tadpoles because that would generate the full ordinary renormalized perturbative series hence lead to a divergent series.

We have therefore to settle for a compromise in the compensation of inner tadpoles. This compromise is provided by the cleaning expansion, which is a systematic expansion along the cycle $\mathcal{C}$ but with a *stoppping rule*. This stopping rule stipulates that if enough non-tadpole ϕ-propagators have been generated at high enough energy we should stop and retransform the remaining counterterms in the rest of the loop $\mathcal{C}$ into an oscillating exponential of the form $e^{2i\sqrt{\lambda}\sigma T_\Lambda}$ type, which can be then bounded by 1. But this backward step however comes with some price : it generates a factor $e^{+\lambda T_\Lambda^2 |\mathcal{V}|}$. However we can taylor our stopping rule to precisely ensure that enough good factors have been generated to pay for this kind of bad term. This is our implementation in this context of the so-called *Nelson's bound* [7].

Unfortunately Nelson's bound is exponential in the volume. This is why we need now to perform a cluster expansion which selects the volume truly occupied by an LVE term. This auxiliary cluster expansion is not necessary for Grosse-Wulkenhaar models or in the case of the ϕ_2^4 theory on a compact Riemannian surface, for which the LVE is therefore more natural.

A.4 The cluster expansion

In this section we consider the connected function in a finite large volume $\mathcal{V}$ made of a union of unit volume squares, and we test which region of $\mathcal{V}$ is really occupied by the σ fields in a given LVE term. For this purpose we need a standard cluster expansion on the regular propagator C ; the ultralocal propagator for σ cannot couple distinct squares.

The formalism is uniform in the limit $\mathcal{V} \to \mathbb{R}^2$. Consider a 2 dimensional finite lattice of unit squares $\mathcal{D} = \{\Delta_0, \cdots, \Delta_{|\mathcal{V}|-1}\} \in \mathcal{V}$ centered on $(1/2, 1/2) + \mathbb{Z}^2$ of finite volume $\mathcal{V}$, where $|\mathcal{V}|$ is an integer. This is convenient since there is in $\mathcal{D}$ a unique square $\Delta_0 = [-1/2, 1/2] \times [-1/2, 1/2]$ centered at the origin. The set of pairs of different squares $b_{ij} = \{\Delta_i, \Delta_j\}$, $\Delta_i \neq \Delta_j$ of $\mathcal{D}$ is called $\mathbb{B}$.

Define $\chi_{\Delta_i}(x)$ as the characteristic function of the square $\Delta_i \in \mathcal{D}$ as

$$\chi_{\Delta_i}(x) = \begin{cases} 1, \text{ if } x \in \Delta_i \\ 0, \text{ otherwise} \end{cases} \tag{1}$$

We can use once more the BKAR forest formula on any LVE term $G_{\mathcal{T},\ell_0}$. The idea is to test whether any C propagator in $G_{\mathcal{T},\ell_0}$ is a tree link between different squares or is a loop.

However here a technical subtlety occurs. It is simpler to perform the cluster expansion on the C propagators ; not on the $C^{1/2}$. But this can be done using the D and R rather than D' and R' representation. Recall we can write any loop vertex either as a trace of D and R operators or as the same trace with D' and R' operators.

Using the non-symmetric representation without prime we consider the C operators as matrix-valued operators between cubes :

$$C_{\Delta,\Delta'}(x,y) = \chi_{\Delta}(x) C(x,y) \chi_{\Delta'}(y) \tag{2}$$

and we apply the forest formula to $G_{\mathcal{T},\ell_0}$ that is we interpolate the off-diagonal terms [2].

The result is

$$G_{\mathcal{T},\ell_0} = \sum_{\mathcal{T}',\Gamma \supset \Delta_0} G_{\mathcal{T},\ell_0,\mathcal{T}',\Gamma} \tag{3}$$

where Γ is any finite set of squares in $\mathcal{D}$ and $\mathcal{T}'$ is a spanning tree joining these squares.

The formula for $G_{\mathcal{T},\ell_0,\mathcal{T}',\Gamma}$ is a bit heavy since it is similar to (17). Let us rather describe it in plain words. In $G_{\mathcal{T},\ell_0,\mathcal{T}',\Gamma}$ for each link $\ell' \in \mathcal{T}'$ between squares $\Delta_{\ell'}$ and $\Delta'_{\ell'}$ we have an explicit propagator C pulled in the numerator of the cycle $\mathcal{C}$ with one end in $\Delta_{\ell'}$ and the other in $\Delta'_{\ell'}$. Furthermore there exists a weakening parameter $w'_{\ell'}$ for each link ℓ' in $\mathcal{T}'$ integrated from 0 to 1. Finally any other remaining propagator C in $G_{\mathcal{T},\ell_0,\mathcal{T}',\Gamma}$ is a function of these w's through the infimum formula between end points, namely $C(x,y)$ for $x \in \Delta$ and $y \in \Delta'$ has to be multiplied by the infimum of the parameters $w'_{\ell'}$ for ℓ' in the unique path of $\mathcal{T}'$ joining Δ to Δ'.

Let us answer a few natural questions at this point :

- Why does the BKAR formula gives directly a result indexed by trees rather than forests ? This is because the cycle $\mathcal{C}$ is connected and the σ measure is ultralocal. The vertex ℓ_0 was fixed at the origin, hence Γ must contain Δ_0. The BKAR formula a priori leads to a result indexed by forests on $\mathcal{D}$, but every forest which is not a tree in this case gives zero contribution because the cycle cannot jump from a square to another without using propagators [3], as these are the only non-local terms in $G_{\mathcal{T},\ell_0}$.
- After the w' parameters have been introduced, is it possible to return to the square root formulation of G in terms of D' and R' operators ? This is a clever question and the good news is that the answer is yes. It is good to return to the D' and

2. Technically the BKAR formula is written for pairs on a *finite set* and this is why we take $\mathcal{V}$ finite ; but the infinite volume limit is uniform.

3. Here the pernickety reader could worry whether our characteristic functions include the boundary of the squares or not, but this is a zero measure hence irrelevant subproblem.

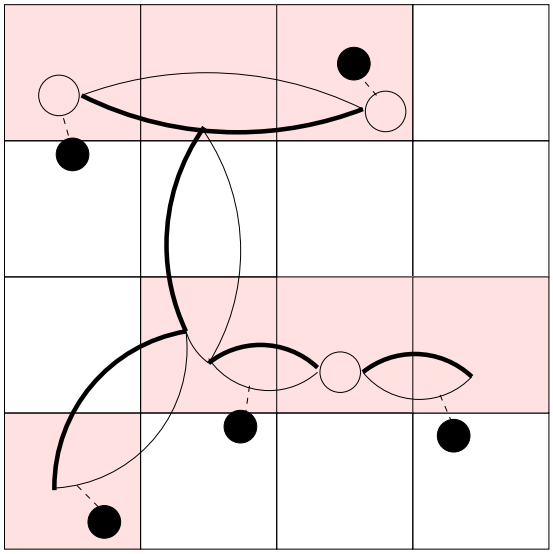

FIGURE A.6: An LVE which spreads over a set Γ. The ultralocal σ propagators that connect the loops are omitted. Each loop has several counterterms attached. The tree $\mathcal{T}'$ is shown in bold.

R' operators rather than keeping the not-hermitian D and R operators because the norm estimates are clearer with these square roots ; it is obvious that $\|(1+iH)\| \leq 1$ if H is Hermitian, but if H is not Hermitian one could worry. Fortunately it is possible to reconstruct square roots after the w' parameters have been added, simply because the matrix of the w' parameters between the squares of Γ, completed by a 1 on the diagonal, is *positive* symmetric, hence has itself a positive square root. The problem is studied at length in eg section 4.2 of [11], so we refer the reader to that paper.

– What about the sigma field measure after the volume has been restricted to Γ ? It can be replaced by a measure which is a white noise on Γ only, that is a Gaussian measure $d\nu_\Gamma(\sigma)$ of propagator

$$\chi_\Gamma(x)\delta(x-y)\chi_\Gamma(y). \tag{4}$$

In other words after the cluster expansion has delimited the volume occupied by all ends of C propagators, the σ field must also live only in that volume, and can be freely replaced by $\chi_\Gamma\sigma$. This is because expanding resolvents, σ fields can be situated only at ends of propagators and this expansion is easily shown Borel-summable at least in any finite volume with a finite ultraviolet cutoff, hence it determines the support properties of σ. This is an important point for the later bounds and for Nelson's argument.

– What about the convergence of the expansion ? Is the new sum over trees $\mathcal{T}'$ not going to jeopardize the combinatoric of this convergence ? Again the answer is no. The true formula behind the BKAR formula is an ordered formula in which the tree $\mathcal{T}'$ can be created in any arbitrary order. The corresponding factorial $|\mathcal{T}'|!$ which comes from the choice of resolvents to be derived is then automatically canceled by the simplex integral over the ordered w's parameters.

– What about the sum over Γ, hence over the cluster ? Any line ℓ of $\mathcal{T}'$ is associated to an explicit C propagator between the two different squares $\Delta_\ell, \Delta'_\ell$. Hence in the estimate of $G_{\mathcal{T},\ell_0,\mathcal{T}',\Gamma}$ we have always tree exponential decay. More precisely defining $\tau(\Gamma)$ as the minimal length of any tree connecting Γ we can extract a factor $e^{-c\tau(\Gamma)}$ from the bound on $|G_{\mathcal{T},\ell_0,\mathcal{T}',\Gamma}|$. This exponential decay ensures that we could sum over the position of all the squares in Γ, starting from the leaves of the tree until we arrive at the root Δ_0 which is fixed. Remark also that the exponential tree decay allows to absorb, through the so called "volume effect", any finite product of the factorials of the coordination number of the tree $\mathcal{T}'$. Indeed large coordination number emanating from one square always lead to many propagators traveling a large distance see eg [7], chapter III.1, Lemma III.1.3. Such factorials easily occur if one uses relatively sloppy bounds.

Remark that as we obtained already connected functions by LVE, we don't follow this cluster expansion with a Mayer expansion as in the traditional method of constructive physics. This is the main advantages of the LVE. We turn now to the main expansion of this paper.

A.5 The Cleaning Expansion

A.5.1 Multiscale analysis

In this section we shall for simplicity study the connected vacuum amplitudes in unit volume. The Schwinger functions and the infinite volume limit will be considered in following sections.

It is convenient to put $\Lambda = M^{j_{max}}$, where $M > 1$ is a constant, $j_{max} \in \mathbb{N}$ and the ultraviolet limit is $j_{max} \to \infty$, so that we can slice the propagator according to renormalization group slices as $C^\Lambda = \sum_{j=0}^{j_{max}}$ with :

$$C_j(x,y) = \int_{M^{-2j}}^{M^{-2j+2}} e^{-\alpha m^2 - \frac{(x-y)^2}{4\alpha}} \frac{d\alpha}{\alpha} \leq K e^{-cM^j|x-y|}. \tag{1}$$

K and c are generic names for inessential constants, respectively large and small.

Considering the square root decomposition of C with a middle point x

$$D'(y,x,z) = 2i\sqrt{\lambda} C^{1/2}(y,x) C^{1/2}(x,z). \tag{2}$$

we have an associated matrix-like decomposition of D as

$$D'_{jk}(x) \equiv 2i\sqrt{\lambda} C_j^{1/2}(.\ ,x) C_k^{1/2}(x,\ .). \tag{3}$$

Similarly the resolvent R writes

$$R'_{jk}(\sigma x) \equiv \frac{1}{1 + 2i\sqrt{\lambda} C_j^{1/2}(.\ ,x)\sigma(x) C_k^{1/2}(x,\ .)}. \tag{4}$$

The relationship between the operators D, D' and R, R' has been explained in section (A.4). While the former representation is more suitable for the cleaning expansion, the latter is more suitable for proving the bounds of the resolvents.

It is convenient to split these matrix-valued operators according to the largest of their two indices :

$$\begin{aligned} D' &= \sum_j D'_j, \quad D'_j = \sum_{k,l \text{ such that } \sup(k,l)=j} D_{kl}, \\ R' &= \sum_j R'_j, \quad R'_j = \sum_{k,l \text{ such that } \sup(k,l)=j} R'_{kl}. \end{aligned} \tag{5}$$

We define also

$$D'_{\bar{j}} = \sum_{k \le j} D'_k, \quad R'_{\bar{j}} = \sum_{k \le j} R'_k, \tag{6}$$

and

$$D_k(x,\sigma) = D_k(x) \cdot \sigma(x). \tag{7}$$

The multiplication rule of two $D'_{ij}(x)$ operators reads :

$$\begin{aligned} \int dz dy' D'_{jk}(y,x,z) D'_{lm}(y',x',z') &= \int dz dy' (2i\sqrt{\lambda})^2 \delta_{kl} \delta(z-y') C_j^{1/2}(y,x) \\ & \quad C_k^{1/2}(x,z) C_l^{1/2}(y',x') C_m^{1/2}(x',z') \\ &= -4\lambda C_j^{1/2}(y,x) C_k(x,x') C_m^{1/2}(x',z'), \end{aligned} \tag{8}$$

so that the integral over z and y' reconstructs C_k in the middle [4].

We decompose also the counterterm T_Λ as :

$$T_\Lambda = \sum_{j \le jmax} T_j, \quad T_{\bar{j}} = \sum_{k \le j} T_k, \tag{9}$$

where

$$T_k = C_k(x,x). \tag{10}$$

In this case the formula of the loop vertex expansion becomes :

Théorème A.5.1 (Multi Scale Loop Vertex Expansion)

$$\log Z(\lambda, \Lambda, \mathcal{V}) = \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{\mathcal{T} \text{ with } n \text{ vertices}} G_{\mathcal{T}} \tag{11}$$

$$G_{\mathcal{T}} = \sum_{j=j_{min}}^{j_{max}} \Big\{ \prod_{\ell \in \mathcal{T}} \int d^2x_\ell d^2y_\ell \big[\int_0^1 dw_\ell \big] \Big\} \int d\nu_{\mathcal{T}}(\{\sigma^v\}, \{w\})$$

$$\Big\{ \prod_{\ell \in \mathcal{T}} \big[\frac{1}{2} \delta(x_\ell - y_\ell) \frac{\delta}{\delta \sigma^{v(\ell)}(x_\ell)} \frac{\delta}{\delta \sigma^{v'(\ell)}(y_\ell)} \big] \Big\} \prod_{v=1}^{n} V_v^j,$$

the notations being straightforward and almost identical to the ones of Theorem A.2.1.

4. Remark that such integrals over middle propagators points are never restricted to a finite volume $\mathcal{V}$, but always performed over all $\mathbb{R}^2$. Only end points of C's are affected by our volume cutoff, if any.

A.5.2 Overview of the Cleaning Expansion

We turn now to the heart of our paper, namely to the cleaning expansion. It develops perturbation theory, and combines different LVE terms in order to compensate any inner tadpole with other LVE's which have counter terms at the exact position of these tadpoles.

In a Feynman graph let's call *good line* at scale j or in short j-line any C_j propagator which is not part of a tadpole. Conversely a j-inner-tadpole is a line C_j enclosed between two σ-half propagators, hence the scale j part of an inner tadpole. The key observation on which the cleaning expansion rests is that when a sufficiently large number of j-lines of high enough j has been produced, we can stop the expansion, re-detach the counter terms from the LVE, and pay the corresponding bad estimate. This is the analog in our context of Nelson's argument.

A characteristic feature of the cleaning expansion is to use the natural canonical cyclic ordering on any LVE provided by its dual representation (10). We start from an arbitrary origin of the cycle which in fact can be chosen as the preferred root point at the origin in (1). Then we use a Taylor formula with integral remainder which forces potential tadpole or non-tadpole propagators to appear, in their natural order "along the cycle $\mathcal{C}$". This formula works along the cycle [5]. We compensate the tadpoles produced, if any, with the other LVE terms with appropriate counterterms at these exact tadpoles positions. This compensation is called the "cleaning".

We cannot clean forever, as this would develop the full perturbation series and ultimately diverge. But we stop and write a Taylor integral remainder when enough cleaning, depending on the scale, has succeeded. If we were not to use the canonical cyclic ordering of the LVE, it would be difficult to find the corresponding "weakening parameters" rule for that Taylor remainder term. But fortunately the cyclic ordering [6] solves nicely this problem ! The beginning of the cycle $\mathcal{C}$ is explicitly and fully cleaned of potential tadpoles ; the rest of the cycle $\mathcal{C}$ has *no weakening parameters* on the remaining potential tadpoles !

The cleaning is done scale by scale along the cycle, starting from j_{max} towards scale 0. We expand the σ fields and contract them, either detecting tadpoles or detecting perturbation lines of scale j which *are not tadpole lines*. If at any scale we find a Taylor remainder term (i.e. the cycle contained more than $a.j$ not-tadpoles lines of scale j), we stop the expansion and *don't test lower scales*. There are two kinds of j-lines in the cleaning expansion : the ones which belong to *crossing* sigma propagators like in Figure 24 B [7], and the ones which appear into *nesting* sigma propagators (see Figure 24 D). In both cases the line is a j-line because we are sure it cannot belong to a tadpole [8].

5. Here the expansion is not canonical (but only in a very slight way) since the two natural orderings along the cycle could be used.

6. Remark that this cyclic ordering is a new feature of the LVE without any analog in the Feynman graphs or in the usual constructive tools, and that it is also the reason why the LVE can build constructively matrix models.

7. Beware that we did not draw explicitly the pure propagator in Figure 24 D.

8. This is a simplifying feature of the ordinary ϕ_2^4 theory ; in the GW_4 theory we expect that nesting situations potentially require renormalization, because they lead to planar graphs ; only crossings give good factors. Hence nesting lines should not be counted in the stopping rule. Nevertheless the expansion

Each crossing j-line of scale j could be simply bounded by

$$\int d^2x C_j(x,y) = \int_{M^{-2j}}^{M^{-2j+2}} e^{-\alpha m^2 - \frac{(x-y)^2}{4\alpha}} \frac{d\alpha}{\alpha} \leq K e^{-cM^j|x-y|} \sim K M^{-2j}, \tag{12}$$

and each nesting j-line is bounded by the integration of the full resolvent exactly in the same way than a crossing j-line

$$\begin{aligned} |\int d^2x C_R^j(\sigma,x,y)| &= |\int d^2x R_j(\sigma) C_j(x,y)| \leqslant \int d^2x |R_j(\sigma)| \cdot |C_j(x,y)| \\ &\leqslant \int d^2x |C_j(x,y)| \sim K M^{-2j}, \end{aligned} \tag{13}$$

where we have used the fact that $|R_j(\sigma)| < 1$.

Hence for any j-line, the amplitude (after bounding all remaining resolvents and oscillating factors of the $e^{ia\sigma}$ type by constants as in (37)-(40) will be that of an *ordinary graph* with that j-line, hence it will provide a small factor KM^{-2j} [7].

A growing number of such compensations must be performed as the scale $j_{max} \to \infty$, because of the worsening of "Nelson's bound" with the ultraviolet cutoff [7].

Hence the result is indexed by new LVE's with a dividing scale j_0. No uncompensated tadpoles of scale higher than j_0 remain. Among tadpoles of scale j_0, the first $a.j_0$ along the cycle are compensated, and the next ones are not. Finally all potential tadpoles of scale lower than j_0 are uncompensated [9].

The reason for this rule will become apparent in section 6.1. Consider the theory in a single unit square Δ. After this cleaning expansion, if the dividing scale is j_0 we have accumulated at least $a.j_0$ good factors M^{-cj_0} from the power counting of non-tadpole lines with scale j_0. This allows to pay both for a sloppy bound on a reexponentiated $e^{2iTr_\Delta \sqrt{\lambda} T_{\bar{j}} \sigma}$ oscillating term (see (10) for the origin of that term) and for the large combinatoric of the expansion. This is the exact analog for the LVE of Nelson's argument as explained in [7].

This ends up the story for the theory in a single square. But in the infinite volume case, Nelson's bound has to be paid in each unit volume square Δ *actually occupied by the LVE term*. That's why we had to introduce the cluster expansion of section 4. Once knowing the exact finite set Γ of squares occupied by the LVE, we need to repeat the cleaning operation once for each square in Γ. This is conceptually easy and clear, but the notations become heavy and obscure the argument. Hence we define below the cleaning expansion only in a single unit square for simplicity and let the generalization to several squares to the reader.

A.5.3 The stopping rule

The expansion should be based on developing first only the resolvent of highest scale and then the lower ones. It uses the formula

should still converge as nesting lines lead to planar graphs, which are very few compared to the non-planar ones and lead to convergent series.

9. There is a last term with no dividing scale, or by convention a kind of dividing scale -1 ; it is the one where the cleaning succeeded at all scales. That one has no longer any tadpole at any scale.

$$\begin{aligned} R^j(\sigma) &= \frac{1}{1+\sum_{k\le j} D^k(\sigma)} = \frac{1}{1+\sum_{k<j} D^k\sigma + D^j\sigma} \\ &= R^{j-1}(\sigma)\frac{1}{1+R^{j-1}(\sigma)D^j\sigma}. \end{aligned} \tag{14}$$

We then decompose the amplitude of the dual tree according to the scale of each object and we can write

$$G_{\bar{\mathcal{T}},\ell_0} = \sum_{j=0}^{j_{max}} G^j_{\bar{\mathcal{T}}} = (-\lambda)^{n-1} \sum_{j}^{j_{max}} \int_{\mathcal{V}} \prod_{\bar{\ell}} d^2x_{\bar{\ell}} \prod_{\bar{b}\in\bar{B}} d^2x_{\bar{b}} Tr_{\mathcal{V}} \left\{ \vec{\prod}_{u\in\mathcal{O}} P^j_u \right\} \tag{15}$$

where the $\vec{\prod}$ means that we take the ordered product of operators along the cycle $\mathcal{C}$ and ℓ_0 is the marked point where we start the expansion.

In this way an algebraic expansion step of the resolvent consists in writing :

$$R^j(\sigma) = R^{j-1}(\sigma) - R^{j-1}(\sigma)\sigma\ D^j\ R^j(\sigma) \tag{16}$$

for any scale index j.

We call formula (12) the cleaning expansion for the resolvent in that we factorize the pure propagator and the remaining resolvents where other possible σ fields are hidden.

We start the cleaning expansion for the resolvent with formula (12) which is on the *right* side of an arbitrary marked point (ℓ_0 for example), and use the integration by parts for the σ fields clockwisely to generate j-lines or tadpoles. In the latter case we search for the corresponding term with a counterterm instead of the tadpole and perform the cancellation ; the result is exactly 0.

The cleaning expansion is conceptually clear but the explicit mathematical notations are heavy, like in any "conditional expansion" where the steps depends on the previous choices. So we propose to consider the case of a relatively simple loop $\mathcal{C}$ of the dual representation with only *two* resolvent $R(x_1,x_2)\ R(y_1,y_2)$ to explain the cleaning expansion, see Figure A.7. It corresponds to an $n=2$ term in the LVE. The reader can convince himself easily that this example generalizes to an arbitrary number of resolvents in $\mathcal{C}$, with just heavier notations.

A.5.4 An Example

We start the cleaning expansion from the highest scale j_{max} and downwards until we gain enough convergent factors. So in this example we shall use the following formula for each resolvent :

$$R^{j_{max}}(\sigma) = R^{j_{max}-1}(\sigma) - R^{j_{max}-1}(\sigma)\sigma\ \cdot D^{j_{max}} R^{j_{max}}(\sigma). \tag{17}$$

The amplitude for this graph reads :

$$\begin{aligned} G &= \int d\mu(\sigma) \int d^2x_1 d^2x_2 d^2y_1 d^2y_2 R^{j_{max}}(\sigma,x_1,x_2) C^{j_{max}}(x_2,y_1) \\ &\quad R^{j_{max}}(\sigma,y_1,y_2) C^{j_{max}}(y_2,x_1). \end{aligned} \tag{18}$$

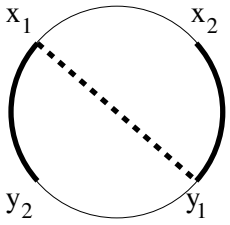

FIGURE A.7: An example for the cleaning expansion. The ordinary line means the resolvent while the thick line means the pure propagator. The bold dash line is the ultralocal σ propagator in the dual representation and corresponds to the term $\delta(x_1 - y_1)$.

We choose x_1 to be the fixed marked point and start the cleaning expansion from the resolvent $R^{j_{max}}(x_1, x_2)$. We have

$$\begin{aligned} R^{j_{max}}(\sigma, x_1, x_2) &= R^{j_{max}-1}(\sigma, x_1, x_2) \\ &- R^{j_{max}-1}(\sigma, x_1, z_1)\sigma(z_1)D^{j_{max}}(z_1, z_2)R^{j_{max}}(\sigma, z_1, x_2). \end{aligned} \tag{19}$$

So the amplitude reads :

$$\begin{aligned} G &= \sum_j^{j_{max}} \int d\mu(\sigma) \int d^2x_1 d^2x_2 d^2y_1 d^2y_2 R^{j_{max}-1}(x_1, x_2)C^{j_{max}}(x_2, y_1) \\ &\quad R^{j_{max}}(y_1, y_2)C^{j_{max}}(y_2, x_1) \; - \; \sum_j j_{max} \int d\mu(\sigma) \\ &\quad \int d^2x_1 d^2x_2 d^2y_1 d^2y_2 d^2z_1 D^{j_{max}}(x_1, z_1)\sigma(z_1)R^{j_{max}}(z_1, z_2) \\ &\quad R^{j_{max}-1}(z_2, x_2)C^{j_{max}}(x_2, y_1)R^{j_{max}}(y_1, y_2)C^{j_{max}}(y_2, x_1), \end{aligned} \tag{20}$$

where (37) is almost the same as the original amplitude except that the scale of the first resolvent is lowered down by 1 and the second term (38) gives the main contribution. For simplicity we consider only the second term. We need to go back to (37) if we have not enough j-lines at scale j_{max} then expand the resolvent of scale $j_{max} - 1$ and so on.

Integrating by parts we have

$$\begin{aligned} &- \sum_j^{j_{max}} \int d\mu(\sigma) \int d^2x_1 d^2x_2 d^2y_1 d^2y_2 d^2z_1 D^{j_{max}}(x_1, z_1)\sigma(z_1)R^{j_{max}}(\sigma, z_1, z_2) \\ &\quad R^{j_{max}-1}(\sigma, z_2, x_2)C^{j_{max}}(x_2, y_1)R^{j_{max}}(\sigma, y_1, y_2)C^{j_{max}}(y_2, x_1) \qquad (21) \\ &= - \sum_j^{j_{max}} \int d\mu(\sigma) \int d^2x_1 d^2x_2 d^2y_1 d^2y_2 d^2z_1 C^{j_{max}}(y_2, x_1)D^{j_{max}}(x_1, z_1) \\ &\quad \frac{\delta}{\delta\sigma(z_1)}[R^{j_{max}}(\sigma, z_1, z_2)R^{j_{max}-1}(\sigma, z_2, x_2)R^{j_{max}}(\sigma, y_1, y_2)]C^{j_{max}}(x_2, y_1). \end{aligned}$$

Note that in order to show how we generate tadpoles and j-lines explicitly, we don't use the derivation formula for the resolvent as shown in (27), but use the induction formula (17) and integration by parts once and again.

For simplicity we consider only the case where the derivation acts only on the resolvents $R^{j_{max}}$ but not $R^{j_{max}-1}$, as the former terms give the main contributions. We shall use the induction formula again.

$$\begin{aligned} & R^{j_{max}}(\sigma, z_1, z_2) = R^{j_{max}-1}(\sigma, z_1, z_2) \\ - \quad & R^{j_{max}-1}(\sigma, z_1)\sigma(z_1)D^{j_{max}}(z_1, z_2)R^{j_{max}}(\sigma, z_2). \end{aligned} \tag{22}$$

Ignoring the $R^{j_{max}-1}$ term, formula (21) reads :

$$\begin{aligned} & \int d\mu(\sigma) \int d^2x_1 d^2x_2 d^2y_1 d^2y_2 d^2z_1 d^2w_1 d^2w_2 C^{j_{max}}(y_2, x_1) D^{j_{max}}(x_1, z_1) \\ & \frac{\delta}{\delta\sigma(z_1)}[D^{j_{max}}(z_1, w_1)\sigma(w_1)R^{j_{max}}(\sigma, w_1, w_2)]R^{j_{max}-1}(\sigma, w_2, z_2) \\ & R^{j_{max}-1}(\sigma, z_2, x_2)R^{j_{max}}(\sigma, y_1, y_2)]C^{j_{max}}(x_2, y_1) \qquad (23) \\ + \quad & \int d\mu(\sigma) \int d^2x_1 d^2x_2 d^2y_1 d^2y_2 d^2z_1 d^2w_1 d^2w_2 C^{j_{max}}(y_2, x_1) D^{j_{max}}(x_1, z_1) \\ & R^{j_{max}}(\sigma, z_1, z_2)R^{j_{max}-1}(\sigma, z_2, x_2)C^{j_{max}}(x_2, y_1)\frac{\delta}{\delta\sigma(z_1)}[R^{j_{max}}(\sigma, y_1, y_2)], \end{aligned}$$

where (23) means that the $\frac{\delta}{\delta\sigma(z_1)}$ term either act on $R^{j_{max}}(\sigma, z_1, z_2)$ and (24) means the derivation acts on $R^{j_{max}}(\sigma, y_1, y_2)$.

Now we first consider the derivation in (23). We have

$$\begin{aligned} & \frac{\delta}{\delta\sigma(z_1)}[D^{j_{max}}(z_1, w_1)\sigma(w_1)R^{j_{max}}(\sigma, w_1, w_2)] \\ = \quad & D^{j_{max}}(z_1, w_1)\delta(z_1 - w_1)R^{j_{max}}(\sigma, w_1, w_2) \qquad (24) \\ + \quad & D^{j_{max}}(z_1, w_1)\sigma(w_1)\frac{\delta}{\delta\sigma(z_1)}R^{j_{max}}(\sigma, w_1, w_2). \qquad (25) \end{aligned}$$

(24) turns out to be the tadpole term after integrating out z_1 and w_1, which should be compensated by the counter term. We need to further analyze (25) by expanding $R^{j_{max}}(\sigma, w_1, w_2)$, which reads :

$$\begin{aligned} & \int d^2p_1 d^2p_2 D^{j_{max}}(z_1, w_1)\sigma(w_1)D^{j_{max}}(w_1, p_1) \\ & \frac{\delta}{\delta\sigma(z_1)}[\sigma(p_1)R^{j_{max}}(\sigma, p_1, p_2)R^{j_{max}-1}(\sigma, p_2, w_2)]. \end{aligned} \tag{26}$$

If the field $\sigma(z_1)$ contracts with $\sigma(p_1)$, then this makes a crossing after $\sigma(w_1)$ contracts with any other σ field ; If $\sigma(z_1)$ contracts with other σ field hidden in $R^{j_{max}}(\sigma, p_1, p_2)$ while $\sigma(w_1)$ contracts with $\sigma(p_1)$, again this makes an inner tadpole and should be compensated.

We have still the possibility that the σ field is contracted with the field hidden in the resolvent $R^{j_{max}}$ but never with other σ field that appear explicitly by the induction formula. There could be infinite number of such "nesting" lines. A corresponding graph is shown in graph D of Figure A.8. The nesting lines are not dangerous. For each such nesting there is a j-line again hence we gain a factor $M^{-2j_{max}}$.

The cleaning process corresponding to (24) and (25) is shown in Figure A.8.

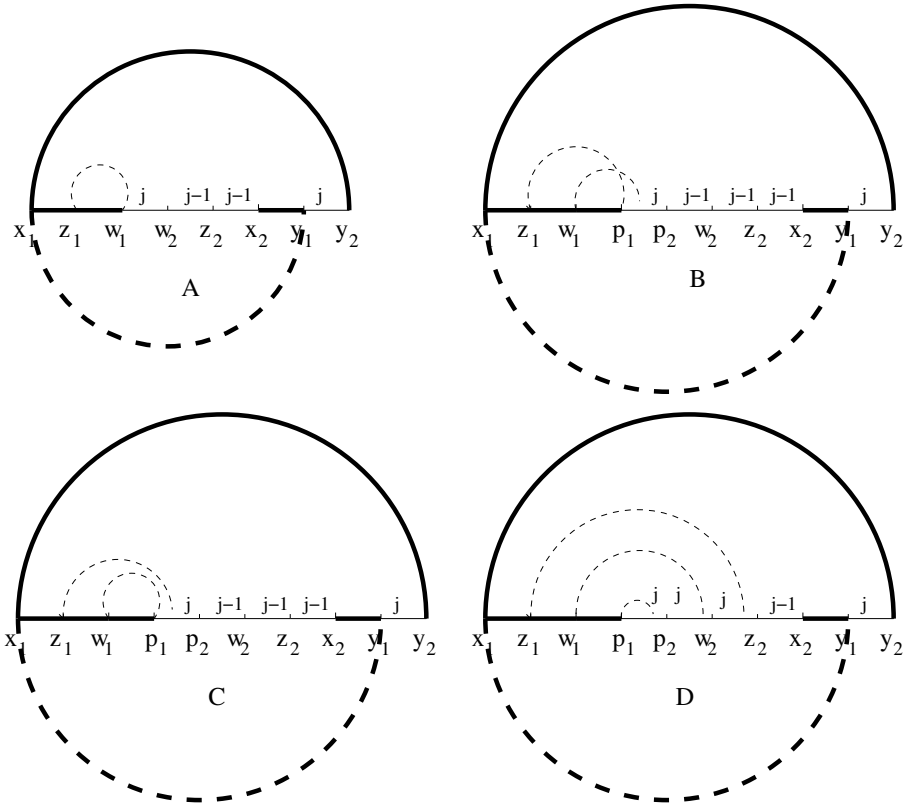

FIGURE A.8: An example of cleaning expansion. x_1 is the marked point and we do the expansion clockwisely. The bold dash line means the term $\delta(x_1 - y_1)$.

For the term (24), we don't use the induction formula to distinguish the tadpole or j-line explicitly but just perform the derivation explicitly :

$$\frac{\delta}{\delta\sigma(z_1)}[R^{j_{max}}(\sigma, y_1, y_2)] = -\int d^2z[R^{j_{max}}(\sigma, y_1, z)]D(z, z_1)[R^{j_{max}}(\sigma, z_1, y_2)]. \tag{27}$$

In fact (24) equals to :

$$\begin{aligned} - \quad & \int d\mu(\sigma) \int d^2x_1 d^2x_2 d^2y_1 d^2y_2 \int d_1^w d^2w_2 d^2z_1 C^{j_{max}}(y_2, x_1) \\ & D^{j_{max}}(x_1, z_1) R^{j_{max}}(\sigma, z_1, z_2) R^{j_{max}-1}(\sigma, z_2, x_2) C^{j_{max}}(x_2, y_1) \\ & R^{j_{max}}(\sigma, y_1, w_1) D(w_1, w_2) \delta(z_1 - w_2) R^{j_{max}}(\sigma, w_2, y_2). \end{aligned} \tag{28}$$

which generates a j-line of the crossing type. A graph corresponding to (28) is shown in Figure A.9.

Then we go back to the resolvent $R^{j_{max}}(\sigma, z_1, z_2)$ and repeat the cleaning expansion again.

A tadpole graph is log divergent, so we stop the expansion whenever we generate an inner tadpole and we compensate it by the graph that has the same structure except that a tadpole is replaced by a counter term. This cancelation is exact, so that no inner tadpoles should appear in the renormalized graph. See section 7.2.

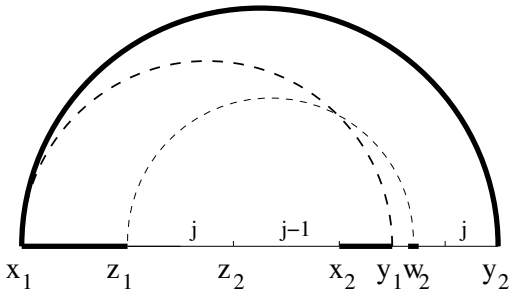

FIGURE A.9: The graph corresponding to (28), which is a graph with one crossing. The bold dash lines correspond to the original σ propagator in the dual graph while the normal dash lines mean the newly generated σ propagator.

On the other hand we could gain a convergent factor M^{-j} for each j-line. But for a given scale j we should not generate an arbitrary number of crossings, since otherwise the expansion would diverge. More precisely, we start from the resolvents of scale j_{max}. Each time a j_{max}-line is generated, we add 1 to a counter. We stop the expansion until the number of j_{max}-lines reaches $N_{j_{max}} = aj_{max}$. In that case we have gained a convergent factor at least $M^{-j^2_{max}} \leq e^{-j^2_{max}}$ for $M > e$. Otherwise, hence if we couldn't generate $N_{j_{max}}$ j_{max}-lines. we turn to the expansion of the resolvents of scale $j_{max} - 1$ and so on.

The combinatoric factor coming from a maximal number of $N_j = aj$ crossings in the loop vertex expansion reads :

$$N_j! \sim e^{N_j \ln N_j} \sim e^{aj(\ln j + \log a)}. \tag{29}$$

This term is not dangerous as it is easily bounded by e^{-j^2}, which is what the N_j j-lines provide.

A.5.5 Reexponentiation of Remaining Counterterms

All inner tadpoles have been canceled against the appropriate counterterms in the cleaned part of the dual graph. But there might still be arbitrary number of counterterms in the uncleaned part and they are divergent. Instead of canceling all of them, we reexponentiate them by using the properties of Gaussian measure and integration by parts. We consider first of all the case of the connected function in a unit square and then consider the general case.

The general formula for the remainder term of order N reads :

$$A^N_{\bar{\mathcal{T}}}|_{|\bar{\mathcal{T}}|=N+1} = \prod_{l,l' \in \bar{\mathcal{T}}} \int d\nu(w,\sigma) \int_0^1 dw_{l'} \lambda^N \prod \hat{R}_l(\sigma) \prod R_{l'}(\sigma) \prod_{m=1}^{N-n} (T_\Lambda)^m. \tag{30}$$

where we have used the fact that the weakening factors for the counterterm leaves equal to one. There are only weakening factors for the propagators between different loop vertices.

Now we consider the unrenormalized amplitude

$$G = \int d\nu(\sigma,w) \prod_{l,l' \in \bar{\mathcal{T}}} \lambda^{|\bar{\mathcal{T}}|} [\hat{R}_l(\sigma)][R_{l'}(\sigma)] e^{\int 2i\sqrt{\lambda}\sigma T_\Lambda}. \tag{31}$$

We use the formula

$$\int d\nu(w,\sigma) f(\sigma) g(\sigma) = e^{\frac{1}{2}\frac{\partial}{\partial\sigma(x)} C(x,x',w) \frac{\partial}{\partial\sigma(x')}} f(\sigma) g(\sigma))|_{\sigma=0}, \tag{32}$$

where $C(x,x',w)$ is the covariance that may or may not depend on the weakening factor w. Hence

$$\begin{aligned}
G &= \int d\nu(w,\sigma) \\
&\quad \sum_{N=0}^{\infty} \frac{1}{N!} [\frac{1}{2} \frac{\partial}{\partial\sigma} \frac{\partial}{\partial\sigma'}]^N \{ \prod_{l,l' \in \bar{\mathcal{T}}} \int_0^1 dw_{l'} [\hat{R}_l(\sigma) R_{l'}(\sigma, l')] e^{\int 2i\sigma T_\Lambda}] \} \\
&= \int d\nu(w,\sigma) \sum_{N=N_1+N_2+N_3}^{\infty} \sum_{N_1}^{\infty} \sum_{N_2}^{\infty} \sum_{N_3}^{\infty} \frac{1}{N!} \frac{N!}{N_1! N_2! N_3!} \prod_{l,l' \in \bar{\mathcal{T}}} \int_0^1 dw_{l'} \\
&\qquad [(\frac{1}{2})^{N_1+N_2} (\frac{\partial}{\partial\sigma})^{N_2} \{ \frac{\partial}{\partial\sigma} \frac{\partial}{\partial\sigma'} \}^{N_1}] [\hat{R}_l(\sigma) R_{l'}(\sigma, w_{l'})] \\
&\qquad\qquad \{ (\frac{\partial}{\partial\sigma'})^{N_2} [\frac{1}{2} \frac{\partial}{\partial\sigma} \frac{\partial}{\partial\sigma'}]^{N_3} e^{\int 2i\sqrt{\lambda}\sigma T_\Lambda} \} |_{\sigma=0} \\
&= \int d\nu(w,\sigma) \sum_{N_1=0}^{\infty} \sum_{N_2=0}^{\infty} \sum_{N_3=0}^{\infty} \prod_{l,\ l' \in \bar{\mathcal{T}}} \int_0^1 dw_{l'} \frac{1}{N_1! N_2!} (\frac{1}{2})^{N_1+N_2} \\
&\qquad [(\frac{\partial}{\partial\sigma})^{N_2} \{ \frac{\partial}{\partial\sigma} \frac{\partial}{\partial\sigma'} \}^{N_1}] [\hat{R}_l(\sigma) R_{l'}(\sigma)] \\
&\qquad\qquad \{ (\frac{\partial}{\partial\sigma'})^{N_2} \frac{1}{N_3!} [\frac{1}{2} \frac{\partial}{\partial\sigma} \frac{\partial}{\partial\sigma'}]^{N_3} e^{\int 2i\sqrt{\lambda}\sigma T_\Lambda} \} |_{\sigma=0}.
\end{aligned} \tag{33}$$

While the N_1 and N_2 derivations generate connected terms, the last derivatives generate N_3 disconnected terms, see Figure A.3.

We sum over the N_3 non-connected terms and we have :

$$\begin{aligned}
&G = \int d\nu(w,\sigma) \sum_{N_1=0}^{\infty} \sum_{N_2=0}^{\infty} \sum_{N_3=0}^{\infty} \prod_{l,\ l' \in \bar{\mathcal{T}}} \int_0^1 dw_{l'} \frac{1}{N_1! N_2!} (\frac{1}{2})^{N_1+N_2} \\
&\{ (\frac{\partial}{\partial\sigma})^{N_2} [\frac{\partial}{\partial\sigma} \frac{\partial}{\partial\sigma'}]^{N_1} \hat{R}_l(\sigma) R_{l'}(\sigma)] \} \{ (\frac{\partial}{\partial\sigma'})^{N_2} e^{\int 2i\sqrt{\lambda}\sigma T_\Lambda} \} e^{-2\lambda T_\Lambda^2} \\
&= A_{\bar{\mathcal{T}}} e^{-2\lambda T_\Lambda^2}.
\end{aligned} \tag{34}$$

Here we have used the fact that the weakening factor for each counterterm is 1. Hence

$$A_{\bar{\mathcal{T}}} = \int d\nu(\sigma, w) \prod_{l,l' \in \bar{\mathcal{T}}} \lambda^{|\bar{\mathcal{T}}|} \hat{R}_l(\sigma) R_{l'}(\sigma) e^{\int 2i\sqrt{\lambda}\sigma T_\Lambda} e^{2\lambda T_\Lambda^2}. \tag{35}$$

A.5.6 The Nelson's Bound

After the resummation of the remaining counterterms we can finally apply bounds to any individual term in the cleaned, reexponentiated expansion The bad factor $e^{2\lambda T_\Lambda^2}$ in

(35) will be compensated by the convergent factors generated by the crossings and nesting lines. More precisely, we have

$$e^{-aj_{max}^2} \cdot j_{max}! \cdot e^{2\lambda T_\Lambda^2} \sim e^{-aj_{max}^2 + j_{max} \ln j_{max} + 2\lambda j_{max}^2} < 1 \tag{36}$$

as long as we choose a properly, for example, let $a > 3\lambda$. We have used the fact that $T_\Lambda \sim j_{max}$. This resummation process is shown in Figure C.17.

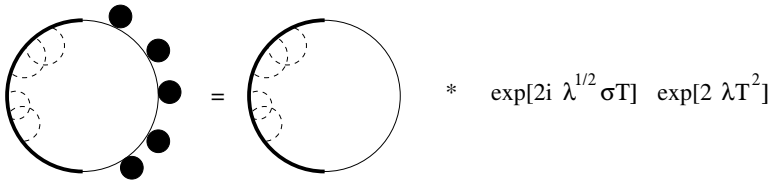

FIGURE A.10: A sketch of the resummation of the counter terms.

So we have

$$\begin{aligned} |G_{\bar{T}}|_{|T|=n-1} &< \int d\nu(\sigma, w) \prod_{l,l' \in \mathcal{T}} |\lambda|^{n-1} |\hat{R}_l(\sigma)| \; |R_{l'}(\sigma, w_{l'})| \; |e^{\int 2i\sigma T_\Lambda}| \\ &\times \; e^{2\lambda T_\Lambda^2} e^{-aj_{max}^2} \leqslant (K|\lambda|)^{n-1}. \end{aligned} \tag{37}$$

where K is an arbitrary constant.

In a finite volume $\mathcal{V}$ this is slightly modified into

$$G_T = \int d\nu(\sigma, w) \prod_{l,l' \in \bar{\mathcal{T}}} \lambda^{|\bar{\mathcal{T}}|} \hat{R}_l(\sigma) R_{l'}(\sigma, w_{l'}) e^{\int 2i\sqrt{\lambda}\sigma T_\Lambda} e^{|\mathcal{V}| 2\lambda T_\Lambda^2}. \tag{38}$$

The amplitude is now divergent due to the term $e^{|\mathcal{V}| 2\lambda T_\Lambda^2}$. The connected function for a cluster Γ could be written as :

$$G(\lambda, w) = \sum_{\Delta \in \Gamma} G_\Delta(\lambda, w), \quad \Gamma = \{\Delta_1, \cdots, \Delta_k\} \subseteq \mathcal{V}. \tag{39}$$

In each square Δ there could be an arbitrary number of σ fields coming from the resolvents (or none). For each square occupied by the σ field attached with $2\sqrt{\lambda} T_j$ the expansion generated aj crossings, where j runs from j_{max} until $\sum_j^{j_{max}} aj > 2\lambda j_{max}^2$, so as to compensate the factor $2\lambda T_\Lambda^2$. Eventually as the σ fields will visit all the squares in $\mathcal{V}$ and we could compensate the factor $e^{2\lambda T_\Lambda^2 \mathcal{V}}$. As we have exponential decay between different squares in the lattice $\mathcal{V}$ we could sum over all squares without generating any divergent factor.

Then we have again

$$|G_{\bar{\mathcal{T}}}^{ren}|_{|\bar{\mathcal{T}}|=n} \leqslant \sum_{\Delta \in T(\Gamma)} (\lambda)^{n-1} \int d\nu(\sigma) \prod_i |\hat{R}_l(\sigma)| |e^{\int d^2x 2i\sigma T_\Lambda}| e^{-\tau(T)} \leqslant (2|\lambda|K)^{n-1}. \tag{40}$$

where $|\hat{R}(\sigma)| \leq 2$, K is an arbitrary constant.

Summing over all trees of loop vertices, we have :

$$|G_n| \leqslant \frac{(2K|\lambda|)^{n-1}}{(n-1)!} \sum_T \sum_\tau \prod_{i=1}^{k_v} \int d\nu(\sigma) \prod_{l \in T} |\hat{R}_l(\sigma)| |e^{\int d^2x 2i\sigma T_\Lambda}|$$

$$\leq (2K|\lambda|)^{n-1} \frac{1}{(n-1)!} \sum_T (k-1)!. \tag{41}$$

By Cayley's theorem the sum over tree gives exactly $\frac{n!}{(k-1)!}$ and this cancels all the factors in above formula.

Then for the connected vacuum function we have :

$$P(\lambda, \mathcal{V}) = \frac{1}{|\mathcal{V}|} \log Z(\lambda, \mathcal{V}) = \sum_n G_n, \tag{42}$$

with $|G_n| \leq (2K|\lambda|)^{n-1}$. Hence $P(\lambda, \mathcal{V})$ is bounded as long as $|\lambda| < \frac{1}{2K}$.

A.6 The Renormalization

In this section we check more explicitly the combinatoric of cancellation of inner tadpoles with counterterms. We know that this must work and that the compensation leads to no remainder. Indeed tadpoles are *exactly* local and the renormalized amplitudes for graphs with tadpoles vanish *exactly* in the ϕ_2^4 theory. But it is interesting to see how the corresponding combinatorics precisely occurs in the context of the loop vertex expansion. We establish three lemmas for this cancelation, in increasing order of complexity.

A.6.1 A Single Loop Vertex

Définition A.6.1 *We define the primary divergent graph of order n as a trivial tree made of a single loop vertex with no crossings or overlaps. More precisely, each σ field in the loop vertex can then contract only either with one of its nearest neighbor or with the counterterm.*

An example of primary divergent graphs and a sketch of the renormalization process is shown in Figure A.11. Here the graph d in this Figure means that there are 3 counterterms attached to the loop vertex. These 3 counterterms could be at any positions and we need to consider all the possibilities in the calculation. So is the case for the other graphs.

To cancel this graph at order n we need to consider all the counterterms up to order n and from scale j_{max} to j_{min}.

Now we consider the amplitude of a primary divergent vacuum graph G of order λ^n with k counterterms attached. In this case we have a single loop vertex with $2n - k$ σ

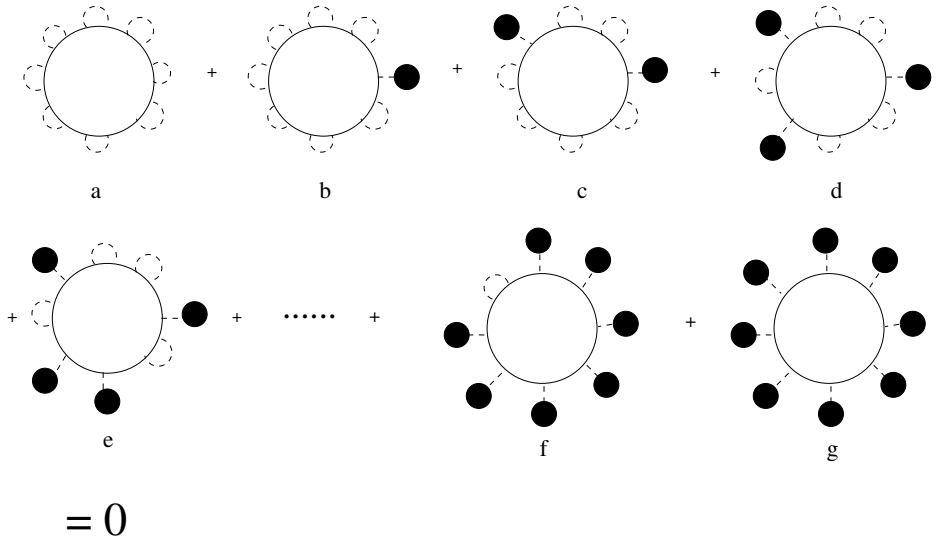

FIGURE A.11: A sketch of renormalization of primary divergent graph at order 8.

fields to contract with k counterterms. The amplitude of G reads :

$$\begin{aligned}\Gamma_G &= -\frac{1}{2}\sum_{k=0}^{n}\int d\mu(\sigma)\frac{(-1)^{2n-k+1}}{(k+1)!}\int d^2x_1 d^2x_2\cdots d^2x_n d^2x_{n+1}\cdots d^2x_k \\ &\times \sum_{\substack{i_1,i_2,\cdots i_N=j_{min},\\ N=2n-k}}^{j_{max}} D_{i_1,i_2}(x_1,x_2,\sigma)D_{i_2,i_3}(x_2,x_3,\sigma)\cdots D_{i_N,i_1}(x_N,x_1,\sigma) \\ &\times \prod_{i=1}^{k}(2i\sum_{j=j_{min}}^{j_{max}}\int T^j\sigma(x_i)), \end{aligned} \tag{1}$$

with $x_{n+1} = x_1$. We can renormalize by using the multiscale analysis. We start with the highest energy scale, where all the propagators and counter terms have the energy scale j_{max}, cancel the tadpole at this scale with the counterterm and then we go to the scale $j_{max} - 1$ and so on. But in our case as the cancelation between the tadpoles and counters are *exact* for all energy scales, we omit the index j during the renormalization process.

Here we have two lemmas concerning the combinatorics of the Wick contractions and the sum of the amplitudes of trivial planar graphs. The first lemma is about combinatorics :

Lemme A.6.1 *The number of planar divergent graph of order n (whose amplitude is proportional to λ^n) with k counterterm attached reads :*

$$f_n^k = \frac{(2n-k)C_n^k}{n}\times k! = \frac{(2n-k)(n-1)!}{(n-k)!}. \tag{2}$$

Preuve A.6.1 *After the contractions of the vertex within itself or contraction with the counterterm, there are exactly n objects which could be either counterterms or inner tadpoles. Then we consider all the combinations of these objects forming a graph. The first position among the $(2n-k)$ ones is special, as this breaks the cyclic ordering. We could first of all choose a line for attaching one counterterm. There are $(2n-k)$ possibilities. Then there are $n-1$ objects left including $k-1$ counterterms. The possibilities of having $k-1$ counterterm attached is $C_{k-1}^{n-1} \times (k-1)!$. By choosing the counterterm we considered also all possibilities of choosing tadpoles.*

So the possibilities of having k counterterms attached at order n reads :

$$\begin{aligned} P_n^k &= (2n-k) \times C_{n-1}^{k-1} \times (k-1)! = (2n-k) \times \frac{(n-1)!(k-1)!}{(k-1)!(n-1-k+1)!} \\ &= (2n-k) \times \frac{(n-1)!}{(n-k)!} = \frac{(2n-k)}{n} \frac{n!}{k!(n-k)!} \times k! \\ &= \frac{(2n-k)C_n^k}{n} \times k!. \end{aligned} \tag{3}$$

The following lemma summarizes the fact that renormalization in this leaves no remainder :

Lemme A.6.2 *The sum of the amplitudes of planar divergent graphs, under the constraint that each line should contract with a nearest neighbor, at arbitrary scale j of order λ^n vanishes exactly :*

$$\begin{aligned} G_n &= \int dx_i \frac{1}{2n} \mathrm{Tr} \prod_{\mathrm{i}=1}^{\mathrm{n}} \mathrm{C_i}(\mathrm{x_i} - \mathrm{x_{i+1}})(\mathrm{A} - \mathrm{B})^{\mathrm{n}} \\ &= \frac{1}{2n}[C^j(x_1 - x_2)C^j(x_2 - x_3) \times \cdots C^j(x_n - x_1)](A-B)^n = 0. \end{aligned} \tag{4}$$

where

$$A = -2\sqrt{\lambda} C_j^{1/2}(x,x) C_j^{1/2}(x,x) = -2\sqrt{\lambda} T^j = B, \tag{5}$$

and $\mathrm{Tr} \prod \mathrm{C^j}(\mathrm{x_i} - \mathrm{x_{i+1}})$ *means the product of all the propagators of the graph according to the cyclic order.*

Preuve A.6.2 *For a planar divergent graph with k counterterm attached at order n, the*

amplitude reads :

$$\begin{aligned} G_n^k &= \frac{1}{(k+1)!} C_{k+1}^1 (-\frac{1}{2}) \frac{(-1)^{(2n-k+1)}}{2n-k} \mathrm{Tr} \prod \mathrm{C}(\mathrm{x_i} - \mathrm{x_{i+1}}) \mathrm{A}^{\mathrm{n-k}} \mathrm{y}^{\mathrm{k}} \mathrm{P}_{\mathrm{n}}^{\mathrm{k}} \\ &= \frac{1}{2n} \frac{(-1)^k}{k!} \mathrm{Tr} \prod_{\mathrm{i}=1}^{\mathrm{n}} \mathrm{C}(\mathrm{x_i} - \mathrm{x_{i+1}}) [\mathrm{C}_{\mathrm{n}}^{\mathrm{k}} \mathrm{A}^{\mathrm{n-k}} \mathrm{B}^{\mathrm{k}} \mathrm{k}!] \\ &= \frac{1}{2n} \mathrm{Tr} \prod_{\mathrm{i}} \mathrm{C}(\mathrm{x_i} - \mathrm{x_{i+1}}) [\mathrm{C}_{\mathrm{n}}^{\mathrm{k}} \mathrm{A}^{\mathrm{n-k}} (-\mathrm{B})^{\mathrm{k}}]. \end{aligned} \tag{6}$$

Then the sum of the amplitude reads :

$$\frac{1}{2n} \mathrm{Tr} \prod_{\mathrm{i}=1}^{\mathrm{n}} \mathrm{C}(\mathrm{x_i} - \mathrm{x_j}) \sum_{\mathrm{k}=0}^{\mathrm{n}} [\mathrm{C}_{\mathrm{n}}^{\mathrm{k}} \mathrm{A}^{\mathrm{n-k}} (-\mathrm{B})^{\mathrm{k}}] = \frac{1}{2\mathrm{n}} \mathrm{Tr} \prod_{\mathrm{i}=1}^{\mathrm{n}} \mathrm{C}(\mathrm{x_i} - \mathrm{x_{i+1}}) (\mathrm{A} - \mathrm{B})^{\mathrm{n}} = 0, \tag{7}$$

as we have $A = B = -2\sqrt{\lambda} T^j$.

A.6.2 A Single Loop Vertex with Crossings

In this subsection we consider the renormalization of the divergent graphs with crossings, see Figure A.12.

Lemme A.6.3 *The divergent graphs with crossings are canceled by the corresponding graphs with counterterms.*

Preuve A.6.3 *Let us consider a divergent graph with order* n $n-m$ *crossings and* k *decorations by counterterms. We note the convergent crossing part by* Q_{m-n}. *The combinatoric factor of such graph reads :*

$$P_m^k = (2m-k) \times C_n^k \times k!. \tag{8}$$

The idea of the proof is the same as the case of primary divergent graphs.

The amplitude of such a graph reads :

$$\begin{aligned} G_m^k &= \frac{1}{(k+1)!} C_{k+1}^1 (-\frac{1}{2}) \frac{(-1)^{(2m-k+1)}}{2m-k} \mathrm{Tr} \prod_{\mathrm{i}=1}^{\mathrm{n}} \mathrm{C}(\mathrm{x_i} - \mathrm{x_{i+1}}) \mathrm{A}^{\mathrm{n-k}} \mathrm{B}^{\mathrm{k}} \mathrm{P}_{\mathrm{n}}^{\mathrm{k}} \\ &\times Q_{m-n}(\lambda^{m-n}) \\ &= \frac{1}{2} \frac{(-1)^k}{k!} \mathrm{Tr} \prod_{\mathrm{i}=1}^{\mathrm{n}} \mathrm{C}(\mathrm{x_i} - \mathrm{x_{i+1}}) [\mathrm{C}_{\mathrm{n}}^{\mathrm{k}} \mathrm{A}^{\mathrm{n-k}} \mathrm{B}^{\mathrm{k}} \mathrm{k}!] \times \mathrm{Q}_{\mathrm{m-n}}(\lambda^{\mathrm{m-n}}) \\ &= \frac{1}{2} \mathrm{Tr} \prod_{\mathrm{i}} \mathrm{C}(\mathrm{x_i} - \mathrm{x_{i+1}}) [\mathrm{C}_{\mathrm{n}}^{\mathrm{k}} \mathrm{A}^{\mathrm{n-k}} (-\mathrm{B})^{\mathrm{k}}] \mathrm{Q}_{\mathrm{m-n}}(\lambda^{\mathrm{m-n}}), \end{aligned} \tag{9}$$

and the sum reads :

$$\begin{aligned} & \mathrm{Tr}\prod_{i=1}^{n} C(x_i - x_{i+1}) \sum_{k=0}^{n} [C_n^k A^{n-k} (-B)^k] Q_{m-n} \\ = \quad & \mathrm{Tr}\prod_{i=1}^{n} C(x_i - x_j)(A-B)^n Q_{m-n}(\lambda^{m-n}) = 0, \end{aligned} \tag{10}$$

as we have $A = B = -2\sqrt{\lambda} T_j$*. Remark that this lemma holds also for the case where there are still resolvents in the loop vertex and they could appear anywhere in the vertex. This is also the case for the crossings. The only requirement is that their positions should be fixed in all graphs so that the combinatorics lemma is still valid.*

Some typical such graphs are shown in Figure A.12. Figure A.12 shows also the process of the renormalization.

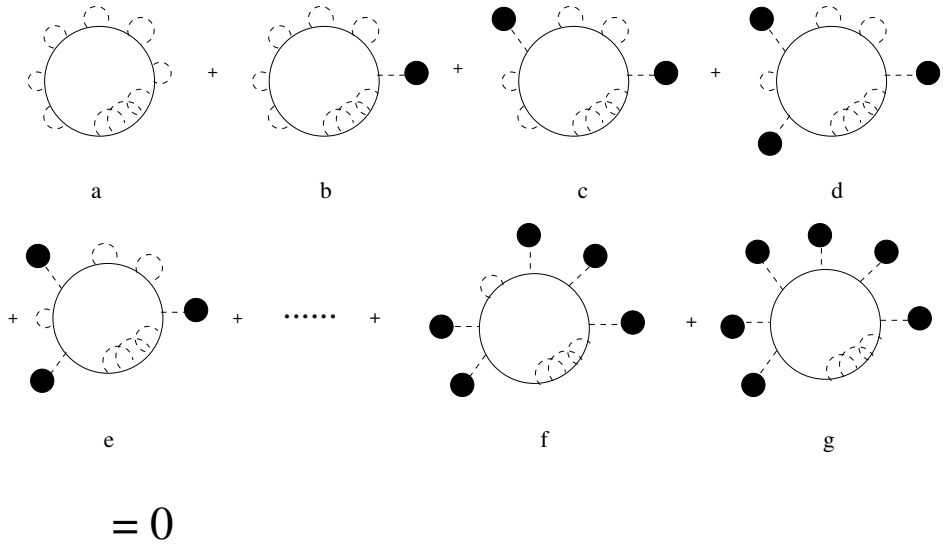

FIGURE A.12: A sketch of renormalization of divergent graph of order 9 with crossings. There could be an arbitrary number of uncleaned resolvents in the loop vertex.

A.6.3 The General Case

A typical tree of loop vertices with divergent part, crossing and counterterms is shown in the Figure A.13.

For the cancelation of the divergences in the tree of loop vertices we go back to the direct representation, as the amplitude for each vertices factorizes. This is the key point for the renormalization of the divergent general tree.

For each divergent vertex we cancel it by the counterterms as shown in last section, which means we use different trees to cancel each divergent term, each tree having exactly the same structure as the remainder vertices of the divergent term.

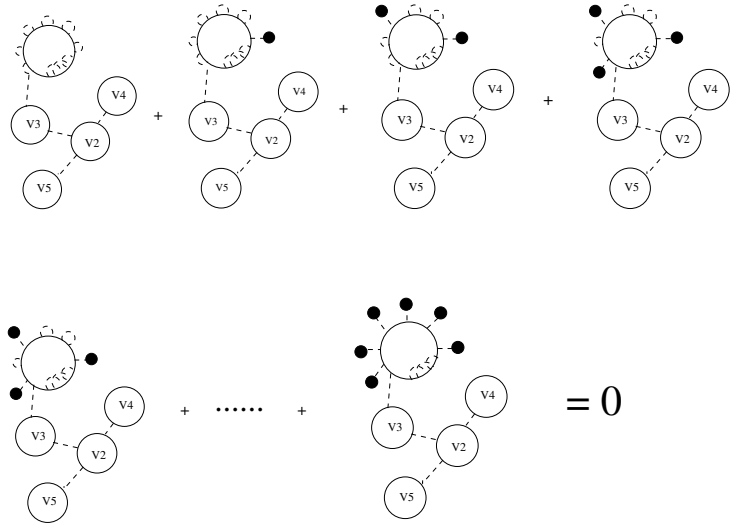

FIGURE A.13: A sketch of renormalization of a tree of loop vertices. The vertices v_i are all complex objects like the one drawn explicitly.

A.7 Borel summability

Let us introduce the N-th order Taylor remainder operator R^N which acts on a function $f(\lambda)$ through

$$R^N f = f(\lambda) - \sum_{n=0}^{N} a_n \lambda^n = \lambda^{N+1} \int_0^1 \frac{(1-t)^N}{N!} f^{(N+1)}(t\lambda) dt. \tag{1}$$

Théorème A.7.1 *(Nevanlinna)[23]*

A series $\sum_{n=0} \frac{a_n}{n!} \lambda^n$ is Borel summable to the function $f(\lambda)$ if the following conditions are met :

- *For some rational number $k > 0$, $f(\lambda)$ is analytic in the domain $C_R^1 = \{\lambda \in C : \Re\lambda^{-1} > R^{-1}\}$.*
- *The function $f(\lambda)$ admits $\sum_{n=0}^{\infty} a_n \lambda^n$ as a strong asymptotic expansion to all orders as $|\lambda| \to 0$ in C_R with uniform estimate in C_R^1 :*

$$\left|R^N f\right| \leqslant A B^N N! |\lambda|^{N+1}. \tag{2}$$

where A and B are some constants.

Then the Borel transform of order k reads :

$$B_f(u) = \sum_{n=0}^{\infty} \frac{a_N}{N!} u^N, \tag{3}$$

it is holomorphic for $|u| < B^{-1}$*, it admits an analytic continuation to the strip* $\{u \in C : |\Im u| < R, \Re u > 0\}$ *and for* $0 \leqslant R$*, one has*

$$f(\lambda) = \frac{1}{\lambda}\int_0^\infty B_f(u) exp[-(u/\lambda)](u/\lambda)du. \tag{4}$$

Théorème A.7.2 *The perturbation series of the connected function for* ϕ_2^4 *theory is Borel summable.*

Proof From the last section and last theorem we know that the analyticity domain for λ is $\mathrm{Re}\lambda > 0$ which means

$$-\frac{\pi}{4} \leq \mathrm{Arg}\sqrt{\lambda} \leq \frac{\pi}{4} \tag{5}$$

and $|\lambda| < \frac{1}{2K}$.

Then we have for each resolvent

$$|\mathcal{R}| = |\frac{1}{1 + 2i\sqrt{\lambda}C\sigma}| \leq \sqrt{2}, \tag{6}$$

and for the leaf tadpole

$$|\hat{\mathcal{R}}| = |\frac{1}{1 + 2i\sqrt{\lambda}C\sigma} - 1| \leq \sqrt{2} + 1, \tag{7}$$

since the covariance C and the field σ are real.

However, since $-\frac{\pi}{4} \leq \mathrm{Arg}\sqrt{\lambda} \leq \frac{\pi}{4}$ in the Borel plane of λ, the term $e^{2i\sqrt{\lambda}\sigma T_\Lambda}$ could be written as :

$$e^{2i\sqrt{\lambda}\sigma} = e^{2i|\sqrt{\lambda}|\sigma\cos\theta T_\Lambda} e^{-2|\sqrt{\lambda}|\sigma\sin\theta T_\Lambda}, \tag{8}$$

where $\theta = \mathrm{Arg}\sqrt{\lambda}$ and $|\sqrt{\lambda}|$ is the norm of $\sqrt{\lambda}$. This term cannot be bounded simply by 1, since the second term in (8) is not oscillating but could diverge for negative values of σ.

So before taking the norm for this problematic term we rewrite the $\sin\theta$ term as :

$$\int d\mu\sigma e^{-1/2\int d^2x\sigma^2} e^{-2\int d^2x|\sqrt{\lambda}|\sin\theta\sigma T_\Lambda} = \int d\mu\sigma e^{-1/2\int d^2x(\sigma+2|\sqrt{\lambda}|T_\Lambda)+2\mathcal{V}\sin^2\theta|\lambda|T_\Lambda^2}, \tag{9}$$

and bound the $\cos\theta$ term simply by 1.

The term $e^{2\mathcal{V}\sin^2\theta|\lambda|T_\Lambda^2}$ could diverge at worst as $e^{\mathcal{V}|\lambda|T_\Lambda^2}$ where $\theta = \pm\pi/4$. But this is not dangerous since we could still choose the coefficient a in the convergent factor $e^{-a\mathcal{V}j_{max}^2} \sim e^{-a\mathcal{V}T_\Lambda^2}$ that we have gained from the j-lines (crossings and/or nesting) in the occupied volume $\mathcal{V}$.

Hence we still have

$$\begin{aligned} |G_N| &= \int d\nu(\sigma, w) \prod_{l,l' \in \mathcal{T}} |\lambda^N||\hat{\mathcal{R}}_l(\sigma)||\mathcal{R}_{l'}(\sigma, w_{l'})| \\ &\cdot \quad |e^{\int i\cos\theta\sigma T_\Lambda}| e^{|\mathcal{V}|(2+1)\lambda T_\Lambda^2} e^{-a\mathcal{V}T_\Lambda^2} \leq (K|\lambda|)^N. \end{aligned} \tag{10}$$

We expand the connected function up to order N in λ by an explicit Taylor formula with integral remainder (1) followed by explicit Wick contractions. We have

$$|\sum_{n=N+1}^{\infty} G_n| < |\lambda|^{N+1} K^N (2N)!! \leq |\lambda|^{N+1} K^N (N)! \tag{11}$$

which leads to the $K^N N!$ term in formula (2), where K is an arbitrary positive constant times the possible factors $\sqrt{2}$ from $\mathcal{R}$ and $\sqrt{2}+1$ from $\hat{\mathcal{R}}$.

Hence this proves the theorem. .

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B

Paper 2

Loop Vertex Expansion for Φ^{2k} Theory in Zero Dimension

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Abstract

In this paper we extend the method of loop vertex expansion to interactions with degree higher than 4. As an example we provide through this expansion an explicit proof that the free energy of ϕ^{2k} scalar theory in zero dimension is Borel-Le Roy summable of order $k-1$. We detail the computations in the case of a ϕ^6 interaction.

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MSC : 81T08, Pacs numbers : 11.10.Cd, 11.10.Ef
Key words : Constructive field theory, Loop vertex expansion, Borel summability.

B.1 Introduction

New constructive Bosonic field theory methods have been recently proposed [16, 17, 53]. The method called Loop vertex expansion or Cactus expansion [16, 17, 54]. is based on applying a canonical forest formula to repackage perturbation theory in a better way. This allows to compute the connected quantities of the theory by the same formula but summed over trees rather than forests. Combining the forest formula with the intermediate field method leads to a convenient resummation of ϕ^4 perturbation theory.

The main advantage of this formalism over previous cluster and Mayer expansions is that connected functions are captured by a single formula, and e.g. a Borel summability theorem for matrix ϕ^4 models can be obtained which scales correctly with the size of the matrix.

In this paper we extend this method, which at first sight looks limited to ϕ^4 interactions, to show that it is in fact suitable for any stable quantum field theory. For simplicity we restrict ourselves to interactions of the $\lambda\phi^{2k}$ type in zero dimension. We introduce several intermediate fields instead of one for the ϕ^4 model. We also take care of the integration contours to bound the integral over intermediate fields. We prove the Borel-Le Roy summability of the right order for this class of theories. Extension to quantum field theories in more than 0 dimension in the line of [17] is devoted to a future publication, but should follow from the method of this paper and the local nature of the interaction.

B.2 The Forest Formula

This formula, a key tool in constructive theory, was perfected along the years by many authors [12, 13]. It is shown here as a Taylor-Lagrange expansion, in which a function of many link variables is expanded around the origin in a careful and symmetric way which stops with an integral remainder before the derivatives create any cycles.

Consider n points. The set of pairs P_n of such points has $n(n-1)/2$ elements $\ell = (i,j)$ for $1 \leq i < j \leq n$. Consider a smooth function f of $n(n-1)/2$ variables x_ℓ, $\ell \in \mathcal{P}_n$. Noting ∂_ℓ for $\frac{\partial}{\partial x_\ell}$, the forest formula is

Théorème B.2.1

$$f(1,\dots,1) = \sum_{\mathcal{F}} \Big[\prod_{\ell\in\mathcal{F}} \int_0^1 dw_\ell\Big] \Big([\prod_{\ell\in\mathcal{F}} \partial_\ell] f\Big) \cdot [X^{\mathcal{F}}(\{w_{\ell'}\})] \tag{1}$$

where

- *the sum over $\mathcal{F}$ is over forests over the n vertices, including the empty one,*
- *$x_\ell^{\mathcal{F}}(\{w_{\ell'}\})$ is the infimum of the $w_{\ell'}$ for ℓ' in the unique path from i to j in $\mathcal{F}$, where $\ell = (i,j)$. If there is no such path, $x_\ell^{\mathcal{F}}(\{w_{\ell'}\}) = 0$ by definition.*
- *The symmetric n by n matrix $X^{\mathcal{F}}(\{w\})$ defined by $X_{ii}^{\mathcal{F}} = 1$ and $X_{ij}^{\mathcal{F}} = x_{ij}^{\mathcal{F}}(\{w_{\ell'}\})$ for $1 \leq i < j \leq n$ is positive.*

Preuve B.2.1 *We do not reproduce here the many proofs of formula (1) [12][13], but we recall the reason for which the matrix $X^{\mathcal{F}}(\{w\})$ is positive. It is because for any ordering of the $\{w\}$ parameters it can be written as a (different!) convex combination of positive block matrices of the I_q type.*

Définition B.2.1 *A block I_q of dimension q is defined as a $q \times q$ matrix with all the elements* 1*. For example, a block of dimension* 3 *is :*

$$I_3 = \begin{pmatrix} 1 & 1 & 1 \\ 1 & 1 & 1 \\ 1 & 1 & 1 \end{pmatrix}. \tag{2}$$

Consider indeed a forest $\mathcal{F}$ with $p \leq n-1$ elements and an ordering

$$0 = w_{p+1} \leq w_p \leq w_{p-1} \leq ... \leq w_1 \leq w_0 = 1, \tag{3}$$

then

$$X^{\mathcal{F}}(\{w\}) = \sum_{k=1}^{p+1} (w_{k-1} - w_k) X^{\mathcal{F},k} \tag{4}$$

where $X_{ij}^{\mathcal{F},k}$ is 1 if i and j are connected by the $k-1$ first lines of the forest, and is 0 otherwise. We have

$$\sum_{k=1}^{p+1} (w_{k-1} - w_k) = 1. \tag{5}$$

Therefore $X^{\mathcal{F},k}$ is a matrix obtained by gluing the blocks corresponding to the connected components of the forest $\mathcal{F}^k$, where $\mathcal{F}^k$ is the subforest of $\mathcal{F}$ made of the $k-1$ first lines of the forest in the ordering.

We need later the fact that the Gaussian measure $d\mu_{I_q}(a_1, ...a_q)$ with covariance I_q really corresponds to a single Gaussian variable, say a_1, with covariance 1, plus $q-1$ delta functions :

$$d\mu_{I_q}(a_1...a_q) = \frac{da_1}{\sqrt{2\pi}} e^{-a_1^2/2} \prod_{i=2}^{q} \delta(a_1 - a_i) da_i \, . \tag{6}$$

B.3 ϕ^6 constructive theory in zero dimension

We consider a massless ϕ^6 scalar theory in zero dimension, where ϕ is simply a number. The Lagrangian reads :

$$\mathcal{L} = -\frac{1}{2}\phi^2 - \lambda\phi^6 \tag{1}$$

and the partition function is

$$Z(\lambda) = \int \frac{d\phi}{\sqrt{2\pi}} e^{-\frac{1}{2}\phi^2} e^{-\lambda\phi^6}. \tag{2}$$

The covariance of the normalized Gaussian measure $\frac{d\phi}{\sqrt{2\pi}} e^{-\frac{1}{2}\phi^2}$ is simply

$$< \phi^2 >= 1. \tag{3}$$

B.3.1 Intermediate Field Representation

We introduce a real intermediate field σ to rewrite the interaction. This leads to

$$Z(\lambda) = \int \frac{d\phi}{\sqrt{2\pi}} e^{-\frac{1}{2}\phi^2} e^{-\lambda\phi^6} = \int \frac{d\phi}{\sqrt{2\pi}} e^{-\frac{1}{2}\phi^2} \int \frac{d\sigma}{\sqrt{2\pi}} e^{-\frac{1}{2}\sigma^2} e^{i\sqrt{2\lambda}\phi^3\sigma}. \tag{4}$$

The induced interaction term could be further transformed as

$$\sqrt{2}\phi^3\sigma = \frac{1}{\sqrt{2}}[(\phi\sigma + \phi^2)^2 - \phi^2\sigma^2 - \phi^4]. \tag{5}$$

We then introduce another three intermediate fields to write the partition function as

$$\begin{aligned} Z(\lambda) &= \int \frac{d\phi}{\sqrt{2\pi}} e^{-\frac{1}{2}\phi^2} \int \frac{d\sigma}{\sqrt{2\pi}} e^{-\frac{1}{2}\sigma^2} \int \frac{da\sqrt{i}}{\sqrt{2\pi}} e^{i[(2\lambda)^{1/4}(\phi\sigma+\phi^2)a - a^2/2]} \\ &\times \int \frac{db}{\sqrt{2i\pi}} e^{-i[(2\lambda)^{1/4}\phi\sigma b - b^2/2]} \int \frac{dc}{\sqrt{2i\pi}} e^{-i[(2\lambda)^{1/4}\phi^2 c - c^2/2]}. \end{aligned} \tag{6}$$

Integrating out the fields ϕ and σ we get :

$$Z(\lambda) = \int \frac{da\sqrt{i}}{\sqrt{2\pi}} \frac{db}{\sqrt{2i\pi}} \frac{dc}{\sqrt{2i\pi}} e^{i(b^2+c^2-a^2)/2} e^V \tag{7}$$

where

$$V = -\frac{1}{2}\mathrm{Tr}\ln[\mathbf{I} + \mathrm{i}(2\lambda)^{1/4} \begin{pmatrix} c-a & b-a \\ b-a & 0 \end{pmatrix}] = -\frac{1}{2}\mathrm{Tr}\ln(\mathbf{I} + \mathrm{i}H), \tag{8}$$

where

$$\mathbf{I} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \quad H = (2\lambda)^{1/4} \begin{pmatrix} c-a & b-a \\ b-a & 0 \end{pmatrix}. \tag{9}$$

Obviously H defined above is Hermitian for $\lambda \geq 0$.

The new resulting integrals (7) over a, b and c are oscillating and still formal, and we have to slightly change the contours of integration to make them well-defined, but this is postponed to the next section.

We use the replica method to write the exponential as :

$$e^V = \sum_n \frac{V^n}{n!} = \sum_n \frac{1}{n!} \prod_{v=1}^{n} V_v, \tag{10}$$

where $V_v = V_v(a^v, b^v, c^v)$.

Then applying the forest formula, the connected function could be written as a sum over trees $\mathcal{T}$ whose nodes are loop vertices, and whose lines are of three different types, corresponding to Wick contractions of a, b and c. Calling $\mathcal{T}_a$, $\mathcal{T}_b$ and $\mathcal{T}_c$ the three corresponding subset of lines of the tree we have

Théorème B.3.1

$$\begin{aligned}
\log Z(\lambda) &= \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{\mathcal{T} \text{ with } n \text{ vertices}} Y_{\mathcal{T}}, \\
Y_{\mathcal{T}} &= \Big\{ \prod_{\ell \in \mathcal{T}} [\int_0^1 dw_\ell] \Big\} \int d\nu_{\mathcal{T}}(\{a^v, b^v, c^v\}, \{w\}) \\
&\times \Big\{ \prod_{\ell \in \mathcal{T}_a} [\delta_{v,v'} \frac{\delta}{\delta a^{v(\ell)}} \frac{\delta}{\delta a^{v'(\ell)}}] \Big\} \Big\{ \prod_{\ell \in \mathcal{T}_b} [\delta_{v,v'} \frac{\delta}{\delta b^{v(\ell)}} \frac{\delta}{\delta b^{v'(\ell)}}] \Big\} \\
&\times \Big\{ \prod_{\ell \in \mathcal{T}_c} [\delta_{v,v'} \frac{\delta}{\delta c^{v(\ell)}} \frac{\delta}{\delta c^{v'(\ell)}}] \Big\} \prod_{v=1}^{n} V_v \qquad (11)
\end{aligned}$$

where

- *each line ℓ of the tree joins two different loop vertices $V^{v(\ell)}$ and $V^{v'(\ell)}$,*
- *the sum is over trees joining n loop vertices, which have therefore $n-1$ lines. These lines can be of type a, b or c.*
- *the normalized "imaginary" Gaussian measure $d\nu_T(\{a^v, b^v, c^v\}, \{w\})$ over the three intermediate fields a^v, b^v and c^v has covariance*

$$< a^v, a^{v'} > = -iw^T(v, v', \{w\}), \qquad (12)$$

$$< b^v, b^{v'} > = iw^T(v, v', \{w\}), \qquad (13)$$

$$< c^v, c^{v'} > = iw^T(v, v', \{w\}), \qquad (14)$$

$$< a^v, b^v > = < b^v, c^v > = < a^v, c^v > = 0 \qquad (15)$$

where $w^T(v, v', \{w\})$ is 1 if $v = v'$, and the infimum of the w_ℓ for ℓ running over the unique path from v to v' in T if $v \neq v'$. This measure will become well-defined since the matrix w^T is positive, if we perform appropriate contour deformations.

If we distinguish the matrix indices which correspond to the former ϕ and σ fields, there are in fact four kinds of half-vertices in the loop vertex expansion and five different kinds of lines. The coupling constant for each half-vertex is $(2\lambda)^{1/4}$, and the coupling constant for each vertex (namely each line of the loop vertex tree) is therefore $(2\lambda)^{1/2}$.

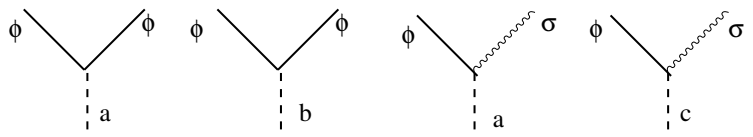

FIGURE B.1: The 4 half-vertices

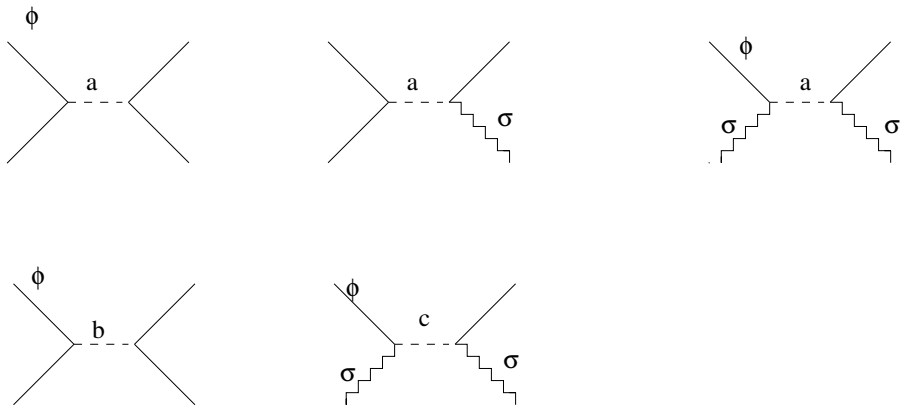

FIGURE B.2: The 5 vertices

B.3.2 Contour Deformation

The integral over the fields a, b and c is not absolutely convergent, so we have to choose the right contour to make it well-defined. As the covariance for the three fields are quite similar, we will consider a first and deform the integration contour. The idea is that, we first of all use the formula (4) to write the field a as an independent sum of $p+1$ fields a_k according to the blocks :

$$a = \sum_{k=1}^{p+1} \sum_{v=1}^{n} a_{k,v} \tag{16}$$

whose covariance is

$$< a_{k,v}, a_{k,v'} >= (w_{k-1} - w_k) X^{\mathcal{F},k}_{v,v'}. \tag{17}$$

Precisely because the covariance of a_k is made of blocks r, we should perform a single contour deformation for each block. We have a formula similar to (6) for each block with variables $a_1, ...a_n$, but now we should remember that the covariances are iI_q, not I_q. Hence we have

$$d\mu_{iI_q} = \frac{da_1}{\sqrt{2\pi}} e^{-ia_1^2/2} \prod_{i=2}^{q} \delta(a_1 - a_i) da_i. \tag{18}$$

In the partition function we have an integration of the type

$$\int_{-\infty}^{\infty} da f(a) e^{ia^2/2} \tag{19}$$

where f is the product of the resolvents which are analytic and bounded in a open neighborhood of the band $\mathcal{B} = \{\Im a \leq A^{-1}\}$ of the real axis, where A is large. This integral is not absolutely convergent. Nevertheless we can bound it in terms of $\sup_{\mathcal{B}} |f|$. Indeed we can deform the integral contour so that the new contour remains in the band $\mathcal{B}$ and the new variable is :

$$a_1' \;=\; a_1 - i\frac{a_1}{A|a_1| + 1},\; a_1' \to a_1 - i\text{sgn } a_1/A \text{ if } a_1 \to \pm\infty. \tag{20}$$

Then the bound of the integral over a_1 becomes :

$$\begin{aligned}
&\left| \int da_1 f(a_1') e^{-ia_1^2/2(w_{k-1}-w_k) - \frac{2a_1^2}{2(w_{k-1}-w_k)(A|a_1|+1)} + i\frac{a_1^2}{2(w_{k-1}-w_k)(A|a_1|+1)^2}} \right| \\
&\leq \sup_{\mathcal{B}} |f| \int da_1 e^{-\frac{2a_1^2}{2(w_{k-1}-w_k)(A|a_1|+1)}} \\
&\leq 2(w_{k-1}-w_k)A \sup_{\mathcal{B}} |f|. \qquad (21)
\end{aligned}$$

So each time we integrate out an intermediate field we get $\sup_{\mathcal{B}} |f|$ times a factor $2(w_{k-1} - w_k)A$ in the bound. Then for the integration of all the intermediate fields a_k we would have at order n a total factor in the bound :

$$\begin{aligned}
&\prod_{k=1}^{p+1}\prod_{v=1}^{n} 2A(w_{k-1}-w_k) \leq \prod_{k=1}^{p+1}\prod_{v=1}^{n} e^{2A(w_{k-1}-w_k)} = \prod_{v=1}^{n} e^{\sum_{k=1}^{p+1} 2A(w_{k-1}-w_k)} \\
\leq \; &\prod_{v=1}^{n} e^{2A} \leq (e^{2A})^n \qquad (22)
\end{aligned}$$

where we have used the fact that

$$\sum_k (w_{k-1} - w_k) \leq 1. \qquad (23)$$

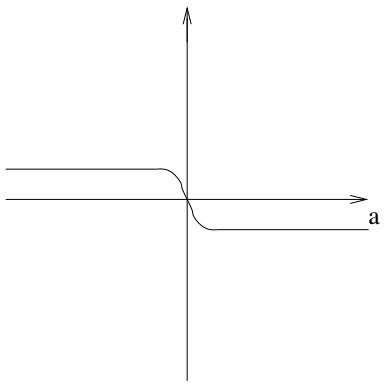

FIGURE B.3: The integral contour for a.

As the signs for b and c are different from a in the covariance, the integral contour for b and c are also different. The contour for b could be chosen as :

$$b_1' = b_1 + i\frac{b_1}{A|b_1|+1}, \; b_1' \to b_1 + i\text{sgn } b_1/A \text{ if } b_1 \to \pm\infty \qquad (24)$$

and the integral contour for c is the same as that for b. Then the bounds proceed exactly like in (21).

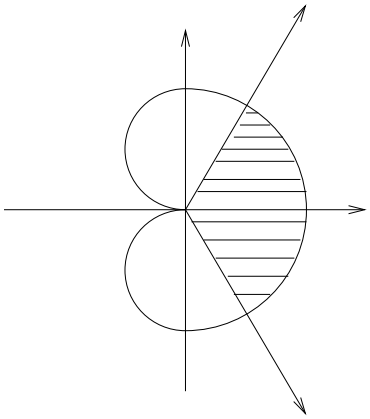

FIGURE B.4: The analyticity domain C_R^2

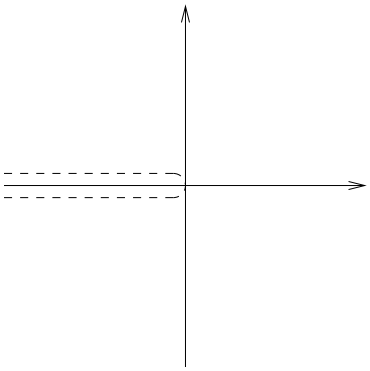

FIGURE B.5: The analyticity domain $\mathcal{D}^2$

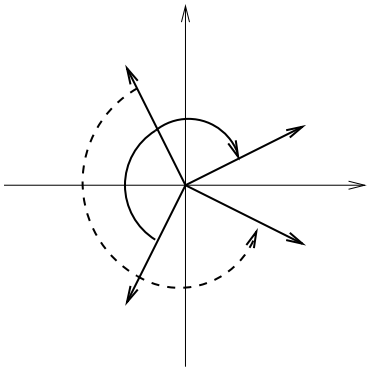

FIGURE B.6: The analytic continuation

The function f is a product of resolvents of the type $(1+iH)^{-1}$ turning around the tree after using the tree formula [16]. On the real axis $\|(1+iH)^{-1}\| \leq 1$. But after contour deformation the bound is slightly altered. $1+iH$ will also be changed into

$$1 + iH - (2\lambda)^{1/4} \begin{pmatrix} \epsilon(a) & \epsilon(a) \\ \epsilon(a) & 0 \end{pmatrix} \tag{25}$$

with $\epsilon = 1/A$ a small number. As $\lambda << 1$, $\|\lambda^{1/4}\begin{pmatrix}\epsilon(a) & \epsilon(a)\\ \epsilon(a) & 0\end{pmatrix}\| << 1$. So after we change the integral contours, the denominators are still bounded by $K = 1+O(1/A)$. This bound changes to $\sqrt{2}+O(1/A)$ if we take $-\pi < Arg\lambda < +\pi$, that is $-\pi/4 < Arg\lambda^{1/4} < +\pi/4$. As essentially the factor $O(1/A)$ doesn't change the bound of the resolvent, hence the power counting of the connected function, we shall forget it in the rest of this paper.

As H is a linear function of b and c, we could use the same method for b and c and the resulting integral is finite.

B.3.3 Borel summability

Let us introduce the N-th order Taylor remainder operator R^N which acts on a function $f(\lambda)$ through

$$R^N f = f(\lambda) - \sum_{n=0}^{N} a_n \lambda^n = \lambda^{N+1} \int_0^1 \frac{(1-t)^N}{N!} f^{(N+1)}(t\lambda) dt. \tag{26}$$

Théorème B.3.2 *(Nevanlinna-Le Roy)[22, 23]*

A series $\sum_{n=0} \frac{a_n}{n!}\lambda^n$ is Borel summable to the function $f(\lambda)$ of order k if the following conditions are met :

- *For some rational number $k > 0$, $f(\lambda)$ is analytic in the domain $C_R^k = \{\lambda \in C : \Re\lambda^{-1/k} > R^{-1}\}$. C_R is a disk for $k = 1$.*
- *The function $f(\lambda)$ admits $\sum_{n=0}^{\infty} a_n\lambda^n$ as a strong asymptotic expansion to all orders as $|\lambda| \to 0$ in C_R with uniform estimate in C_R^k :*

$$\left|R^N f\right| \leqslant AB^N \Gamma(kN+1)|\lambda|^{N+1}. \tag{27}$$

where A and B are some constants.

Then the Borel-Le Roy transform of order k reads :

$$B_f^{(k)}(u) = \sum_{n=0}^{\infty} \frac{a_n}{\Gamma(kn+1)} u^n, \tag{28}$$

it is holomorphic for $|u| < B^{-1}$, it admits an analytic continuation to the strip $\{u \in C : |\Im u| < R, \Re u > 0\}$ and for $0 \leqslant R$, one has

$$f(\lambda) = \frac{1}{k\lambda} \int_0^{\infty} B_f^{(k)}(u) exp[-(u/\lambda)^{1/k}](u/\lambda)^{(1/k-1)} du. \tag{29}$$

Théorème B.3.3 *The partition function $Z(\lambda)$ for ϕ^6 theory is Borel-Le Roy summable of order 2.*

Preuve B.3.1 *The remainder after the Taylor expansion of ϕ^6 at Nth order reads :*

$$R^N Z(\lambda) = (-\lambda)^{N+1} \int_0^1 dt \int d\phi \frac{(1-t)^N}{N!} \phi^{6(N+1)} e^{-t\lambda\phi^6 - \frac{\phi^2}{2}}. \tag{30}$$

We use the Cauchy-Schwarz inequality :

$$\begin{aligned} |R^N Z(\lambda)| &= |\lambda|^{N+1} \int_0^1 dt \int d\phi \frac{(1-t)^N}{N!} [\phi^{12(N+1)} e^{-2t\lambda\phi^6 - \phi^2}]^{1/2} \qquad (31) \\ \leqslant\ & |\lambda|^{N+1} \int_0^1 dt \frac{(1-t)^N}{N!} (\int d\phi \phi^{12(N+1)} e^{-\phi^2/2})^{1/2} (\int d\phi e^{-2t\lambda\phi^6} e^{-\phi^2/2})^{1/2}. \end{aligned}$$

The first term is bounded by $[(12(N+1))!!]^{1/2}/N! \sim (6N)!!/N! \sim (2N)!$, where $\sim \cdots$ means $\leq K^N \times \cdots$. Now consider the second term. We perform a scaling on ϕ as

$$\lambda^{1/6}\phi = u \tag{32}$$

then

$$\int d\phi e^{-2t\lambda\phi^6} e^{-\phi^2/2} = \int_{-\infty}^{\infty} e^{-2tu^6 - \lambda^{-1/3}u^2} \frac{du}{\lambda^{1/6}}. \tag{33}$$

For $-\pi < Arg\lambda < \pi$, we have $-\pi/3 < Arg(\lambda^{1/3}) < \pi/3$. Let us define $\mathcal{D}^2 = \{\lambda | -\pi < Arg\lambda < \pi\}$. We have $C_R^2 \subset \mathcal{D}^2$. The corresponding analytic domains are shown in figure B.4 and B.5. We shall prove analyticity and Taylor remainder bounds in $\mathcal{D}^2$ rather than C_R^2.

In $\mathcal{D}^2$ the integrand of (33) is analytic in λ and we always have $\Re\lambda^{-1/3} > 0$. Moreover we have uniform convergence

$$|\int_{-\infty}^{\infty} e^{-2tu^6 - \lambda^{-1/3}u^2} \frac{du}{\lambda^{1/6}}| \leq \int_{-\infty}^{\infty} e^{-(\Re\lambda^{-1/3})u^2} \frac{du}{|\lambda^{1/6}|} \leq \sqrt{\pi}. \tag{34}$$

This proves that the partition function is Borel summable of order 2.

The rest of this section is devoted to prove the following more difficult results :

Théorème B.3.4 *The connected function $\log Z(\lambda)$ with potential $\lambda\phi^6$ is Borel-Le Roy summable of order 2.*

Preuve B.3.2 *We use the loop vertex representation (11) of $\log Z(\lambda)$. We shall first prove uniform convergence of this loop vertex representation in the domain $\mathcal{D}_\epsilon^2 = \{\lambda | -\pi < Arg\lambda < \pi$ and $|\lambda| < \epsilon\}$ and then prove the Taylor remainder bound.*

Lemme B.3.1 *In the domain $\mathcal{D}_\epsilon^2$ each term $Y_{\mathcal{T}}(\lambda)$ is bounded by $\epsilon^{(n-1)/2} K^n$.*

Preuve B.3.3 *In the loop vertex expansion remember there are 4 different kinds of half-vertices, as shown in Figure B.1, and 5 different types of tree lines after contraction of the a b or c intermediate fields, as shown in Figure (B.2).*

We shall first of all prove that the resolvents are bounded. Consider

$$\frac{1}{1+iH} = \frac{1}{\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + i(2\lambda)^{1/4} \begin{pmatrix} c-a & b-a \\ b-a & 0 \end{pmatrix}}. \tag{35}$$

The denominator could always be diagonalized and the result reads :

$$\frac{1}{1+iH} = \frac{1}{\begin{pmatrix} 1+i(2\lambda)^{1/4}\omega_+ & 0 \\ 0 & 1+i(2\lambda)^{1/4}\omega_- \end{pmatrix}}, \tag{36}$$

where

$$\begin{aligned} \omega_+ &= \big(c-a+\sqrt{(c-a)^2+4(b-a)^2}\,\big)/2 > 0 \\ \omega_- &= \big(c-a-\sqrt{(c-a)^2+4(b-a)^2}\,\big)/2 < 0. \end{aligned} \tag{37}$$

The analyticity domain for λ contains at least $\mathcal{D}^2$. Hence

$$-\pi/4 < \mathrm{Arg}(\lambda^{1/4}) < \pi/4. \tag{38}$$

It implies

$$|(1+i\lambda^{1/4}\omega_+)^{-1}| < \sqrt{2}\,, \quad |(1+i\lambda^{1/4}\omega_-)^{-1}| < \sqrt{2}. \tag{39}$$

So each resolvent is bounded as

$$\|\frac{1}{1+iH}\| \leq \sqrt{2}(1+O(1/A)). \tag{40}$$

Again we shall forget the inessential factor $O(1/A)$. Now we know that the resolvents multiply around the tree in each contribution $Y_{\mathcal{T}}$ [?]. Hence for a tree of order n, the product of all the $2(n-1)$ resolvents in the tree is bounded by $\sqrt{2}^{2(n-1)}$. The global trace adds a factor 2 to the bound so that

$$\left| \mathrm{Tr} \prod_{\text{around } \mathcal{T}} \frac{1}{1+\mathrm{iH}} \right| \leq 2 \cdot \sqrt{2}^{2(n-1)} = 2^n. \tag{41}$$

Now we consider the vertices. A tree $\mathcal{T}$ at n-th order has $n-1$ vertices. Each vertex contributes a factor $\sqrt{\lambda}$, hence we have a factor $\lambda^{(n-1)/2}$ in $Y_{\mathcal{T}}$. There are 5 different kinds of vertices, in the loop vertex expansion, but when considering the trace over the products of the resolvents, we only have 3 choice each time which corresponding to whether the intermediate field is a, b or c. So the choice over the type of the vertices is bounded by an additional factor 3^{n-1}. Don't forget that we have also a factor $(e^{2A})^n$ from the contour deformation. So, composing this bound with the resolvent bound we have

$$|Y_{\mathcal{T}}(\lambda)| \leq 2^n 3^{n-1} |\lambda|^{(n-1)/2} (e^{2A})^n \leq \epsilon^{(n-1)/2} K^n \tag{42}$$

where $K = 6e^{2A}$. So we have proved this Lemma.

Cayley's theorem states that the number of labeled trees over n vertices is n^{n-2}. Hence combining it with the Lemma we get convergence and analyticity of the loop vertex representation in the domain $\mathcal{D}^2_\epsilon$:

$$\begin{aligned} \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{\mathcal{T} \text{ with } n \text{ vertices}} |Y_{\mathcal{T}}(\lambda)| &\leq \sum_{n=1}^{\infty} \frac{n^{n-2}}{n!} \epsilon^{(n-1)/2} K^n \\ &\leq \sum_{n=1}^{\infty} \epsilon^{(n-1)/2} (eK)^n \end{aligned} \tag{43}$$

where we used Stirling's formula. This converges for small enough ϵ. Actually since $K = 6e^{2A}$, $\epsilon = e^{-2A-2}/36$ works.

We now turn to the Taylor remainder bound. The remainder formula reads :

$$R^N \log Z(\lambda) = \sum_{n=1}^{\infty} \frac{1}{n!} \sum_{\mathcal{T} \text{ with } n \text{ vertices}} R^N Y_{\mathcal{T}}(\lambda). \tag{44}$$

For trees with $\mathcal{T}$ with $n \geq 2N+3$ we have $R^N[Y_{\mathcal{T}}(\lambda)] = Y_{\mathcal{T}}$, hence inserting the estimate of the previous Lemma

$$\begin{aligned} |\sum_{n=2N+3}^{\infty} \frac{1}{n!} \sum_{\mathcal{T} \text{ with } n \text{ vertices}} R^N Y_{\mathcal{T}}(\lambda)| &\leq |\lambda|^{N+1} \sum_{n=2N+3}^{\infty} \epsilon^{(n-1)/2-(N+1)} (eK)^n \\ &\leq |\lambda|^{N+1} K^N \end{aligned} \tag{45}$$

for $\lambda \in \mathcal{D}^2_\epsilon$.

So we need only to consider now trees with $n \leq 2N+2$ vertices. Defining $\bar{Y}$ through $Y_{\mathcal{T}}(\lambda) = \lambda^{(n-1)/2} \bar{Y}_{\mathcal{T}}(\lambda)$ we have for such trees

$$R^N Y_{\mathcal{T}} = \lambda^{(n-1)/2} R^{N-(n-1)/2} \bar{Y}_{\mathcal{T}}, \tag{46}$$

and the following bound :

Lemme B.3.2 *In the domain $\mathcal{D}^2_\epsilon$ we have for trees $\mathcal{T}$ with $n \leq 2N+2$*

$$|\lambda^{(n-1)/2} R^{N-(n-1)/2} \bar{Y}_{\mathcal{T}}| \leq |\lambda|^{N+1} K^N \Gamma(2N-n+1). \tag{47}$$

Preuve B.3.4 *The $R^{N-(n-1)/2}$ operator now acts on the product of resolvents*

$$\mathrm{Tr} \prod_{\text{around } \mathcal{T}} \frac{1}{1+\mathrm{iH}}. \tag{48}$$

We can evaluate it through a Taylor-Lagrange integral formula, and this formula brings intermediate fields a, b or c to the numerator. The choice of which resolvent is derived gives a factorial but which is compensated by the factorial in the Taylor formula itself, so these choices contribute only K^N to the bound. Since each half vertex contributes a coupling constant $\lambda^{1/4}$, the number of such fields brought to the numerator by the Taylor Lagrange formula must obey

$$(n_a + n_b + n_c)/4 = N - (n-1)/2 \tag{49}$$

as this should be compatible with the fact that we expand to order λ^{N+1}.

Therefore we have :

$$\begin{aligned} |R^{N-(n-1)/2} \bar{Y}_{\mathcal{T}}| \;&\leq\; K^N |\lambda|^{N-(n-1)/2} \\ &\sum_{\substack{n_a,n_b,n_c \\ n_a+n_b+n_c=4N-2n+2}} \int d\mu(a) d\mu(b) d\mu(c) a^{n_a} b^{n_b} c^{n_c} \end{aligned} \tag{50}$$

where $d\mu(a)d\mu(b)d\mu(c)$ are the oscillating Gaussian measures with contour deformation for the fields a, b and c respectively. After using the usual bound on the resolvents and Wick contraction for the intermediate fields we get :

$$|\lambda^{(n-1)/2} R^{N-(n-1)/2} \bar{Y}_{\mathcal{T}}| \leq |\lambda|^{N+1} K^N (n_a + n_b + n_c)!! \tag{51}$$

So the remainder is bounded in the worst case $n = 0$ by :

$$\begin{aligned} & |\lambda|^{N+1} K^N (n_a + n_b + n_c)!! \\ =\; & |\lambda|^{N+1} K^N (4N-2n+2)!! \leq \lambda^{N+1} K (4N+2)!! \leq \lambda^{N+1} K^N (2N)! \end{aligned} \tag{52}$$

where K is a generic name for a constant.

Combining lemmas B.3.1 and B.3.2 together with (44) and (45) proves that $\log Z(\lambda)$ *is analytic in some* $\mathcal{D}_\epsilon^2$ *domain, hence in some* C_R^2 *domain and that the Taylor remainder at order* N *is bounded by* $|\lambda|^{N+1}K^N\Gamma(2N+1)$*. This completes the proof of Theorem B.3.4.*

B.4 ϕ^{2k} theory in zero dimension

B.4.1 The intermediate fields for ϕ^{2k} theory

In the general case of a $\lambda\phi^{2k}$ interaction, we could introduce the intermediate fields inductively, and in each step we attribute to the interaction term of a field ϕ with an intermediate field a coupling constant $\lambda^{\frac{1}{2k}}$. In the first step we introduce a first intermediate field σ_1 and forgetting the inessential normalizing factor, and the result reads :

$$e^{-\lambda\phi^{2k}} = \int d\sigma_1 e^{-\sigma_1^2/2+i\sqrt{\lambda}\phi^k\sigma_1} \tag{1}$$

and

$$2\sqrt{\lambda}\phi^k\sigma_1 = [(\lambda^{\frac{1}{2k}}\phi\sigma_1 + \lambda^{\frac{k-1}{2k}}\phi^{k-1})^2 - \lambda^{\frac{1}{k}}\phi^2\sigma_1^2 - \lambda^{\frac{k-1}{k}}\phi^{2k-2}]. \tag{2}$$

For the first term in the r.h.s. we shall introduce another intermediate field σ_2 and we have :

$$e^{i(\lambda^{\frac{1}{2k}}\phi\sigma_1+\lambda^{\frac{k-1}{2k}}\phi^{k-1})^2} = \int d\sigma_2 e^{i(\lambda^{\frac{1}{2k}}\phi\sigma_1+\lambda^{\frac{k-1}{2k}}\phi^{k-1})\sigma_2} e^{-i\sigma_2^2/2}. \tag{3}$$

For the second term we have simply

$$e^{-i\lambda^{\frac{1}{k}}\phi^2\sigma_1^2} = \int d\sigma_3 e^{-i\lambda^{\frac{1}{2k}}\sigma_3\phi\sigma_1} e^{i\sigma_3^2/2}. \tag{4}$$

The third term has the potential ϕ^{2k-2} which means that we have the same type of interaction but with the degree of the potential lowered by 2 and the coupling constant lowered by degree $\lambda^{\frac{1}{k}}$. We could repeat this process inductively until the final form is linear at most in each of the final $[3(k-2)+1]$ intermediate fields σ_i, quadratic at most in ϕ, and trilinear in all fields taken together, which means the field ϕ and all the intermediate fields Remark that we can maintain imaginary factors throughout the induction, by using imaginary Gaussian integrals. Again we integrate out some of the intermediate fields and the initial field ϕ. The result could always be written in the form (up to inessential normalization constants)

$$Z(\lambda) = \int \prod_r da_r \prod_s db_s \prod dc \; e^{i(a_1^2-a_2^2-a_3^2+b_1^2-b_2^2-b_3^2\pm c^2\cdots)/2} e^V \tag{5}$$

where

$$V = -\frac{1}{2}\mathrm{Tr}\ln[\mathrm{A} + \mathrm{iH}(\{\mathrm{a}\},\{\mathrm{b}\},\{\mathrm{c}\}...)]. = -\frac{1}{2}\mathrm{Tr}\ln[\mathrm{G}]. \tag{6}$$

Here $A = diag(1,1,i,-i...)$ where the number of 1 depends on whether k is even or odd : if k is even, there is only one 1 in A and the other diagonal elements are $\pm i$; if k is

odd the first two diagonal elements are 1s and the other diagonal elements are $\pm i$. a_i, b_i, c_i... represent the remaining intermediate fields. H is a symmetric matrix with nonzero elements appearing only in the first line and the fist column, for example :

$$H = \lambda^{\frac{1}{2k}} \begin{pmatrix} \lambda^{\frac{1}{2k}} g_1(a_i, b_i, c_i...) & g_2(a_i, b_i, c_i...) & g_3(a_i, b_i, c_i...) & ... \\ g_2(a_i, b_i, c_i...) & 0 & 0 & ... \\ g_3(a_i, b_i, c_i...) & 0 & 0 & ... \\ ... & ... & ... & ... \end{pmatrix}. \tag{7}$$

Here $g_j(a_i)$ is a sum of *linear* terms in the intermediate fields that appears in the determinant.

We take the $e^{-\lambda\phi^8}$ model for example. In this case $k = 4$, so we associate to each field ϕ a coupling constant $\lambda^{\frac{1}{2k}} = \lambda^{1/8}$. The interaction form can also be written as

$$\int d\sigma db_i dX e^{-XGX^t} e^{-\frac{1}{2}\sigma^2 - \frac{i}{2}(a_1^2 - a_2^2 - a_3^2 + b_1^2 - b_2^2 - b_3^2)} \tag{8}$$

where

$$X = \big(\phi, a_1, a_2, a_3\big) \tag{9}$$

and

$$\begin{aligned} G = A + iH = & \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & i & 0 & 0 \\ 0 & 0 & -i & 0 \\ 0 & 0 & 0 & -i \end{pmatrix} \\ + \quad i\lambda^{1/8} & \begin{pmatrix} -\lambda^{1/8}(b_1 - b_3) & -(b_1 + \sigma) & \sigma & b_1 - b_2 \\ -(b_1 + \sigma) & 0 & 0 & 0 \\ \sigma & 0 & 0 & 0 \\ b_1 - b_2 & 0 & 0 & 0 \end{pmatrix} \end{aligned} \tag{10}$$

where a_i, b_i and σ are the intermediate fields. It is not surprising that we have an element with a coupling constant $\lambda^{1/4}$ in the matrix, as this term corresponds to the interaction term $\phi^2(b_1 - b_3)$, and we associate to each ϕ a factor $\lambda^{1/8}$: We have a similar situation for all other ϕ^{2k} case, see (7).

Then we consider a more complicated example, the $\exp(-\lambda\phi^{10})$ model. In this case we have $k = 5$ and the coupling constant for each field ϕ is $\lambda^{1/10}$. The interaction form can be written as

$$\int da_i dc_i dX e^{-XGX^t} e^{-\frac{i}{2}(a_1^2 - a_2^2 - a_3^2 + c_1^2 - c_2^2 - c_3^2)}, \tag{11}$$

where

$$X = \big(\phi, \sigma, b_1, b_2, b_3\big) \tag{12}$$

and

$$G = A + iH = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & i & 0 & 0 \\ 0 & 0 & 0 & -i & 0 \\ 0 & 0 & 0 & 0 & -i \end{pmatrix} \tag{13}$$
$$+ \quad i\lambda^{1/10} \begin{pmatrix} -\lambda^{1/10}(c_1 - c_3) & -(a_1 - a_2) & -(a_1 + \sigma) & a_1 - a_3 & c_1 - c_2 \\ -(a_1 - a_2) & 0 & 0 & 0 & 0 \\ -(a_1 + \sigma) & 0 & 0 & 0 & 0 \\ a_1 - a_3 & 0 & 0 & 0 & 0 \\ c_1 - c_2 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

where a_i, b_i, c_i and σ are the intermediate fields.

Lemme B.4.1 *The inverse of the matrix G exists and is bounded by $\sqrt{2}$.*

Preuve B.4.1 *The matrix $G = A + iH$ is a symmetric matrix that has only non zero elements in the first line, the first row and the diagonal. We have*

$$G = A + iH = A(\mathbf{I} + iA^{-1}H). \tag{14}$$

As A is a diagonal matrix whose elements are either 1 or $\pm i$, the inverse of A is bounded and has a similar structure. So in the following we consider only the inverse of the matrix $\mathbf{I} + iA^{-1}H$. For a general ϕ^{2k} theory we have

$$iA^{-1}H = i\lambda^{\frac{1}{2k}} \begin{pmatrix} \lambda^{\frac{1}{2k}} d_1 & d_2 & d_3 & ... & ... & d_n \\ \pm i d_2 & 0 & 0 & ... & ... & 0 \\ \pm i d_3 & 0 & 0 & 0 & ... & 0 \\ ... & ... & .. & .... & ... & ... \\ \pm i d_n & 0 & 0 & 0 & 0 & 0 \end{pmatrix}. \tag{15}$$

where d_i is an arbitrary element of $A^{-1}H$ which has non vanishing elements only in the first row and first column. The matrix $A^{-1}H$ has only two non vanishing eigenvalues, each of multiplicity 1. The exact formula for the eigenvalues depends also on whether k is even or not. For k even, we have

$$\omega_\pm = \frac{-\lambda^{\frac{1}{2k}} d_1 (1 \pm \sqrt{1 - \frac{4i\lambda^{-1/k} B}{d_1^2}})}{2} \tag{16}$$

where

$$B = \pm d_2^2 \pm d_3^2 \pm ... \pm d_n^2 \tag{17}$$

is a combination of the squares of the intermediate fields.

When k is odd, we have

$$\omega_{\pm} = \frac{-\lambda^{\frac{1}{2k}} d_1 (1 \pm \sqrt{1 + \frac{4\lambda^{-1/k}}{d_1^2}(d_2^2 - iB')})}{2} \tag{18}$$

and in this case

$$B' = \pm d_3^2 \pm ... \pm d_n^2. \tag{19}$$

Through some basic calculation we can easily find that in both case we have

$$|1 + i\omega_{\pm}| \geq \frac{1}{\sqrt{2}}. \tag{20}$$

So we have

$$\mathbf{I} + iA^{-1}H = P \begin{pmatrix} 1+i\omega_+ & 0 & 0 & 0 & ... & 0 \\ 0 & 1+i\omega_- & 0 & 0 & ... & 0 \\ 0 & 0 & 1 & 0 & ... & 0 \\ ... & ... & ... & ... & ... & ... \\ 0 & 0 & 0 & 0 & ... & 1 \end{pmatrix} P^{-1} \tag{21}$$

(22)

where P is the diagonalizing matrix. Therefore $\mathbf{I} + iA^{-1}H$ *is invertible and its inverse has eigenvalues* $(1 + i\omega_{\pm})^{-1}$ *and 1 so that we have*

$$\| \, [\mathbf{I} + iA^{-1}H]^{-1} \, \| \leqslant \sqrt{2}. \tag{23}$$

This proves this lemma.

B.4.2 The analytic domain and contour deformation

In the $\lambda\phi^{2k}$ theory, the analytic domain for the coupling constant λ is

$$-\frac{(k-1)\pi}{2} \leqslant \text{Arg } \lambda \leqslant -\frac{(k-1)\pi}{2}. \tag{24}$$

As for each ϕ we have a coupling constant $\lambda^{\frac{1}{2k}}$, and as in the matrix G each element is linear in ϕ, we have for each element a_i in G the relation :

$$-\frac{(k-1)\pi}{4k} \leqslant \text{Arg } a_i \leqslant \frac{(k-1)\pi}{4k}. \tag{25}$$

Similarly we could prove that the inverse of the matrix G is bounded for all λ in its analytic domain. To be more precisely, we have

$$|1 + i\omega_{\pm}| \geq c \sin \frac{\pi}{2k} \tag{26}$$

for either k even or odd, with c a small constant. So we proved that the inverse of matrix G exists and is bounded.

The contour deformations then proceeds as in the previous section, since all the intermediate fields integrals are of the same type and we get again a bound of the type (22).

B.4.3 Borel summability

The proof of the Borel summability for ϕ^{2k} theory is quite similar to the ϕ^6 theory. We shall first of all consider the Borel summability for the partition function Z and then the connected function $\log Z$.

Théorème B.4.1 *The partition function of a field theory with potential $\lambda\phi^{2k}$ is Borel summable or order $k-1$.*

Preuve B.4.2 *This theorem is easy and do not need any loop vertex expansion. In this case the analytic region for λ would become $\mathcal{D}^{k-1} = \{-\frac{(k-1)\pi}{2} < Arg(\lambda) < \frac{(k-1)\pi}{2}\}$. The argument for analyticity and Taylor bounds is the same as above, replacing 2 by $(k-1)$.*

Théorème B.4.2 *The connected function $\log Z(\lambda)$ for the theory with potential $\lambda\phi^{2k}$ is Borel-Le Roy summability at order $k-1$.*

Preuve B.4.3 *The argument is quite similar to the $\lambda\phi^6$ case and we need the loop vertex expansion. In (2) we have shown that the general potential $\lambda\phi^{2k}$ could always be expressed in terms of intermediate fields. After each step we have a new potential $i\phi^{2k-2}$. We could get the bound for the connected function with the same method as in the ϕ^6 case. Now we consider the factorials. In the intermediate fields expression, each intermediate field is linearly interacting with a field ϕ and coupling constant $\lambda^{\frac{1}{2k}}$, so after the expansion to N-th order of the coupling constant λ, and Wick contraction, we get a factor*

$$\frac{[2kN]!!}{N!} \sim (k-1)N!. \tag{27}$$

Combining all the arguments above we find that the remainder of the Taylor expansion is bounded by

$$|\lambda|^{N+1} K^N \Gamma[(k-1)N+1] \tag{28}$$

So the connected function is Borel -Le Roy summable of order $k-1$.

B.5 Conclusion and Perspectives

It is now clear that the traditional constructive tool of decomposing space into an *ad hoc* lattice of cubes and performing cluster expansions is not fundamental and can be replaced by better techniques. The loop vertex expansion [16] and [17] is the first of these. A different but related approach is proposed in [53]. The fundamental idea of the loop vertex expansion is to decompose an interaction of arbitrary degree until trilinear or "three body" interactions are reached, since these are the most "basic". The basic objects are loops made out of a subfamily of the corresponding fields. The loops are made of resolvents, which are uniformly bounded in the case of stable interactions, and they are joined by explicit propagators into cacti structures. This technique reconciles constructive theory and the spirit of Feynman's perturbative theory. The essential mathematical problem of field theory is to iteratively compute connected functions in order to perform renormalization. In Feynman's graphical representation of field theory, connected functions were very easy to compute since they were written as sums of connected graphs, but the corresponding formulas have no mathematical meaning since the expansion diverges. In the loop vertex expansion formalism connected functions are still very easy to identify as they are written as sums of connected cacti, but these sums are now convergent, hence the corresponding formulas are mathematically meaningful.

It will become increasingly necessary in our opinion to develop advanced constructive techniques such as loop vertex expansions to understand nonperturbatively new field theories such as non commutative field theories or group field theories of quantum gravity. Indeed these theories include non-local aspects which, up to our knowledge, cannot be treated through lattice of cubes decomposition and traditional cluster expansions.

Still a long road is to be performed to validate this new constructive philosophy and to push it up to the level where we can reproduce with it all the previous results of the constructive literature over the last decades. This paper accomplished a small but significant step in showing that the decomposition into trilinear interactions does not work solely for the ϕ^4 interaction but also for interactions of any degree. But clearly the limitation to zero dimension must now be lifted. The three main steps ahead are the construction of models in single renormalization group slice, the construction of matrix models with arbitrarily high degree interaction and correct scaling as the size of the matrix gets large, and finally the inclusion of renormalization.

B.5.1 Sliced ϕ^{2k}model in any dimension

We could easily generalize the loop vertex expansion method to a ϕ^{2k} theory in any dimension in a single renormalization group slice, by following [17]. We only sketch the general idea in this paper, details being devoted to a future publication. For instance In D dimensions the propagator in a single renormalization group slice reads :

$$C_j(x,y) = \int_{M^{-2j}}^{M^{-2j+2}} e^{-\alpha m^2} e^{-(x-y)^2/4\alpha} \alpha^{-D/2} d\alpha \leqslant K M^{(D-2/2)j} e^{-cM^j|x-y|} \qquad (1)$$

In ϕ^{2k} we associate to each field ϕ a coupling constant $\phi^{1/2k}$ and an operator $D_j = C_j^{1/2}$. And we still have

$$\int d_{\mu_{C_j}}(\phi) e^{-\int \lambda\phi^{2k}} = \int d\nu(\sigma_i) e^{-\frac{1}{2}log(A+iH)} \tag{2}$$

where A is the same as in the formula (6), and

$$H = \lambda^{\frac{1}{2k}} \begin{pmatrix} \lambda^{\frac{1}{2k}} D_j g_1(a_i, b_i, c_i...) D_j & g_2(a_i, b_i, c_i...) D_j & g_3(a_i, b_i, c_i...) D_j & ... \\ D_j g_2(a_i, b_i, c_i...) & 0 & 0 & ... \\ D_j g_3(a_i, b_i, c_i...) & 0 & 0 & ... \\ ... & ... & ... & ... \end{pmatrix}. \tag{3}$$

We find that the form of H is quite similar to the formula (7) except that to each factor $\lambda^{\frac{1}{2k}}$ is now associated a factor D_j and we require that the first column is the transpose conjugate to the first row, as D_j are all operators now.

Then the proof of the uniform Borel summability and the decay of connected functions should follow in the same way as in [17].

B.5.2 Matrix models

A very interesting property of loop vertex expansions is to allow uniform Borel summability theorems on matrix models with the right scaling of the Borel radius as the matrix gets large [16].

To extend this result to a single matrix model with ϕ^{2k} interaction and matrix of size N we should prove Borel-Le Roy summability with a radius that scales like $N^{-(k-1)}$. This seems doable but all the intermediate fields are now matrix like and one should carefully control the normalization factors associated to contour deformation of the corresponding fields in section B.3.2.

B.5.3 Renormalization

This is the most difficult part. The first goal should be eg to construct very simple models such as the ϕ_2^4 model which requires only Wick ordering with the loop vertex expansion technique. Then we expect the loop vertex expansion should be applied to just renormalizable models such as infrared ϕ_4^4 [55, 56] and ultimately it should be a key tool for the hopefully full construction of an interacting field theory in four dimensions, namely the Grosse-Wulkenhaar model [28, 29, 39, 40].

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C

Paper 3

Construction of 2-dimensional Grosse-Wulkenhaar Model

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Abstract In this paper we construct the noncommutative Grosse-Wulkenhaar model on 2-dimensional Moyal plane with the method of loop vertex expansion. We treat renormalization with this new tool, adapt Nelson's argument and prove Borel summability of the perturbation series. This is the first non-commutative quantum field theory model to be built in a non-perturbative sense.

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Key words : Noncommutative quantum field theory, Renormalization, Constructive field theory, Loop vertex expansion, Borel summability.

C.1 Introduction

Quantum field theories on noncommutative space time became popular after the discovery that they may arise as effective regimes of string theory either due to the compactification [60] or due to the presence of the Green-Schwarz B field for open strings [61, 62]. The simplest non-commutative space is the Moyal space, which could be considered also as a low energy limit of open string theory. However the quantum field theories on the Moyal space are non-renormalizable due to a phenomenon called UV/IR mixing, namely when we integrate out the high scale fields for non planar graphs, there exist still infrared divergences and the divergent terms cannot be compensated by counterterms [25].

Several years ago H. Grosse and R. Wulkenhaar made a breakthrough by introducing a harmonic oscillator term in the propagator so that the theory fully obeys a new symmetry

called the Langmann-Szabo duality. They have proved in a series of papers [27, 26, 28] (see also [29]) that the noncommutative $\phi_4^{\star 4}$ field theory possessing the Langmann-Szabo duality (which we call the GW_4 model hereafter) is perturbatively renormalizable to all orders.

After their work many other QFT models on Moyal space [63, 64, 31, 34, 32, 33, 65] or degenerate Moyal space [35, 36] have also been shown to be *perturbatively* renormalizable. More details could be found in [24, 66, 67].

The GW_4 model is not only perturbatively renormalizable but also asymptotically safe due to the vanishing of the β function in the ultraviolet regime [37], [38], [39], and the renormalization flow of the coupling constant is bounded, which means that it is even better behaved than its commutative counterpart. In the commutative case we either have the Landau ghost for the ϕ_4^4 theory and QED in the ultraviolet regime or we have confinement for non-Abelian gauge theory in the infrared regime. This makes the non-perturbative construction of these commutative field theories either difficult or impossible ; it also means that GW_4 is a prime natural candidate for a full construction of a four dimensional just renormalizable quantum field theory [7].

Constructive field theory or constructive renormalization theory builds the exact Green's functions whose Taylor expansions correspond to perturbative quantum field theory. The traditional techniques for Bosonic constructive theories are the cluster and Mayer expansions which divide the space into cubes and require locality of the interaction. However due to the noncommutativity of the coordinates in Moyal plane, the naive division of space into cubes seems unsuited in this case. What's more, due to the non-locality of the interaction vertex, it is not clear to which cube an interaction vertex should belong. So it seems very difficult, if not impossible, to construct the Grosse-Wulkenhaar model with the traditional methods of cluster and Mayer expansion.

In this paper we shall construct the Grosse-Wulkenhaar model in the 2 dimensional Moyal plane or GW_2, as a warm-up towards building non-perturbatively the full GW_4 model. The method we use is the Loop Vertex Expansion (LVE) which was invented precisely for overcoming these difficulties of the traditional methods [16] and has been applied also to other QFT models with cutoffs [17, 18, 57]. This method has been tested first on the ordinary ϕ_2^4 commutative case in a companion paper [20]. Another approach towards the construction of GW_4, based on the combination of a Ward identity and Schwinger-Dyson equations, is given in [40].

The loop vertex expansion is a combination of the intermediate field techniques with the BKAR forest formula [12, 13]. In the case of a $\lambda\phi^4$ interaction, the first step is to introduce an intermediate field σ and integrate out the original field ϕ. After that we get a non local interaction $(1/2)\mathrm{Tr}\log(1+\mathrm{i}\sqrt{\lambda}\mathrm{C}^{1/2}\sigma\mathrm{C}^{1/2})$, where Tr means summing over all indices for the matrix model (or integrating over the whole space for the commutative case) while keeping the cyclic order, and C is the covariance of the original fields ϕ. After this step we have a new representation of the same theory where the dynamical variables are the intermediate fields σ. The new features are that the interaction vertex becomes non-local (even when the initial vertex, like in ordinary ϕ^4 was local) while the covariance of the σ fields becomes ultralocal, which means that the σ fields have trivial propagation.

Then we apply the BKAR tree formula to obtain the (unrenormalized) connected Green's functions. But this is not the whole story. Some parts of the Wick ordering counterterms can be easily compensated with the corresponding terms within the LVE

formalism. But the Green's functions obtained after the LVE are still divergent due to other remaining counter-terms from the Wick ordering of the interaction which correspond to terms which are not explicitly visible within the LVE. This is why we need a second expansion to generate terms called *inner tadpoles* and to compensate them with their associated counter-terms. We call this second expansion the cleaning expansion ; it was introduced and applied to the commutative ϕ_2^4 case in [20].

After this cleaning expansion, using analogs of the Nelson's bounds for ϕ_2^4 (see [7] and references therein), we obtain convergence of the expansion hence rigorous construction of the Green's functions and we can prove their Borel summability. In fact due to the harmonic Grosse-Wulkenhaar potential these bounds are *easier* in the GW_2 case.

This paper is organized as follows. In section 2 we introduce the basic properties of the noncommutative Moyal space, the matrix basis for functions defined on this space, the GW_2 model in the matrix basis and the combinatorial factors for its Wick ordering. In section 3 we introduce the intermediate field techniques and the BKAR tree formula. In section 4 we consider the amplitude, the graph representation and the dual representation of the connected functions obtained by the LVE. In section 5 we define the cleaning expansion. In section 6 we consider the renormalization. Sections 7 and 8 are devoted to the resummation of the counter-terms and the Borel-sumability. Two examples for the first and second order perturbative renormalizations are given in the Appendix.

C.2 Moyal space and Grosse-Wulkenhaar Model

C.2.1 The Moyal space

The D-dimensional Moyal space $\mathbb{R}_\theta^D$ for D even is generated by the non-commutative coordinates x^μ that obey the commutation relation $[x^\mu, x^\nu] = i\Theta^{\mu\nu}$, where Θ is a $D \times D$ non-degenerate skew-symmetric matrix such that $\Theta^{\mu\nu} = -\Theta^{\nu\mu}$. It is the simplest and best studied model of non-commutative space (see [42, 24] for more details).

Consider the Moyal algebra of smooth and rapidly decreasing functions $\mathcal{S}(\mathbb{R}^D)$ defined on $\mathbb{R}_\theta^D$. $\mathcal{S}(\mathbb{R}^D)$ is equipped with the non-commutative *Groenwald-Moyal* product defined by : $\forall f, g \in \mathcal{S}_D := \mathcal{S}(\mathbb{R}^D)$,

$$(f \star_\Theta g)(x) = \int_{\mathbb{R}^D} \frac{d^D k}{(2\pi)^D} d^D y \, f(x + \tfrac{1}{2}\Theta \cdot k) g(x+y) e^{\imath k \cdot y} \tag{1}$$

$$= \frac{1}{\pi^D |\det \Theta|} \int_{\mathbb{R}^D} d^D y d^D z \, f(x+y) g(x+z) e^{-2\imath y \Theta^{-1} z} \, . \tag{2}$$

We now set $D = 2$ in this paper. The creation and annihilation operators are, in terms of the coordinates of the Moyal plane $\mathbb{R}_\theta^2$ [26, 42] :

$$a = \frac{1}{\sqrt{2}}(x_1 + \mathrm{i} x_2) \, , \qquad \bar{a} = \frac{1}{\sqrt{2}}(x_1 - \mathrm{i} x_2) \, ,$$
$$\frac{\partial}{\partial a} = \frac{1}{\sqrt{2}}(\partial_1 - \mathrm{i}\partial_2) \, , \qquad \frac{\partial}{\partial \bar{a}} = \frac{1}{\sqrt{2}}(\partial_1 + \mathrm{i}\partial_2) \, . \tag{3}$$

For any function f defined above we have

$$\begin{aligned}
(a \star f)(x) &= a(x)f(x) + \frac{\theta_1}{2}\frac{\partial f}{\partial \bar{a}}(x) \ , & (f \star a)(x) &= a(x)f(x) - \frac{\theta_1}{2}\frac{\partial f}{\partial \bar{a}}(x) \ , \\
(\bar{a} \star f)(x) &= \bar{a}(x)f(x) - \frac{\theta_1}{2}\frac{\partial f}{\partial a}(x) \ , & (f \star \bar{a})(x) &= \bar{a}(x)f(x) + \frac{\theta_1}{2}\frac{\partial f}{\partial a}(x) \ . \qquad (4)
\end{aligned}$$

With the creation and annihilation operators we could build the matrix basis for the functions $\phi(x) \in \mathcal{S}(\mathbb{R}^2)$ defined on $\mathbb{R}^D_\theta$, such that :

$$\phi(x) = \sum_{mn} \phi_{mn} f_{mn}(x), \qquad (5)$$

where the matrix basis $f_{mn}(x)$ is defined as :

$$\begin{aligned}
f_{mn}(x) &:= \frac{1}{\sqrt{n!m!\,\theta_1^{m+n}}}\, \bar{a}^{\star m} \star f_0 \star a^{\star n} \qquad (6) \\
&= \frac{1}{\sqrt{n!m!\,\theta_1^{m+n}}} \sum_{k=0}^{\min(m,n)} (-1)^k \binom{m}{k}\binom{n}{k} k!\, 2^{m+n-2k}\, \theta^k\, \bar{a}^{m-k}\, a^{n-k} f_0 \ ,
\end{aligned}$$

and $f_0(x) = 2\mathrm{e}^{-\frac{1}{\theta}(x_1^2+x_2^2)}$.

The matrix basis defined above has the following properties :

$$(f_{mn} \star f_{kl})(x) = \delta_{nk} f_{ml}(x) \ . \qquad (7)$$

and

$$\int d^2x f_{mn}(x) = 2\pi\theta\delta_{mn}. \qquad (8)$$

C.2.2 Ribbon graphs

Matrix models or quasi-matrix models use *ribbon* graphs, which were first introduced by 't Hooft for QCD [69]. A ribbon graph $G = (V, E)$ is an orientable surface with boundary represented as a set of closed disks, also called vertices, V and a set of ribbons, also called edges E, such that all disks are attached to each other by the ribbons glued to their boundaries. Figure C.1 is an example of a non-planar ribbon graph. The interested reader could go to [70, 71] for a more general introduction.

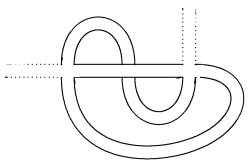

FIGURE C.1: An example of a ribbon graph.

C.2.3 The 2-dimensional Grosse-Wulkenhaar Model

The 2-dimensional Grosse-Wulkenhaar model or GW_2 model is defined by the Wick-ordered action :

$$\begin{aligned} S &= \int d^2x \Big[\frac{1}{2}\partial_\mu\phi \star \partial^\mu\phi + \frac{\Omega^2}{2}(\tilde{x}_\mu\phi)\star(\tilde{x}^\mu\phi) + \frac{\mu^2}{2}\phi\star\phi \\ &+ : \frac{\lambda}{4}\phi\star\phi\star\phi\star\phi : \Big]. \end{aligned} \tag{9}$$

where $\phi(x)$ is a real function defined over the noncommutative coordinates x_μ, $\tilde{x}_\mu = 2(\Theta^{-1})_{\mu\nu}x^\nu$ and the Euclidean signature has been used. Remark that this model is not translation invariant due to the harmonic oscillator term.

The action (9) has a remarkable symmetry such that if we exchange the position and momentum operator :

$$p_\mu \leftrightarrow \tilde{x}_\mu, \quad \hat{\phi}(p) \leftrightarrow \pi\sqrt{|\det\theta|}\phi(x), \tag{10}$$

where

$$\hat{\phi}(p) := \int d^2x e^{-ip\Theta x}\phi(x), \quad p\Theta x := p_\mu\Theta^{\mu\nu}x_\nu, \tag{11}$$

the action (9) is invariant up to a scalar factor :

$$S[\phi;\mu,\lambda,\Omega] \mapsto \Omega^2 S[\phi;\frac{\mu}{\Omega},\frac{\lambda}{\Omega},\frac{1}{\Omega}]. \tag{12}$$

This symmetry is called the Langman-Szabo duality [141]. It is essential for solving the UV/IR mixing problem hence for proving the renormalisabiliy of the Grosse-Wulkenhaar model.

The renormalisability of the GW model has been studied extensively in the matrix representation [**?**, **?**], the direct space representation [41] and the parametric representation [**?**]. In this paper we shall use the matrix representation.

In the matrix basis the GW_2 action could be written as :

$$\begin{aligned} & S[\phi] = 2\pi\theta \mathrm{Tr}\Big[\frac{1}{2}\phi\Delta\phi + : \frac{\lambda}{4}\phi_\star^4 : \Big] \\ = \; & 2\pi\theta \sum_{m,n,k,l} \Big[\frac{1}{2}\phi_{mn}\Delta_{mn;kl}\phi_{kl} + \frac{\lambda}{4} : \phi_{mn}\phi_{nk}\phi_{kl}\phi_{lm} : \Big]. \end{aligned} \tag{13}$$

where ϕ_{mn} is a real symmetric matrix. In this basis the Laplacian reads :

$$\Delta_{mn,kl} = [\mu^2 + \frac{2}{\theta}(m+n+1)]\delta_{ml}\delta_{nk} - \frac{2}{\theta}(1-\Omega^2) \tag{14}$$

$$\times \quad \big[\sqrt{(m+1)(n+1)}\delta_{m+1,l}\delta_{n+1,k} + \sqrt{mn}\delta_{m-1,l}\delta_{n-1,k}\big]. \tag{15}$$

We could obtain the covariance $C_{sr,kl}$ by the following relation [**?**] :

$$\sum_{r,s\in N} \Delta_{mn,rs}C_{sr,kl} = \delta_{ml}\delta_{nk}. \tag{16}$$

When $\Omega = 1$, the kinetic matrix reduces to a much simpler form :

$$\Delta_{mn} = \Big[\mu^2 + \frac{2(1+\Omega^2)}{\theta}(m+n+1)\Big] = \Big[\mu^2 + \frac{4}{\theta}(m+n+1)\Big], \tag{17}$$

and the covariance reads :

$$C_{mn} = \frac{1}{\mu^2 + \frac{4}{\theta}(m+n+1)}. \tag{18}$$

$\Omega = 1$ is the ultraviolet fixed point of this theory [38], hence we shall for simplicity take $\Omega = 1$ in the rest of this paper.

The Wick ordering of the interaction with ultraviolet cutoff Λ reads :

$$: \phi_{mn}\phi_{nk}\phi_{kl}\phi_{lm} := \phi_{mn}\phi_{nk}\phi_{kl}\phi_{lm} - 8\phi_{mp}\phi_{pm}T_m + 6\mathrm{Tr_m T_m^2}. \tag{19}$$

where

$$T_\Lambda = T_m^\Lambda = \sum_{q=0}^{\Lambda} \frac{1}{q+m} = \log \frac{\Lambda + m}{m} \sim \log \Lambda, \text{for } 1 \leqslant m \ll \Lambda, \tag{20}$$

and

$$T_\Lambda^2 = \mathrm{Tr_m}(\mathrm{T_m^\Lambda})^2 = \sum_{\mathrm{m}} (\sum_{\mathrm{p}} \frac{1}{\mathrm{m+p}})(\sum_{\mathrm{q}} \frac{1}{\mathrm{m+q}}) \sim (2\ln^2 2 + \frac{\pi^2}{6})\Lambda. \tag{21}$$

The counter terms $\mathrm{Tr}\phi^2\mathrm{T}$ and T_Λ^2 and an explanation to the combinatorial factors are shown in Figures C.2 and C.3.

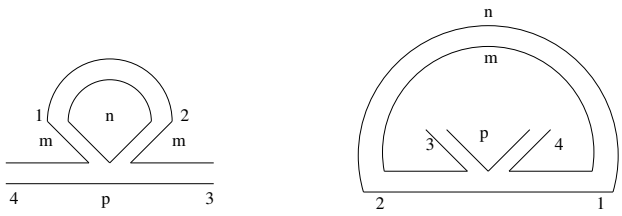

FIGURE C.2: The counter-term $\mathrm{Tr}(\phi^2\mathrm{T})$ from the Wick ordering. There are 4 possibilities of contracting the nearest neighbor fields, for example, 1 with 2, 2 with 3 , 3 with 4 or 4 with 1 ; And each contraction has additional two possibilities of whether the two uncontrcted fields are in the external face of the inner face, as shown in this Figure. In total this gives a factor 8.

Note that since the infinite volume limit is hidden into the matrix basis, we don't need any more the cluster expansion. In this sense the Grosse-Wulkenhaar model is even simpler than the commutative ϕ_2^4 theory.

C.3 The intermediate field representation and the Loop vertex expansion

C.3.1 The intermediate field representation

The partition function for the matrix model reads :

$$Z(\lambda) = \int d\mu(\phi_{mn}) e^{-S[\phi_{mn}]}, \tag{1}$$

where

$$d\mu(\phi_{mn}) = \pi^{-N(N-1)/2} e^{-1/2[\mathrm{Tr}\phi_{\mathrm{mn}}\Delta_{\mathrm{mn}}\phi_{\mathrm{nm}}]} \prod_m d\phi_{mm} \prod_{n<m} d\phi_{nm}. \tag{2}$$

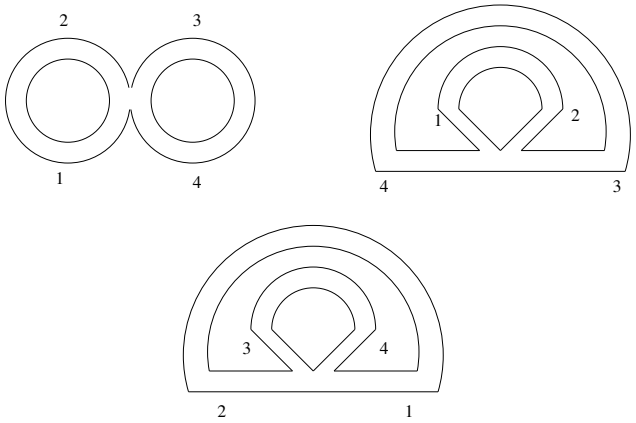

FIGURE C.3: The counter term T^2_Λ and combinatorics. There are 2 possibilities of forming a vacuum graph, for example 1 contracts with 2 and 3 contracts with 4, or 1 contracts with 4 and 2 contracts with 3. For each case there are an additional 3 possibilities shown in this Figure. In total this gives a factor 6.

is the normalized Gaussian measure of covariance C given by (18) and $S[\phi_{mn}]$ is the Wick ordered interaction term.

We introduce the real symmetric matrix σ_{mn} as the intermediate field and the partition function reads :

$$Z(\lambda) = \int d\sigma_{mn} d\phi_{mn} e^{-\frac{1}{2}\sigma_{mn}\sigma_{nm} - \frac{1}{2}\phi_{mn}\Delta_{mn}\phi_{nm} - i\sqrt{\lambda/2}\sigma_{km}(\phi_{mn}\phi_{nk} - 4\delta_{mk}T_m) + \frac{5}{2}\lambda T^2_\Lambda}. \tag{3}$$

After integrating out the matrix fields ϕ_{mn} we have :

$$Z(\lambda) = \int d\mu(\sigma) e^{2i\sqrt{2\lambda}\mathrm{Tr_m T_m}\sigma_{\mathrm{mm}} - \frac{1}{2}\mathrm{Tr}\log[1+\mathrm{i}\sqrt{2\lambda}(\mathrm{C}^{1/2})_{.\mathrm{n}}\sigma_{\mathrm{nk}}(\mathrm{C}^{1/2})_{\mathrm{k.}}] + \frac{5}{2}\lambda \mathrm{T}^2_\Lambda}, \tag{4}$$

where $\mathrm{Tr}\log[1 + \mathrm{i}\sqrt{2\lambda}(\mathrm{C}^{1/2})_{.\mathrm{n}}\sigma_{\mathrm{nk}}(\mathrm{C}^{1/2})_{\mathrm{k.}}]$ is defined by the expansion :

$$\begin{aligned} & \mathrm{Tr}\log[1 + \mathrm{i}\sqrt{2\lambda}(\mathrm{C}^{1/2})_{.\mathrm{n}}\sigma_{\mathrm{nk}}(\mathrm{C}^{1/2})_{\mathrm{k.}}]_{\mathrm{mm}} \\ = \; & (i\sqrt{2\lambda}) \sum_{mnk} (C^{1/2})_{mn}\sigma_{nk}(C^{1/2})_{km} - \frac{1}{2}(i\sqrt{2\lambda})^2 \sum_{mnklpq} \\ & (C^{1/2})_{mn}\sigma_{nk}(C^{1/2})_{kl}(C^{1/2})_{lp}\sigma_{pq}(C^{1/2})_{qm} + \cdots \end{aligned} \tag{5}$$

Remark that in this formalism the matrix $C^{1/2}\sigma C^{1/2}$ is Hermitian.

The multiplication rules or the *fusion* rules for covariances read :

$$\sum_n (C^{1/2})_{mn}(C^{1/2})_{nl} = C_{ml}, \quad (C^{1/2})_{mn}(C^{1/2})_{kl} = \delta_{nk}\delta_{ml}C_{mn}. \tag{6}$$

Lemme C.3.1 *The interaction vertices generated by* (4) *are planar cycles.*

Unlike in the commutative case, we might generate non planar-graphs if we apply the fusion rule of the covariance naively. We take the second order expansion for a single loop vertex as example. We have

$$-\frac{1}{2}\mathrm{Tr}[-\frac{(\mathrm{i}\sqrt{2\lambda})^2}{2}(\mathrm{C}^{1/2}\sigma\mathrm{C}^{1/2})^2] = \frac{(\mathrm{i}\sqrt{2\lambda})^2}{4}\mathrm{Tr}[\mathrm{C}^{1/2}_{\mathrm{nm}}\sigma_{\mathrm{ml}}\mathrm{C}^{1/2}_{\mathrm{ln}} \cdot \mathrm{C}^{1/2}_{\mathrm{rq}}\sigma_{\mathrm{qp}}\mathrm{C}^{1/2}_{\mathrm{pr}}]. \tag{7}$$

There are two possibilities of multiplying the covariance :

$$[C^{1/2}_{nm}\sigma_{ml}C^{1/2}_{ln}C^{1/2}_{rq}\sigma_{qp}C^{1/2}_{pr}], \quad or \quad [C^{1/2}_{nm}\sigma_{ml}C^{1/2}_{ln}C^{1/2}_{pr}C^{1/2}_{rq}\sigma_{qp}]. \tag{8}$$

The first term means we first contract $C^{1/2}_{ln}$ with $C^{1/2}_{rq}$ and $C^{1/2}_{pr}$ with $C^{1/2}_{nm}$ while the second term means first we contract $C^{1/2}_{ln}$ with $C^{1/2}_{pr}$, and then contract the rest terms. The two cases are shown in Figure C.4. But we should notice that the second case couldn't occur when we take into account the global Trace operator.

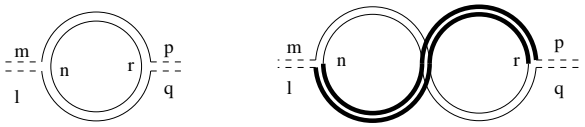

FIGURE C.4: The planar and non planar graphs from the fusion of the covariance. A is the planar graph and B is the non-planar one.

So that although there do exist non planar graphs in the Grosse-Wulkenhaar model, all graphs generated from the Loop Vertex expansion are planar, and the non-planar graphs could only be generated by the crossings in the cleaning expansions that we shall introduce in section C.5.

There exist still non planar graphs at first order (see Figure C.5) that come from the convergent terms of the Wick ordering. We have ignored them in formula (19), in Figure C.2 and Figure C.3, and in the rest of this paper, as they are convergent and do not cause any problem for the renormalisation or the construction of this model.

FIGURE C.5: The convergent non planar graphs from the Wick ordering.

We first expand the loop vertex term as :

$$\begin{aligned} & -\frac{1}{2}\mathrm{Tr}\log[1+\mathrm{i}\sqrt{2\lambda}\mathrm{C}^{1/2}\sigma\mathrm{C}^{1/2}] \\ = \; & -\frac{i}{2}\mathrm{Tr}(\sqrt{2\lambda}\mathrm{C}^{1/2}\sigma\mathrm{C}^{1/2}) - \frac{1}{2}\mathrm{Tr}\log_2[1+\mathrm{i}\sqrt{2\lambda}\mathrm{C}^{1/2}\sigma\mathrm{C}^{1/2}], \end{aligned} \tag{9}$$

where $\log_n(x)$ for $n \geq 2$ is defined as the n-th Taylor remainder of the function $\log(x)$:

$$\log_n(x) = \log(x) - [x - x^2/2 + x^3/3 \cdots + (-1)^{n+1}x^n/n]. \tag{10}$$

The first term in formula (9) reads :

$$-\frac{i}{2}\mathrm{Tr}(\sqrt{2\lambda}\mathrm{C}^{1/2}\sigma\mathrm{C}^{1/2}) = -2\cdot\frac{\mathrm{i}}{2}\sqrt{2\lambda}\mathrm{Tr}_{\mathrm{m}}\mathrm{T}_{\mathrm{m}}\sigma_{\mathrm{mm}}, \tag{11}$$

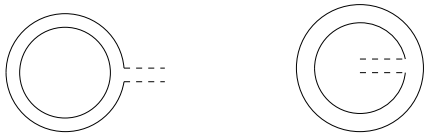

FIGURE C.6: Tadpoles.

where the additional factor 2 is due to that there are two kinds of tadpoles (see Figure C.6) when taking the $Trace$ after using the fusion rule for the covariances $C^{1/2}$. Hence the leaf tadpole term (11) could cancel partially the linear counter term of the Wick ordering, see formula (4).

So we could define the new interaction vertex as

$$\begin{aligned} V(\sigma) &= -\frac{1}{2}\mathrm{Tr}\log_2[1+\mathrm{i}\sqrt{2\lambda}(\mathrm{C}^{1/2})_{.\mathrm{n}}\sigma_{\mathrm{nk}}(\mathrm{C}^{1/2})_{\mathrm{k}.}]_{\mathrm{mm}} \\ &+ i\sqrt{2\lambda}\mathrm{Tr}_{\mathrm{m}}\mathrm{T}_{\mathrm{m}}\sigma_{\mathrm{mm}} + \frac{5}{2}\lambda\mathrm{T}_{\Lambda}^2, \end{aligned} \tag{12}$$

where the leaf tadpoles have been already canceled by the counter term and the partition function could be written as :

$$Z(\lambda) = \int d\nu(\sigma) e^{i\sqrt{2\lambda}\mathrm{Tr}_{\mathrm{m}}\mathrm{T}_{\mathrm{m}}\sigma_{\mathrm{mm}} - \frac{1}{2}\mathrm{Tr}\log_2[1+\mathrm{i}\sqrt{2\lambda}(\mathrm{C}^{1/2})\sigma(\mathrm{C}^{1/2})] + \frac{5}{2}\lambda\mathrm{T}_{\Lambda}^2}. \tag{13}$$

where

$$d\nu(\sigma) = \pi^{-N(N-1)/2} e^{-\mathrm{Tr}\sigma^2} \prod_{ij} d\sigma_{ij}. \tag{14}$$

and the covariance reads :

$$< \sigma_{mn}, \sigma_{kl} >= \int d\mu(\sigma)\sigma_{mn}\sigma_{kl} = \delta_{nk}\delta_{ml}. \tag{15}$$

C.3.2 The BKAR Tree formula and the expansion

The most interesting quantities in our model are the connected Schwinger's functions. We shall derive them by the BKAR tree formula.

Let us first of all expand the exponential as $\sum_n \frac{V(\sigma)^n}{n!}$. To compute the connected function while avoiding an additional factor $n!$, we give a kind of $fictitious$ index v, $v = 1, \cdots, n$ to each field σ in vertex $V(\sigma)$ and we could rewrite the expanded interaction term as $\sum_n \prod_{v=1}^n \frac{V(\sigma^v)}{n!}$. This means that we consider n different copies σ_v of σ with a degenerate Gaussian measure

$$d\nu(\{\sigma_v\}) = d\nu(\sigma_{v_0}) \prod_{v' \neq v_0}^{n} \delta(\sigma'_v - \sigma_{v_0}) d\sigma_{v'} \ , \tag{16}$$

where v_0 is an arbitrarily marked vertex. The vacuum Schwinger's function is given by :

Théorème C.3.1 (Loop Vertex Expansion [16])

$$\log Z(\lambda,\Lambda,\mathcal{V}) = \sum_{n=1}^{\infty}\frac{1}{n!}\sum_{\mathcal{T}\text{ with }n\text{ vertices}} G_{\mathcal{T}} \tag{17}$$

$$G_{\mathcal{T}} = \Big\{\prod_{\ell\in\mathcal{T}}\sum_{m_\ell,n_\ell,p_\ell,q_\ell}[\int_0^1 dw_\ell]\Big\}\int d\nu_{\mathcal{T}}(\{\sigma^v\},\{w\})$$

$$\Big\{\prod_{\ell\in\mathcal{T}}[\frac{1}{2}[\delta_{m_\ell p_\ell}\delta_{n_\ell q_\ell}+\delta_{m_\ell q_\ell}\delta_{n_\ell p_\ell}]\frac{\delta}{\delta\sigma^{v(\ell)}_{m_\ell,n_\ell}}\frac{\delta}{\delta\sigma^{v'(\ell)}_{p_\ell,q_\ell}}]\Big\}\prod_{v=1}^{n}V_v,$$

where

- *each line ℓ of the tree joins two different loop vertices $V^{v(\ell)}$ and $V^{v'(\ell)}$ which are identified through the function $[\delta_{m_\ell p_\ell}\delta_{n_\ell q_\ell}+\delta_{m_\ell q_\ell}\delta_{n_\ell p_\ell}]$, since the propagator of σ is ultra-local.*
- *the sum is over spanning trees joining all n loop vertices. These trees have therefore $n-1$ lines, corresponding to σ propagators.*
- *the normalized Gaussian measure $d\nu_{\mathcal{T}}(\{\sigma_v\},\{w\})$ over the fields σ_v has now a covariance*

$$<\sigma^v_{mn},\sigma^{v'}_{kl}>=\delta_{nk}\delta_{ml}w^{\mathcal{T}}(v,v',\{w\}), \tag{18}$$

which depend on the "fictitious" indices. Here $w^{\mathcal{T}}(v,v',\{w\})$ equals to 1 if $v=v'$, and equals to the infimum of the w_ℓ for ℓ running over the unique path from v to v' in $\mathcal{T}$ if $v\neq v'$.

The proof is a standard consequence of the BKAR formula (see [16, 17]).

The n-point Euclidean Green's functions or the Schwinger's functions, with $n=2p$ even, could be derived similarly by introducing the (matrix) source fields $j=[j_{mn}]$. The partition function with source fields reads :

$$Z(\Lambda,j(x)) = \int d\mu(\phi)e^{-\frac{\lambda}{4}\mathrm{Tr}[(\phi^2-4\mathrm{T}_\Lambda)^2+10\mathrm{T}_\Lambda^2]+\mathrm{Trj}\phi} \tag{19}$$

$$= \int d\mu(\phi)\int d\nu(\sigma)e^{2i\sqrt{2\lambda}\mathrm{Tr}(\mathrm{T}_\Lambda\sigma)+\frac{5}{2}\mathrm{T}_\Lambda^2}e^{-\int d^2x\frac{1}{2}tr[M(\phi+M^{-1}j)^2]+\frac{1}{2}jM^{-1}j}$$

$$= \int d\nu(\sigma)e^{\frac{5}{2}T_\Lambda^2+\mathrm{Tr}\left(2\mathrm{i}\sqrt{2\lambda}\mathrm{T}_\Lambda\sigma-\frac{1}{2}\log(1+\mathrm{i}\sqrt{2\lambda}\mathrm{C}^{1/2}\sigma\mathrm{C}^{1/2})\right)}e^{\mathrm{Tr}[\frac{1}{2}\mathrm{jC}^{1/2}\frac{1}{1+2\mathrm{i}\sqrt{\lambda}\mathrm{C}^{1/2}\sigma\mathrm{C}^{1/2}}\mathrm{C}^{1/2}\mathrm{j}]}$$

$$= \int d\nu(\sigma)e^{\frac{5}{2}\lambda T_\Lambda^2+\mathrm{Tr}\left(2\mathrm{i}\sqrt{2\lambda}\mathrm{T}_\Lambda\sigma-\frac{1}{2}\log(1+\mathrm{i}\sqrt{2\lambda}\mathrm{C}^{1/2}\sigma\mathrm{C}^{1/2})\right)+\frac{1}{2}\mathrm{Tr}[\mathrm{jC}^{1/2}\mathrm{RC}^{1/2}\mathrm{j}]}$$

where

$$M = 1+\mathrm{Tr}[\mathrm{i}\sqrt{2\lambda}\mathrm{C}^{1/2}\sigma\mathrm{C}^{1/2}] = 1+\mathrm{i}\sqrt{2\lambda}\sum_{\mathrm{mnp}}\mathrm{C}^{1/2}_{\mathrm{mn}}\sigma_{\mathrm{np}}\mathrm{C}^{1/2}_{\mathrm{pm}}, \tag{20}$$

and

$$R = \frac{1}{1+[i\sqrt{2\lambda}C^{1/2}\sigma C^{1/2}]},\ R_{mn}(\sigma) = [C^{1/2}\frac{1}{1+[i\sqrt{2\lambda}C^{1/2}\sigma C^{1/2}]}C^{1/2}]_{mn}. \tag{21}$$

The connected n-point function is given by :

$$S(l_1r_1,\cdots,l_{2p}r_{2p}) = \frac{1}{(2p)}\frac{\partial^{2p}}{\partial j_{l_1r_1}\cdots\partial j_{l_{2p}r_{2p}}}\log Z(\Lambda,j)|_{j=0}. \tag{22}$$

More precisely, we have the following theorem for the connected Schwinger's function :

Théorème C.3.2

$$S^c = \sum_{\pi} \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{\mathcal{T} \text{ with } n \text{ vertices}} \Big\{ \prod_{\ell \in \mathcal{T}} [\int_0^1 dw_\ell] \Big\}$$
$$\int d\nu_{\mathcal{T}}(\{\sigma^v\}, \{w\}) \Big\{ \prod_{\ell \in \mathcal{T}} [\frac{1}{2}[\delta_{m_\ell p_\ell} \delta_{n_\ell q_\ell} + \delta_{m_\ell q_\ell} \delta_{n_\ell p_\ell}] \frac{\delta}{\delta \sigma_{mn}^{v(\ell)}} \frac{\delta}{\delta \sigma_{pq}^{v'(\ell)}}] \Big\}$$
$$\prod_{v=1}^{n} V_v \prod_{k=1}^{2p} R_{l_{1\pi(k)}, r_{1,\pi(k)}}(\sigma), \tag{23}$$

where the sum over π runs over the parings of the $2p$ external variables according to the cyclic order and $R_{l_{1\pi(k)}, r_{1,\pi(k)}}(\sigma)$ are the resolvent matrices whose explicit form are given by formula (21).

So in this paper we consider only the vacuum connected Schwinger's function, as the n-point function could be easily obtained by derivation w.r.t. the source fields.

C.4 The grapdhs and the amplitudes of the LVE

C.4.1 Direct representation of the LVE

Now we consider the basic graph structure of the Loop vertex expansion, which we call the *direct graphs* or the LVE in the direct representation.

There are three basic line structures or propagators in the LVE :

– The full resolvent R_{mn} is defined as :

$$R_{mn} = R_{mn}(\sigma, \lambda) = \frac{\partial}{\partial \sigma_{mn}} [-\frac{1}{2} \mathrm{Tr} \log(1 + \mathrm{i}\sqrt{2\lambda} \mathrm{C}_{\mathrm{pq}}^{1/2} \sigma_{\mathrm{ql}} \mathrm{C}_{\mathrm{lp}}^{1/2})]$$
$$= -\frac{1}{2} i\sqrt{2\lambda} \sum_p C_{mp}^{1/2} \frac{1}{1 + i\sqrt{2\lambda} C^{1/2} \sigma C^{1/2}} C_{pn}^{1/2}. \tag{1}$$

Define

$$R = R(\sigma, \lambda) = \frac{1}{1 + i\sqrt{2\lambda} C^{1/2} \sigma C^{1/2}} = \frac{1}{1 + i\sqrt{2\lambda} \sum_{mnp} C_{mn}^{1/2} \sigma_{mp} C_{pm}^{1/2}}, \tag{2}$$

so that we have

$$R_{mn} = [C^{1/2} \; R \; C^{1/2}]_{mn}. \tag{3}$$

This coincides with the terms in formula (21).

– The propagators C_{mn} between the original fields ϕ_{mn},

– The propagators between the σ fields.

The propagators are shown in Figure C.7.

There are three kinds of interaction vertices in the LVE : the counter terms, the leaf terms K with coordination number 1 and the general interaction vertices V. A leaf vertex K is generated by deriving once w.r.t the σ field on the $\log_2$ term :

$$K_{mn} = \frac{\partial}{\partial \sigma_{mn}} [-\frac{1}{2} \mathrm{Tr} \log_2(1 + \mathrm{i}\sqrt{2\lambda} \mathrm{C}_{\mathrm{pq}}^{1/2} \sigma_{\mathrm{ql}} \mathrm{C}_{\mathrm{lp}}^{1/2})]$$
$$= -\frac{1}{2} i\sqrt{2\lambda} \sum_p C_{mp}^{1/2} [\frac{1}{1 + i\sqrt{2\lambda} C^{1/2} \sigma C^{1/2}} - 1] C_{pn}^{1/2}. \tag{4}$$

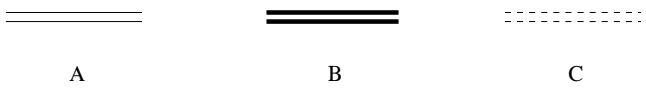

FIGURE C.7: The propagators in LVE. A stands for the resolvent R, B is the pure propagator C, and C is the propagator of the σ fields.

A general loop vertex could be obtained by deriving twice or more with respect to the σ fields :

$$\begin{aligned} V_{m_1 m_2 \cdots m_{p-1} m_p}(\lambda, \sigma) &= \frac{\partial}{\partial \sigma_{m_1 m_2}} \cdots \frac{\partial}{\partial \sigma_{m_{p-1} m_p}} [-\frac{1}{2} \mathrm{Tr} \log(1 + \mathrm{i}\sqrt{2\lambda} \mathrm{C}^{1/2} \sigma \mathrm{C}^{1/2})] \\ &= -\frac{1}{2} (i\sqrt{2\lambda})^p \sum_{\tau} R_{m_1 m_{\tau(1)}}(\sigma, \lambda) \cdots R_{m_{\tau(p)} m_1}(\sigma, \lambda). \end{aligned} \tag{5}$$

with $p \geq 2$ and the sum over τ is over the p cyclic permutations of the resolvents.

The basic interaction vertices are shown in Figure 12, where we didn't show explicitly the pure propagator $C^{1/2}$.

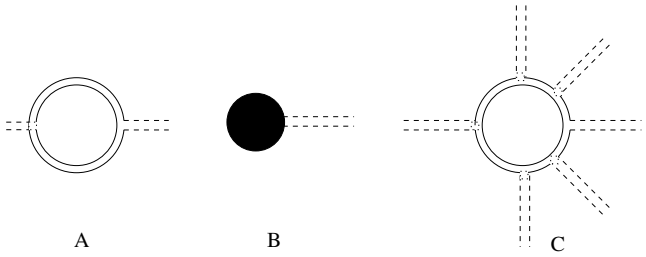

FIGURE C.8: The basic graph elements of LVE. Graph A means the leaf K, graph B means the counter term and graph C means the most general loop vertex which has several σ fields attached.

After integrating out the σ fields we obtain the vacuum graphs which are planar ribbon graphs without broken faces. The first order graphs whose amplitudes are proportional to λ are shown in Figure C.9. Graph C in this figure means the counter term $\frac{5}{2}\lambda T_\Lambda^2$ that appears only in the first order expansion.

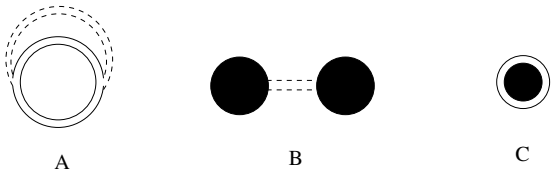

FIGURE C.9: The graphs of first order expansion. We have taken account of all possible configurations of graph A in the calculation.

Since we consider a real symmetric matrix model, the leaf loop vertices and the counter terms could appear either in the outer side of a loop vertex or in the inner side, we call the graph of the first case the normal direct tree and the second case the inner direct tree. The normal direct tree and the inner direct tree are shown in Figure C.10 and Figure C.11. The amplitude of the two graphs are exactly the same, so in this paper we only consider

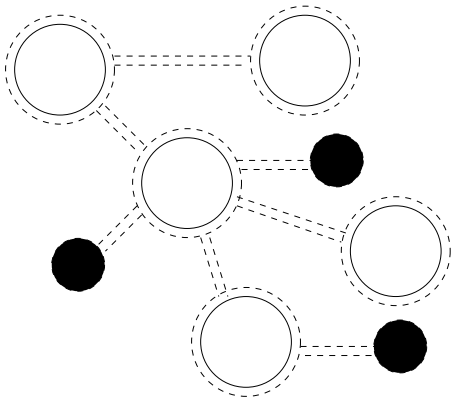

FIGURE C.10: A general graph of Loop vertex expansion. Here the leaf means the leaf loop vertex K.

explicitly the first case for the direct graph representations and the dual representations. The contributions of the inner direct trees to the amplitude should be taken into account when calculating correctly the combinatorial factors. A general graph of LVE is shown in

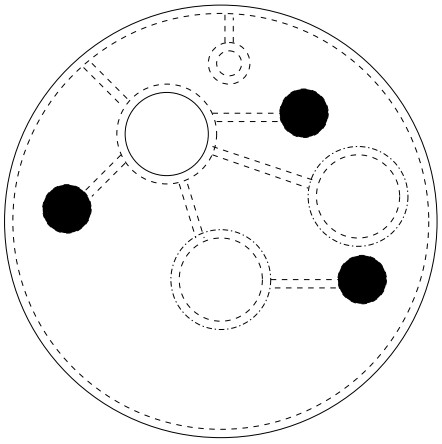

FIGURE C.11: A tree of loop vertices where the counter terms or leaf terms are attached to the inner part of the loop vertices. Here the leaf means the leaf loop vertex K.

Figure C.10 and Figure C.11.

Now we have the following definitions for the objects in $\mathcal{T}$: To each counterterm $\mathrm{Tr}\sigma \mathrm{T}_{\Lambda} \mathrm{D}_{\mathrm{mn}}$, where D_{mn} is the pure propagator generated by the derivation $\frac{\partial}{\partial \sigma_{mn}}$ on the loop vertex, we associate a tree line index ℓ_c, to each leaf term $K_{mn}(\sigma)$ we associate a tree line index ℓ and to each full resolvent $R(\sigma)_{mn}$ sandwiched by the two tree lines ℓ_l and ℓ_r, such that ℓ_l is on the left side and ℓ_r is on the right side, we associate two tree line indices ℓ_l and ℓ_r .

So the amplitude for a tree of loop vertices reads :

$$G_{\mathcal{T}} = \prod_{\ell,\ell_c,\ell_l,\ell_r \in \mathcal{T}} \int d\mu(\sigma) \mathrm{Tr}[\mathrm{TrK}^{\ell}_{\mathrm{mn}}(\sigma)][\mathrm{Tr}(\sqrt{2\lambda}\mathrm{T}_{\Lambda}\mathrm{D}^{\ell_c}_{\mathrm{mn}})][\mathrm{TrR}^{\ell_l.\ell_r}_{\mathrm{mn}}(\sigma)], \tag{6}$$

where the global Tr means that we need to take the cyclic order of all the objects. The indices ℓ_l or ℓ_r may coincide with the indices ℓ or ℓ_c.

C.4.2 The dual representation

Since a LVE graph in the direct representation is *planar*, the notion of duality is globally well-defined. In the dual representation we have a canonical (up to an orientation choice) and more explicit cyclic ordering of all ingredients occurring in the expansion [20].

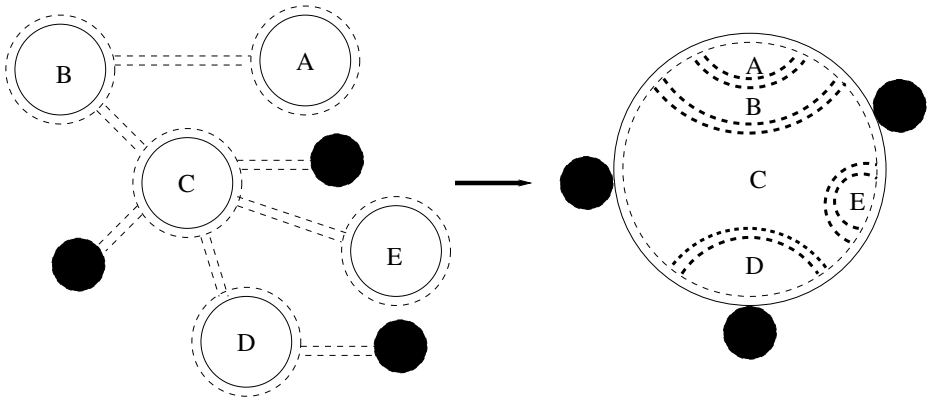

FIGURE C.12: The dual graph of a LVE. The area enclosed by the bold dash ribbons correspond to the original loop vertices.

In the dual representation the original tree of loop vertices $\mathcal{T}$ is replaced by the dual tree $\bar{\mathcal{T}}$, which consists of a single huge loop vertex $\mathcal{C}$ with dual tree line $\bar{\ell}$. Each dual tree line $\bar{\ell} \in \bar{\mathcal{T}}$ corresponds to a line $\ell \in \mathcal{T}$ in the direct picture, and is pictured as a bold dash line. See Figure C.12 for a dual tree.

The important fact reflecting the tree character of $\mathcal{T}$ is that these bold dash lines *cannot cross*. Hence they divide this disk $\mathcal{C}$ into different connected regions $\bar{v} \in \bar{V}$, each of which corresponds to a single loop vertex $v \in V$ of the direct expansion. This explains how the two pictures, dual of each other, are equivalent and how anyone can be reconstructed from the other.

This cycle $\mathcal{C}$ contains all the ordinary full lines (ϕ resolvents) of the direct representation and the counter terms, but reread in the cyclic order obtained by turning around the tree. The order of all the objects in the dual representation should be the same as in the direct representation.

Here is the notation for the objects $\mathcal{O} = \bar{B} \cup \bar{S} \cup \mathcal{T} \cup \mathcal{L}$ in the dual representation :

- The counter-terms that are pictured as black vertices $\bar{b} \in \bar{B}$. More exactly, each counter-term is considered as a black vertex attached to a pure propagator, so that the amplitude of each counter-term equals to the product of the amplitudes of the two objects. See formula (7). To each counterterm we associate an index $\bar{\ell}_c$.
- The bold dash lines $\bar{\ell} \in \bar{\mathcal{T}}$ that correspond to the original σ propagators. We also call them the dual tree line.
- Simple leaves $\bar{v}_0 \in \bar{S}$, i.e resolvent lines surrounded by a single dual tree line $\bar{\ell}$. For example, the normal lines in region A and E in Figure C.12 correspond to two simple leaves. To each leaf we associate an index $\bar{\ell}$.
- Non-leaf resolvent lines $\bar{v}_1 = \{\bar{\ell}_l, \bar{\ell}_r\} \in \mathcal{L}$, ie resolvent lines not in $\bar{S}$ that correspond to the full resolvents $R_{mn}^{\ell_l,\ell_r}(\sigma)$ in the direct representation. We follow the clockwise direction in the dual graph and use two indices $\bar{\ell}_l$ and $\bar{\ell}_r$ to indicate each non-leaf resolvent line sandwiched between the two different dual tree lines $\bar{\ell}_l$ and $\bar{\ell}_r$.

The beauty of the LVE representation is that all the various traces of the loop vertices in the direct representation give rise to a *single* trace in the dual representation. This is the fundamental observation which made the representation suited for constructive matrix model and non commutative quantum field theories [16].

We need a label u to describe the various objects met when turning around the cycle $\mathcal{C}$. Then to each object $u \in \mathcal{O}$ is associated an operator P_u, with value

$$P_u = T_\Lambda D^{\bar{\ell}_c}_{mn} \text{ if } u = \bar{b} \in \bar{B}, \tag{7}$$

$$P_u = K^{\bar{\ell}}_{mn}(\sigma) \text{ if } u = \bar{v}_0 \in \bar{S}, \tag{8}$$

$$P_u = R^{\bar{\ell}_l,\bar{\ell}_r}_{mn}(\sigma) \text{ if } u = \bar{v}_1 \in \mathcal{L}, \tag{9}$$

In this dual picture the measure $d\nu(\sigma, w)$ corresponds to the following rule : the weakening factor between a $\sigma_{\bar{v}}$ and a $\sigma_{\bar{v}'}$ is the minimum of the w parameters of the lines $\bar{\ell}$ that have to be *crossed* to join the two regions $\bar{v}$ and $\bar{v}'$.

Hence the amplitude for a tree of loop vertices reads :

$$P_\Lambda = \log Z(\lambda, \Lambda, \mathcal{V}) = \sum_{\bar{\mathcal{T}} \text{ lines}, \ |\bar{\mathcal{T}}|=n-1} G_{\bar{\mathcal{T}}}$$

$$G_{\bar{\mathcal{T}}} = (-\lambda)^{n-1} \int d\nu(\sigma, w) \mathrm{Tr}\Big\{ \tilde{\prod}_{u\in\mathcal{O}} \mathrm{P_u} \Big\}. \tag{10}$$

where the operator Tr means that we sum over all matrix indices following the cyclic ordering of all the objects P_u.

Théorème C.4.1 *The vacuum Green's function P_Λ is absolutely convergent and defines an analytic function in the half-disk $\mathcal{D}_\Lambda = \{\lambda \mid \Re(\lambda^{-1}) \geq K \log \Lambda$, where K is a large constant.*

Preuve C.4.1 *This is just an application of the techniques of [16, 17] which we recall briefly.*

- *The number of trees is n^{n-2}, by Cayley's theorem,*
- *each full resolvent $R(\sigma)_{mn}$ is bounded by 1 in the disk,*
- *each decorating counter term is bounded by $\log \Lambda$.*

So the connected function is bounded by

$$P_\Lambda \leq \sum_n \frac{n^{n-2}}{n!} \lambda^n (\ln \Lambda)^n \leq \sum (\lambda K)^n (\ln \Lambda)^n. \tag{11}$$

which is a convergent geometric series as long as $\Re(\lambda^{-1}) \geq K \log \Lambda$ but divergent when $\Lambda \to \infty$. So the loop vertex expansion alone couldn't tame all divergences and we still need renormalization. Remark that the role of the divergent tadpole terms and counter-terms are in some sense changed. The divergences come from the linear counter terms due to the Wick ordering of the interaction. We need to compensate them, at least partially, by the inner tadpoles and crossings generated by expanding and contracting the σ fields hidden in the resolvents. So we shall put the perturbation series and the remainders on the same footing, which is another interesting property of the loop vertex expansion. We shall discuss this point in more detail in the next section.

C.5 The Cleaning Expansion

In this section we shall introduce another expansion, which we call the cleaning expansion, to generate and contract the σ fields hidden in the resolvents so as to compensate the divergent counter terms. We shall work in the dual representation of the LVE, since in this representation the cyclic ordering of different objects is more explicit. Instead of having a trace operator for every objects, we have a single global trace operator. We shall use the multi-scale representation of the propagators and the resolvents.

We impose also the stopping rule for the cleaning expansion to make sure that we don't expand forever (this would lead to divergence). We stop the expansion until we have gained enough convergent factors to compensate the divergent Nelson's factor.

C.5.1 The sliced propagator and resolvents

We introduce the Schwinger parameter representation the propagator :

$$C_{mn} = \int_0^\infty d\alpha e^{-\alpha(\frac{\mu^2}{\theta}+m+n+1)} = K\int_0^1 d\alpha e^{-\alpha(\frac{\mu^2}{\theta}+m+n+1)}. \tag{1}$$

We decompose the propagator as :

$$C_{mn} = \sum_{j=0}^{\infty} C^j_{mn}, \tag{2}$$

where

$$C^j_{mn} = \int_{M^{-2i}}^{M^{-(2i-2)}} d\alpha e^{-\alpha(\mu^2+\frac{4}{\theta}(m+n+1))}, \tag{3}$$

and M is an arbitrary positive constant. We could easily find that

$$|C^j_{mn}| \leq KM^{-2j}e^{-M^{-2j}||\mu^2+\frac{4}{\theta}(m+n+1)||}. \tag{4}$$

We have the associated decomposition for the amplitude of the connected function :

$$G_T = \sum_j G^j_T. \tag{5}$$

Due to the cyclic order of the global Trace operator, we could rewrite the resolvent defined in formula (2) as :

$$R_{mn} = \frac{\partial}{\partial\sigma_{mn}}[-\frac{1}{2}\mathrm{Tr}\log(1+\mathrm{i}\sqrt{2\lambda}\mathrm{C}\sigma)] = -\mathrm{i}\frac{1}{2}\sqrt{2\lambda}\mathrm{C}_{\mathrm{mn}}\mathrm{Tr}\frac{1}{1+\mathrm{i}\sqrt{2\lambda}\mathrm{C}\sigma}. \tag{6}$$

It is convenient to introduce further notations to write down in more compact form the contribution, or amplitude of a tree in the LVE. We put

$$D_{mn} = i\sqrt{2\lambda}C_{mn}, \tag{7}$$

so we have

$$R(\sigma) = \mathrm{Tr}\frac{1}{1+\mathrm{D}_{\mathrm{mn}}\sigma_{\mathrm{nm}}}, \text{ and } \mathrm{R}(\sigma)_{\mathrm{mn}} = \mathrm{R}(\sigma)\mathrm{D}_{\mathrm{mn}}, \tag{8}$$

where D_{mn} and $R(\sigma)$ should be thought of as a matrix operator. The existence of $R(\sigma)$ is ensured by the i factor in (7) [16]. More precisely, since σ and C are real symmetric matrices, all eigenvalues of the matrix $C\sigma$ should also be real. Hence there are no poles in the denominator of $1/(1+i\sqrt{2\lambda}C\sigma)$.

Then we define the sliced resolvent as :

$$R^j(\sigma) = \mathrm{Tr}\frac{1}{1+\sum_{\mathrm{k}\leqslant\mathrm{j}} \mathrm{i}\sqrt{2\lambda}\mathrm{C}^{\mathrm{k}}_{\mathrm{nm}}\sigma_{\mathrm{mn}}} = \mathrm{Tr}\frac{1}{1+\sum_{\mathrm{k}\leqslant\mathrm{j}} \mathrm{D}^{\mathrm{k}}_{\mathrm{nm}}\sigma_{\mathrm{mn}}}, \tag{9}$$

and the full resolvent as

$$R^j_{mn} = D^j_{mn}R^j. \tag{10}$$

For each resolvent we have the algebraic equation

$$\begin{aligned} R^j(\sigma) &= \frac{1}{1+\sum_{k\leq j} D^k\sigma} = \frac{1}{1+\sum_{k<j} D^k\sigma + D^j\sigma} \\ &= R^{j-1}(\sigma)\frac{1}{1+R^{j-1}(\sigma)D^j\sigma}. \end{aligned} \tag{11}$$

C.5.2 The cleaning expansion

In this section we shall expand the resolvents and contract the σ fields to generate the inner tadpoles for canceling (partly) the counter terms and to generate the crossings and nesting lines for gaining convergent factors. This is called the cleaning expansion [20].

The canonical cyclic ordering of the LVE is also essential for the cleaning expansion, as otherwise it would be difficult to find the corresponding "weakening parameters" rule for that Taylor remainder term. But fortunately the cyclic ordering solves nicely this problem ! The beginning of the cycle $\mathcal{C}$ is explicitly and fully cleaned of potential tadpoles ; the rest of the cycle $\mathcal{C}$ has *no weakening parameters* on the remaining potential tadpoles !

We won't just expand each resolvent naively as power series of $i\sqrt{\lambda}C^{1/2}\sigma C^{1/2}$, since that would generate unnecessarily complicated weakening factors when we contract the σ fields of different regions in dual graph, and what's more, this naive expansion is not convenient for the multi-scale analysis of the amplitude, which is crucial to analyze the Taylor remainder terms.

We apply the cleaning expansion in two steps :

In step 1, we start from an arbitrary marked point in the dual graph and apply the algebraic induction formula (12)

$$R^j(\sigma) = R^{j-1}(\sigma) - R^{j-1}(\sigma)\sigma_{mn}D^j_{nm}R^j(\sigma)\ , \tag{12}$$

which rewrites the multi-scale representation of each resolvent of scale j into two parts, for the resolvent we meet clockwisely. The first part is resolvent of a lower scale $j-1$, and the second part is a product form of a pure propagator of scale j, a resolvent of scale j and a resolvent of scale $j-1$. We start form the highest scale j_{max}. Due to the cyclic ordering the pure propagator term appears only on the left side of the product.

So by using the formula (12) inductively we could lower down the scale of the resolvents and propagators on the one hand and decompose the resolvents into a cleaning part which is the pure propagator with σ field attached and the uncleaned resolvents, on the other hand.

In step 2, we use the integration by parts of the σ fields with respect to the Gaussian measure. For example, for each term in the second part of formula (12) we have :

$$\int d\mu(\sigma) R^{j_{max}-1}(\sigma)\sigma_{mn} D^{j_{max}}_{nm} R^{j_{max}}(\sigma) \times \text{ other resolvents of scale } \mathrm{j_{max}}$$
$$= D^{j_{max}}_{mn} \int d\mu(\sigma) \frac{\partial}{\partial \sigma_{nm}} [R^{j_{max}}(\sigma) R^{j_{max}-1}(\sigma) \times \text{ other resolvents}]. \quad (13)$$

We ignore the case where $\frac{\partial}{\partial\sigma}$ acts on the $R^{j_{max}-1}$ term as this gives minor contribution and we consider only the case $\frac{\partial}{\partial\sigma} R^{j_{max}}$ for the first resolvent $R^{j_{max}}$ we meet. For better explaining the cleaning expansion and showing explicitly the tadpoles and crossings generated by the cleaning expansion, we don't use the derivation formula (5) directly but we apply the induction again to test whether the terms in the derivations are resolvents of lower scale, pure propagators with σ field attached or the resolvents of scale j. A typical cleaning expansion step is shown in Figure C.13. We ignore the counter terms which could be anywhere in the dual graph, just for better explanation, as they don't change the cleaning process.

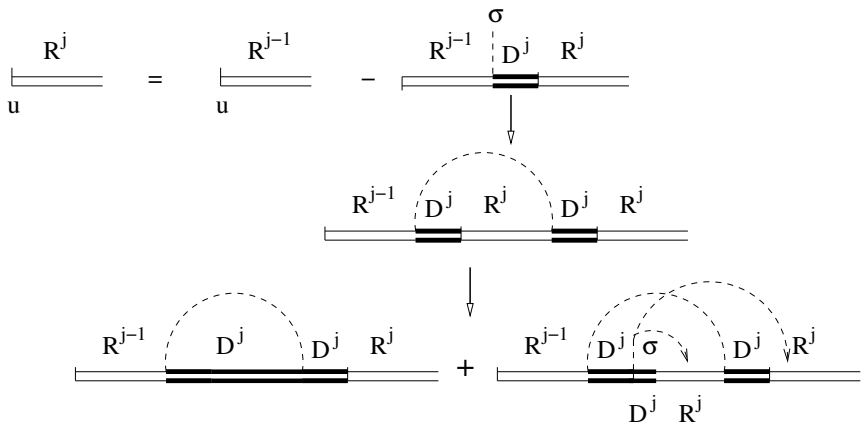

FIGURE C.13: The cleaning expansion. The ordinary ribbon stands for the resolvents and the bold double line stands for the pure propagator. The dashed lines should be envisioned also as double lines and these stand for the σ fields. u is the marked point.

The second line in Figure C.13 shows the graph of leading order where we apply the induction formula (12) for 4 times. There are the following possibilities of the contraction of the σ fields :

- The field σ_1 is contracted with σ_2 ; this generates an inner tadpole of scale j which should be compensated by the counter term.
- σ_1 is contracted with σ_3 ; This generates a crossing of order j from each of which we gain a convergent factor M^{-2j}.
- σ_1 is contracted with σ_4 or other field hidden in the resolvents and σ_2 is contracted with σ_3 ; this generates an inner tadpole which overlaps with some other loop. Again the inner tadpole formed by σ_2 and σ_3 fields should be compensated by the counter-term.
- there exists the possibility that each σ field of the induction formula (12) contracts only with another one hidden in other resolvents and never generate crossings or tadpoles but only nesting lines. This process could last forever. This case is not

dangerous, since for such overlaps we could gain a convergent factor M^{-2j}. So an overlap plays the same role as a crossing.
– if the field σ_1 is contracted with other resolvents than the first one in formula (13), this would generate a crossing since different regions of the dual graph would be connecetd by that contraction, as could be easily seen in Figure C.15.

Now we consider the amplitudes for crossing lines (see graph B in Figure (C.16)) and nesting lines (see graph D in Figure C.16). A crossing means a crossing of two σ propagators that correspond to δ functions and are pictured as dash lines, and the σ propagator that overlaps another is attached to a *pure* propagator. Since a crossing line doesn't form a tadpole, the amplitude of each crossing line equals to the bound of the sliced propagator (see formula (4)) :

$$||C^j_{mn}|| \leqslant |KM^{-2j} e^{-M^{-2j}||\mu^2 + \frac{4}{\theta}(m+n+1)||}| < KM^{-2j}, \tag{14}$$

Nesting lines means many σ propagators that each one is overlapped by another. Each nesting line is made up of a σ propagator with two pure propagator attached to each end of the σ propagator, and many (at least one) resolvents or /and pure propagators being overlapped by the σ propagator.

The amplitude of a nesting line is bounded by :

$$|R^{\bar{\ell}_l,\bar{\ell}_r\, j}_{mn}[\prod R^{\bar{\ell}'_l,\bar{\ell}'_r\, j}_{m'n'}]| < |R^{\bar{\ell}_l,\bar{\ell}_r\, j}_{mn} C^j_{mn}| \leqslant ||R^j|| |C_{mn}|| < KM^{-2j}, \tag{15}$$

where $[\prod R^{\bar{\ell}'_l,\bar{\ell}'_r\, j}_{m'n'}]$ means the full resolvents overlapped by the nesting line that we consider and C^j_{mn} means the pure propagator attached to either side of the σ propagator. Since they don't form any closed tadpole, it doesn't matter which C^j_{mn} we consider ; we have used the fact that each full resolvent $R^{\bar{\ell}'_l,\bar{\ell}'_r\, j}_{m'n'}$ is bounded by 1.

So we gain a convergent factor M^{-2j} for each of the two cases.

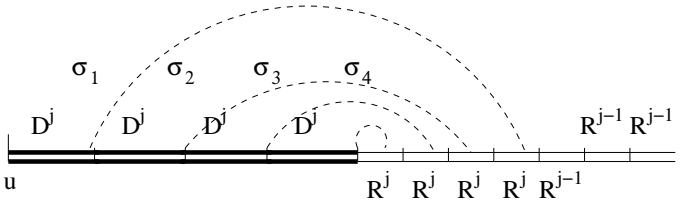

FIGURE C.14: The cleaning expansion with 4 nesting lines. u is the marked point.

Finally we have an example of a dual graph with two initial resolvents, as shown in Figure C.16. Graphs A and C stand for the tadpoles, graph B shows two crossing lines and graph D shows the nesting lines.

C.5.3 The stopping rules

We cannot do the cleaning expansion forever, as this would develop the full perturbation series and ultimately diverge. More exactly, for each scale j the number of crossings shouldn't exceeds $n(j) = (aj)!$, where a is a positive constant to be fixed, otherwise we would no longer be able to compensate for that divergent factorial with the convergent factors coming from the crossings and/or nestings.

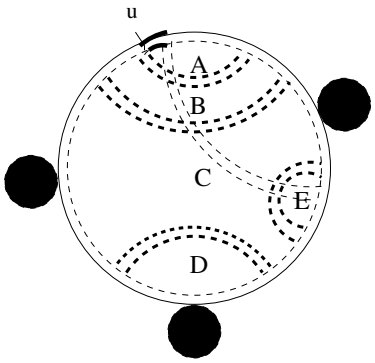

FIGURE C.15: The crossing generated by the case that the field σ_1 is contracted with other resolvents which is not the first one in formula (13), for a more general case ; The bold double dash line means the original dual σ propagator, the normal double dash line means the σ propagator generated by the cleaning expansion. Clearly this forms three crossings. u is the marked point.

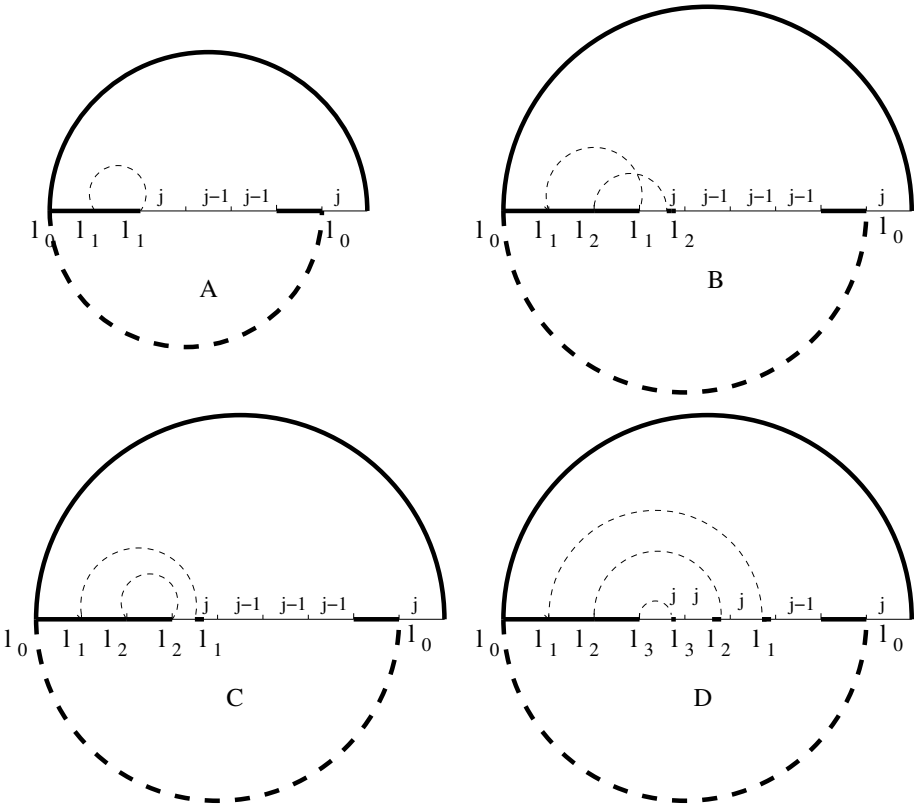

FIGURE C.16: An example of a dual graph with two resolvents where each ordinary line should be considered as a ribbon. The bold dash lines mean the original σ propagator and the normal dash lines are the σ propagators generated by the cleaning expansion. The normal lines are the resolvents while the bold lines are the pure propagators.

So we impose the following stopping rules for the cleaning expansion and write the rest terms as Taylor integral remainder when enough cleaning, depending on the scale, has succeeded :

– We start from resolvents and pure propagators of scale j_{max} and go on the expansion

towards lower scales.
- We stop the expansion whenever we generate an inner tadpole and we cancel it with its corresponding counter-term.
- We go on the expansion when we generate a crossing or overlapping until the total number of crossings and nestling lines reaches $n(j_{max}) = aj_{max}$, so that in total we could gain a convergent factor $M^{-aj_{max}^2}$ from the crossings and the nestling lines while lose a combinatorial factor $(aj_{max})! \sim e^{j_{max} \ln j_{max}}$, which is not dangerous. Then we stop the expansion and don't test the lower scale resolvents.
- It is very possible that we couldn't generate enough crossings by expanding a single resolvent in the dual tree. Then we need to go to the next resolvent and repeat the same process.

After the cleaning expansion, no inner tadpole terms nor counter-terms exist in the cleaned part in the dual graph. We shall check this nontrivial fact in section C.6. However the amplitude could still be divergent due to the possible arbitrarily large number of counter terms attached to the uncleaned part of the graph. Instead of further generating inner tadpoles we choose to re-sum them. After the resummation we would generate a divergent factor which we call the Nelson's factor. But this factor is not dangerous since the convergent factors from the crossings would be enough to cancel it. We shall go back to this point in section C.7.

C.6 The renormalization

C.6.1 The power counting theorem.

Théorème C.6.1 *The amplitude of a graph of V vertices, B broken faces and genus g is proportional to*

$$\Lambda^{4-2V-4g-2B} \tag{1}$$

where Λ is the ultraviolet cutoff [27], g is the genus and B is the number of broken faces.

Remark that unlike the commutative ϕ_2^4 model, the power counting for the vacuum Schwinger's functions is different from that for n-point Schwinger's functions due to the number of broken faces.

We could prove the power counting theorem in at least three representations : the matrix base representation [27] [29], the direct space representation and [41] and the parametric representation [68].

We shall not prove this theorem in this paper, as what we need essentially are the topological and combinatorial properties of the ribbon graphs. Since the LVE doesn't change the theory, it doesn't change the power counting either.

C.6.2 The renormalization

Since the leaf tadpoles could be compensated by the counter terms before we use the LVE, the only possible divergences are the counter terms, and we use the inner tadpoles, the crossings and the nesting lines to compensate them. To be more precise, we use the inner tadpole to compensate the counter terms attached to the cleaned part and use the crossings and the nesting lines to compensate the uncleaned tadpoles.

We have the following theorem for the renormalization, by which we mean the compensation between inner tadpoles and counter terms :

Théorème C.6.2 *Each inner tadpole should be canceled by a counter-term.*

The compensation of the uncleaned tadpoles will be discussed in the next section.

Preuve C.6.1 *We consider an inner tadpole generated from an arbitrary resolvent $R(\sigma)$. We Taylor expand this resolvent as*

$$R(\sigma) = \frac{1}{1 + i\sqrt{2\lambda}C\sigma} = 1 - i\sqrt{2\lambda}C\sigma + (i\sqrt{2\lambda}C\sigma)^2 + \text{Remainder}, \tag{2}$$

and the amplitude for an inner tadpole reads :

$$T_{tadpole} = \int d\nu(\sigma)(\sum_{mn}^{\Lambda} i\sqrt{2\lambda}C_{mn}\sigma_{nm})^2 = -2\lambda T_{\Lambda}^2. \tag{3}$$

The amplitude for a counter term attached to this resolvent reads :

$$T_c = \int d\nu(\sigma)\frac{1}{2!} \times 2 \times (i\sqrt{2\lambda}\sum_{mn} C_{mn}\sigma_{nm})\mathrm{Tr_p}\sigma_{\mathrm{pp}}\mathrm{T_p} = 2\lambda \mathrm{T}_{\Lambda}^2. \tag{4}$$

So we have

$$T_{tadpole} + T_c = 0. \tag{5}$$

Remark that since the cancelation is exact, this theorem is true also for the sliced inner tadpoles and counter terms at any scale j. For proving this, we could just use the same method while replacing each propagator C by $\sum_j C^j$ and each counter term T_Λ by $\sum_j T_j$.

So we have proved the remormalization theorem. We could also use the combinatorial renormalization (see the appendix) to prove the cancelation of the inner tadpoles with the counter terms. We don't discuss this point in detail in this paper as it is not essential. The interested reader could also look at [20] and [57] for more details for the combinatorial properties for the LVE.

C.7 Nelson's argument and the bound of the connected function

In the last section we introduced the cleaning expansion, after which all inner tadpoles should be canceled by the counter-terms in the cleaned part of the dual graph. But there might still be arbitrarily many counter terms in the uncleaned part and they are divergent. Instead of canceling all of them, we resum them by using the inverse formula of the Gaussian integral and integration by parts.

We write more explicitly, but loosely, the amplitude of the connected function after the LVE and the cleaning expansion as :

$$A_T^N = \prod_{\bar{\ell},\bar{\ell}_c\bar{\ell}_l,\bar{\ell}_r\in\mathcal{T}} \int d\mu(w,\sigma) \int_0^1 dw_l \mathrm{Tr} \overrightarrow{\prod} [\mathrm{K}_{\mathrm{mn}}^{\bar{\ell}}(\sigma)\mathrm{R}_{\mathrm{mn}}^{\bar{\ell}_\mathrm{l},\bar{\ell}_\mathrm{r}}(\sigma)(\mathrm{D}^{\bar{\ell}_\mathrm{c}}\mathrm{T}_\Lambda)], \tag{1}$$

where $R_{mn}(\sigma)$ is the full resolvent that contains the pure propagator C, Tr needs to follow the cyclic order according to the real positions of the leaf terms K, the resolvents R and the counter terms $T_\Lambda D$. We have used the fact that all weakening factors for the counter terms equal to one, as they are leaves in the graph. There are only weakening factors for the σ propagators that cross different regions in the dual graph.

Now we consider the function G for which we haven't expanded the counter-term :

$$G = \int d\nu(\sigma, w) \prod_{l\in\mathcal{T}} \mathrm{Tr}[\mathrm{K_{mn}R_{mn}}(\sigma)]\mathrm{e}^{\mathrm{Tri}\sqrt{2\lambda}\sigma\mathrm{T}_\Lambda}. \tag{2}$$

We use the formula

$$\int d\nu(w,\sigma) f(\sigma)g(\sigma) = e^{\frac{1}{2}\frac{\partial}{\partial\sigma}C(\sigma,\sigma',w)\frac{\partial}{\partial\sigma}} f(\sigma)g(\sigma))|_{\sigma=0}, \tag{3}$$

where $C(\sigma, \sigma', w)$ is the covariance that might depend on the weakening factor w or not. Hence

$$\begin{aligned} G &= \int d\nu(\sigma, w) \sum_{N=0}^{\infty} \frac{1}{N!}[\frac{1}{2}\frac{\partial}{\partial\sigma}\frac{\partial}{\partial\sigma'}]^N\{ \prod_{\bar{l}\in\bar{\mathcal{T}}} \int_0^1 dw_{l'} \mathrm{Tr}[\mathrm{K}^{\bar{\ell}}(\sigma)\mathrm{R}^{\bar{\ell}_\mathrm{l},\bar{\ell}_\mathrm{r}}(\sigma)]\mathrm{e}^{\mathrm{Tri}\sqrt{2\lambda}\sigma\mathrm{T}_\Lambda}]\} \\ &= \sum_{N_1=0}^{\infty}\sum_{N_2=0}^{\infty}\sum_{N_3=0}^{\infty}\prod_{l\in\mathcal{T}}\int_0^1 dw_l \frac{1}{N_1!N_2!}(\frac{1}{2})^{N_1+N_2}[(\frac{\partial}{\partial\sigma})^{N_2}\{\frac{\partial}{\partial\sigma}\frac{\partial}{\partial\sigma'}\}^{N_1}][K(\sigma)R(\sigma)] \\ &\quad \{(\frac{\partial}{\partial\sigma'})^{N_2}\frac{1}{N_3!}[\frac{1}{2}\frac{\partial}{\partial\sigma}\frac{\partial}{\partial\sigma'}]^{N_3} e^{\mathrm{Tri}\sqrt{2\lambda}\sigma\mathrm{T}_\Lambda}\}|_{\sigma=0}. \end{aligned} \tag{4}$$

While the N_1 and N_2 derivations generate connected terms, the last derivatives generate N_3 disconnected terms, see graph B in Figure C.8.

We sum over the N_3 non-connected terms and we have :

$$\begin{aligned} G &= \int d\nu(\sigma, w) \sum_{N_1=0}^{\infty}\sum_{N_2=0}^{\infty}\sum_{N_3=0}^{\infty}\prod_{\bar{l}\in\bar{\mathcal{T}}}\int_0^1 dw_{\bar{l}} \frac{1}{N_1!N_2!}(\frac{1}{2})^{N_1+N_2} \\ &\quad \{(\frac{\partial}{\partial\sigma})^{N_2}[\frac{\partial}{\partial\sigma}\frac{\partial}{\partial\sigma'}]^{N_1}[K^{\bar{\ell}}_{mn}(\sigma)R^{\bar{\ell}_l,\bar{\ell}_r}_{mn}(\sigma)]\}\{(\frac{\partial}{\partial\sigma'})^{N_2}e^{\mathrm{Tri}\sqrt{2\lambda}\sigma\mathrm{T}_\Lambda}\}e^{-\lambda T_\Lambda^2} \\ &= A_T e^{-\lambda T_\Lambda^2}. \end{aligned} \tag{5}$$

Here we have used the fact that the weakening factor for each counter-term is 1.

Hence

$$A_T = \int d\nu(\sigma, w) \prod_{\bar{\ell}\in\bar{\mathcal{T}}} \mathrm{Tr}\overrightarrow{\prod}[\mathrm{K}^{\bar{\ell}}_{\mathrm{mn}}(\sigma)\mathrm{R}^{\bar{\ell}_\mathrm{l},\bar{\ell}_\mathrm{r}}_{\mathrm{mn}}(\sigma)]\mathrm{e}^{\mathrm{Tri}\sqrt{2\lambda}\sigma\mathrm{T}_\Lambda}\mathrm{e}^{\lambda\mathrm{T}_\Lambda^2}, \tag{6}$$

which means we generated a divergent factor $e^{\lambda T_\Lambda^2}$, which we call the Nelson's factor, after the resummation of the disconnected counter-terms. This divergent factor is not really dangerous since we could compensate it by the convergent factors generated by the crossings and nesting lines. More precisely, we have

$$e^{-aj_{max}^2}\cdot j_{max}!\cdot e^{2\lambda T_\Lambda^2} \sim e^{-aj_{max}^2 + j_{max}\ln j_{max} + 2\lambda j_{max}^2} < 1 \tag{7}$$

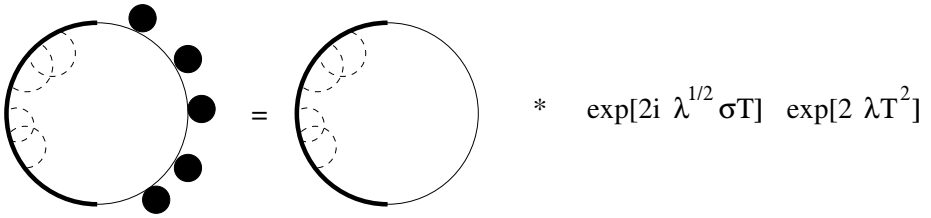

FIGURE C.17: A sketch of the resummation of the counter terms.

as long as we choose a properly, for example $a > 3\lambda$. We have used the fact that $T_\Lambda \sim j_{max}$. This resummation process is shown in Figure C.17.

So we have

$$\begin{aligned} |A_T|_{|T|=n} &< \int d\nu(\sigma, w) \prod_{l,l' \in \mathcal{T}} \lambda^{n-1} |\mathrm{Tr}\{||\mathrm{K}^{\bar{\ell}}_{\mathrm{mn}}(\sigma)||\ ||\mathrm{R}^{\bar{\ell}_\mathrm{l},\bar{\ell}_\mathrm{r}}_{\mathrm{mn}}(\sigma)||\ |\mathrm{e}^{\mathrm{Tri}\sqrt{2\lambda}\sigma \mathrm{T}_\Lambda}|\}| \\ &\times e^{\lambda T_\Lambda^2} e^{-a j_{max}^2} \leqslant (K\lambda)^{n-1}. \end{aligned} \tag{8}$$

where the global trace operator Tr will result in a cutoff index Λ which could be easily bounded by the exponential $e^{-aj^2_{max}}$ without generating any divergent factor, and K is an arbitrary constant times the possible factors $\sqrt{2}$ or $\sqrt{2}+1$ from the bound of the resolvent R and the leaf K respectively.

This is the Nelson's argument in this context.

C.8 Borel summability

Théorème C.8.1 *The perturbation series of the connected function for ϕ^4_2 theory is Borel summable.*

Proof For the perturbation series $\sum_{n=0} a_k \lambda^k$ to be Borel summable to the function G, we need to have

$$G(\lambda) = \sum_{n=0} a_k \lambda^k + R^{(n+1)}(\lambda), \tag{1}$$

where $R^{(n+1)}(\lambda)$ is the Taylor remainder. The analyticity domain C_λ for G should be at least $|\lambda| < \frac{1}{K}$ and $\mathrm{Re}\lambda > 0$ [23, 16], which means

$$-\frac{\pi}{4} \leq \mathrm{Arg}\sqrt{\lambda} \leq \frac{\pi}{4}. \tag{2}$$

We rewrite the resolvent as

$$R = \frac{1}{1 + i\sqrt{2\lambda}\mathrm{Tr}[\mathrm{C}^{1/2}\sigma \mathrm{C}^{1/2}]}. \tag{3}$$

Since the matrix $C^{1/2}\sigma C^{1/2}$ is Hermitian, the eigenvalues are real, hence there are no poles in the denominator. In the analytic domain of G we have

$$\| R \| = |\frac{1}{1 + i\sqrt{2\lambda}\mathrm{Tr}[\mathrm{C}^{1/2}\sigma \mathrm{C}^{1/2}]}| \leqslant \sqrt{2}, \tag{4}$$

and

$$\| K \| = \| R - 1 \| \leqslant 1 + \sqrt{2}. \tag{5}$$

However in the analytic domain C_λ the linear counter term becomes :

$$e^{\mathrm{Tri}\sqrt{2\lambda}\sigma \mathrm{T}_\Lambda} = e^{\mathrm{Tri}|\sqrt{2\lambda}|\cos\theta\sigma \mathrm{T}_\Lambda} e^{-\mathrm{Tr}|\sqrt{2\lambda}|\sin\theta\sigma \mathrm{T}_\Lambda}, \tag{6}$$

where $\theta = \mathrm{Arg}\sqrt{\lambda}$. We could bound the first term in (6) by 1, but the second term would diverge for negative σ.

We rewrite this term as :

$$\begin{aligned} & \int d\mu(\sigma) e^{-1/2\mathrm{Tr}\sigma^2} e^{-\mathrm{Tr}|\sqrt{2\lambda}|\sin\theta\sigma \mathrm{T}_\Lambda} \\ = & \int d\mu(\sigma) e^{-1/2\mathrm{Tr}(\sigma + \sqrt{2\lambda}\sin\theta \mathrm{T}_\Lambda)^2} e^{\sin^2\theta T_\Lambda^2}. \end{aligned} \tag{7}$$

The term $e^{\sin^2\theta T_\Lambda^2}$ could diverge at worst as $e^{1/2T_\Lambda^2}$ for $\theta = \pm\pi/4$. But this is not dangerous, since we could still bound it with the convergent factor $e^{-aj_{max}^2} \sim e^{-aT_\Lambda^2}$ that we gained from the crossings and nesting lines.

We use simply the Taylor expansion with remainder for the connected function (1) :

$$G(t\lambda)|_{t=1} = \frac{G^{(n)}(\lambda)}{n!}|_{t=0} + \int_0^1 dt \frac{(1-t)^n}{n!} G^{(n+1)}(t\lambda), \tag{8}$$

followed by explicit Wick contractions. We have for the reaminder

$$||R^{n+1}|| < |\lambda|^{n+1} K^n (2n)!! \leq |\lambda|^{n+1} [K']^n (n!), \tag{9}$$

where K and K' are positive numbers including the possible factors $\sqrt{2}$ or $\sqrt{2}+1$ from the bound of the resolvent R_{mn} and of the leaf K_{mn} respectively. Hence we have proved the Borel summability of the perturbation series.

C.9 Appendix

C.9.1 The order 1 case.

There are only three terms for order 1 expansion and the corresponding graphs are shown in Figure C.9. The amplitude for graph A reads :

$$\begin{aligned} G_A^1 &= -\frac{1}{2}[-\frac{1}{2}(i\sqrt{2\lambda})^2] \int d\nu(\sigma) \sum_{mnp} [(C_{mn}\sigma_{np} C_{mp}\sigma_{pn})] \times 3 \\ &= -\frac{3}{2} T_\Lambda^2, \end{aligned} \tag{1}$$

where the factor 3 comes from the number of contractions that can form graph A. The amplitude for graph B reads :

$$G_B^1 = \frac{1}{2!}(i\sqrt{2\lambda})^2 \int d\nu(\sigma) [\mathrm{Tr_m T_m} \sigma_{\mathrm{mm}} \mathrm{Tr_n T_n} \sigma_{\mathrm{nn}}] = -\lambda \mathrm{T}_\Lambda^2. \tag{2}$$

Graph C is the constant counter-term and has amplitude $\frac{5}{2}T_\Lambda^2$. So we have

$$G_A^1 + G_B^1 + G_C^1 = 0. \tag{3}$$

C.9.2 The order 2 case

There are three kinds of graphs of order 2 which are shown in Figure C.18 with nontrivial combinatorial factors. The amplitude of graphs A reads :

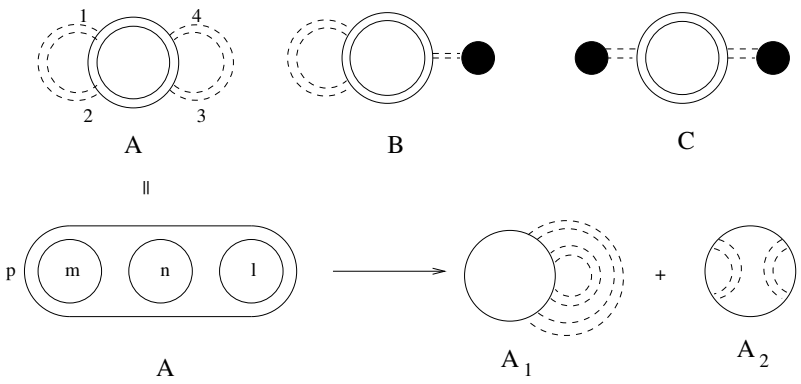

FIGURE C.18: The renormalization of the divergent graphs at order 2.

$$G_A^{(2)} = -\frac{1}{2}\frac{-(i\sqrt{2\lambda})^4}{4}\sum_{mnlp}^{\Lambda}\frac{1}{m+p+1}\frac{1}{n+p+1}\frac{1}{l+p+1}\times D_A^{(2)}$$
$$= T_\Lambda^3, \tag{4}$$

where

$$T_\Lambda^3 = \sum_{mnlp}^{\Lambda}\frac{1}{m+p+1}\frac{1}{n+p+1}\frac{1}{l+p+1}, \tag{5}$$

and $D_A^{(2)}$ is the combinatorial factor for graph A. For generating graph A we could either contract the half line 1 with 2 and 3 with 4, or 1 with 4 and 3 with 2. If the sigma fields are attached to the inner face like graph A_2, these are all possible cases. If the σ fields are attached to the outer faces, there are two additional cases for each kind of contraction, take A_1 for example. So in total there are 8 possibilities and $D_A^{(2)} = 8$. Similarly we have $D_B^{(2)} = 9$ and $D_C^{(2)} = 4$.

So the amplitude of graphs B reads :

$$G_B^{(2)} = \frac{1}{2!}\cdot 2\cdot\left[-\frac{1}{2}\frac{(i\sqrt{2\lambda})^3}{3}(i\sqrt{2\lambda})\sum_{nl}^{\Lambda}\mathrm{Tr_p T_p}\frac{1}{\mathrm{n+p+1}}\frac{1}{\mathrm{l+p+1}}\right]\times \mathrm{D_B^{(2)}}$$
$$= -\frac{3}{2}T_\Lambda^3, \tag{6}$$

and the amplitude for graphs C reads :

$$G_C^{(2)} = \frac{1}{3!}\cdot 3\cdot\left[-\frac{1}{2}\frac{-(i\sqrt{2\lambda})^2}{2}(i\sqrt{2\lambda})^2\sum_{n}^{\Lambda}\mathrm{Tr_p T_p Tr_p T_p}\frac{1}{\mathrm{n+p+1}}\right]\times \mathrm{D_C^{(2)}}$$
$$= \frac{1}{2}T_\Lambda^3, \tag{7}$$

So again we have

$$G_A^{(2)} + G_B^{(2)} + G_C^{(2)} = 0. \tag{8}$$

This checks explicitly the renormalization at low orders.

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D

Paper 4

How are Feynman graphs resummed by the Loop Vertex Expansion ?

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Abstract The purpose of this short letter is to clarify which set of pieces of Feynman graphs are resummed in a Loop Vertex Expansion, and to formulate a conjecture on the ϕ^4 theory in non-integer dimension.

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Key words : Constructive field theory, Loop vertex expansion.

D.1 Introduction

In quantum field theory (hereafter QFT) any connected (i.e. interesting) quantity is written as a sum of amplitudes for a certain category of connected graphs

$$S = \sum_G \mathcal{A}_G \tag{1}$$

but this formula is not a valid definition of S since usually

$$\sum_G |\mathcal{A}_G| = \infty. \tag{2}$$

This phenomenon, known since [10] is basically due to the very large number of elements at order n in the species [43] of Feynman graphs. Accordingly the *generating functional*

for the Feynman graphs species, namely the series $\sum_n \frac{\lambda^n a_n}{n!}$, where a_n is the number of Feynman graphs at order n, has zero radius of convergence as power series in λ. We call such a species a *proliferating species*. In zero space-time dimension, quantum field theory reduces to this generating functional, hence to graphs counting. In higher dimensions quantum field theory is in fact a *weighted* such species, that is Feynman graphs have to be pondered with weights, called Feynman amplitudes. For an introduction to the structure of Feynman graphs, see [44]. Nevertheless these Feynman amplitudes tend to behave as K^n at order n (at least in low dimensions). Hence the perturbation series eg for the ϕ^4 Euclidean Bosonic quantum field theory has be proved to have zero radius of convergence in one, two and three dimensions ([45, 46]). Nothing is yet known for sure in dimension 4 but there are strong reason to expect also the renormalized Feynman series to diverge there as well (see [7] and references therein).

In contrast Cayley's theorem, which states that the total number of labeled trees at order n is n^{n-2}, implies that the species of *trees* is *not* proliferating. This fact can be related to the local existence theorems for flows in classical mechanics, since classical perturbation theory is indexed by trees [47]. These theorems have no quantum counterpart, but constructive theory can be seen as various recipes to replace the ordinary divergent Feynman graph expansions by convergent ones, usually indexed by trees rather than graphs [48]. It can therefore be considered a bridge between QFT and classical mechanics, since it repacks the loops which are the fundamental feature of QFT, and brings the expansion closer to the more familiar trees of classical mechanics. Historically constructive theory used cumbersome non canonical tools borrowed from lattice statistical mechanics, such as cluster expansions which did not respect the rotational invariance of the underlying theory [9, 7]. The Loop Vertex Expansion [16, 17] in contrast is a more canonical way to exchange the ordinary perturbative divergent expansion for a convergent expansion, hence to really compute quantities in quantum field theory in principle to arbitrary accuracy.

One of us (VR) was recently asked exactly which (pieces of) Feynman graphs are resummed by this expansion. The answer is contained in the initial papers, but perhaps not easy to extract. The purpose of this little note is therefore to explain more explicitly exactly which pieces of which Feynman graphs of different orders are combined together by the loop vertex expansion to create a convergent expansion. This reshuffling is fully explicited up to third order for the simplest of all possible examples, namely the ϕ^4_0 quantum field theory. Finally we propose a conjecture, which, if true, would allow to define QFT in non-integer dimensions of space-time.

D.2 Relative Tree Weights in a Graph

A graph may contain many (spanning) forests, and a forest can be extended into many graphs with cycles. So the relationship between graphs and their spanning forests is not trivial.

The forest formula which we use [12, 13] can be viewed as a tool to associate to any pair made of a graph G and a spanning forest $F \subset G$ a unique rational number or weight $w(G, F)$ between 0 and 1, called the relative weight of T in G. These barycentric weights obey for any G

$$\sum_{F \subset G} w(G, F) = 1 \tag{1}$$

The numbers $w(G,F)$ are multiplicative over disjoint unions [1]. Hence it is enough to give the formula for (G,F) only when G is *connected* and $F=T$ is a spanning tree in it [2].

The definition of these weights is

Définition D.2.1

$$w(G,T)=\prod_{\ell\in T}\int_0^1\prod_{\ell\in T}dw_\ell\prod_{\ell\notin T}x_\ell^T(\{w\}) \tag{2}$$

where $x_\ell^T(\{w\})$ is the infimum over the $w_{\ell'}$ parameters over the lines ℓ' forming the unique path in T joining the ends of ℓ.

For a fixed spanning tree inside a graph, we call *loop lines* the lines not in the tree.

D.2.1 Examples

Consider the graph G of Figure D.1. There are 5 spanning trees inside this graph :

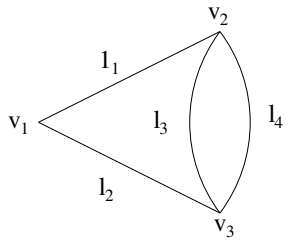

FIGURE D.1: A first example

$$T_{12}=\{l_1,l_2\},\ T_{13}=\{l_1,l_3\},\ T_{14}=\{l_1,l_4\},\ T_{23}=\{l_2,l_3\},\ T_{24}=\{l_2,l_4\}.$$

For example, for the tree $T_{12}=\{l_2,l_4\}$, the loop lines are l_1 and l_3.

To take into account the weakening factors $x_\ell^T(\{w\})$ of (1) for each loop line ℓ, it is convenient to decompose the integration domain $[0,1]^{|T|}$ into $|T|!$ sectors corresponding to complete orderings of the w_ℓ parameters for $\ell\in T$.

Let us compute in this way the relative weights of the five trees of G. First consider the contribution of the tree T_{12}. In this case the loop lines are l_3 and l_4. For each of them we have a factor $\inf(w_1w_2)$. Hence

$$\begin{aligned} w(G,T_{12}) &= \int_0^1\int_0^1 dw_1dw_2[\inf(w_1,w_2)]^2 \\ &= 2\int_0^1 dw_2\int_0^{w_2}dw_1w_1^2=\frac{2}{12}=\frac{1}{6}. \end{aligned}$$

Next we consider the spanning tree T_{13}. In this case the "loop lines" are l_2 which connects

1. And also over vertex joints of graphs, just as in the universality theorem for the Tutte polynomial.
2. It is enough in fact to compute such weights for 1-particle irreducible and 1-vertex-irreducible graphs, then multiply them in the appropriate way for the general case.

the vertices v_1 and v_3 and l_4 which connects v_2 and v_3. So we have :

$$\begin{aligned} w(G,T_{13}) &= \int_{w_1<w_3} dw_1 \int dw_3 \inf(w_1 w_3) w_3 \\ &+ \int_{w_3<w_1} dw_1 \int dw_3 \inf(w_1 w_3) w_3 \\ &= \int_0^1 dw_3 \int_0^{w_3} dw_1 w_1 w_3 + \int_0^1 dw_1 \int_0^{w_1} dw_3 w_3^2 = \frac{1}{8} + \frac{1}{12} = \frac{5}{24}. \end{aligned} \tag{3}$$

With the same method we find that

$$w(G,T_{14}) = w(G,T_{24}) = w(G,T_{23}) = \frac{5}{24}, \tag{4}$$

and we have

$$\sum_{T\in G} w(G,T) = \frac{1}{6} + 4.\frac{5}{24} = 1. \tag{5}$$

Let us treat a second example. Consider the graph G' of Fig. D.2, which has 6 edges :

$$\{l_1, l_2, l_3, l_4, l_5, l_6\} \tag{6}$$

To each edge l_i we associate a factor w_i. There are 12 spanning trees :

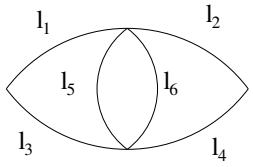

FIGURE D.2: Example 2-the eye graph

$$\begin{aligned} &\{l_1,l_2,l_3\},\{l_1,l_2,l_4\},\{l_1,l_3,l_4\},\{l_2,l_3,l_4\},\{l_1,l_2,l_5\},\{l_1,l_2,l_6\}, \\ &\{l_3,l_4,l_5\},\{l_3,l_4,l_6\},\{l_1,l_5,l_4\},\{l_1,l_6,l_4\},\{l_3,l_5,l_2\},\{l_3,l_6,l_2\}. \end{aligned} \tag{7}$$

Let us compute the relative weight for each of these spanning trees in G'. First of all consider $T_{123} = \{l_1, l_2, l_3\}$. The other edges are drawn in dotted lines. See figure(D.3) As

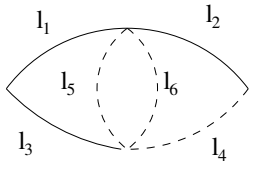

FIGURE D.3: The spanning tree $\{l_1, l_2, l_3\}$

is easily seen the corresponding loop lines are l_4, l_5 and l_6. The weakening factor for l_5

and l_6 $\inf(w_1, w_3)$and the weakening factor for l_4 is $\inf(w_1, w_2, w_3)$. Therefore we have

$$\begin{aligned} & w(G', T_{123}) = \int_{0<w_1<w_2<w_3<1} dw_1 dw_2 dw_3 \inf(w_1, w_3)^2 \inf(w_1, w_2, w_3) \\ + \quad & \text{other permutations of } \mathrm{w}_1, \mathrm{w}_2, \mathrm{w}_3 \\ = \quad & \int_{w_1<w_2<w_3} dw_1 dw_2 dw_3 w_1^3 + \int_{w_2<w_3<w_1} dw_1 dw_2 dw_3 w_3^2 w_2 \\ + \quad & \int_{w_3<w_1<w_2} dw_1 dw_2 dw_3 w_3^3 + \int_{w_2<w_1<w_3} dw_1 dw_2 dw_3 w_1^2 w_2 \\ + \quad & \int_{w_3<w_2<w_1} dw_1 dw_2 dw_3 w_3^3 + \int_{w_1<w_3<w_2} dw_1 dw_2 dw_3 w_1^3. \end{aligned}$$

We just calculate two of the integrals explicitely as others could be get just by changing the names of variables.

$$\int_{w_1<w_2<w_3} dw_1 dw_2 dw_3 \; w_1^3 = \int_0^1 dw_3 \int_0^{w_3} dw_2 \int_0^{w_2} dw_1 w_1^3 = \frac{1}{120}, \tag{8}$$

$$\int_{w_1<w_2<w_3} dw_1 dw_2 dw_3 \; w_3^2 \; w_2 = \frac{1}{60} \tag{9}$$

So we have

$$w(G', T_{123}) = \frac{1}{120} \times 4 + \frac{1}{60} \times 2 = \frac{1}{15}. \tag{10}$$

The relative weights in G' of the spanning trees T_{124}, T_{134} and T_{234} are the same.

Now we consider the tree $\{l_1, l_2, l_5\}$. (See figure D.4). To the loop line l_3 is associ-

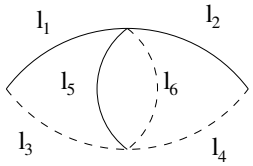

FIGURE D.4: The spanning tree $\{l_1, l_2, l_5\}$

ated a weakening factor $\inf(w_1, w_5)$. To the loop line l_4 is associated a weakening factor $\inf(w_2, w_5)$. To the loop line l_6 is associated a weakening factor w_5. So we have

$$\begin{aligned} & w(G', T_{125}) = \int_{w_1<w_2<w_5} dw_1 dw_2 dw_5 \inf(w_1, w_5) \inf(w_2, w_5) w_5 \qquad (11) \\ + \quad & \text{other permutations of } \mathrm{w}_1, \mathrm{w}_2, \mathrm{w}_5 \\ = \quad & \int_{w_1<w_2<w_5} dw_1 dw_2 dw_5 w_1 w_2 w_5 + \int_{w_5<w_1<w_2} dw_1 dw_2 dw_5 w_5^3 \\ + \quad & \int_{w_2<w_5<w_1} dw_1 dw_2 dw_5 w_5^2 w_2 + \int_{w_2<w_1<w_5} dw_1 dw_2 dw_5 w_1 w_2 w_5 \\ + \quad & \int_{w_1<w_5<w_2} dw_1 dw_2 dw_5 w_5^2 w_1 + \int_{w_5<w_2<w_1} dw_1 dw_2 dw_5 w_5^3. \end{aligned}$$

We have

$$\int_{w_1<w_2<w_5} dw_1 dw_2 dw_5 w_1 w_2 w_5 = \frac{1}{48}, \tag{12}$$

$$\int_{w_5<w_1<w_2} dw_1 dw_2 dw_5 w_5^3 = \frac{1}{120}, \tag{13}$$

$$\int_{w_2<w_5<w_1} dw_1 dw_2 dw_5 w_2 w_5^2 = \frac{1}{60}. \tag{14}$$

Other integrations could be got from the above three integrals by changing the name of variables and we get

$$w(G', T_{125}) = \frac{1}{120} \times 2 + \frac{1}{60} \times 2 + \frac{1}{48} \times 2 = \frac{11}{120} \tag{15}$$

By the same method we find that this is also the relative weight of trees T_{126}, T_{345}, T_{346}, T_{125}, T_{145}, T_{146}, T_{235} and T_{236}.

We have again

$$\sum_{T \in G'} w(G', T) = 4.\frac{1}{15} + 8.\frac{11}{120} = 1. \tag{16}$$

D.3 Resumming Feynman Graphs according to the forest Formula

D.3.1 Naive Repacking

Consider the expansion (1) of a connected quantity S. The most naive way to reorder Feynman perturbation theory according to trees rather than graphs is to insert for each graph the relation (1)

$$S = \sum_G A_G = \sum_G \sum_{T \subset G} w(G, T) \mathcal{A}_G \tag{1}$$

and exchange the order of the sums over S and T. Hence it writes

$$S = \sum_T \mathcal{A}_T, \quad \mathcal{A}_T = \sum_{G \supset T} w(G, T) \mathcal{A}_G. \tag{2}$$

This rearranges the Feynman expansion according to trees, but each tree has the same number of vertices as the initial graph. Hence it resshuffles the various terms of a *given, fixed* order of perturbation theory. Remark that if the initial graphs have say degree 4 at each vertex, only trees with degree less than or equal to 4 occur in the rearranged tree expansion.

For Fermionic theories this is typically sufficient and one has for small enough coupling

$$\sum_T |\mathcal{A}_T| < \infty \tag{3}$$

because Fermionic graphs essentially mostly compensate each other at a fixed order by Pauli's principle ; mathematically this is because these graphs form a determinant and the

size of a determinant is much less than what its permutation expansion usually suggests. This is well known [50, 51, 52].

But this repacking fails completely for Bosonic theories, because the only compensations there occur between graphs of different orders. Hence if we perform this naive reshuffling, eg on the ϕ_0^4 theory we still have

$$\sum_T |\mathcal{A}_T| = \infty. \tag{4}$$

D.4 The Loop Vertex Expansion

The loop vertex expansion overcomes this difficulty by exchanging the role of vertices and propagators before applying the forest formula. The corresponding regrouping is completely different and each tree resums an infinite number of pieces of the previous graphs. It relies on a technical tool which physicists call the intermediate field representation. In general terms it decomposes any interaction of higher order in terms of simpler three-body interactions. This is particularly natural for the 4-body interaction, but can be generalized to other interactions as well [18].

This quite universal and powerful trick is linked to various deep physical and mathematical tools, such as the color 1/N expansion and the Matthews-Salam and Hubbard-Stratonovich methods in physics and the Kaufmann bracket of a knot and many similar ideas in mathematics.

It is easy to describe the intermediate field method in terms of functional integrals, as it is a simple generalization of the formula

$$e^{-\lambda\phi^4/2} = \int e^{-\sigma^2/2} e^{i\sqrt{\lambda}\sigma\phi^2} d\sigma \tag{1}$$

But in this paper we would like to rather explain the graphical procedure equivalent to this formula.

In the case of a ϕ^4 graph G each vertex has exactly four half-lines hence there are exactly three ways to pair these half-lines into two pairs. Hence each fully labeled (vacuum) graph of order n (with labels on vertices and half-lines), which has $2n$ lines can be decomposed exactly into 3^n labeled graphs G' with degree 3 and two different types of lines

- the $2n$ old ordinary lines
- and n dotted lines which indicate the pairing chosen at each vertex (see Figure 5).

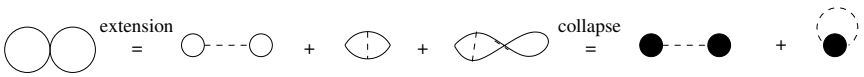

FIGURE D.5: The extension and collapse for order 1 graph

Such graphs G' are called the 3-body extensions of G and we write G' ext G when G' is an extension of G. Let us introduce for each such expansion G' an amplitude $A'_G = 3^{-n} A_G$ so that

$$A_G = \sum_{G' \text{ ext } G} A_{G'} \tag{2}$$

when G' is an extension of G.

Now the ordinary lines of any extension G' of any G must form cycles. These cycles are joined by dotted lines.

Définition D.4.1 *We define the collapse $\bar{G}'$ of such a G' graph as the graph obtained by contracting each cycle to a "bold" vertex (see Figure 5). We write $\bar{G}'$ coll G' if $\bar{G}'$ is the collapse of G'. and define the amplitude of the collapsed graph $\bar{G}'$ as equal to that of G'.*

Remark that collapsed graphs, made of bold vertices and dotted lines, can have now arbitrary degree at each vertex. Remark also that several different extensions of a graph G can have different collapsed graphs, see Figure ...

Now the loop vertex expansion rewrites

$$S = \sum_{G} A_G = \sum_{G' \text{ ext } G} A_{G'} = \sum_{\bar{G}' \text{ coll } G' \text{ ext } G} A_{\bar{G}'} \tag{3}$$

Now we perform the tree repacking according to the graphs $\bar{G}'$ with the n dotted lines and *not* with respect to G. This is a completely different repacking :

$$A_{\bar{G}'} = \sum_{\bar{T} \in \bar{G}'} w(\bar{G}', \bar{T}) A_{\bar{G}'} \tag{4}$$

so that

$$S = \sum_{G' \text{ ext } G} A_{\bar{G}'} = \sum_{\bar{T} \in \bar{G}'} A_{\bar{T}} \tag{5}$$

$$A_{\bar{T}} = \sum_{\bar{G}' \supset \bar{T}} w(\bar{G}', \bar{T}) A_{\bar{G}'} \tag{6}$$

The "miracle" is that

Théorème D.4.1 *For λ small*

$$\sum_{\bar{T}} |A_{\bar{T}}| < \infty \tag{7}$$

the result being the Borel sum of the initial perturbative series [49].

The proof of the theorem will not be recalled here (see [16, 17, 49]) but it relies on the positivity property of the $x_\ell^T(\{w\})$ symmetric matrix, and the representation of each $A_{\bar{T}}$ amplitude as an integral over a corresponding normalized Gaussian measure of a product of resolvents bounded by 1. This convergence would not be true if we had chosen naive $w5T, G)$ barycentric weights such as $1/5$ for each of the five trees of the graph in Figure 1.

This method is valid for any ϕ^4 model in any dimension with cutoffs [17]. It is not limited to ϕ^4 but works eg for any stable interaction at the cost of introducing more intermediate particles until three body elementary interactions are reached [18]. It also reproduces correctly the large N behaviour of ϕ^4 matrix models, which was the key property for which this expansion was found [16].

D.5 Examples

In this section we give the extension and collapse of the Feynman graphs for Z and $\log Z$ for the ϕ^4 model in dimension zero and up to order 3. We also recover the same combinatorics of those graphs by computing them through the ordinary functional integral formula for the loop vertex expansion formula of [49].

The graphs of extension and collapse of order 1 are shown in Figure (D.6). In this case the tree structure is easy, namely there is the trivial "empty" tree with one vertex and no line and the "almost trivial" tree with two vertices and a single edge.

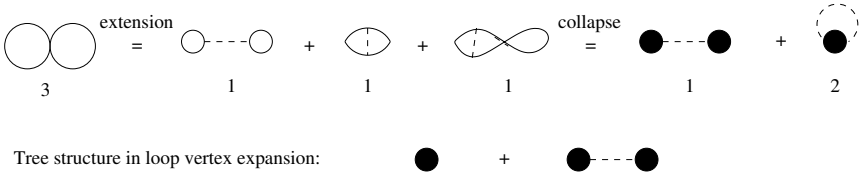

FIGURE D.6: The extension and collapse for order 1 graph and tree structure

At second order we have one diconnected Feynman graph and two connected ones ; Only the connected ones survive in the expansion of $\log Z$.

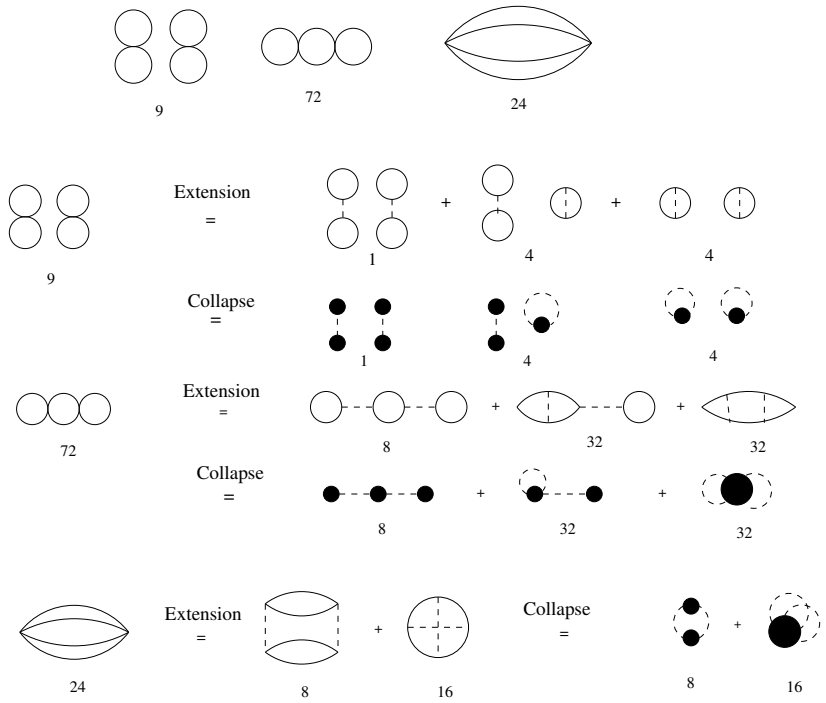

FIGURE D.7: The extension and collapse for order 2 graph and the number of graphs.

The correponding graphs and tree structures are shown in Figure (D.7) and Figure(D.8). Using the loop vertex expansion formula we begin to see that graphs that come from different order of the expansion of λ are associated to the same trees by the loop vertex expansion. Indeed we recover contributions for the trivial and almost trivial trees of the previous figure. But we find also a new contribution belonging to a tree with two lines.

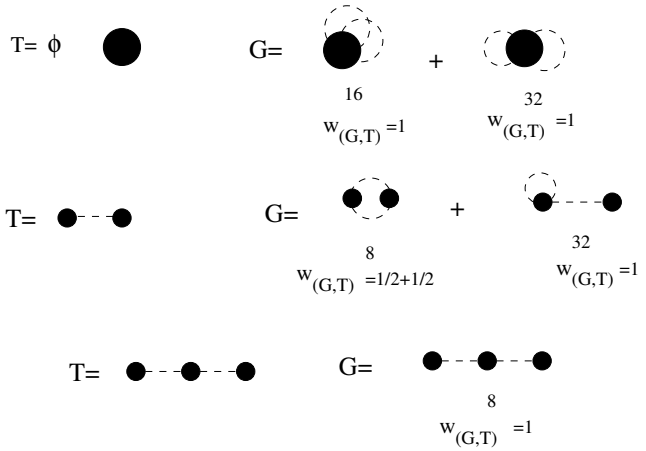

FIGURE D.8: The connected graphs and the tree structure from the Loop vertex expansion.

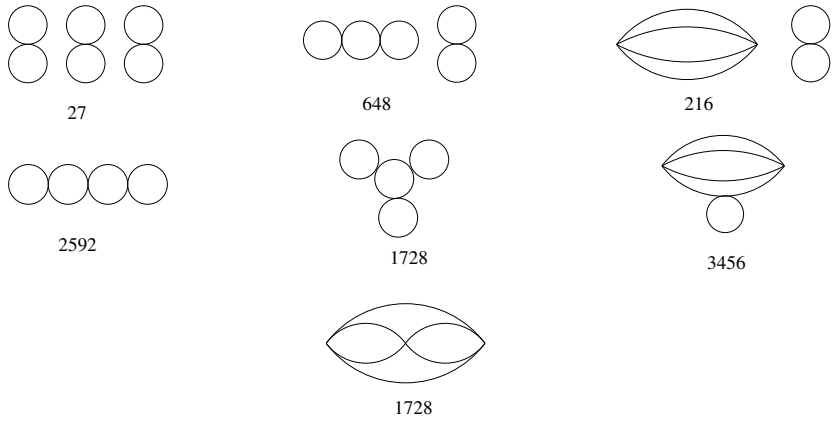

FIGURE D.9: The order 3 vacuum graph and the number of graphs.

At order three the computation becomes a bit more involved but the process is clear. We could start from the ordinary Feynman graphs and get the graphs of loop vertex expansion by extansion and collapse. This is shown in Figure (D.9). The number under each collapsed graph means the number of the corresponding graphs.The tree structure is shown in Figure(D.12). We could also get the structure and combinatorics by using directly the loop vertex expansion, as shown in the appendix and the result is shown in Figure (D.11). In this process we expand both $\exp V$ and the vertex $V = \mathrm{tr}\log(1 + 2i\sqrt{2\lambda}\sigma)$. From this expansion we could see the structure of interactions, namely loop vertices V with various attached σ fields. This is shown on the left hand of this figure. For example, the symbol 123 means we expand $\exp V$ to three orders V^3. We then expand one of the V to the order $\lambda^{1/2}$, namely we have one σ field attached, one to the order λ, namely 2 σ fields attached and the third one to $\lambda^{3/2}$, namely 3 σ fields. Then we contrat the sigma fields with respect to the gaussian measure and we get all the structures of the contracted graphs. The total number of this contraction is the result of the Gaussian integration. For getting the number of each graph we need to calculate the relative weight which is the same as in the ordinary Feynman graph case. For example we consider the 123 case in Figure (D.11) shown more explicitly in Figure(D.13). We use $a, b, \cdots, f$ to label the σ fields attached to the vertices. After the wick contractions we get three different graphs

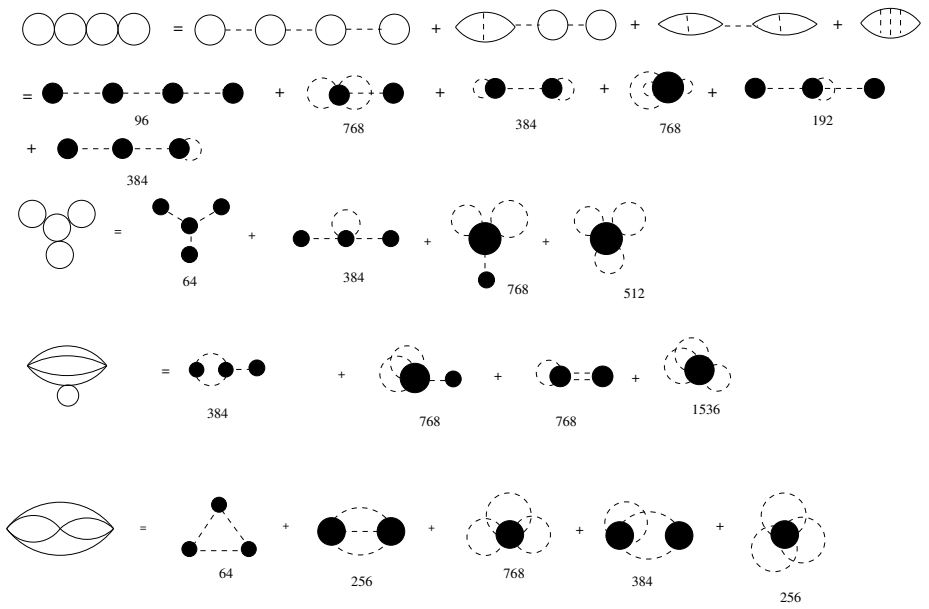

FIGURE D.10: The extension and collapse for order 3 graph.

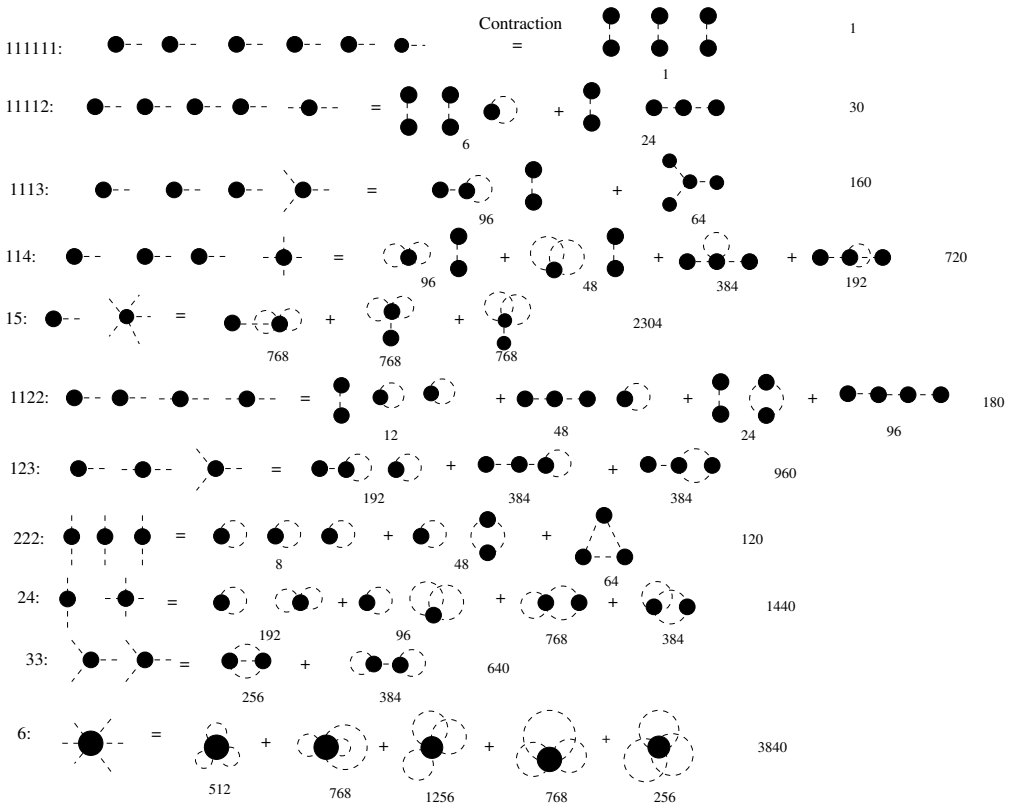

FIGURE D.11: The graph structure and combinatorics from the loop vertex expansion at order 3. The symbols like 1122 means we have 4 loop vertices V, two of them have one σ field each and two of them have two σ feilds each, as we could read directly from this figure.

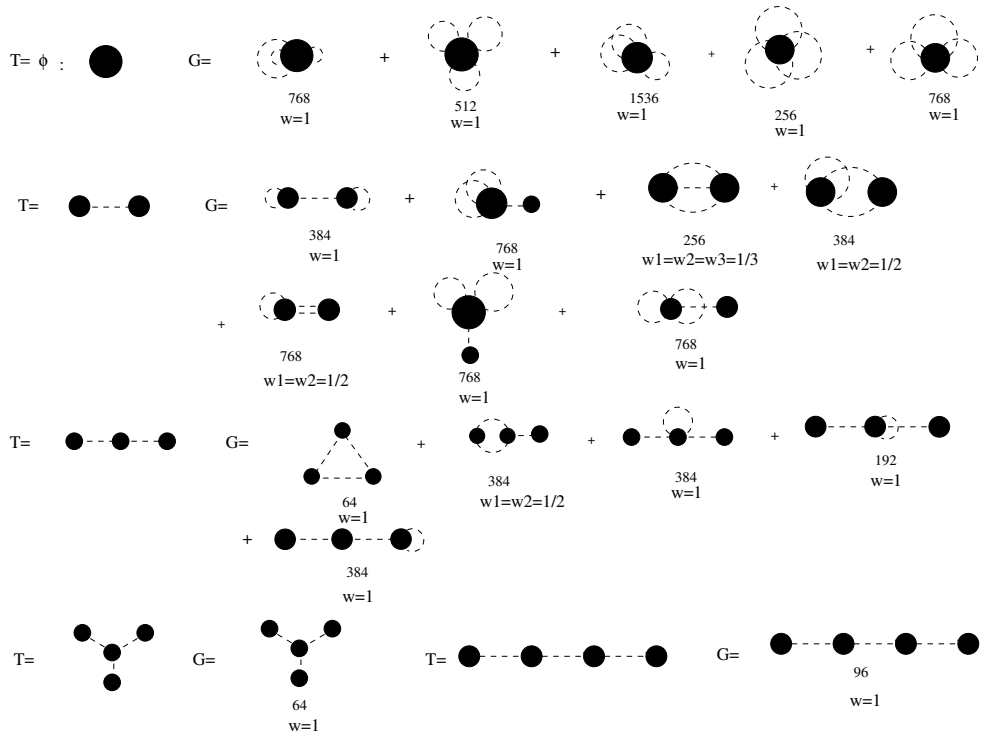

FIGURE D.12: The tree structure of order 3 graphs.

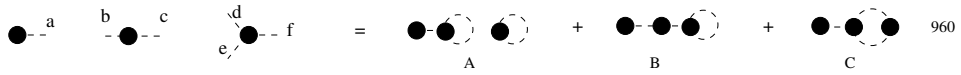

FIGURE D.13: The example of '123' contractions.

A, B and C. The number of possiblities to get A is 3, the number of B is $2 \times 3 = 6$ and the number of C is also 6. So the relative weight for graph A is $3/(3+6+6) = 1/5$. the relatively weight for B and C are both $6/(3+6+6) = 2/5$. As we could read directly from the loop vertex formula that the total number of 123 contraction graphs is 960, we get finally the number of graph A is $960 \times 1/5 = 192$, the number of graph B and C are $960 \times 2/5 = 384$. This result agrees with the one from the Feynman graph calculation.

From these exampls we find that the loop vertex expansion is totally different from the Feynman graph calculations. At each order of the loop vertex expansion we combine the terms in different orders of λ. But the result is the same as the Feynman graph calculation. We could find it from the combinatorics. So that the loop vertex expansion method is not only suitable for constructive renormalization, but is also suitable for pertubative renormalization to some finite orders.

D.6 Non-integer Dimension

Let us consider, eg for $0 < D \leq 2$ the Feynman amplitude for the $\phi 4_D$ theory ; They are given by the folllowing convergent parametric representation

$$A_{D,G} = \int_0^\infty d\alpha \frac{e^{-\sum \alpha}}{U_G^{D/2}} \tag{1}$$

where U_G is the Kirchoff-Symanzik polynomial for G

$$U_G = \sum_{T \in G} \prod_{\ell \notin T} \alpha_\ell \tag{2}$$

All the previous decompositions working at the level of graphs, they are independent of the space-time dimension. We can therefore repack the series of Feynman amplitudes in non integer dimension to get the D dimensional tree amplitude :

$$A_{D,\bar{T}} = \sum_{G \supset T} w(T,G) A_{D,G} \tag{3}$$

We know that for $D = 0$ and $D = 1$ the loop vertex expansion is convergent. Therefore it is tempting to conjecture , for instance at least for D real and $0 \leq D < 2$ (that is when no ultraviolet divergences require renormalization)

Conjecture D.6.1 *For λ small*

$$\sum_{\bar{T}} |A_{D,\bar{T}}| < \infty \tag{4}$$

the result being the Borel sum of the initial perturbative series.

Progress on this conjecture would be extremely interesting as it would allow to bridge quantum field theories in various dimensions of space time, and ie perhaps justify the Wilson-Fisher $4 - \epsilon$ expansion that allows good numerical approximate computations of critical indices in 3 dimensions.

We know however that when renormalization is needed ; ie for $D \geq 2$ this approach needs to be completed with the correct introduction of counterterms. presumably in this case the tree expansion should be adapted to select optimal trees with respect to renormalization group scales. This is work in progress.

An other possible approach to quantum field theory in non integer dimension, also based on the forest formula but more radical, is proposed in [53].

D.7 Conclusion

From the physics point of view the lessons we might draw from the Loop Vertex Expansion are

- Interactions should be decomposed into three body elementary interactions. The corresponding fields might be more fundamental than the initial ones ;
- Tree formulas solve the constructive problem ie resum perturbation theory at the cost of loosing locality of the new vertices.

It may be also interesting to further uderstand why trees are central both in the parametric formulas (1) for *single* Feynman amplitudes and in the non-perturbative treatment of the theory. The answer might imply a complete refoundation of quantum field theory around the notion of trees, rather than Feynman graphs or even functional integrals [53]

Finally from the combinatorics point of view the lesson might be to also investigate more systematically how generating functionals for proliferating species might be repacked into other (convergent) functionals for non-proliferating species.

D.8 Appendix

The combinatorics of graphs from the loop vertex expansion

In this Appendix we compute the weight of collapsed Feynman graphs from the intermediate field functional integral.

For the ϕ_0^4 model we have :

$$Z = \frac{1}{\sqrt{2\pi}} \int d\phi e^{-\frac{1}{2}\phi^2 - \lambda\phi^4} = \frac{1}{\sqrt{2\pi}} \int d\sigma e^{-\frac{1}{2}\sigma^2 - \frac{1}{2}\log(1+2i\sqrt{2\lambda}\sigma)} \tag{1}$$

We define

$$V = -\frac{1}{2}\log(1 + 2i\sqrt{2\lambda}\sigma). \tag{2}$$

In what follows we calculate the number of vacuum graphs un to order 3. We have

$$\begin{aligned} \log(1 + 2i\sqrt{2\lambda}\sigma) &= 2i\sqrt{2\lambda}\sigma + 4\lambda\sigma^2 - \frac{16i}{3}\lambda^{3/2}\sigma^3 - 8\lambda^2\sigma^4 \\ &+ \frac{64i}{5}\lambda^{5/2}\sigma^5 + \frac{64}{3}\lambda^3\sigma^6 \end{aligned} \tag{3}$$

We expand Z into powers of V :

$$Z = \frac{1}{\sqrt{2\pi}} \int d\sigma e^{-\frac{1}{2}\sigma^2}[1 - \frac{1}{2}V + \frac{1}{2!}V^2 - \frac{1}{3!}V^3 + \frac{1}{4!}V^4 - \frac{1}{5!}V^5 + \frac{1}{6!}V^6] \tag{4}$$

The first term

$$\frac{1}{\sqrt{2\pi}} \int d\sigma e^{-\frac{1}{2}\sigma^2} 1 = 1 \tag{5}$$

is trivial.

The order V terms give :

$$\begin{aligned} &\frac{1}{\sqrt{2\pi}} \int d\sigma e^{-\frac{1}{2}\sigma^2} V = \frac{1}{\sqrt{2\pi}} \int d\sigma e^{-\frac{1}{2}\sigma^2}[-2\lambda\sigma^2 + 8\lambda^2\sigma^4 - \frac{128}{3}\lambda^3\sigma^6] \\ = \; & -2\lambda + 24\lambda^2 - 640\lambda^3 \end{aligned} \tag{6}$$

The V^2 terms give :

$$\begin{aligned} &\frac{1}{2!}\frac{1}{\sqrt{2\pi}} \int d\sigma e^{-\frac{1}{2}\sigma^2} V^2 = \frac{1}{2!}(\frac{1}{2})^2 \frac{1}{\sqrt{2\pi}} \int d\sigma e^{-\frac{1}{2}\sigma^2}[-4\lambda\sigma^2 + 16\lambda^2\sigma^4 \\ - \; & \frac{64 \times 8}{9}\lambda^3\sigma^6 + \frac{128}{3}\lambda^2\sigma^4 - \frac{128 \times 8}{5}\lambda^3\sigma^6 - 128\lambda^3\sigma^6] \\ = \; & -\lambda + 22\lambda^2 - \frac{320}{3}\lambda^3 - 624\lambda^3 \end{aligned} \tag{7}$$

The V^3 terms give :

$$\begin{aligned} &-\frac{1}{3!}\frac{1}{\sqrt{2\pi}} \int d\sigma e^{-\frac{1}{2}\sigma^2} V^3 = -\frac{1}{3!}(\frac{1}{2})^3 \frac{1}{\sqrt{2\pi}} \int d\sigma e^{-\frac{1}{2}\sigma^2}[64\lambda^3\sigma^6 - 96\lambda^2\sigma^4 \\ + \; & 384\lambda^3\sigma^6 + 512\lambda^3\sigma^6] \\ = \; & 6\lambda^2 - 300\lambda^3 \end{aligned} \tag{8}$$

The V^4 terms give :

$$\begin{aligned} & \frac{1}{4!}(\frac{1}{2})^4 \frac{1}{\sqrt{2\pi}} \int d\sigma e^{-\frac{1}{2}\sigma^2} [64\lambda^2\sigma^4 - \frac{2048}{3}\lambda^3\sigma^6 - 768\lambda^3\sigma^6] \\ = \quad & \frac{1}{2}\lambda^2 - \frac{80}{3}\lambda^3 - 30\lambda^3 \end{aligned} \tag{9}$$

The V^5 terms give :

$$-\frac{1}{5!}(\frac{1}{2})^5 \frac{1}{\sqrt{2\pi}} \int d\sigma e^{-\frac{1}{2}\sigma^2}\ 1280\lambda^3\sigma^6 = -5\lambda^3 \tag{10}$$

The V^6 term gives :

$$-\frac{1}{6!}(\frac{1}{2})^6 \frac{1}{\sqrt{2\pi}} \int d\sigma e^{-\frac{1}{2}\sigma^2}\ 512\lambda^3\sigma^6 = -\frac{1}{6}\lambda^3 \tag{11}$$

So up to 3rd order of λ we have

$$Z = -3\lambda + \frac{105}{2}\lambda^2 - \frac{10395}{6}\lambda^3 = -4!!\lambda + \frac{8!!}{2!}\lambda^2 - \frac{12!!}{3!}\lambda^3 \tag{12}$$

which of course coincide with the last numbers, namely the number of ordinary Wick contractions derived by the regular $\lambda\phi^4$ Feynman expansion.

Acknowledgments We thank H. Knörrer for asking the question which lead to writing this paper.

E

Paper 5

Topological Graph Polynomials and Quantum Field Theory Part I : Heat Kernel Theories

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Abstract

We investigate the relationship between the universal topological polynomials for graphs in mathematics and the parametric representation of Feynman amplitudes in quantum field theory. In this first paper we consider translation invariant theories with the usual heat-kernel-based propagator. We show how the Symanzik polynomials of quantum field theory are particular multivariate versions of the Tutte polynomial, and how the new polynomials of noncommutative quantum field theory are particular versions of the Bollobás-Riordan polynomials.

E.1 Introduction

Quantum field theory lies at the root of modern physics. After the success of the standard model in describing particle physics, one of the most pressing open question is how to derive an extended version of field theory which encompasses the quantization of gravity. There are several attempts for this, among which string theory, loop gravity and

noncommutative geometry are the best known. In each of these attempts one of the key problem is to relax the constraints that formulate quantum field theory on a particular space-time geometry.

What is certainly more fundamental than geometry is topology and in particular discrete structures on finite sets such as the species of combinatorists [43]. The most prominent such species in field theory is the species of Feynman graphs. They were introduced by Feynman to label quantum field perturbation theory and to automatize the computation of *connected* functions. Feynman graphs also became an essential tool in *renormalization*, the structure at the heart of quantum field theory.

There are two general canonical operations on graphs namely the deletion or contraction of edges. Accordingly perhaps the most important quantity to characterize a graph is its Tutte polynomial [106, 107]. This polynomial obeys a simple recursion rule under these two basic operations. It exists in many different variations, for instance multivariate versions, with possible decorations at vertices. These polynomials have many applications, in particular to statistical physics. For recent reviews see [108, 109, 110].

In recent years the Tutte polynomial has been generalized to the category of ribbon graphs, where it goes under the name of the Bollobás-Riordan polynomial [111, 112, 110]. Around the same time physicists have increasingly turned their attention to quantum field theory formulated on noncommutative spaces, in particular flat vector spaces equipped with the Moyal-Weyl product [113]. This type of quantum field theory is hereafter called NCQFT. It happens that perturbation theory for such NCQFT's is no longer labeled by ordinary graphs but by ribbon graphs, suggesting a possible connection to the work of Bollobás-Riordan.

Quantum field perturbation theory can be expressed in several representations. The momentum representation is the most common in the text books. The direct space representation is closer to physical intuition. However it is the parametric representation which is the most elegant and compact one. In this representation, after the integration of internal position and/or momentum variables has been performed explicitly, the result is expressed in terms of the Symanzik polynomials. There is an extensive literature on these polynomials (see e.g. [114, 115] for classical reviews). These polynomials only depend on the Schwinger parameters. Space time no longer enters explicitly into that representation except through its dimension which appears simply as a parameter.

This observation is crucial for several key applications in QFT which rely on dimensional interpolation. Dimensional regularization and renormalization was a crucial tool in the proof by 't Hooft and Veltmann that non-Abelian gauge theories are renormalizable [116]. The Wilson-Fisher ϵ expansion [117] is our best theoretical tool to understand three dimensional phase transitions. Dimensional regularization is also used extensively in the works of Kreimer and Connes [118, 119] which recast the recursive BPHZ forest formula of perturbative renormalization into a Hopf algebra structure and relate it to a new class of Riemann-Hilbert problems [120].

Following these works, renormalizability has further attracted considerable interest in the recent years as a pure mathematical structure. The renormalization group ambiguity reminds mathematicians of the Galois group ambiguity for roots of algebraic equations [121]. Hence the motivations to study quantum field theory and renormalization come no longer solely from physics but also at least partly from number theory.

The fact that the parametric representation is relatively independent of the details

of space time makes it also particularly appealing as a prototype for the tools we need in order to quantize gravity. The point of view of loop gravity is essentially based on the diffeomorphism invariance of general relativity. In the spin foam or group field theory formalism amplitudes are expressed as discrete sums associated to combinatoric structures which generalize Feynman graphs. They are in fact generalizations of ribbon graphs. To extend the parametric representation and eventually the theory of renormalization to this context is a major challenge, in which some preliminary steps have been performed [122].

In this paper we uncover the relationship between universal polynomials of the Tutte and Bollobás-Riordan type and the parametric representation in quantum field theory. The Symanzik polynomials that appear in ordinary commutative QFT are particular multivariate versions of Tutte polynomials. The relation between Bollobás-Riordan polynomials and the non commutative analogs of the Symanzik polynomials uncovered in [123, 68, 124] is new. This establishes a relation between NCQFT, combinatorics and algebraic topology. Recently the relation between renormalization and topological polynomials was explored in [125], and in [126]. We intend also to explore in the future the relation between Feynman amplitudes and knot polynomials.

The plan of this paper is as follows. In the next section we give a brief introduction to graph theory and to Tutte-like polynomials. In the third section we derive the parametric representation of Feynman amplitudes of QFT and give a new method to compute the corresponding Symanzik polynomials. The deletion/contraction property (11) of these polynomials is certainly not entirely new [127, 128]. But our method which starts from the phase-space representation of Feynman amplitudes is inspired by earlier work on NCQFT [68, 124] and introduces two main technical improvements. One is the use of Grassmann variables to exploit the quasi-Pfaffian structure of Feynman amplitudes. This quasi-Pfaffian structure was discovered in [68] in the context of NCQFT but to our knowledge was never applied to the simpler case of ordinary QFT. The second improvement is that we do not factor out as usual the delta functions expressing global momentum conservation, because this requires a noncanonical choice of a root for every connected graph. Instead we introduce an infrared regularization in the form of a small harmonic potential at each vertex which leads to more elegant and canonical formulas. The corresponding generalized Symanzik polynomials obey a transparent deletion/contraction relation which allows to identify them with particular multivariate Tutte polynomials. These polynomials are close but not identical to the polynomials of [108] ; we show how they both derive from a more general "categorified" polynomial. The usual Symanzik polynomials are simply recovered as the leading terms when the small harmonic potentials tend to zero.

For completeness we also include a more standard way to compute the Symanzik polynomials through x space representation and the tree matrix theorem.

In the fourth section we introduce ribbon graphs and Bollobás-Riordan polynomials. In the fifth and last section we define the first and second Symanzik polynomials of NCQFT and relate them to the Bollobás-Riordan polynomials, using again the Pfaffian variables. Formulas for such polynomials were first sketched in [123], but without proofs, nor relation to the Bollobás-Riordan polynomials.

In a companion paper we shall discuss generalizations of the Tutte and Bollobás-Riordan polynomials that occur for non-translation invariant theories with propagators based on the Mehler rather than the heat kernel. These theories appeared as the first examples of renormalizable NCQFT's [28, 27, 29, 41, 24] and they are the most promising

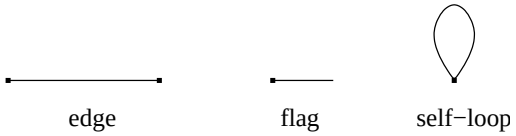


FIGURE E.1: Basic building blocks of a graph

candidates for a fully non-perturbative construction of a field theory in four dimensions [130, 38, 39, 16]. In this case the harmonic potentials on the vertices are no longer needed as the Mehler kernel already contains an harmonic potential for the propagators of the graphs.

E.2 Tutte Polynomial

E.2.1 Graph Theory, Notations

A graph G is defined as a set of vertices V and of edges E together with an incidence relation between them. The number of vertices and edges in a graph will be noted also V and E for simplicity, since our context always prevents any confusion. Graph theorists and field theorists usually have different words for the same objects so a little dictionary may be in order. We shall mostly use in this review the graph theorists language. In subsection E.2.4 we introduce also some enlarged notion of graphs, with decorations called *flags* which are attached to the vertices of the graph to treat the external variables of physicists, plus other decorations also attached to vertices called (harmonic) weights to regularize infrared divergences. Generalizations to ribbon graphs will be described in section E.4.

Edges in physics are called lines (or propagators). Edges which start and end at the same vertex are definitely allowed, and called (self)-loops in graph theory and tadpoles in physics. A proper graph, i.e. a graph G without such self-loops, together with an arrow orienting each edge, can be fully characterized through its incidence matrix ϵ_{ve}. It is the rectangular E by V matrix with indices running over vertices and edges respectively, such that

- ϵ_{ve} is $+1$ if e starts at v,
- ϵ_{ve} is -1 if e ends at v,
- ϵ_{ve} is 0 otherwise.

It is also useful to introduce the absolute value $\eta_{ve} = |\epsilon_{ve}|$ These quantities can be then generalized to graphs with self-loops by defining $\epsilon_{ev} = 0$ for any self-loop e and vertex v *but* $\eta_{ev} = 2$ for a self-loop attached at vertex v and $\eta_{ev} = 0$ otherwise. The number of half-edges at a vertex v is called the degree of v in graph theory, noted $d(v)$. Physicists usually call it the coordination number at v. A self-loop counts for 2 in the degree of its vertex, so that $d(v) = \sum_e \eta_{ev}$.

An edge whose removal increases (by one) the number of connected parts of the graph is called a bridge in graph theory and a one-particle-reducible line in physics.

A forest is an acyclic graph and a tree is a connected forest. A cycle in graph theory is a connected subset of n edges and n vertices which cannot be disconnected by removing any edge. It is called a loop in field theory.

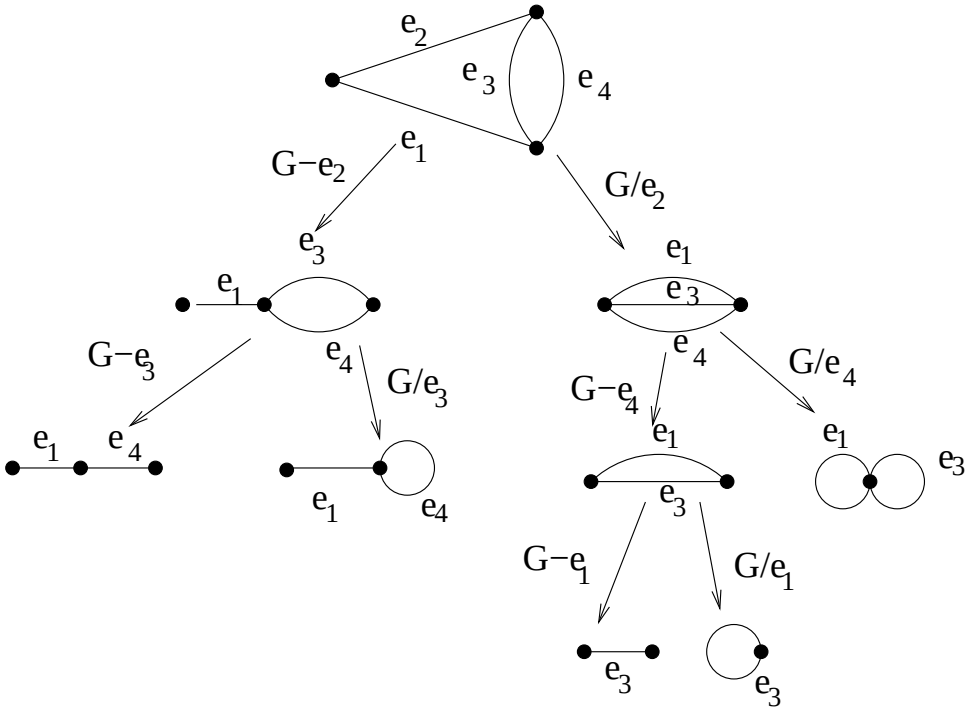


FIGURE E.2: The contraction-deletion of a graph

Physicists understood earlier than graph theorists that half-edges (also called flags in graph theory [131]) are more fundamental than edges. This is because they correspond to integrated fields through the rule of Gaußian integration, which physicists call Wick's theorem. Feynman graphs form a category of graphs with external flags decorating the vertices. They occur with particular weights, in physics called amplitudes. These weights depend on the detail of the theory, for instance the space-time dimension. A quantum field theory can be viewed the generating functional for the species of such weighted Feynman graphs. In this paper we shall reserve the convenient word flag exclusively for the "external fields" decorations and always use the word half-edge for the "internal half-edges".

An edge which is neither a bridge nor a self-loop is called regular. We shall call *semi-regular* an edge which is not a self-loop, hence which joins two distinct vertices.

There are two natural operations associated to an edge e of a graph G, pictured in Figure E.2 :

- the deletion, which leads to a graph noted $G - e$,
- the contraction, which leads to a graph noted G/e. If e is not a self-loop, it identifies the two vertices $v_1$and v_2 at the ends of e into a new vertex v_{12}, attributing all the flags (half-edges) attached to v_1 and v_2 to v_{12}, and then it removes e. If e is a self-loop, G/e is by definition the same as $G - e$.

A subgraph G' of G is a subset of edges of G, together with the attached vertices. A *spanning forest* of G is an acyclic subgraph of G that contains all the vertices of G. If G is connected a spanning forest is in fact a tree of G and any such *spanning tree* has $|V| - 1$ vertices.

As explained in the introduction a topological graph polynomial is an algebraic or combinatoric object associated with a graph that is usually invariant under at least graph

homeomorphism. It encodes information of the graph and so enables combinatoric and algebraic method to deal with graphs.

The Tutte polynomial [106] is one of the most general polynomial to characterize a graph. It is defined under a simple rule through the deletion and contraction of edges. It can be generalized to the larger theory of matroids [132].

The original Tutte polynomial which is a function of two variables can be generalized in various ways to multi-variable polynomials which have many applications, in particular in statistical mechanics where it evaluates the Potts model on graphs [108, 109, 110]. These applications shall not be reviewed here.

We present first the two main equivalent definitions of the Tutte polynomial. One direct way is to specify its linear recursion form under contraction of regular edges (which are neither loops nor bridges), together with an evaluation on terminal forms solely made of bridges and self-loops. Another definition is as a rank-nullity generating function. By induction these definitions can be proved equivalent.

E.2.2 Tutte Polynomial

The definition through a recursion relation is a reduction rule on edges together with an evaluation for the terminal forms. The Tutte polynomial may be defined by such a linear recursion relation under deleting and contracting regular edges. The terminal forms, i.e. the graphs without regular edges are forests (i.e. graphs made of bridges) decorated with an additional arbitrary number of self-loops at any vertex. The Tutte polynomial evaluated on these terminal forms simply counts separately the number of bridges and loops :

Définition E.2.1 (Deletion-Contraction) *If $G = (V, E)$ is a graph, and e is a regular edge, then*

$$T_G(x, y) = T_{G/e}(x, y) + T_{G-e}(x, y). \tag{1}$$

For a terminal form G with m bridges and n self-loops the polynomial is defined by

$$T_G(x, y) = x^m y^n. \tag{2}$$

It is not obvious that Definition E.2.1 is a definition at all since the result might depend on the ordering in which different edges are suppressed through deletion/contraction, leading to a terminal form. The best proof that T_G is unique and well-defined is in fact through a second definition of the Tutte polynomial as a global sum over subgraphs. It gives a concrete solution to the linear deletion/contraction recursion which is clearly independent on the order in which edges are suppressed :

Définition E.2.2 (Sum overs subsets) *If $G = (V, E)$ is a graph, then the Tutte polynomial of G, $T_G(x, y)$ has the following expansion :*

$$T_G(x, y) = \sum_{A \subset E} (x-1)^{r(E)-r(A)} (y-1)^{n(A)}, \tag{3}$$

where $r(A) = |V| - k(A)$ is the rank of the subgraph A and $n(A) = |A| + k(A) - |V|$ is its nullity or cyclomatic number. In physicists language $n(A)$ is the number of independent loops in A.

Remark that $r(A)$ is the number of edges in any spanning forest of A, and $n(A)$ is the number of remaining edges in A when a spanning forest is suppressed, so it is the number of *independent cycles* in A.

Théorème E.2.1 *These two definitions are equivalent.*

One can show that the polynomial defined by the sum over subsets obeys the deletion-contraction recursion. One can also evaluate it directly and show that it coincides with the first definition on the terminal forms with only loops and bridges.

There is a third definition of the Tutte polynomial through spanning trees (see eg [109]). This third definition involves ordering the edges of the graph. We think it may be also relevant in the context of field theory, in particular in relation with the ordered trees or forests formulas of constructive theory [12, 13, 53], but this point of view will not be developed here.

E.2.3 Multivariate Tutte polynomials

Multivariate Tutte polynomials can also be defined through linear recursion or global formulas.

The ordinary multivariate Tutte polynomial $Z_G(q, \{\beta\})$ has a different variable β_e for each edge e, plus another variable q to count vertices. We also write it most of the time as $Z_G(q, \beta)$ for simplicity. It is defined through a completely general linear deletion-contraction relation :

Définition E.2.3 (Deletion-Contraction) *For any edge e (not necessarily regular)*

$$Z_G(q, \{\beta\}) = \beta_e Z_{G/e}(q, \{\beta - \{\beta_e\}\}) + Z_{G-e}(q, \{\beta - \{\beta_e\}\}). \tag{4}$$

This relation together with the evaluation on terminal forms completely defines $Z_G(q, \beta)$, since the result is again independent of the order of suppression of edges. The terminal forms are graphs without edges, and with v vertices ; for such graphs $Z_G(q, \beta) = q^v$.

We can also define $Z_G(q, \beta)$ as a sum over subsets of edges :

Définition E.2.4 (Sum over subsets)

$$Z_G(q, \beta) = \sum_{A \subset E} q^{k(A)} \prod_{e \in A} \beta_e, \tag{5}$$

where $k(A)$ is the number of connected components in the subgraph (V, A).

One can prove as for the two variables Tutte polynomial that this definition is equivalent to the first. In [108] this multivariate polynomial is discussed in detail.

To understand the relation between this multivariate and the ordinary Tutte polynomial with two variables we multiply Z_G by q^{-V}, we set $\beta_e = y - 1$ and $q = (x-1)(y-1)$ and get

$$\left[q^{-V} Z_G(q, \beta)\right]|_{\beta_e = y-1, q=(x-1)(y-1)} \;=\; (x-1)^{k(E)-|V|} T_G(x, y). \tag{6}$$

We consider also

$$q^{-k(G)} Z_G(q, \beta). \tag{7}$$

Taking the limit $q \to 0$ that is retaining only the constant term in q we obtain a sum over maximally spanning subgraphs A, that is subgraphs with $k(A) = k(G)$:

$$S_G(\beta) = \sum_{A \text{ maximally spanning } E} \prod_{e \in A} \beta_e. \tag{8}$$

If we now retain only the lowest degree of homogeneity in β we obtain a sum over maximally spanning graphs with lowest number of edges, ie maximally spanning acyclic graphs or *spanning forests* of G.

$$F_G(\beta) = \sum_{\mathcal{F} \text{ maximally spanning forest of } G} \prod_{e \in \mathcal{F}} \beta_e. \tag{9}$$

Finally if we divide $F_G(\beta)$ by $\prod_{e \in E} \beta_e$ and change variables to $\alpha_e = \beta_e^{-1}$ we obtain the "(Kirchoff-Tutte)-Symanzik" polynomial. This polynomial is usually defined for connected graphs, in which case the sum runs over spanning trees $\mathcal{T}$ of G.

$$U_G(\alpha) = \sum_{\mathcal{T} \text{ spanning tree of } G} \prod_{e \notin \mathcal{T}} \alpha_e. \tag{10}$$

This polynomial satisfies the deletion contraction-recursion

$$U_G(\alpha) = U_{G/e}(\alpha) + \alpha_e U_{G-e}(\alpha) \tag{11}$$

for any regular edge e, together with the terminal form evaluation

$$U_G(\alpha) = \prod_{e \text{ self-loop}} \alpha_e, \tag{12}$$

for any G solely made of self-loops and bridges. The deletion-contraction (11) can be extended to general edges if we define U for disconnected graphs as the product over the connected components of the corresponding U's and put the contraction of any self-loop to 0.

The polynomial U appears in a key computation of QFT, namely that of the parametric representation of the Feynman amplitude associated to the graph G. We give a proof of this fact based on a new Pfaffian representation of Feynman amplitudes together with harmonic weights at vertices so as to make the deletion/contraction rule (11)-(12) particularly transparent.

But to define the second (Kirchoff-Tutte)-Symanzik polynomial as well as to make the computation of the first Symanzik polynomial more canonical, we need first to enlarge slightly our category of graphs to include some decorations at the vertices.

E.2.4 Decorated graphs

Decorations are essential in physics to represent the concept of *external variables*, which are ultimately those connected to actual experiments and observations.

Graphs with integers attached to each vertex and their corresponding multivariate polynomials $W_G(\alpha_e, N_v)$ have been considered in [135]. But to represent external variables

we need to replace the integer N_v by a set of N_v disjoint objects [1], hereafter called *flags* (see subsection E.2.1).

Each flag is attached to a single vertex. A momentum variable p_f in $\mathbb{R}^d$ is associated to each such flag. The incidence matrix can be extended to the flags, that is we define ϵ_{fv} as $+1$ if the flag f is associated to the vertex v and 0 otherwise. The total momentum incident to a subset S of the graph is then defined as $\sum_f \sum_{v\in S} \epsilon_{fv} p_f$. Remark that this momentum is defined for subgraphs S which may contain connected components reduced to single vertices. For translation invariant QFT's, global momentum conservation means that the condition $p_G = 0$ must be fulfilled.

Similarly we attach to each vertex a number $q_v > 0$ called the (harmonic) weight of the vertex. The total weight of a subgraph S is $\sum_{v\in S} q_v$.

The deletion/contraction relation is then extended to this category of graphs. The deletion is easy but the contraction is a bit non trivial. For a semi-regular edge joining vertices v_1 and v_2 it collapses the two vertices into a single one v_{12}, attaching to v_{12} all half-edges of v_1 and v_2. But it also attaches to v_{12} the union of all the flags attached to v_1 and v_2, so that the total momentum incoming to v_{12} is the sum of the momenta incoming to v_1 and to v_2. Finally the new weight of v_{12} is the sum $q_{v_1} + q_{v_2}$ of the weights of v_1 and v_2.

These decorated graphs are the natural objects on which to define generalized Symanzik polynomials in field theory.

Remaining for the moment in the context of graph theory we can define the second (Kirchoff-Tutte)-Symanzik polynomial for a connected graph as :

$$V_G(\alpha, p) = -\frac{1}{2} \sum_{v\neq v'} p_v \cdot p_{v'} \sum_{\mathcal{T}_2 \ 2-\text{tree separating } v \text{ and } v'} \prod_{e\notin \mathcal{T}_2} \alpha_e, \tag{13}$$

where a two tree $\mathcal{T}_2$ means a tree minus one edge, hence a forest with two disjoint connected components G_1 and G_2 ; the separation condition means that v and v' must belong one to G_1 the other to G_2.

For any pair of distinct vertices v and v' we can build a canonical graph $G(v, v')$ first by joining vertices v and v' in G with a new edge and then *contracting that edge*. This operation could be called the contraction of the pair of vertices v and v'. The following result goes back to Kirchhoff [133].

Proposition E.2.1 *The second Symanzik polynomial is a quadratic form in the total momenta p_v at each vertex, whose coefficients are the $U_{G(v,v')}$ polynomials :*

$$V_G(\alpha, p) = -\frac{1}{2} \sum_{v\neq v'} p_v \cdot p_{v'} \ U_{G(v,v')}. \tag{14}$$

Proof The graph $G(v, v')$ has $V - 1$ vertices, hence its spanning trees have $V - 2$ edges. They cannot make cycles in G because they would make cycles in $G(v, v')$. They are therefore two-trees in G, which must separate v and v', otherwise they would make a cycle in $G(v, v')$.

On the submanifold of flag variables satisfying the *momentum conservation* condition $p_G = \sum_f p_f = 0$ there is an alternate less symmetric definition of a similar polynomial :

1. In mathematics such a replacement is called a categorification of the integers N_v.

Définition E.2.5

$$\bar{V}_G(\alpha, p) = \sum_{\mathcal{T}_2 \ 2-\text{tree}} p_{G_1}^2 \prod_{e \notin \mathcal{T}_2} \alpha_e \tag{15}$$

where $\mathcal{T}_2$ is again a two-tree with two disjoint connected components G_1 and G_2.

Indeed this is an unambiguous definition. On the submanifold $p_G = 0$ we have $p_{G_1} = -p_{G_2}$, hence equation (15) does not depend of the choice of G_1 rather than G_2.

Proposition E.2.2 *On the manifold of flag variables satisfying the momentum conservation condition $p_G = \sum_f p_f = 0$ one has $V_G(\alpha, p) = \bar{V}_G(\alpha, p)$.*

Proof We simply commute the sums over v, v' and $\mathcal{T}_2$ in (13). For a given $\mathcal{T}_2$ the condition that v and v' are separated allows to separate the p_v with $v \in G_1$ from the $p_{v'}$ with $v' \in G_2$; one gets therefore $-\frac{1}{2}\ 2p_{G_1} \cdot p_{G_2}$ which is nothing but $p_{G_1}^2$ or $p_{G_2}^2$ on the manifold $p_G = 0$.

We shall give in subsection E.3.4 a definition of generalized first and second Symanzik polynomials for any graph, connected or not from which U_G, V_G or $\bar{V}_G$ can be easily derived in certain limits. Before actually performing these computations we include a brief interlude on Grassmann representation of determinants and Pfaffians. The reader familiar with this topic can jump directly to the next section.

E.2.5 Grassmann representations of determinants and Pfaffians

Independent Grassmann variables $\chi_1, ..., \chi_n$ satisfy complete anticommutation relations

$$\chi_i \chi_j = -\chi_j \chi_i \quad \forall i, j \tag{16}$$

so that any function of these variables is a polynomial with highest degree one in each variable. The rules of Grassmann integrations are then simply

$$\int d\chi = 0, \quad \int \chi d\chi = 1. \tag{17}$$

The determinant of any n by n matrix can be then expressed as a Grassmann Gaußian integral over $2n$ independent Grassmann variables which it is convenient to name as $\bar{\psi}_1, \ldots, \bar{\psi}_n, \psi_1, \ldots, \psi_n$, although the bars have nothing yet at this stage to do with complex conjugation. The formula is

$$\det M = \int \prod d\bar{\psi}_i d\psi_i e^{-\sum_{ij} \bar{\psi}_i M_{ij} \psi_j}. \tag{18}$$

The Pfaffian $\mathrm{Pf}(A)$ of an *antisymmetric* matrix A is defined by

$$\det A = [\mathrm{Pf}(A)]^2. \tag{19}$$

Proposition E.2.3 *We can express the Pfaffian as :*

$$\mathrm{Pf}(A) = \int d\chi_1 ... d\chi_n e^{-\sum_{i<j} \chi_i A_{ij} \chi_j} = \int d\chi_1 ... d\chi_n e^{-\frac{1}{2}\sum_{i,j} \chi_i A_{ij} \chi_j}. \tag{20}$$

Proof Indeed we write

$$\det A = \int \prod_i d\bar{\psi}_i d\psi_i e^{-\sum_{ij} \bar{\psi}_i A_{ij} \psi_j}. \tag{21}$$

Performing the change of variables (which a posteriori justifies the complex notation)

$$\bar{\psi}_i = \frac{1}{\sqrt{2}}(\chi_i - i\omega_i), \quad \psi_i = \frac{1}{\sqrt{2}}(\chi_i + i\omega_i), \tag{22}$$

whose Jacobian is i^{-n}, the new variables χ and ω are again independent Grassmann variables. Now a short computation using $A_{ij} = -A_{ji}$ gives

$$\begin{aligned} \det A &= i^{-n} \int \prod_i d\chi_i d\omega_i e^{-\sum_{i<j} \chi_i A_{ij} \chi_j - \sum_{i<j} \omega_i A_{ij} \omega_j} \\ &= \int \prod_i d\chi_i e^{-\sum_{i<j} \chi_i A_{ij} \chi_j} \prod_i d\omega_i e^{-\sum_{i<j} \omega_i A_{ij} \omega_j}, \end{aligned} \tag{23}$$

where we used that $n = 2p$ has to be even and that a factor $(-1)^p$ is generated when changing $\prod_i d\chi_i d\omega_i$ into $\prod_i d\chi_i \prod_i d\omega_i$. Equation (23) shows why $\det A$ is a perfect square and proves (20).

Lemme E.2.1 *The determinant of a matrix $D + A$ where D is diagonal and A antisymmetric has a "quasi-Pfaffian" representation*

$$\det(D + A) = \int \prod_i d\chi_i d\omega_i e^{-\sum_i \chi_i D_{ii} \omega_i - \sum_{i<j} \chi_i A_{ij} \chi_j + \sum_{i<j} \omega_i A_{ij} \omega_j}. \tag{24}$$

Proof The proof consists in performing the change of variables (22) and canceling carefully the i factors.

Tree-Matrix Theorem

Let A be an $n \times n$ matrix such that

$$\sum_{i=1}^{n} A_{ij} = 0 \ \ \forall j \ . \tag{25}$$

Obviously $\det A = 0$. The interesting quantities are eg the diagonal minors $\det A^{ii}$ obtained by deleting the i-th row and the i-th column in A. The "Kirchoff-Maxwell" matrix tree theorem expresses these minors as sums over trees :

Théorème E.2.2 (Tree-matrix theorem)

$$\det A^{ii} = \sum_{T \text{ spanning tree of} A} \prod_{e \in T} (-A_e), \tag{26}$$

where the sum is over spanning trees on $\{1, ...n\}$ oriented away from root i.

Proof We give here a sketch of the Grassmann proof given in [139]. We can assume without loss of generality that $i = 1$. For any matrix A we have :

$$\det A^{11} = \int [\prod_{i=1}^{n} d\bar{\psi}_i d\psi_i] \psi_1 \bar{\psi}_1 e^{-\sum_{i,j} \bar{\psi}_i A_{ij} \psi_j}. \tag{27}$$

The trick is to use (25) to write

$$\bar{\psi} A \psi = \sum_{i,j=1}^{n} (\bar{\psi}_i - \bar{\psi}_j) A_{ij} \psi_j, \tag{28}$$

hence

$$\begin{aligned} \det A^{11} &= \int \mathrm{d}\bar{\psi}\mathrm{d}\psi \ (\psi_1 \bar{\psi}_1) \exp\left(- \sum_{i,j=1}^{n} A_{ij} (\bar{\psi}_i - \bar{\psi}_j) \psi_j \right) \\ &= \int \mathrm{d}\bar{\psi}\mathrm{d}\psi \ (\psi_1 \bar{\psi}_1) \left[\prod_{i,j=1}^{n} \left(1 - A_{ij} (\bar{\psi}_i - \bar{\psi}_j) \psi_j \right) \right] \end{aligned} \tag{29}$$

by the Grassmann rules. We now expand to get

$$\det A^{11} = \sum_{\mathcal{G}} \left(\prod_{\ell=(i,j)\in\mathcal{G}} (-A_{ij}) \right) \Omega_{\mathcal{G}} \tag{30}$$

where $\mathcal{G}$ is *any* subset of $[n] \times [n]$, and we used the notation

$$\Omega_{\mathcal{G}} \equiv \int \mathrm{d}\bar{\psi}\mathrm{d}\psi \ (\psi_1 \bar{\psi}_1) \left(\prod_{(i,j)\in\mathcal{G}} \left[(\bar{\psi}_i - \bar{\psi}_j) \psi_j \right] \right). \tag{31}$$

Then the theorem follows from the following

Lemme E.2.2 $\Omega_{\mathcal{G}} = 0$ *unless the graph* $\mathcal{G}$ *is a tree directed away from 1 in which case* $\Omega_{\mathcal{G}} = 1$.

Proof Trivially, if (i, i) belongs to $\mathcal{G}$, then the integrand of $\Omega_{\mathcal{G}}$ contains a factor $\bar{\psi}_i - \bar{\psi}_i = 0$ and therefore $\Omega_{\mathcal{G}}$ vanishes.

But the crucial observation is that if there is a loop in $\mathcal{G}$ then again $\Omega_{\mathcal{G}} = 0$. This is because then the integrand of $\Omega_{\mathcal{F},\mathcal{R}}$ contains the factor

$$\bar{\psi}_{\tau(k)} - \bar{\psi}_{\tau(1)} = (\bar{\psi}_{\tau(k)} - \bar{\psi}_{\tau(k-1)}) + \cdots + (\bar{\psi}_{\tau(2)} - \bar{\psi}_{\tau(1)}). \tag{32}$$

Inserting this telescoping expansion of the factor $\bar{\psi}_{\tau(k)} - \bar{\psi}_{\tau(1)}$ into the integrand of $\Omega_{\mathcal{F},\mathcal{R}}$, the latter breaks into a sum of $(k-1)$ products. For each of these products, there exists an $\alpha \in \mathbb{Z}/k\mathbb{Z}$ such that the factor $(\bar{\psi}_{\tau(\alpha)} - \bar{\psi}_{\tau(\alpha-1)})$ appears *twice* : once with the + sign from the telescopic expansion of $(\bar{\psi}_{\tau(k)} - \bar{\psi}_{\tau(1)})$, and once more with a + (resp. −) sign if $(\tau(\alpha), \tau(\alpha-1))$ (resp. $(\tau(\alpha-1), \tau(\alpha))$) belongs to $\mathcal{F}$. Again, the Grassmann rules entail that $\Omega_{\mathcal{G}} = 0$.

To complete the proof of (26) every connected component of $\mathcal{G}$ must contain 1, otherwise there is no way to saturate the $d\psi_1$ integration.

This means that $\mathcal{G}$ has to be a directed tree on $\{1, ...n\}$. It remains only to see that $\mathcal{G}$ has to be directed away from 1, which is not too difficult.

The interlude is over and we now turn to perturbative QFT and to the parametric representation of Feynman amplitudes.

E.3 Parametric Representation of Feynman Amplitudes

In this section we will give a brief introduction to the parametric representation of ordinary QFT on a commutative vector space $\mathbb{R}^d$. We may take the example of ϕ^4 bosonic theory but the formalism is completely general.

E.3.1 Green and Schwinger functions in QFT

In particle physics the most important quantity is the diffusion matrix S whose elements or cross sections can be measured in particle experiments. The S matrix can be expressed from the Green functions through the reduction formulas. Hence they contain all the relevant information for that QFT.

These Green functions are time ordered vacuum expectation values of the fields ϕ, which are operator-valued and act on the Fock space :

$$G_N(z_1, ..., z_N) = \langle \psi_0, T[\phi(z_1)...\phi(z_N)]\psi_0 \rangle. \tag{1}$$

Here ψ_0 is the vacuum state and the T-product orders $\phi(z_1)...\phi(z_N)$ according to increasing times.

In the functional integral formalism the Green functions can be written as :

$$G_N(z_1, ..., z_N) = \frac{\int \prod_{j=1}^N \phi(z_j) e^{i\int \mathcal{L}(\phi(x))dx} D\phi}{\int e^{i\int \mathcal{L}(\phi(x))dx} D\phi}. \tag{2}$$

Here $\mathcal{L} = \mathcal{L}_0 + \mathcal{L}_{int}$ is the full Lagrangian of the theory. The Green functions continued to Euclidean points are called the Schwinger functions and are given by the Euclidean Feynman-Kac formula :

$$S_N(z_1, ..., z_N) = Z^{-1} \int \prod_{j=1}^N \phi(z_j) e^{-\int \mathcal{L}(\phi(x))dx} D\phi, \tag{3}$$

$$Z = \int e^{-\int \mathcal{L}(\phi(x))dx} D\phi. \tag{4}$$

For instance for the ϕ^4 theory, $\mathcal{L}_{int} = \frac{\lambda}{4!}\phi(x)^4$ and we have

$$\mathcal{L}(\phi) = \frac{1}{2}\partial_\mu \phi(x)\partial^\mu \phi(x) + \frac{1}{2}m\phi(x)^2 + \frac{\lambda}{4!}\phi(x)^4 \tag{5}$$

where

- λ is the (bare) coupling constant, which characterizes the strength of the interaction, the traditional factor $1/4!$ is inessential but slightly simplifies some computations.
- m is the (bare) mass,
- Z is the normalization factor,
- $D\phi$ is an ill-defined "flat" product of Lebesgue measures $\prod_x d\phi(x)$ at each space time point.

The coefficient of the Laplacian is set to 1 in (5) for simplicity. Although this coefficient actually in four dimensions flows through renormalization, it is possible to exchange this flow for a rescaling of the field ϕ.

To progress towards mathematical respectability and to prepare for perturbation theory, we combine the $e^{-\int \mathcal{L}_I(\phi(x))dx} D\phi$ and the free normalization factor $Z_0 = \int e^{-\int \mathcal{L}_I(\phi(x))dx} D\phi$ into a normalized Gaußian measure $d\mu_C(\phi)$ which is well-defined on some subspace of the Schwartz space of distributions $S'(R^d)$ [9]. The covariance of this measure is the (free) translation invariant propagator $C(x,y) = \int \phi(x)\phi(y)d\mu_C(\phi)$, which by slight abuse of notation we also write as $C(x-y)$ and whose Fourier transform is

$$C(p) = \frac{1}{(2\pi)^d} \frac{1}{p^2 + m^2}. \tag{6}$$

In this way the Schwinger functions are rewritten as

$$S_N(z_1, ..., z_N) = Z^{-1} \int_{R^d} \prod_{j=1}^{N} \phi(z_j) e^{-\int_{R^d} \mathcal{L}_{int}(\phi)} d\mu_C(\phi), \tag{7}$$

$$Z = \int e^{-\int_{R^d} \mathcal{L}_{int}(\phi(x))dx} d\mu_C(\phi). \tag{8}$$

However this expression is still formal for two reasons ; for typical fields the interaction factor is not integrable over R^d so that $\int_{R^d} \mathcal{L}_{int}(\phi)$ is ill-defined (infrared or thermodynamic problem) and in dimension more than 2 even when the interaction factor is restricted to a finite volume it is still ill-defined because for typical distributions ϕ, products such as $\phi^4(x)$ are also ill-defined. This is the famous ultraviolet problem which requires renormalization (see [136]), but this problem is not addressed here, as we discuss solely the structure of the integrands in Feynman parametric representations, not the convergence of the integrals. The reader worried by ill-defined integrals in the rest of this paper for space-time dimension d larger than 2 should impose a ultraviolet regulator. This means he should replace replace $C(p)$ by a better behaved $C_\kappa(p)$ such as

$$C_\kappa(p) = \frac{1}{(2\pi)^d} \frac{e^{-\kappa(p^2+m^2)}}{p^2 + m^2} = \int_\kappa^\infty e^{-\alpha(m^2+p^2)} d\alpha, \tag{9}$$

so that

$$C_\kappa(x,y) = \int_\kappa^\infty e^{-\alpha m^2 - (x-y)^2/4\alpha} \frac{d\alpha}{\alpha^{D/2}}. \tag{10}$$

We now turn to perturbation theory in which the factor $e^{-\int_{R^d} \mathcal{L}_{int}(\phi)}$ is expanded as a power series. This solves the thermodynamic problem, at the cost of introducing another problem, the divergence of that perturbation expansion. This divergence which in the good cases can be tackled by constructive field theory [9, 7, 137, 138] will not be treated in this paper.

E.3.2 Perturbation theory, Feynman Graphs

Wick theorem is nothing but the rule of pairing which computes the moments of a Gaußian measure. It allows to integrate monomials of fields

$$\int \phi(x_1)...\phi(x_n) d\mu_C(\phi) = \sum_G \prod_{e \in G} C(x_{i_e}, x_{j_e}) \tag{11}$$

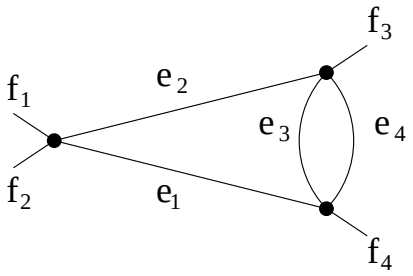


FIGURE E.3: A ϕ^4 graph

where the sum over G is over all contraction schemes (i.e. pairings of the fields) and $C(x_{i_e}, x_{j_e})$ is the propagator kernel joining the arguments of the two fields $\phi(x_{i_e})$ and $\phi(x_{j_e})$ paired into the edge e by the contraction scheme G.

It was Feynman's master stroke to represent each such contraction scheme by a particular *graph* in which edges represent pairs of contracted fields and vertices stand for the interaction.

In the case of a ϕ^4 theory, remark that these interaction vertices have degree 4. Indeed the Schwinger functions after perturbative expansion are

$$S_N(z_1...z_N) = \frac{1}{Z}\sum_{n=0}^{\infty}\frac{(-\lambda)^n}{4^n n!}\int [\int \prod_{v=1}^{n}\phi^4(x_v)dx_v]\phi(z_1)...\phi(z_N)d\mu(\phi). \tag{12}$$

The pairings of Wick's theorem therefore occur between n internal vertices each equipped with four fields and N external vertices or sources corresponding to the single fields $\phi(z_1)$, ... , $\phi(z_N)$.

Schwinger function are therefore expressed as sums over Feynman graphs of associated quantities or weights called the Feynman amplitudes. In this position space representation the Feynman graphs have both n *internal vertices* corresponding to the $\mathcal{L}_{int}$ factors, plus N external vertices of degree 1 corresponding to the fields $\phi(z_1), ..., \phi(z_N)$. In the case of the ϕ^4 theory each internal vertex has degree 4.

The Feynman amplitudes are obtained by integrating over all positions of internal vertices the product of the propagator kernels for all the edges of the graphs

$$A_G(z_1, ..., z_N) = \int \prod_v dx_v \prod_{e\in G} C(x_{i_e}, x_{j_e}), \tag{13}$$

where the product $\prod_v$ runs over the *internal* vertices v.

The quantities that are relevant to physical experiments are the *connected* Schwinger functions which can be written as :

$$\Gamma_N(z_1, ..., z_N) = \sum_{\phi^4 \text{ connected graphs } G \text{ with } N(G)=N} \frac{(-\lambda)^{n(G)}}{S(G)} A(G)(z_1, ..., z_N), \tag{14}$$

where $S(G)$ is a combinatoric factor (symmetry factor).

The momentum space representation corresponds to a Fourier transform to momenta variables called $p_1, ..., p_N$:

$$\Gamma_N(p_1, ..., p_N) = \int dz_1...dz_N e^{2i\sum p_f z_f}\Gamma_N(z_1, ..., z_N), \tag{15}$$

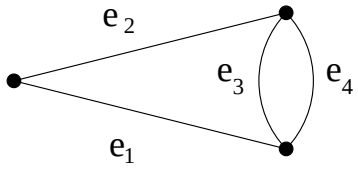


FIGURE E.4: A truncated ϕ^4 graph

where the factor 2 is convenient and we forget inessential normalization factors. This is a distribution, proportional to a global momentum conservation $\delta(\sum_{f=1}^{N} p_f)$. From now on we use an index f to label external momenta to remember that they are associated to corresponding graph-theoretic *flags*. Usually one factors out this distribution together with the external propagators, to obtain the expansion in terms of truncated amputated graphs :

$$\Gamma_N^T(p_1,...,p_N) = \sum_{\phi^4 \text{ truncated graphs } G \text{ with } N(G)=N} \frac{(-\lambda)^{n(G)}}{S(G)} \delta(\sum_{f=1}^{N} p_f) \prod_{f=1}^{N} \frac{1}{p_f^2+m^2} A_G^T(p_1,...,p_N). \quad (16)$$

In this sum we have to describe in more detail the *truncated graphs* G with N external flags. Such truncated graphs are connected, but they may contain bridges and self-loops. They no longer have external vertices of degree 1. Instead, they still have N external variables p_f, no longer associated to edges but to flags (N in total), which decorate the former internal vertices. For instance for the ϕ^4 theory the degree of a truncated graph G is no longer 4 at each internal vertex. It is the total degree, that is the number of half-edges plus flags which remains 4 at every vertex.

Ordinary Schwinger functions can be expressed as sums over partitions of the arguments of products of the corresponding truncated functions. We now give the explicit form of the corresponding truncated amplitudes $A_G^T(p_1,...,p_N)$.

E.3.3 Parametric representation

We shall first consider a fixed truncated oriented diagram G and compute the corresponding contribution or amplitude A_G^T as given by Feynman rules.

We denote again by E and V the number of edges and vertices respectively, and by N the number of flags. Since G is connected its incidence matrix has rank $V-1$.

Now consider a Feynman graph G contributing to some truncated Schwinger function $\Gamma^T(p_1,...,p_N)$. The usual way to take into account the global δ function in (16) is to restrict to configurations such that $\sum_f p_f = 0$. Extraction of this global delta function in (16) for the amplitude of a particular graph can be done provided we do not integrate the position of one of the vertices in (13), but rather fix it at an arbitrary point, eg the origin. From now on we suppose this vertex is $\bar{v}$ the one with last index. It provides a *root* in the graph G. However this standard procedure requires the non-canonical choice of that root vertex, and the final result does not depend on that choice.

Another possibility is to modify the interaction $\lambda\phi^4(x)$ into $\lambda e^{-qx^2}\phi^4(x)$, in which case there is no longer global momentum conservation. One can compute modified amplitudes $B_G^T(p_1,...p_N;q)$ without factoring out the global $\delta(\sum_{f=1}^N p_f)$ factor, so that

$$\Gamma_N^T(p_1,...,p_N;q) = \sum_{\phi^4 \text{ truncated graphs } G \text{ with } N(G)=N} \frac{(-\lambda)^{n(G)}}{S(G)} \prod_{f=1}^N \frac{1}{p_f^2+m^2} B_G^T(p_1,...,p_N;q). \tag{17}$$

The momentum conserving usual amplitudes are recovered when $q \to 0$:

$$\lim_{q\to 0} B_G^T(p_1,...,p_N;q) = \delta(\sum_{f=1}^N p_f) A_G^T(p_1,...,p_N). \tag{18}$$

This is the procedure we shall follow in subsection E.3.4, because it avoids the choice of a noncanonical root. But for the moment let us complete the standard presentation of $A_G^T(p_1,...,p_N)$.

The momentum representation of A_G^T, forgetting from now on inessential factors of 2π, is :

$$A_G^T(p_1,...,p_N) = \int \prod_{e=1}^E d^dk_e \frac{1}{k_e^2+m^2} \prod_{v=1}^{V-1} \delta(\epsilon_{fv}p_f + \epsilon_{ev}k_e). \tag{19}$$

in which we use the convention that repeated indices are summed, so that $\epsilon_{fv}p_f + \epsilon_{ev}k_e$ stands for the total momentum $\sum_f \epsilon_{fv}p_f + \sum_e \epsilon_{ev}k_e$ incoming at vertex v.

To obtain the parametric representation we have first to rewrite the propagators as :

$$\frac{1}{k^2+m^2} = \int_0^\infty d\alpha e^{-\alpha(k^2+m^2)}. \tag{20}$$

We obtain the momentum parametric representation

$$A_G^T(p_1,...,p_N) = \int \prod_{e=1}^E d\alpha_e d^dk_e e^{-\alpha_e(k_e^2+m^2)} \prod_{v=1}^{V-1} \delta(\epsilon_{fv}p_f + \epsilon_{ev}k_e). \tag{21}$$

Fourier transforming the $V-1$ Dirac distributions into oscillating integrals we obtain, up to some inessential global factors the phase-space parametric representation

$$A_G^T(p_1,...,p_N) = \int \prod_{e=1}^E \left[d\alpha_e e^{-\alpha_e m^2} d^dk_e\right] \prod_{v=1}^{V-1} d^dx_v e^{-\alpha_e k_e^2 + 2i(p_f\epsilon_{fv}x_v + k_e\epsilon_{ev}x_v)}, \tag{22}$$

where again $k_e\epsilon_{ev}x_v$ means $\sum_{e=1}^E \sum_{v=1}^{V-1} k_e\epsilon_{ev}x_v$ etc, and the factor 2 is convenient.

Finally integrating out the edge momenta whose dependence is Gaußian leads to the x or direct space parametric representation :

$$A_G^T(p_1,...,p_N) = \int \prod_{e=1}^E d\alpha_e \frac{e^{-\alpha_e m^2}}{\alpha_e^{d/2}} \prod_{v=1}^{V-1} d^dx_v e^{2ip_f\epsilon_{fv}x_v - x_v\cdot x_{v'}\epsilon_{ve}\epsilon_{v'e}/\alpha_e}. \tag{23}$$

Remember this amplitude is only defined on the submanifold $p_G = 0$, because it is only there that the formula gives a result independent of the choice of the root not integrated out in (23)

The parametric representation consists in integrating out fully the x or p variables in (21), (22) or (23). One obtains the parametric representation, which is an integral on α parameters only :

$$A_G^T(p_1, ..., p_N) = \int \prod_{e=1}^{E} \left[d\alpha_e e^{-\alpha_e m^2} \right] \frac{e^{-V_G(p,\alpha)/U_G(\alpha)}}{U_G(\alpha)^{d/2}}, \tag{24}$$

where U_G and V_G are called the first and second Symanzik's polynomials.

Théorème E.3.1 *The first Symanzik polynomial U_G in (24) is the multivariate Tutte polynomial (10). On the submanifold $p_G = 0$, the only one where it is unambiguously defined, the second polynomial V_G of (24) coincides with (13) and (15).*

We are going to give two proofs of this classic theorem of quantum field theory, one relying directly on contraction-deletion and on the phase-space representation (22) the other more standard and relying on the direct representation (23) and on the tree-matrix theorem.

Indeed in order to compute the Symanzik's polynomials, let us remark first that the momentum representation mostly used in textbooks is not very convenient. To use (21) we should "solve" the δ functions, that is rewrite each edge momentum in terms of independent momenta for cycles. In physics this is called a momentum routing. But such a momentum routing is linked to the choice of a particular spanning tree of G. The momenta of the edges not in this tree are kept as independent variables and the tree edges momenta are recursively computed in terms of those by progressing from the leaves of the tree towards the root which is the fixed vertex v_n. This is not a canonical prescription, as it depends on the choice of the tree.

The representations (22) or (23) are more convenient to integrate the space or momentum variables because the dependence in variables x and k is Gaußian so that the result is a determinant to a certain power times a Gaußian in the external variables. In fact (22) is the best as we shall argue below. However there is still a small noncanonical choice, the one of the root. This is why we prefer to compute the regularized amplitudes

$$B_G^T(p_1, ..., p_N; q) = \int \prod_{e=1}^{E} \left[d\alpha_e e^{-\alpha_e m^2} d^d k_e \right] \prod_{v=1}^{V} d^d x_v e^{-\alpha_e k_e^2 - q \sum_{v=1}^{V} x_v^2 + 2i(p_f \epsilon_{fv} x_v + k_e \epsilon_{ev} x_v)} \tag{25}$$

and to deduce the ordinary amplitudes from a limit $q \to 0$.

The last modification we perform is to attribute a different weight q_v to each vertex regulator. This is more natural from the point of view of universal polynomials. So we define

$$B_G^T(p_1, ..., p_N; \{q_v\}) = \int \prod_{e=1}^{E} \left[d\alpha_e e^{-\alpha_e m^2} d^d k_e \right] \prod_{v=1}^{V} d^d x_v e^{-\alpha_e k_e^2 - q_v x_v^2 + 2i(p_f \epsilon_{fv} x_v + k_e \epsilon_{ev} x_v)}. \tag{26}$$

These amplitudes are Gaußian in the external variables p_f and no longer involve any noncanonical choice. We shall now compute their generalized Symanzik polynomials and deduce the ordinary Symanzik polynomials from these as leading terms when all q_v's are sent to 0.

E.3.4 Generalized Symanzik Polynomials

We consider the phase space representation (26). We have to perform a Gaußian integral in $E+V$ variables (each of which is d-dimensional). We consider these momentum and position variables as a single vector. We also forget the label ^{T} for truncation as it is no longer needed in this section. The graph we consider may be connected or not.

We introduce the condensed notations :

$$B_G(p_f, q_v) = \int \prod_e d\alpha_e e^{-\alpha_e m^2} d^d k_e \int \prod_v d^d x_v e^{-Y X_G Y^t} \tag{27}$$

where X_G is a $d(E+V+N)$ by $d(E+V+N)$ square matrix, namely

$$X_G = \begin{pmatrix} \alpha_e & -i\epsilon_{ev} & 0 \\ -i\epsilon_{ev} & q_v & -i\epsilon_{fv} \\ 0 & -i\epsilon_{fv} & 0 \end{pmatrix} \tag{28}$$

where α_e and q_v are short notations for diagonal matrices $\alpha_e \delta_{e,e'}$ and $q_v \delta_{v,v'}$. Y is an $E+V+N$ by 1 line, namely $Y = \begin{pmatrix} k_e & x_v & p_f \end{pmatrix}$.

We can further decompose X_G as

$$X_G = \begin{pmatrix} Q_G & -iR_G^t \\ -iR_G & 0 \end{pmatrix} . \tag{29}$$

where $Q_G = \begin{pmatrix} \alpha_e & -i\epsilon_{ev} \\ -i\epsilon_{ev} & q_v \end{pmatrix}$ is a $d(E+V)$ by $d(E+V)$ square matrix and R_G is the real rectangular N by $E+V$ matrix made of a dN by dE zero block and the dN by dV "incidence flag" matrix ϵ_{fv}^{μ}. The dimensional indices μ being quite trivial we no longer write them down from now on.

Note P the line p_f, hence the last part of the line Y. Gaußian integrations can be performed explicitly and the result is a Gaußian in external variables. Therefore up to inessential constants

$$\begin{aligned} B_G(p_f, q_v) &= \int \prod_e d\alpha_e e^{-\alpha_e m^2} \frac{1}{\det Q_G^{d/2}} e^{-P R_G Q_G^{-1} R_G^t P^t} \\ &= \int \prod_e d\alpha_e e^{-\alpha_e m^2} d^d k_e \frac{e^{-\mathcal{V}/\mathcal{U}}}{\mathcal{U}^{d/2}} \end{aligned} \tag{30}$$

for some polynomial $\mathcal{U}_G$ in α's and q's and a quadratic form in the p variable $\mathcal{V}_G$ with polynomial coefficients in α's and q's.

Définition E.3.1 *The generalized Symanzik polynomials with harmonic regulators are the polynomials appearing in (30), namely*

$$\mathcal{U}_G(\alpha_e, q_v) = \det Q_G, \tag{31}$$

$$\mathcal{V}_G(\alpha_e, q_v, p_f)/\mathcal{U}_G(\alpha_e, q_v) = P R_G Q_G^{-1} R_G^t P^t. \tag{32}$$

These polynomials can be computed explicitly :

Théorème E.3.2

$$\mathcal{U}_G(\alpha_e, q_v) = \sum_{\mathcal{F}} \prod_{e \notin \mathcal{F}} \alpha_e \prod_{\mathcal{C}} q_{\mathcal{C}}, \tag{33}$$

$$\mathcal{V}_G(\alpha_e, q_v, p_f) = \sum_{\mathcal{F}} \prod_{e \notin \mathcal{F}} \alpha_e \sum_{\mathcal{C}} p_{\mathcal{C}}^2 \prod_{\mathcal{C}' \neq \mathcal{C}} q_{\mathcal{C}'}, \tag{34}$$

where the sum over $\mathcal{F}$ runs over all forests of the graph, and the indices $\mathcal{C}$ and $\mathcal{C}'$ means any connected component of that forest (including isolated vertices if any). The variables $p_{\mathcal{C}}$ and $q_{\mathcal{C}}$ are the natural sums associated to these connected components.

In order to prove this theorem we introduce now the quasi-Grassmann representations of $\mathcal{U}_G$ and $\mathcal{V}_G$ of Lemma E.2.1.

Let's calculate first $\mathcal{U}$, hence the determinant of Q_G. Factoring out powers of i we get :

$$\det Q_G = \det \begin{pmatrix} \alpha_e & -\epsilon_{ev} \\ \epsilon_{ev} & q_v \end{pmatrix} \tag{35}$$

which can be written as sum of a diagonal matrix D, with coefficients $D_{ee} = \alpha_e$ and $D_{vv} = q_v$ and of an antisymmetric matrix A with elements ϵ_{ev}, that is, $Q = D + A$.

By Lemma E.2.1

$$\mathcal{U}_G(\alpha_e, q_v) = \int \prod_{v,e} d\chi_v d\omega_v d\chi_e d\omega_e e^{-\alpha_e \chi_e \omega_e} e^{-q_v \chi_v \omega_v} e^{-\chi_e \epsilon_{ev} \chi_v + \omega_e \epsilon_{ev} \omega_v}. \tag{36}$$

Similarly $\mathcal{V}$ which is a minor related to the Q_G matrix is given by a Grassmann integral but with sources

$$\begin{aligned} \mathcal{V}_G(\alpha_e, q_v, p_f) \quad &= \quad \int \prod_{v,e} d\chi_v d\omega_v d\chi_e d\omega_e e^{-\alpha_e \chi_e \omega_e} e^{-q_v \chi_v \omega_v} e^{-\chi_e \epsilon_{ev} \chi_v + \omega_e \epsilon_{ev} \omega_v} \\ &\quad p_f \cdot p_{f'} \epsilon_{fv} \epsilon_{f'v'} (\chi_v \omega_{v'} + \chi_{v'} \omega_v), \end{aligned} \tag{37}$$

where we have expanded $\bar{\psi}_v \psi_{v'}$ as $\frac{1}{2}[\chi_v \chi_{v'} + \omega_v \omega_{v'} + i(\chi_v \omega_{v'} + \chi_{v'} \omega_v)]$ and canceled out the $\chi_v \chi_{v'} + \omega_v \omega_{v'}$ term which must vanish by symmetry and the i factors.

Now we can prove directly that these polynomials obey a deletion-contraction rule.

Théorème E.3.3 *For any semi-regular edge e*

$$\mathcal{U}_G(\alpha_e, q_v) = \alpha_e \, \mathcal{U}_{G-e}(\alpha_e, q_v) + \mathcal{U}_{G/e}(\alpha_e, q_v), \tag{38}$$

$$\mathcal{V}_G(\alpha_e, q_v, p_f) = \alpha_e \mathcal{V}_{G-e}(\alpha_e, q_v, p_f) + \mathcal{V}_{G/e}(\alpha_e, q_v, p_f). \tag{39}$$

Moreover we have the terminal form evaluation

$$\mathcal{U}_G(\alpha_e, q_v) = \prod_e \alpha_e \prod_v q_v, \tag{40}$$

$$\mathcal{V}_G(\alpha_e, q_v, p_f) = \prod_e \alpha_e \sum_v p_v^2 \prod_{v' \neq v} q_v \tag{41}$$

for G solely made of self-loops attached to isolated vertices.

Proof If G is not a terminal form we can pick up any semi-regular edge e connecting vertices v_1 and v_2 with $\epsilon_{v_1} = +1, \epsilon_{v_2} = -1$. We expand

$$e^{-\alpha_e \chi_e \omega_e} = 1 + \alpha_e \omega_e \chi_e. \tag{42}$$

For the first term, since we must saturate the χ_e and ω_e integrations, we must keep the $\chi_e(\chi_{v_1} - \chi_{v_2})$ term in $e^{\sum_v \chi_e \epsilon_{ev} \chi_v}$ and the similar ω term, hence we get a contribution

$$\begin{aligned} \det Q_{G,e,1} \;=\; & \int \prod_{e' \neq e,v} d\chi_{e'} d\omega_{e'} d\chi_v d\omega_v (\chi_{v_1} - \chi_{v_2})(\omega_{v_1} - \omega_{v_2}) \\ & e^{-\sum_{e' \neq e} \alpha'_e \chi_{e'} \omega_{e'}} e^{-q_v \chi_v \omega_v} e^{-\sum_{e' \neq e,v} \chi_{e'} \epsilon_{e'v} \chi_v + \sum_{e' \neq e,v} \omega_{e'} \epsilon_{e'v} \omega_v}. \end{aligned} \tag{43}$$

Performing the trivial triangular change of variables with unit Jacobian :

$$\hat{\chi}_{v_1} = \chi_{v_1} - \chi_{v_2}, \quad \hat{\chi}_v = \chi_v \;\; for \;\; v \neq v_1, \tag{44}$$

and the same change for the ω variables we see that the effect of the $(\chi_{v_1} - \chi_{v_2})(\omega_{v_1} - \omega_{v_2})$ term is simply to change the v_1 label into v_2 and to destroy the edge e and the vertex v_1. This is exactly the contraction rule, so $\det Q_{G,e,1} = \det Q_{G/e}$. The second term $\det Q_{G,e,2}$ with the $\alpha_e \omega_e \chi_e$ factor is even easier. We must simply put to 0 all terms involving the e label, hence trivially $\det Q_{G,e,2} = \alpha_e \det Q_{G-e}$. Remark that during the contraction steps the weight factor $q_{v_1} \chi_{v_1} \omega_{v_1}$ is just changed into $q_{v_1} \chi_{v_2} \omega_{v_2}$. That's why we get the new weight $q_{v_1} + q_{v_2}$ for the new vertex v_2 which represent the collapse of former vertices v_1 and v_2.

Note that the source terms in $\mathcal{V}$ do not involve χ_e and ω_e variables. Therefore the argument goes through exactly in the same way for the second polynomials. The only remark to make is that like weights, flag momenta follow contraction moves.

The evaluation on terminal forms is easy. For a graph with only vertices and self-loops the matrix Q_G is diagonal, because ϵ_{ev} is always 0. Hence $\mathcal{U}_\mathcal{G}$ is the product of the diagonal elements $\prod_e \alpha_e \prod_v q_v$. The second polynomial can be analyzed through the Grassmann representation, but it is simpler to use directly (32) and the fact that Q_G is diagonal to get (41). This completes the proof of Theorem E.3.3, hence also of Theorem E.3.2.

We turn now to the limit of small regulators q_v to show how for a connected graph G the ordinary amplitude $\delta(\sum_f p_f) A_G$ and the ordinary polynomials U_G and V_G emerge out of the leading terms of the regularized amplitude B_G and the generalized polynomials $\mathcal{U}_G$ and $\mathcal{V}_G$.

When all q's are sent to zero there is no constant term in $\mathcal{U}_G$ but a constant term in $\mathcal{V}_G$. Up to second order in the q variables we have :

$$\mathcal{U}_G(\alpha_e, q_v) = q_G \sum_{\mathcal{T}} \prod_{e \notin \mathcal{T}} \alpha_e + O(q^2), \tag{45}$$

$$\mathcal{V}_G(\alpha_e, q_v, p_f) = p_G^2 \sum_{\mathcal{T}} \prod_{e \notin \mathcal{T}} \alpha_e + \sum_{\mathcal{T}_2} (p_{G_1}^2 q_{G_2} + p_{G_2}^2 q_{G_1}) \prod_{e \notin \mathcal{T}_2} \alpha_e + O(q^2), \tag{46}$$

where the sum over $\mathcal{T}$ runs over trees and the sum over $\mathcal{T}_2$ runs over two trees separating the graph into two connected components G_1 and G_2. Hence we find

$$\frac{e^{-\mathcal{V}/\mathcal{U}}}{\mathcal{U}^{d/2}} = \frac{e^{-p_G^2/q_G}}{q_G^{d/2}} \frac{e^{-\sum_{\mathcal{T}_2} (p_{G_1}^2 q_{G_2} + p_{G_2}^2 q_{G_1}) \prod_{e \notin \mathcal{T}_2} \alpha_e / q_G \sum_{\mathcal{T}} \prod_{e \notin \mathcal{T}} \alpha_e + p_G^2 O(1) + O(q)}}{[\sum_{\mathcal{T}} \prod_{e \notin \mathcal{T}} \alpha_e + O(q)]^{d/2}}. \tag{47}$$

Up to inessential normalization factors the first term tends to $\delta(p_G)$ and the second one tends to $e^{-V/U}/U^{d/2}$ if we use the fact that $\delta(p_G)f(p_G) = \delta(p_G)f(0)$, that is if we use the delta distribution to cancel the $p_G^2 O(1)$ term and to simplify $(p_{G_1}^2 q_{G_2} + p_{G_2}^2 q_{G_1})$ into $q_G p_{G_1}^2 = q_G p_{G_2}^2$. This proves (18).

The U_G and V_G polynomials are in fact easy to recover simply from the $\mathcal{U}_G$ polynomial alone :

Théorème E.3.4 *For any connected G and any vertex v*

$$U_G(\alpha_e) = \frac{\partial}{\partial q_v} \mathcal{U}_G(\alpha_e, q_v) \;\; |_{q_{v'}=0 \;\forall v'}. \tag{48}$$

On the submanifold $p_G = 0$ we further have

$$V_G(\alpha_e, p_f) = -\frac{1}{2} \sum_{v \neq v'} p_v \cdot p_{v'} \;\; \frac{\partial^2}{\partial q_v \partial q_{v'}} \mathcal{U}_G(\alpha_e, q_v) \;\; |_{q_{v''}=0 \;\forall v''}. \tag{49}$$

Proof It is an easy consequence of Theorem E.3.2.

We can also prove an analog of Proposition E.2.1 between $\mathcal{V}_G$ and $\mathcal{U}_{G(vv')}$ but only on the submanifold $p_G = 0$.

E.3.5 Relation to discrete Schrödinger Operator

As an aside, it is worthwhile to notice that there is a relation with discrete Schrödinger operators on graphs [143]. Recall that given a graph $G = (V, E)$, the discrete Laplacian is defined as follows. We first introduce the 0-forms $\Omega_0(G) = \mathbb{R}^V$ as the real functions on the set of vertices and 1-forms $\Omega_0(G) = \mathbb{R}^E$ as functions on the edges. Then, the discrete differential $\mathrm{d} : \Omega_0(G) \to \Omega_1(G)$ is defined as

$$\mathrm{d}\psi(e) = \sum_v \epsilon_{ev} \, \psi_v, \tag{50}$$

where we recall the convention that for a self-loop $\epsilon_{ev} = 0$ and an arbitrary orientation is chosen on the edges. Next, given strictly positive weights β_e associated to the edges, we define $\mathrm{d}^* : \Omega_1(G) \to \Omega_0(G)$ by

$$\mathrm{d}^*\phi(v) = \sum_e \beta_e \epsilon_{ev} \, \phi_e. \tag{51}$$

Note that d^* is precisely the adjoint of d for the scalar product on $\mathbb{R}^E$ defined by the weights β_e and the Euclidean one on $\mathbb{R}^V$. Accordingly, the 0-form Laplacian $\Delta : \Omega_0(G) \to \Omega_0(G)$ is

$$\Delta = \mathrm{d}^*\mathrm{d}, \tag{52}$$

or, in terms of its action on functions $\psi \in \mathbb{R}^V$,

$$\Delta\psi(v') = \sum_{e,v} \beta_e \epsilon_{ev'} \epsilon_{ev} \, \psi_v. \tag{53}$$

Note that there is exactly one zero mode per connected component, as follows from the equivalence between $\Delta\psi = 0$ and $\mathrm{d}\psi = 0$. Finally, the weights q_v associated to the

vertices [2] define a function V from the vertices to $\mathbb{R}$ acting multiplicatively on $\Omega_0(G)$ so that we define the discrete Schrödinger operator (Hamiltonian in the quantum mechanics language) on the graph by

$$H = -\Delta + V. \tag{54}$$

Turning back to the parametric representation, if we perform the Gaußian integration over the momenta we are left with

$$\frac{\pi^{D/2}}{(\alpha_1 \cdots \alpha_e)^{D/2}} \int \prod_v dx_v \, \mathrm{e}^{-\sum_{v,v'} x_v H_{v,v'} x_{v'} + 2\mathrm{i} \sum_v x_v \cdot p_v}, \tag{55}$$

with weights $\beta_e = \frac{1}{\alpha_e}$. In particular, the first Symanzik polynomial with regulators q_v is expressed in terms of the determinant of H,

$$\mathcal{U}_G(\alpha, q) = (\textstyle\prod_e \alpha_e) \det H = (\prod_e \alpha_e) \displaystyle\int \prod_v d\overline{\psi}_v d\psi_v \, \mathrm{e}^{-\sum_{v,v'} \overline{\psi}_v H_{v,v'} \psi_{v'}}, \tag{56}$$

with $\overline{\psi}_v, \psi_v$ Grassmann variables. By the same token, the ratio appearing in the Feynman amplitude is expressed in terms of its inverse G (Green's function in the quantum mechanics language),

$$\frac{\mathcal{V}_G(\alpha, q, p)}{\mathcal{U}_G(\alpha, q)} = \sum_{v,v'} G_{v,v'} \, p_v \cdot p_v, \tag{57}$$

where the Green's function can also be expressed using Grassmann integrals. As a byproduct, it turns out that it can also be computed by contraction/deletion.

E.3.6 Categorified Polynomials

We have up to now considered two seemingly unrelated graph polynomials obeying contraction/deletion rules, the multivariate Tutte polynomial $Z_G(\beta_e, q)$ and $\mathcal{U}_G(\alpha_e, q_i)$, from which the Symanzik polynomials can be recovered by various truncations. Therefore, it is natural to wonder wether there is a single graph polynomial, obeying contraction/deletion rules too, from which both $Z_G(\beta_e, q)$ and $\mathcal{U}_G(\alpha_e, q_i)$ can be recovered. In this subsection for simplicity we shall consider only the first Symanzik polynomial, and the flags considered in this subsection no longer bear external momenta, but an abstract index.

Such a polynomial is an invariant of graphs with flags, i.e. labeled half-edges attached to the vertices. In order to make the contraction possible, it is necessary to allow each vertex to have several flags, all carrying distinct labels. The requested polynomial, $\mathcal{W}_G(\beta_e, q_I)$ depends on edge variables β_e as well as on independent variables q_I for each non empty subset I of the set of labels of the flags, with the proviso that, for each vertex, the subsets I contain all the flags attached to the vertex or none of them. Thus, for a diagram with V' vertices carrying flags there are $2^{V'} - 1$ variables q_I.

Définition E.3.2 *For a graph G with flags, $\mathcal{W}_G(\beta_e, q_I)$ is defined by the expansion*

$$\mathcal{W}_G(\beta_e, q_I) = \sum_{A \subset E} \Big(\prod_{e \in E} \beta_e \prod_{\substack{\mathcal{C}_n \\ \text{connected components}}} q_{I_n} \Big), \tag{58}$$

where I_n are the sets of flags attached to the vertices of the connected component $\mathcal{C}_n$ of the spanning graph (V, A).

2. Strictly speaking, the latter are associated to the flags and q_v is the sum the weights of the flags attached to v.

For example, for the bubble graph on two vertices with two edges between these vertices and flags $1, 2$ attached to one of vertex, and flag 3 to the other one, we have

$$\mathcal{W}_G(\beta_e, q_I) = (\beta_1\beta_2 + \beta_1 + \beta_2)q_{123} + q_{12}q_3. \tag{59}$$

Since the variables q_I are defined using the flags, the contraction/deletion rule for $\mathcal{W}_G(\beta_e, q_I)$ requires us to properly define how the flags follow the contraction/deletion rule for any edge of $G - e$ and G/e. Because the vertices and the flags of $G - e$ are left unchanged, the same variables q_I appear in G and $G - e$. For G/e, we restrict the q_I to those associated with subsets that contain either all the flags attached to the two vertices merged by the contraction of e, either none of them. This is best formulated using flags : the new vertex simply carries the flags of the two vertices that have been merged. Then, the contraction/deletion identity simply follows from grouping the terms in $\mathcal{W}_G(\beta_e, q_I)$ that contain β_e and those that do not.

Proposition E.3.1 *The polynomial $\mathcal{W}_G(\beta_e, q_I)$ obeys the contraction/deletion rule for any edge*

$$\mathcal{W}_G(\beta_e, q_I) = \beta_e \mathcal{W}_{G/e}(\beta_{e'\neq e}, q_I|_{G/e}) + \mathcal{W}_{G-e}(\beta_{e'\neq e}, q_I). \tag{60}$$

The multivariate Tutte polynomial is easily recovered by setting $q_I = q$ for any I,

$$Z_G(\beta_e, q) = \mathcal{W}_G(\beta_e, q_I = q). \tag{61}$$

In this case, all the information about the flags is erased and so that the latter may be omitted. To recover $\mathcal{U}_G(\alpha_e, q_i)$, it is convenient to introduce as an intermediate step the polynomial

$$\Upsilon_G(\alpha_e, q_i) = \sum_{A\subset E} \prod_{e\notin E} \alpha_e \prod_{\mathcal{C}_n} \Big(\sum_{i\in I_n} q_i \Big), \tag{62}$$

where as before I_n are the flags included in the connected component $\mathcal{C}_n$ of the spanning graph (V, A). By its very definition, $\Upsilon_G(\alpha_e, q_i)$ is related to $\mathcal{W}_G(\beta_e, q_I)$ by setting $q_I = \sum_{i\in I} q_i$,

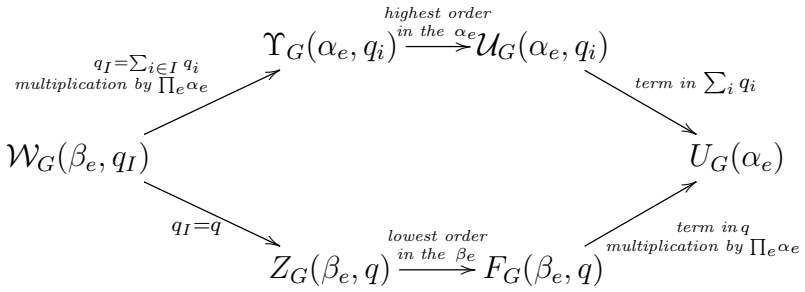
$$\Upsilon_G(\alpha_e, q_i) = \Big(\prod_e \alpha_e \Big) \mathcal{W}_G(\beta_e = 1/\alpha_e, q_I = \textstyle\sum_{i\in I} q_i). \tag{63}$$

Then, the polynomial $\mathcal{U}_G(\alpha_e, q_i)$ is obtained from $\Upsilon_G(\alpha_e, q_i)$ by keeping only the highest degree terms in the α_e's for each term in $\prod_{\mathcal{C}_n} \sum_{i\in I_n} q_i$. Indeed, $\mathcal{U}_G(\alpha_e, q_i)$ is obtained from $\Upsilon_G(\alpha_e, q_i)$ by truncating its expansion to those subsets $A \subset E$ that are spanning forests, i.e. that obey $0 = |A| - V + k(A)$. Since the number of connected components $k(A)$ is fixed by the global degree in the q_i's, the forests are obtained with $|A|$ minimal, so that the global degree in the α_e's must be maximal. Note that a truncation to the spanning forests may also be performed at the level of the multivariate Tutte polynomial by restricting, at fixed degree in q, to the terms of minimal degree in the β_e's. This yields an expansion over spanning forests [108] (see also [134]).

$$F_G(\beta_e, q) = \sum_{\substack{A\subset E \\ \text{spanning forest}}} \Big(\prod_{e\in A} \beta_e \Big) q^{k(A)}. \tag{64}$$

This, as well as the relation to the Symanzik polynomial, is conveniently summarized by the following diagram.

Proposition E.3.2 *The previous polynomials may be obtained from $\mathcal{W}(\alpha_e, q_I)$ by the following series of substitutions and truncations,*



(65)

where $\alpha_e = 1/\beta_e$.

Alternatively, the polynomial $\mathcal{W}_G(\alpha_e, q_I)$ can be seen as an extension of the polynomial $W_G(\xi_a, y)$ introduced by Noble and Welsh in [135].

Définition E.3.3 *For a graph with weights $\omega_v \in \mathbb{N}^*$ assigned to the vertices, the W polynomial is defined as*

$$W_G(\xi_a, y) = \sum_{A\subset E} (y-1)^{|A|-r(A)} \prod_{\substack{\mathcal{C}_1,\dots,\mathcal{C}_{k(A)} \\ \text{connected components of } (V,A)}} \xi_{a_n} \tag{66}$$

with $a_n = \sum_{v\in\mathcal{C}_n} \omega_v$ the sum of the weights of the vertices in the connected component $\mathcal{C}_n$.

This polynomial also obeys the contraction/deletion rule if we add the weights of the two vertices that are merged after the contraction of an edge. Alternatively, weights may be assigned to flags, with the convention that the weight of a vertex is the sum of the weights of the flags attached to it. Then, $W(\xi_a, y)$ is naturally extended to diagrams with flags and results from a simple substitution in $\mathcal{W}_G(\xi_a, y)$.

Proposition E.3.3 *For a graph with weights $\omega_i \in \mathbb{N}^*$ assigned to the flags,*

$$W_G(\xi_a, y) = (y-1)^{-|V|} \mathcal{W}_G\big(\beta_e = y-1, q_I = (y-1)\xi_{a_I}\big), \tag{67}$$

with $a_I = \sum_{i\in I} \omega_i$ the sum of the weights of the flags in I.

The polynomial $W_G(\xi_a, y)$ only encodes the sum of the weights of the flags in each connected component and erases information about their labels. In particular, if we weight each flag by $\omega_i = 1$, then the expansion of W only counts the number flags per component whereas that of $\mathcal{W}_G(\beta_e, q_I)$ keeps track of the associated set of labels. In a more sophisticated language, the latter may be considered as the simplest categorification of the former : integers, understood as finite sets up to isomorphisms, have been replaced by the category of finite sets.

E.3.7 Symanzik Polynomials through the tree matrix theorem in x-space

In this section we provide a sketch of a more standard proof of Theorem E.3.1 through the x space representation and the tree matrix theorem. The reason we include it here is for completeness and because we have not been able to find it in the existing literature, in which the same computation is usually performed through the Binet-Cauchy theorem.

The $V \times V$ matrix $Q_G(\alpha)$ analog in this case of (29) is defined as

$$[Q_G(\alpha)]_{v,v'} = \sum_e \epsilon_{ev} \frac{1}{\alpha_e} \epsilon_{ev'}. \tag{68}$$

It has vanishing sum over lines (or columns) :

$$\sum_{v'} [Q_G(\alpha)]_{v,v'} = \sum_{v'} \sum_e \epsilon_{ve} \frac{1}{\alpha_e} \epsilon_{ev'} = 0. \tag{69}$$

Therefore by the tree matrix theorem the determinant of the $(V-1) \times (V-1)$ matrix $Q_G(\alpha)$ defined as its principal minor with the line and column for the root vertex number V deleted is :

$$\Delta_G(\alpha) = \det[Q_G(\alpha)] = \sum_{\mathcal{T}} \prod_{e \in \mathcal{T}} \frac{1}{\alpha_e} \tag{70}$$

where the sum is over all trees of G. Since every tree of G has $V-1$ edges, Δ_G is clearly a homogenous polynomial in the α_e^{-1}. For $\alpha > 0$, Δ is positive. The remaining $(V-1)$ vectors z may then be integrated over and the result is

$$A_G(p) = \int_0^\infty \prod_l (d\alpha_e e^{-\alpha_e m^2}) \frac{\exp\{-p_v [Q_G^{-1}(\alpha)]_{v,v'} p_{v'}\}}{[\alpha_1 ... \alpha_E \Delta_G(\alpha)]^{d/2}}. \tag{71}$$

This formula expresses $A_G(p)$ as a function of the invariant scalar product of external momenta $p_v \cdot p_{v'}$. The denominator

$$U_G(\alpha) \equiv \alpha_1 ... \alpha_E \Delta_G(\alpha) = \sum_{\mathcal{T}} \prod_{e \notin \mathcal{T}} \alpha_e \tag{72}$$

is a homogenous polynomial of degree $V-1$. This gives an alternative proof of (24). The second Symanzik polynomial can also be obtained through this method and the corresponding computation is left to the reader. Of course harmonic regulators can also be included if one wants to avoid the noncanonical choice of a root, but the Pfaffian structure of the phase space representation is lost. Also this x-space method does not generalize easily to noncommutative field theory to which we now turn our attention.

E.4 Bollobás-Riordan Polynomials

E.4.1 Ribbon graphs

A ribbon graph $G = (V, E)$ is an orientable surface with boundary represented as the union of V closed disks, also called vertices, and E ribbons also called edges, such that :

– the disks and the ribbons intersect in disjoint line segments,
– each such line segment lies on the boundary of precisely one disk and one ribbon,
– every ribbon contains two such line segments.

So one can think of a ribbon graph as consisting of disks (vertices) attached to each other by thin stripes (edges) glued to their boundaries (see Figures E.5-E.6). For any such ribbon graph G there is an underlying ordinary graph $\bar{G}$ obtained by collapsing the disks to points and the ribbons to edges.

Two ribbon graphs are isomorphic if there is a homeomorphism from one to the other mapping vertices to vertices and edges to edges. A ribbon graph is a graph with a fixed cyclic ordering of the incident half-edges at each of its vertices.

A face of a ribbon graph is a connected component of its boundary as a surface. If we glue a disk along the boundary of each face we obtain a closed Riemann surface whose genus is also called the genus of the graph. The ribbon graph is called planar if that Riemann surface has genus zero.

Generalized ribbon graphs that can also incorporate Moebius strips and correspond to nonorientable surface can be defined but will not be considered in this paper.

There is a duality on ribbon graphs which preserves the genus but exchanges faces and vertices, keeping the number of edges fixed. It simply considers the disks glued along faces as the vertices of a dual graph and changes the ends of each ribbon into borders of the dual ribbon.

Extended categories of ribbon graphs with flags can be defined. Flags can be represented as ribbons bordered by dotted lines to distinguish them from ordinary edges (see Figures E.5-E.6). Beware that the cyclic ordering of flags and half-edges at each vertex is very important and must be respected under isomorphisms. The genus of an extended graph is defined as the genus of the graph obtained by removing the flags and closing the corresponding segments on their vertices. The number of broken faces is the number of faces which do contain at least one flag. It is an important notion in noncommutative field theory.

We define for any ribbon graph
– $V(G)$ as the number of vertices ;
– $E(G)$, the number of edges,
– $k(G)$, the number of connected components,
– $r(G) = V(G) - k(G)$, the rank of G,
– $n(G) = E(G) - r(G)$, the nullity of G,
– $bc(G) = F(G)$, the number of components of the boundary of G [3],
– $g(G) = k - (V - E + bc)/2$ is the genus of the graph,
– $f(G)$ the number of flags of the graph.

A graph with a single vertex hence with $V = 1$ is called a *rosette*.

A subgraph H of a ribbon graph G is a subset of the edges of G.

The Bollobás-Riordan polynomial, which is a generalization of Tutte polynomial, is a algebraic polynomial that is used to incorporate new topological information specific to ribbon graphs, such as the genus and the number of "broken" or "external" faces. It is a polynomial invariant of the ribbon graph.

3. This is the number of *faces* of G when G is connected.

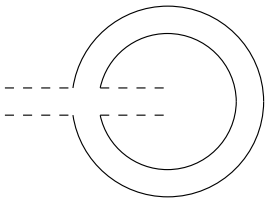


FIGURE E.5: A planar ribbon graph with $V = E = 1$. $bc = 2$ and two flags.

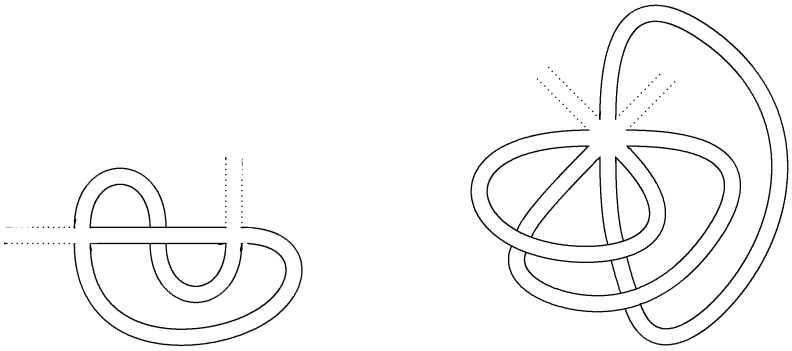


FIGURE E.6: A non-planar ribbon graph without flags, with $V = 2$, $E = 3$, $bc = 1$, $g = 1$, $f = 2$, and its dual graph with $V = 1$, $E = 3$, $bc = 2$, $g = 1$, $f = 2$.

E.4.2 Bollobás-Riordan Polynomial

Définition E.4.1 (Global definition) *The Bollobás-Riordan polynomial is defined by :*

$$R_G = R_G(x, y, z) = \sum_{H \subset G} (x-1)^{r(G)-r(H)} y^{n(H)} z^{k(H)-bc(H)+n(H)}. \tag{1}$$

The relation to the Tutte polynomial for the underlying graph $\bar{G}$ is $R_G(x-1, y-1, 1) = T_{\bar{G}}(x, y)$. Remark also that if G is planar we have $R_G(x-1, y-1, z) = T_{\bar{G}}(x, y)$.

When H is a spanning graph of G, we have $k(H) - k(G) = r(G) - r(H)$. So we can rewrite the R polynomial as :

$$R_G = (x-1)^{-k(G)} \sum_{H \subset G} M(H), \tag{2}$$

where

$$M(H) = (x-1)^{k(H)} y^{n(H)} z^{k(H)-bc(H)+n(H)} \tag{3}$$

so that $M(H)$ depends only on H but not on G.

E.4.3 Deletion/contraction

The deletion and contraction of edges in a ribbon graph are defined quite naturally : the deletion removes the edge and closes the two scars at its end ; the contraction of a semi-regular edge creates a new disk out of the two disks at both ends of the ribbon with a new boundary which is the union of the boundaries of the two disks and of the ribbon

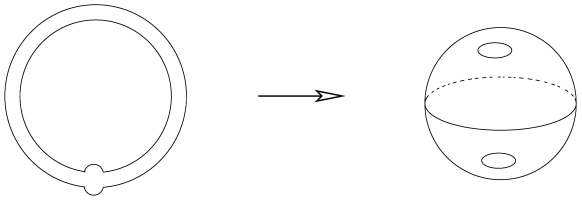


FIGURE E.7: Contraction of the single self-loop G_1.

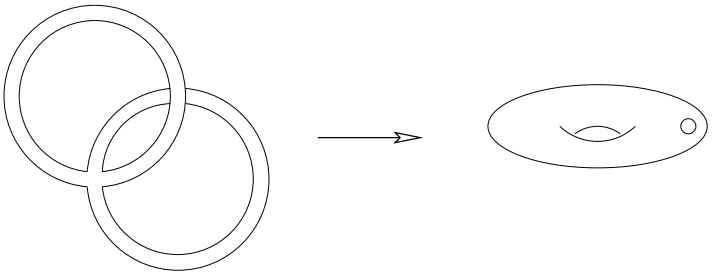


FIGURE E.8: Contraction of the two self loops non-planar G_2.

(see Figure E.10). An interesting property is that deletion and contraction of edges are exchanged in the dual graph.

The deletion of a self-loop is standard. However the natural contraction of a self-loop creates a surface with a new border. Iterating, we may get surfaces of arbitrary genus with an arbitrary number of disks removed, a category also called disk-punctured surfaces. The ribbons can now join any puncture to any other. For instance the contraction of the self-loop on the graph G_1 of Figure E.7 leads to a cylinder ie to a single vertex which is a sphere with two disks removed. The contraction of the two self-loops in graph G_2 of Figure E.8 corresponds to the cylinder with a ribbon gluing the two ends, hence to a torus.

Deletion and contraction defined in this extended category of graphs can be iterated until the graph has no longer any edge, ie is a collection of disk-punctured Riemann surfaces. These punctured Riemann surfaces are very natural objects both in the context of string theory and in NCQFT. However we do not consider them in this paper.

In this paper we remain in the category of ordinary ribbon graphs with disk-like vertices. The contraction/deletion of semi-regular edges leads to rosettes as terminal forms. To treat them we introduce the notion of *double contraction* on *nice crossings*. Nice crossings were introduced in [68] :

Définition E.4.2 *A nice crossing pair of edges in a rosette is a pair of crossing edges e_1 and e_2 which are adjacent on the cycle of the rosette. Adjacency means that one end of e_1 is consecutive with an end of e_2 (see Figure E.9).*

It is proved in [68] that any rosette $\mathcal{R}$ of genus $g > 0$ contains at least one nice crossing.

The double contraction of such a nice crossing pair consists in deleting e_1 and e_2 and interchanging the half-edges encompassed by e_1 with the ones encompassed by e_2, see Figure E.9. This *double contraction* was defined in [68] under the name of "3rd Filk move". It decreases the genus by one and the number of edges by 2.

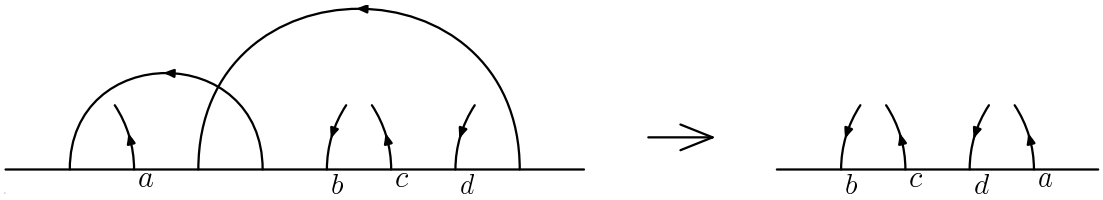


FIGURE E.9: When deleting the two edges of a nice pair crossing on some contracted vertex, one also needs to interchange the half-edges encompassed by the first edges with those encompassed by the second one. Beware that the horizontal line in this picture is a part of the rosette cycle.

In the next section iterating this double contraction until we reach planarity allows us to compute the $U^\star$ Symanzik polynomial by remaining in the category of ordinary ribbon graphs.

Théorème E.4.1 (Bollobás-Riordan polynomial, contraction/deletion)

$$R_G = R_{G/e} + R_{G-e} \tag{4}$$

for every ribbon graph G and any regular edge e of G and

$$R_G = xR_{G/e} \tag{5}$$

for every bridge of G.

Therefore the R polynomial satisfy contraction-deletion relations as the Tutte polynomial. However to complete its definition we also need to define the R polynomial for single vertex graphs, namely the rosettes, which can be read off from (3). For such a rosette $\mathcal{R}$, $k(\mathcal{R}) = V(\mathcal{R}) = k(H) = V(H) = 1$, so that the R polynomial does not depend on x and

$$R_{\mathcal{R}}(y,z) = \sum_{H\subset\mathcal{R}} y^{E(H)} z^{2g(H)}. \tag{6}$$

For $z = 1$ we recover $R_{\mathcal{R}}(y-1,1) = y^{E(\mathcal{R})}$.

E.4.4 The multivariate Bollobás-Riordan polynomial

Like in the case of Tutte polynomial, we can generalize the Bollobás-Riordan polynomial to a multivariate case. As before,we associate to each edge e a variable β_e.

Définition E.4.3 *The multivariate Bollobás-Riordan polynomial of a ribbon graph analog of the multivariate polynomial (4) is :*

$$Z_G(x,\{\beta_e\},z) = \sum_{H\subset G} x^{k(H)} \Big(\prod_{e\in H} \beta_e\Big)\, z^{bc(H)}. \tag{7}$$

It obeys again a deletion/contraction relation similar to Theorem (E.4.1) for any semi-regular edge.

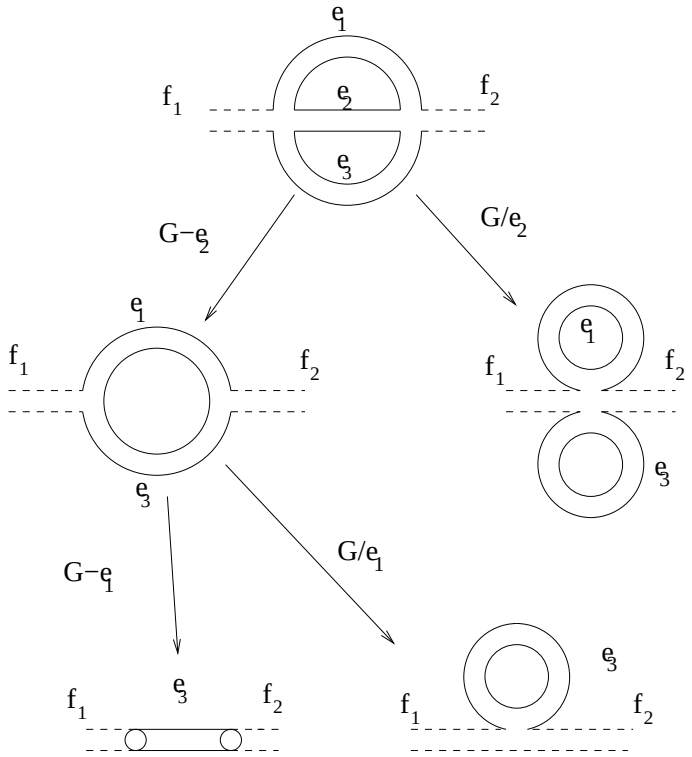


FIGURE E.10: The contraction-deletion for a ribbon graph.

E.5 Translation-invariant NCQFT

E.5.1 Motivation

Noncommutative quantum field theory, hereafter called NCQFT, has a long story. Schrödinger, Heisenberg [144] and Yang [145] tried to extend the noncommutativity of phase space to ordinary space. Building on their ideas Snyder [146] formulated quantum field theory on such noncommutative space in the hope that it might behave better than ordinary QFT in the ultraviolet regime.

Right from the start another motivation to study noncommutative quantum field theory came from the study of particles in strong magnetic fields. It was early recognized that non zero commutators occur for the coordinates of the centers of motion of such quantum particles, so that noncommutative geometry of the Moyal type should be the proper setting for many body quantum physics in strong external field. This includes in condensed matter the quantum Hall effect (see the contribution of Polychronakos in [147]), or other strong field situations.

An other motivation comes from particle physics. After initial work by Dubois-Violette, Kerner and Madore, Connes, Lott, Chamseddine and others have forcefully advocated that the *classical* Lagrangian of the current standard model arises naturally on a simple noncommutative geometry. For a review see Alain Connes's contribution in [147] and references therein.

Still an other motivation came from the search of new regularizations of non-Abelian gauge theories that may throw light on their difficult mathematical structure. After

't Hooft proposed the large N limit of matrix theory, in which planar graphs dominate, as relevant to the subject [69], the Eguchi-Kawai model was an important attempt for an explicit solution. These ideas have been revived in connection with the ultraviolet behavior of NCQFT on the Moyal-Weyl geometry, which also leads to the domination of planar graphs. Seiberg and Witten proposed in [148] a mapping between ordinary and noncommutative gauge fields which does not preserve the gauge groups but preserve the gauge equivalent classes.

The interest for non commutative geometry also stems from string theory. Open string field theory may be recast as a problem of noncommutative multiplication of string states [149]. It was realized in the late 90's that NCQFT is an effective theory of strings [60]. Roughly this is because in addition to the symmetric tensor $g_{\mu\nu}$ the spectrum of the closed string also contains an antisymmetric tensor $B_{\mu\nu}$. There is no reason for this antisymmetric tensor not to freeze at some lower scale into a classical field, inducing an effective non commutative geometry of the Moyal type. There might therefore be some intermediate regime between QFT and string theory where NCQFT is the relevant formalism. The ribbon graphs of NCQFT may be interpreted either as "thicker particle world-lines" or as "simplified open strings world-sheets" in which only the ends of strings appear but not yet their internal oscillations.

E.5.2 Scalar models on the Moyal space

The noncommutative Moyal space is defined in even dimension d by

$$[x^\mu, x^\nu]_\star = \imath\Theta^{\mu\nu}, \tag{1}$$

where Θ is an antisymmetric $d/2$ by $d/2$ block-diagonal matrix with blocks :

$$\begin{pmatrix} 0 & \theta \\ -\theta & 0 \end{pmatrix} \tag{2}$$

and we have denoted by $\star$ the Moyal-Weyl product

$$(f \star g)(x) = \int \frac{d^4k}{(2\pi)^4} d^4y \, f(x + \tfrac{1}{2}\Theta \cdot k) g(x+y) e^{\imath k \cdot y}. \tag{3}$$

Note that in the limit $\theta \to 0$ this product becomes the ordinary commutative product of functions.

The "naive" model

The simplest field theory on this space consists in replacing the ordinary commutative local product of fields by the Moyal-Weyl product

$$S[\phi] = \int d^dx (\frac{1}{2}\partial_\mu\phi \star \partial^\mu\phi + \frac{1}{2}\mu^2\phi\star\phi + \frac{\lambda}{4}\phi\star\phi\star\phi\star\phi). \tag{4}$$

In momentum space the action (4) writes

$$S[\phi] = \int d^dp (\frac{1}{2}p_\mu\phi p^\mu\phi + \frac{1}{2}\mu^2\phi\phi + V(\phi,\theta)). \tag{5}$$

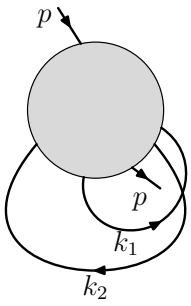


FIGURE E.11: An example of a rosette with two flags. The crossings of edges k_1 and k_2 indicate the non trivial genus (here $g = 1$).

where $V(\phi, \theta)$ is the corresponding potential.

An important consequence of the use of the non-local product $\star$ is that the interaction part no longer preserves the invariance under permutation of external fields. This invariance is restricted to cyclic permutations. Furthermore, there exists a basis - the matrix base - of the Moyal algebra where the Moyal-Weyl product takes the form of an ordinary (infinite) matrix product. For these reasons the associated Feynman graphs are ribbon graphs, that is propagators should be drawn as ribbons.

In [150] several contractions on such a Feynman graph were defined. In particular the "first Filk move" is the contraction introduced in subsection E.4.3. Repeating this operation for the $V - 1$ edges of a spanning tree, one obtains a *rosette* (see Figure E.11).

Note that the number of faces or the genus of the graph does not change under contraction. There is no crossing between edges for a planar rosette. The example of Figure E.11 corresponds thus to a non-planar graph (one has crossings between the edges k_1 and k_2). This pair is called a *nice crossing* pair.

The notions expressed in the previous section (namely the Green and Schwinger functions or the perturbation theory concepts) remain the same as in QFT. Usual Feynman graphs are simply replaced by ribbon Feynman graphs.

Recall that this "naive model" (4) is not renormalizable in $d = 4$. This is due to a new type of non-local divergence at the level of the $2-$point function - the UV/IR mixing [123].

A translation-invariant renormalizable scalar model

In order to restore renormalizability at $d = 4$, the propagator can be modified in the following way [151]

$$S_{GMRT}[\phi] = \int d^d p \, (\frac{1}{2} p_\mu \phi p^\mu \phi + \frac{1}{2} m^2 \phi\phi + \frac{1}{2} a \frac{1}{\theta^2 p^2} \phi\phi + \frac{\lambda}{4} \phi \star \phi \star \phi \star \phi), \tag{6}$$

where a is some dimensionless parameter which is taken in the interval $0 < a \leq \frac{1}{4}\theta^2 m^4$.

The corresponding propagator writes in momentum space

$$C_{GMRT} = \frac{1}{p^2 + \mu^2 + \frac{a}{\theta^2 p^2}} \, . \tag{7}$$

In [151], this model was proved to be renormalizable at any order in perturbation theory. Furthermore, its renormalization group flows [152] were calculated ; a mechanism for tak-

ing the commutative limit has been proposed [153] (for a review on all these developments, see [154]).

E.5.3 The NC Parametric representation

In this subsection we present the implementation of the parametric representations for the noncommutative scalar models introduced in the previous subsection.

To keep track of the cyclic ordering at the vertex it is convenient to detail the incidence matrix ε_{ev} into a more precise incidence tensor ε^v_{ei} where $i = 1, ..., 4$ indexes the four corners of the Moyal vertex. As before it is 1 if the edge e starts at corner i of vertex v, -1 if it exits at that corner, and 0 otherwise.

To implement the parametric representation we follow subsection E.3.3. The propagator remains the same as in QFT, but the contribution of a vertex v now corresponds to a Moyal kernel. In momentum space it writes using again summation over repeated indices

$$\delta(\sum_{i=1}^{4} \varepsilon^v_{ei} k_e) e^{-\frac{i}{2}\sum_{1\leq i<j\leq 4} \varepsilon^v_{ei} k_e \Theta \varepsilon^v_{ej} k_e}. \tag{8}$$

By $k_i \Theta k_j$ we denote $k_i^\mu \Theta_{\mu\nu} k_j^\nu$. The $\delta-$function appearing in the vertex contribution (8) is nothing but the usual momentum conservation. It can be written as an integral over a new variable $\tilde{x}_v$, called *hyperposition*. One associates such a variable to any Moyal vertex, even though this vertex is non-local :

$$\delta(\sum_{i=1}^{4} \varepsilon^v_{ei} k_e) = \int \frac{d\tilde{x}'_v}{(2\pi)^4} e^{i\tilde{x}'_v(\sum_{i=1}^4 \varepsilon^v_{ei} k_e)} = \int \frac{d\tilde{x}_v}{(2\pi)^4} e^{\tilde{x}_v \sigma(\sum_{i=1}^4 \varepsilon^v_{ei} k_e)}. \tag{9}$$

where σ is a $d/2$ by $d/2$ block-diagonal matrix with blocks :

$$\sigma = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}. \tag{10}$$

Note that to pass from the first to the second line in (9), the change of variables $i\tilde{x}'_v = \tilde{x}_v \sigma$ has Jacobian 1.

The "naive" model

Putting now together the contributions of all the internal momenta and vertices, one has the following parametric representation :

$$\mathcal{A}^T_G(p_1, \cdots, p_N) = K^T_G \int \prod_{e,e'=1}^{E} d^d k_e d\alpha_e e^{-\alpha_e(k_e^2+m^2)} \tag{11}$$

$$\prod_{v=1}^{V-1} \int d^d \tilde{x}_v e^{i\tilde{x}_v(\sum_{i=1}^4 \varepsilon^v_{ei} k_e)} e^{-\frac{i}{2}\sum_{i<j} \varepsilon^v_{ei} k_e \Theta \varepsilon^v_{e'j} k_{e'}} \tag{12}$$

where we have denoted by K^T_G some inessential normalization constant. Furhermore note that in the integrand above we have denoted, to simplify the notations, by k_e or $k_{e'}$ momenta which can be both internal or external.

The translation-invariant model

The parametric representation of the model (6) was analyzed in [155]. This representation is intimately connected to the one of the model (4) (see the previous subsubsection) for the following reason. One can rewrite the propagator (7) as

$$\frac{1}{A+B} = \frac{1}{A} - \frac{1}{A}B\frac{1}{A+B} \tag{13}$$

for

$$A = p^2 + m^2, \quad B = \frac{a}{\theta^2 p^2}. \tag{14}$$

Thus, the propagator (7) writes

$$\begin{aligned} C_{GMRT} &= \frac{1}{p^2+m^2} - \frac{1}{p^2+m^2}\frac{a}{\theta^2 p^2(p^2+m^2)+a}, \\ &= \frac{1}{p^2+m^2} - \frac{1}{p^2+m^2}\frac{a}{\theta^2(p^2+m_1^2)(p^2+m_2^2)} \end{aligned} \tag{15}$$

where $-m_1^2$ and $-m_2^2$ are the roots of the denominator of the second term in the LHS (considered as a second order equation in p^2, namely $\frac{-\theta^2 m^2 \pm \sqrt{\theta^4 m^4 - 4\theta^2 a}}{2\theta^2} < 0$. Note that the form (15) allows us already to write an integral representation of the propagator $C(p, m, \theta)$. Nevertheless, for the second term one would need a triple integration over some set of Schwinger parameters :

$$\begin{aligned} C_{GMRT} &= \int_0^\infty d\alpha e^{-\alpha(p^2+m^2)}, \\ &- \frac{a}{\theta^2}\int_0^\infty \int_0^\infty d\alpha d\alpha^{(1)} d\alpha^{(2)} e^{-(\alpha+\alpha^{(1)}+\alpha^{(2)})p^2} e^{-\alpha m^2} e^{-\alpha^{(1)} m_1^2} e^{-\alpha^{(2)} m_2^2}. \end{aligned} \tag{16}$$

Instead of that one can use the following formula :

$$\frac{1}{p^2+m_1^2}\frac{1}{p^2+m_2^2} = \frac{1}{m_2^2-m_1^2}\left(\frac{1}{p^2+m_1^2} - \frac{1}{p^2+m_2^2}\right). \tag{17}$$

This allows to write the propagator (15) as

$$C_{GMRT} = \frac{1}{p^2+m^2} - \frac{a}{\theta^2(m_2^2-m_1^2)}\frac{1}{p^2+m^2}\left(\frac{1}{p^2+m_1^2} - \frac{1}{p^2+m_2^2}\right). \tag{18}$$

This form finally allows to write down the following integral representation :

$$\begin{aligned} C_{GMRT} &= \int_0^\infty d\alpha e^{-\alpha(p^2+m^2)} - \frac{a}{\theta^2(m_2^2-m_1^2)}\int_0^\infty \int_0^\infty d\alpha d\alpha_1 e^{-(\alpha+\alpha_1)p^2 - \alpha m^2 - \alpha_1 m_1^2} \\ &+ \frac{a}{\theta^2(m_2^2-m_1^2)}\int_0^\infty \int_0^\infty d\alpha d\alpha_2 e^{-(\alpha+\alpha_2)p^2} e^{-\alpha m^2} e^{-\alpha_2 m_2^2}. \end{aligned} \tag{19}$$

Let us also remark that the noncommutative propagator C_{GMRT} is bounded by the "usual" commutative propagator $C(p, m)$

$$C_{GMRT} \leq C(p, m). \tag{20}$$

Using now (16), the parametric representation of the model (6) is thus a sum of 2^E terms coming from the development of the E internal propagators. Each of these terms has the same form of the one of polynomials in the previous subsection. The only differences comes from

– the proper substitution of the set of Schwinger α parameters
– the mass part.

One has

$$\mathcal{A}_G^T = K_G^T \left(\int \prod_{i=1}^{L} d\alpha_i \frac{1}{[U(\alpha)]^{\frac{D}{2}}} e^{\frac{-V(\alpha,p)}{U(\alpha)}} e^{-\sum_{i=1}^{L} \alpha_i m^2} \right. \tag{21}$$

$$+(-\frac{a}{\theta^2})^{L-1} \sum_{j_1=1}^{L} \int d\alpha_{j_1} \prod_{i\neq j_1,\, i=1}^{L} d\alpha_i d\alpha_i^{(1)} d\alpha_i^{(2)} \frac{1}{[U(\alpha_i+\alpha_i^{(1)}+\alpha_i^{(2)}, \alpha_{j_1})]^{\frac{d}{2}}}$$

$$e^{-\frac{V(\alpha_i+\alpha_i^{(1)}+\alpha_i^{(2)}, \alpha_{j_1}, p)}{U(\alpha_i+\alpha_i^{(1)}+\alpha_i^{(2)}, \alpha_{j_1})}} e^{-\sum_{i=1}^{L} \alpha_i m^2} e^{-\sum_{i\neq j_1,\, i=1}^{L} \alpha_i^{(1)} m_1^2} e^{-\sum_{i\neq j_1,\, i=1}^{L} \alpha_i^{(2)} m_2^2}$$

$$+(-\frac{a}{\theta^2})^{L-2} \sum_{j_1<j_2,\, j_1, j_2=1}^{L} \int d\alpha_{j_1} d\alpha_{j_2} \prod_{i\neq j_1, j_2,\, i=1}^{L} d\alpha_i d\alpha_i^{(1)} d\alpha_i^{(2)}$$

$$\frac{1}{[U(\alpha_i+\alpha_i^{(1)}+\alpha_i^{(2)}, \alpha_{j_1}, \alpha_{j_2})]^{\frac{d}{2}}}$$

$$e^{-\frac{V(\alpha_i+\alpha_i^{(1)}+\alpha_i^{(2)}, \alpha_{j_1}, \alpha_{j_2}, p)}{U(\alpha_i+\alpha_i^{(1)}+\alpha_i^{(2)}, \alpha_{j_1}, \alpha_{j_2})}} e^{-\sum_{i=1}^{L} \alpha_i m^2} e^{-\sum_{i\neq j_1 j_2,\, i=1}^{L} \alpha_i^{(1)} m_1^2} e^{-\sum_{i\neq j_1, j_2\, i=1}^{L} \alpha_i^{(2)} m_2^2}$$

$$+ \ldots +$$

$$+(-\frac{a}{\theta^2})^{L} \int \prod_{i=1}^{L} d\alpha_i d\alpha_i^{(1)} d\alpha_i^{(2)} \frac{1}{[U(\alpha_i+\alpha_i^{(1)}+\alpha_i^{(2)})]^{\frac{d}{2}}} e^{-\frac{V(\alpha_i+\alpha_i^{(1)}+\alpha_i^{(2)}, p)}{U(\alpha_i+\alpha_i^{(1)}+\alpha_i^{(2)})}}$$

$$\left. e^{-\sum_{i=1}^{L} \alpha_i m^2} e^{-\sum_{i=1}^{L} \alpha_i^{(1)} m_1^2} e^{-\sum_{i=1}^{L} \alpha_i^{(2)} m_2^2} \right) .$$

E.5.4 Deletion/contraction for the NC Symanzik polynomials

In this subsection we give some results relating the Bollobás-Riordan polynomial and the parametric representations of the noncommutative scalar models introduced here.

The "naive" model

As in the commutative case, we have to perform a Gaußian integral in a $d(E+V-1)$ dimensional space. Consider a ribbon graph G with a root $\bar{v}$.

We introduce the condensed notations analog to (27)-(29)

$$A_G(p) = \int \prod_e d\alpha_e e^{-\alpha_e m^2} \int d^d\tilde{x} d^d p e^{-YXY^t} \tag{22}$$

where

$$Y = \begin{pmatrix} k_e & \tilde{x}_v & p_e & \tilde{x}_{\bar{v}} \end{pmatrix} \quad , \quad X = \begin{pmatrix} Q & -iR^t \\ -iR & M \end{pmatrix} . \tag{23}$$

Q is an $d(E+V-1)$-dimensional square matrix. We have denoted by p_e the external momenta and by $\tilde{x}_{\bar{v}}$ the hyperposition associated to the root vertex $\bar{v}$. The matrix R is a $d(N+1)\times d(E+V-1)$ dimensional matrix and M is a $d(N+1)$ dimensional square matrix representing the Moyal couplings between the external momenta and the root vertex.

Gaußian integration gives, up to inessential constants :

$$A_G(p) = \int \prod_e d\alpha_e e^{-\alpha_e m^2} \frac{1}{\det Q^{d/2}} e^{-PRQ^{-1}R^tP^t} \tag{24}$$

where P is a line matrix regrouping the external momenta (and the hyperposition associated to the root vertex).

The determinant of the matrix Q defines therefore the first Symanzik NC-polynomial $U^\star$ and the product of the matrices R and inverse of Q defines the quotient of the second Symanzik polynomial $V^\star$ by $U^\star$ where the star recalls the Moyal product used to define this NCQFT.

Let us calculate first the determinant of Q. One has

$$Q = D \otimes 1_d + A \otimes \sigma \tag{25}$$

where D is a diagonal matrix with coefficients $D_{ee} = \alpha_e$, for $e = 1, \ldots, E$ and $D_{vv} = 0$ for the rest, $v = 1, \ldots, V-1$. A is an antisymmetric matrix. In [68] it was noted that, for such a matrix

$$\det Q = \det(D + A)^d. \tag{26}$$

This implies, as in the commutative case, that

$$U^\star = \det(D + A). \tag{27}$$

Factoring out powers of i one has

$$\det(D+A) = \det \begin{pmatrix} \alpha_1 & f_{12} & & -\sum_{i=1}^4 \epsilon_{ei}^v \\ -f_{12} & \alpha_2 & & \\ & & \ldots & \\ & & & \ldots \\ \sum_{i=1}^4 \epsilon_{ei}^v & & & 0 \end{pmatrix}. \tag{28}$$

The difference with the commutative case comes from the non-trivial antisymmetric coupling between the E edges variables. It corresponds to an E dimensional square matrix F with matrix elements

$$f_{ee'} = -\frac{\theta}{2} \sum_{v=1}^{n} \sum_{i,j=1}^{4} \omega(i,j) \varepsilon_{ei}^v \varepsilon_{e'j}^v, \ \forall e < e', \ e, e' = 1, \ldots, E \tag{29}$$

where ω is an antisymmetric matrix such that $\omega(i,j) = 1$ if $i < j$. This matrix takes into account the antisymmetric character of Θ in $k_\mu \Theta^{\mu\nu} p_\nu$.

Using again Lemma E.2.1

$$\begin{aligned} \det(D+A) &= \int \prod_{i,e} d\omega_i d\chi_i d\omega_e d\chi_e \\ &\quad e^{-\sum_e \alpha_e \chi_e \omega_e} e^{-\sum_{e,v} \chi_e \epsilon_{ev} \chi_v + \chi \leftrightarrow \omega} e^{-\sum_{e,e'} \chi_e f_{ee'} \chi_{e'} + \chi \leftrightarrow \omega}. \end{aligned} \tag{30}$$

Note that the last term above represents the difference with the commutative case.

We have the exact analog of Theorem E.3.3 to prove a deletion-contraction rule.

Théorème E.5.1 *For any semi-regular edge e*

$$\det(D+A)_G = \alpha_e \det(D+A)_{G-e} + \det(D+A)_{G/e}. \tag{31}$$

Proof We pick up a semi-regular edge e entering v_1 and exiting v_2. Thus it exists some i and j with $\epsilon_{ei}^{v_1} = +1, \epsilon_{ej}^{v_2} = -1$. We expand

$$e^{-\alpha_e \chi_e \omega_e} = 1 + \alpha_e \omega_e \chi_e. \tag{32}$$

leading to two contributions, which we denote respectively by $\det Q_{G,e,1}$ and $\det Q_{G,e,2}$. For the first term, since one must saturate the χ_e and ω_e integrations, one has to keep the $\chi_e(\chi_{v_1} - \chi_{v_2} + \sum_{\tilde{e}} f_{e\tilde{e}}\chi_{\tilde{e}})$ term and the similar ω term. Note that the sum is done on all the edges $\tilde{e}$ hooking to any of the vertices v_1 and v_2 and with whom the edge e has no trivial Moyal oscillation factor. One has

$$\begin{aligned} \det Q_{G,e,1} \;=\; & \int \prod_{e' \neq e, v} d\chi_{e'} d\chi_v d\omega_{e'} d\omega_v \\ & (\chi_{v_1} - \chi_{v_2} + \sum_{\tilde{e}} f_{e\tilde{e}}\chi_{\tilde{e}})(\omega_{v_1} - \omega_{v_2} + \sum_{\tilde{e}} f_{e\tilde{e}}\omega_{\tilde{e}}) \\ & e^{-\sum_{e' \neq e} \alpha'_e \chi_{e'} \omega_{e'}} e^{-\frac{1}{4}\sum_{e' \neq e, v} \chi_{e'} \epsilon_{e'v} \chi_v + \chi \leftrightarrow \omega}. \end{aligned} \tag{33}$$

As in the commutative case, we now perform the trivial triangular change of variables with unit Jacobian :

$$\hat{\chi}_{v_1} = \chi_{v_1} - \chi_{v_2} + \sum_{\tilde{e}} f_{e\tilde{e}}\chi_{\tilde{e}}, \quad \hat{\chi}_v = \chi_v \;\; \textit{for} \;\; v \neq v_1, \tag{34}$$

and the same change for the ω variables. What happens now is analogous to the commutative case, with the difference that the last term in the definition of $\hat{\chi}_{v_1}$ will lead to the reconstruction of the Moyal oscillation factors of the edges hooking to v_1 with the edges hooking to v_2. This completes the ribbon contraction, thus $\det Q_{G,e,1} = \det Q_{G/e}$. The second term $\det Q_{G,e,2}$ with the $\alpha_e \omega_e \chi_e$ factor is even easier. We must simply put to 0 all terms involving the e label, hence trivially $\det Q_{G,e,2} = \alpha_e \det Q_{G-e}$.

We need now to compute $U^\star$ on terminal forms after contracting/deleting all semi-regular edges, that is compute $U^\star$ on a rosette graph $\mathcal{R}$. This is done by using the double contraction introduced in the previous section.

Consider a nice crossing of $\mathcal{R}$ between two edges e_1 and e_2 with parameters α_1 and α_2. It leads to a contribution

$$U^\star_{\mathcal{R}} = (\alpha_1\alpha_2 + \frac{1}{4}\theta^2) U_{\mathcal{R}/e_1e_2} \tag{35}$$

where we recall that the contracted rosette $\mathcal{R}/e_1e_2$ is obtained by deleting e_1 and e_2 from $\mathcal{R}$ and interchanging the half-edges encompassed by e_1 with the ones encompassed by e_2, see Figure E.9. The procedure can be iterated on $\mathcal{R}/e_1e_2$ until after $g(\mathcal{R})$ double contractions a planar rosette with $2E(\mathcal{R}) - 2g(\mathcal{R})$ is reached, for which $F = 0$ and for which the terminal form is $\prod_e \alpha_e$ as in the commutative case.

Remark that the main difference with the commutative case is the inclusion of the θ^2 term in the terminal form evaluation (35). This type of *genus-term* has no analog in the commutative case.

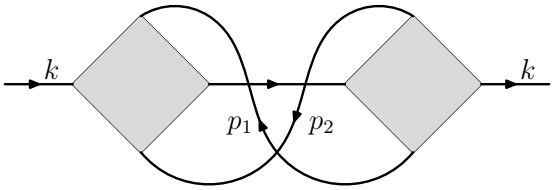


FIGURE E.12: An example of a non-planar graph, $g = 1$.

Exemple E.5.1 *Consider the graph of Figure E.12. Its first Symanzik polynomial is [155]*

$$\alpha_1\alpha_2 + \alpha_1\alpha_3 + \alpha_2\alpha_3 + \frac{1}{4}\theta^2. \tag{36}$$

Choosing α_3 as a regular edge leads to a contracted graph where the pair of edges α_1 and α_2 realizes a nice crossing. We thus have a contribution to the first polynomial

$$\alpha_1\alpha_2 + \frac{1}{4}\theta^2. \tag{37}$$

The deleted part then follows as in the commutative case leading to a contribution

$$\alpha_3\alpha_1 + \alpha_3\alpha_2. \tag{38}$$

Putting together (37) *and* (38) *leads to the expected result* (36).

Let us now give the following definition :

Définition E.5.1 *A $\star$-tree of a connected graph G is a subset of edges with one boundary.*

This definition allows to write a $\star$-tree in some graph of genus g as an ordinary tree plus at most g pairs of "genus edges" (where by "genus edges" we understand pairs of edges which make a recursive succession of nice crossings under double contractions on the rosette obtained after contracting the edges of the tree in the graph).

Exemple E.5.2 *For the graph of Figure E.12, the $\star$-trees are the ordinary trees $\{1\}$, $\{2\}$, $\{3\}$ and the tree plus one pair of genus edges, namely $\{1, 2, 3\}$ which is the whole graph.*

In [123], the following general expression for the first polynomial U of the "naive" noncommutative model was given

$$U^\star(\alpha_1, \ldots, \alpha_E) = \left(\frac{\theta}{2}\right)^b \sum_{\mathcal{T}^\star \ \star-tree} \prod_{e \notin \mathcal{T}^\star} 2\frac{\alpha_e}{\theta}, \tag{39}$$

where we have denoted by

$$b = F - 1 + 2g \tag{40}$$

the number of loops of G. Note that the factor 2 above is the one which matches our conventions.

Let us now give a proof of the formula (39). Consider the following lemma :

Lemme E.5.1 (Lemma III.2 of [68]) *Let $D = (d_i\delta_{ij})_{i,j\in\{1,\dots,D\}}$ be diagonal and $A = (a_{ij})_{i,j\in\{1,\dots,D\}}$ be such that $a_{ii} = 0$. Then*

$$\det(D + A) = \sum_{K\subset\{1,\dots,N\}} \det(B_{\hat{K}}) \prod_{i\in K} a_i \tag{41}$$

where $A_{\hat{K}}$ is the matrix obtained from A by deleting the lines and columns with indices in K.

The particular form (25) of the matrix Q allows thus to use this Lemma to calculate its determinant (*i.e.* the polynomial U). Factoring out $\frac{\theta}{2}$ on the first E lines and then $\frac{2}{\theta}$ on the last $V-1$ columns, one has

$$U^\star(\alpha) = \left(\frac{\theta}{2}\right)^b \sum_{K\subset\{1,\dots,E\}} \det A_{\hat{K}} \prod_{e\in K} 2\frac{\alpha_e}{\theta} \tag{42}$$

where we have used that

$$b - E = -(V-1).$$

Note that the set K on which one sums up corresponds to a set of edges of the graph ; this comes from the fact that the last $V-1$ entries on the diagonal of the matrix A are equal to 0. In [68] (see Lemma III.4) it is proven, using a non trivial triangular change of Grassmanian variables that a determinant of type $A_{\hat{K}}$ is not vanishing if and only if it corresponds to a graph with only one face. This means that the complement of the subset of edges K must be a $\star-$tree, $\bar{K} = \mathcal{T}^*$. Furthermore, one has

$$\prod_{\bar{\mathcal{T}}^\star} \alpha_e = \prod_{e\notin\mathcal{T}^\star} \alpha_e. \tag{43}$$

The translation-invariant model

The relation of the parametric representation of the model (6) to the Bollobás-Riordan polynomials follows the one of the "naive" model (4) presented above. This is an immediate consequence of the intimate relationship between the parametric representation of these two noncommutative models, a relationship explained in the previous subsection.

E.5.5 The second polynomial for NCQFT

In this section we prove the form of the second polynomial for the model (4) (both its real and imaginary part, as we will see in the sequel). We then relate this second polynomial to the Bollobás-Riordan polynomial.

From (24) it follows directly that

$$\frac{V^\star(\alpha, p)}{U^\star(\alpha)} = -PRQ^{-1}R^tP^t \tag{44}$$

where we have left aside the matrix M coupling the external momenta to themselves. Note that the matrix R couples the external momenta (and the hyperposition associated to the root vertex) to the internal momenta and the remaining $V-1$ hyperpositions. This

coupling is done in an analogous way to the coupling of the internal momenta with the respective variables.

We can thus state that the V polynomial is given, as in the commutative case, by the inverse Q^{-1} of the matrix Q giving the U polynomial. The particular form (25) of the matrix Q leads to

$$\begin{aligned} Q^{-1} &= \frac{1}{2}\left((D+A)^{-1}+(D-A)^{-1}\right)\otimes 1_d \\ &+ \frac{1}{2}\left((D+A)^{-1}-(D-A)^{-1}\right)\otimes \sigma \end{aligned} \tag{45}$$

Thus, the polynomial V has both a real V^R and an imaginary part V^I.

In the case of the commutative theories, the imaginary part above disappears. This is a consequence of the fact that the matrix F, coupling through the Moyal oscillations the internal momenta, vanishes for $\theta = 0$.

We have the following definition :

Définition E.5.2 *A two $\star$-tree is a subset of edges with two boundaries.*

Furthermore, let K a subset of lines of the antisymmetric matrix A. Let $\mathrm{Pf}(A_{\hat{K}\hat{\tau}})$ be the Pfaffian of the antisymmetric matrix obtained from A by deleting the edges in the set $K \cup \{\tau\}$ for $\tau \notin K$. We also define $\varepsilon_{K,\tau}$ to be the signature of the permutation obtained from $(1,\dots,E)$ by extracting the positions belonging to $K \cup \{\tau\}$ and replacing them at the end in the order :

$$1,\dots,E \to 1,\dots,\hat{i}_1,\dots,\hat{i}_p,\dots,\hat{i}_\tau,\dots,E,i_\tau,i_p,\dots,i_1.$$

We now prove a general form for both the real and the imaginary part of the polynomial $V^\star$, noted $\mathcal{X}^\star$ and $\mathcal{Y}^\star$.

Théorème E.5.2 *The real part of the second Symanzik polynomial writes*

$$\mathcal{X}^\star = \left(\frac{\theta}{2}\right)^{b+1} \sum_{\mathcal{T}_2^\star \ 2-\star \text{ tree}} \prod_{e\notin\mathcal{T}_2^\star} 2\frac{\alpha_e}{\theta}(p_{\mathcal{T}_2^\star})^2, \tag{46}$$

where $p_{\mathcal{T}_2^\star}$ is the sum of the momenta entering one of the two faces of the 2 $\star-$tree $\mathcal{T}_2^\star$.

Note that by momentum conservation, the choice of the face in the above theorem is irrelevant. Furthermore, let us emphasize on the fact that, being on the submanifold $p_G = 0$, an equivalent writing of (46) is

$$\mathcal{X}^\star = -\frac{1}{2}\left(\frac{\theta}{2}\right)^{b+1} \sum_{\mathcal{T}_2^\star \ 2-\star \text{ tree}} \prod_{e\notin\mathcal{T}_2^\star} 2\frac{\alpha_e}{\theta} p_v \cdot p_{v'}, \tag{47}$$

where p_v (and resp. $p_{v'}$) is the total momenta entering one of the two faces of the 2-$\star$ tree.

Proof. We base our proof on the following lemma :

Lemme E.5.2 *(**Lemma IV.1 of [68]**)*
The real part of the polynomial $V^\star$ writes

$$\mathcal{X}^\star = \sum_K \prod_{i\notin K} d_i \left(\sum_{e_1} p_{e_1} \sum_{\tau\notin K} R_{e_1\tau}\varepsilon_{K\tau}\mathrm{Pf}(A_{\hat{K}\hat{\tau}}) \right)^2 \tag{48}$$

where d_i are the elements on the diagonal of the matrix Q. Furthermore, when $|K| \in \{E-1, E\}$ the matrix with deleted lines is taken to be the empty matrix, with unit Pfaffian.

Note that, as before, since the matrix Q has vanishing entries on the diagonal for the last $V-1$ entries the subsets K are nothing but subsets of edges. The empty matrix obtained from deleting all the first E edges in the graph corresponds to the graph with no internal edges but only disconnected vertices. Each of these disconnected components has one boundary ; hence the Pfaffian is non-vanishing.

Note that the Pfaffian in (48) disappears iff the corresponding graph has 1 boundary (see above). This means that $K \cup \{\tau\}$ is the complement of a $\star-$tree $\mathcal{T}^\star$:

$$\overline{K \cup \{\tau\}} = \mathcal{T}^\star. \tag{49}$$

Hence the subset K is the complement of a $\star$-tree plus an edge (just like in the commutative case). Adding an extra edge to a $\star$-tree represents an increase of the number of boundaries by one unit. Hence, the subset of edges K above is the complement of some 2 $\star-$tree $\mathcal{T}_2^\star$

$$\bar{K} = \mathcal{T}_2^\star. \tag{50}$$

As before, one has

$$\prod_{e\in K} \alpha_e = \prod_{e\notin \mathcal{T}_2^\star} \alpha_e.$$

The diagonal terms in the matrix Q are again the parameters α_e. Factoring out $\frac{\theta}{2}$ factors on the lines of the matrices corresponding to the edges of the graph and then $\frac{2}{\theta}$ for the lines of the matrices corresponding to the vertices. The extra factor $\theta/2$ corresponds to the extra edge τ.

Let us now investigate the square root of the momenta combination entering (48). Note that the matrix element $R_{e_1\tau}$ is not vanishing only for external momentum p_{e_1} which has a Moyal oscillation with the internal momenta associated to the edge τ. It is this edge τ which actually creates the extra boundary. Thus the sum on the external momenta in (48) is nothing but the sum of the momenta entering one of the two boundaries. By a direct verification, one can explicitly check the signs of the respective momenta in (48), which concludes the proof.

Exemple E.5.3 *For the graph of Figure E.12, the second polynomial is*

$$V^\star(\alpha, p) = \alpha_1\alpha_2\alpha_3 p^2 + \frac{1}{4}(\alpha_1+\alpha_2+\alpha_3)\theta^2 p^2. \tag{51}$$

Let us now investigate the form of the imaginary part $\mathcal{Y}^\star$. One has the following theorem :

Théorème E.5.3 *The imaginary part of the second Symanzik polynomial writes*

$$\mathcal{Y}^\star(\alpha,p)=\left(\frac{\theta}{2}\right)^b\sum_{\mathcal{T}^\star\ \star\ \text{tree}}\prod_{e\notin\mathcal{T}^\star}2\frac{\alpha_e}{\theta}\psi(p), \tag{52}$$

where $\psi(p)$ is the phase obtained by following the momenta entering the face of the $\star-$tree $\mathcal{T}^\star$ as if it was a Moyal vertex.

Proof. The proof follows closely the one of Theorem E.5.2. Nevertheless, the equivalent of (48) is now (see again [68])

$$\mathcal{Y}^\star(\alpha,p) \;=\; \sum_K\prod_{e\notin K}d_i\epsilon_K\mathrm{Pf}(A_{\hat{K}})\left(\sum_{e_1,e_2}\left(\sum_{\tau,\tau'}R_{e_1\tau}\epsilon_{K\tau\tau'}\mathrm{Pf}(A_{\hat{K}\hat{\tau}\hat{\tau}'})R_{e_2\tau'}\right)p_{e_1}\sigma p_{e_2}\right) \tag{53}$$

where d_i are the elements on the diagonal of the matrix Q. Since we look this time for sets such that $\mathrm{Pf}(B_{\hat{K}})$ is non-vanishing, this implies as above that K is the complement of some $\star$-tree $\mathcal{T}^\star$. Furthermore one needs to consider the two extra edges τ and τ'. It is possible from the initial $\star-$tree above to erase these two more edges such that the Pfaffian $PfB_{\hat{K}\hat{\tau}\hat{\tau}'}$ is non-vanishing. Indeed, if the $\star$-tree is a tree, by erasing two more edges of it we obtain a graph with 3 disconnected components, each of it with a single boundary ; the corresponding Pfaffian will be non-vanishing. Summing up on all these possibilities leads to the Moyal oscillations of the external momenta (the one which disappears when truncating the graph). If the $\star-$tree is formed by a tree and some pair of genus edges we can always delete further the pair of genus edges and remain with the regular tree. Obviously the corresponding Pfaffian is again non-vanishing (since it corresponds to a graph with only one boundary).

Note that the form of the real part and of the imaginary one of the polynomial $V^\star$ are qualitatively different. Indeed, the real part contains some square of a sum of incoming external momenta, while the imaginary one contains a phase involving the external momenta.

Let us end this section by stating that the second noncommutative Symanzik polynomial also obeys the deletion-contraction rule. The proof is exactly like in the commutative case, a straightforward rereading of Theorem E.5.1.

E.5.6 Relation to multivariate Bollobás-Riordan polynomials

In the previous subsections, we have identified the first Symanzik polynomial of a connected graph in a scalar NCQFT as the first order in w of the multivariate Bollobás-Riordan polynomial,

$$U^\star_G(\alpha,\theta)=(\theta/2)^{E-V+1}\Big(\prod_{e\in E}\alpha_r\Big)\times\lim_{w\to0}w^{-1}Z_G\Big(\tfrac{\theta}{2\alpha_e},1,w\Big). \tag{54}$$

Recall that the multivariate Bollobás-Riordan polynomial (see [156]) is a generalization of the multivariate Tutte polynomial to orientable ribbon graphs defined by the expansion,

$$Z_G(\beta,q,w)=\sum_{A\subset E}\Big(\prod_{e\in A}\beta_e\Big)q^{k(A)}w^{b(A)}, \tag{55}$$

with $q(A)$ the number of connected components and $b(A)$ the number of boundaries of the spanning graph (V, A).

In order to deal with the second Symanzik polynomial in the noncommutative case, we now introduce an extension of $Z_G(\beta, q, w)$ for ribbon graphs with flags at $q = 1$. In the case of ribbon graphs, the flags are attached to the vertices and the cyclic order of flags and half-edges at each vertex matters. For each cyclically oriented subset I of the set of labels of the flags, we introduce an independent variable w_I. Cyclically ordered subsets I are defined as sequences of different labels up to a cyclic permutation. Then, each boundary of a graph with the orientation induced by the graph, defines a cyclically ordered subset of the set of labels of the flags, by listing the flags in the order they appear on the boundary. Accordingly, a variable w_I is attached to each boundary.

Définition E.5.3 *For an orientable ribbon graph G with flags $\Xi_G(\beta_e, w_I)$ is defined by the expansion*

$$\Xi_G(\alpha_e, \beta_e, w_I) = \sum_{A \subset E} \Big(\prod_{e \notin E} \alpha_e \prod_{e \in E} \beta_e \prod_{boundaries} w_{I_n} \Big), \tag{56}$$

where I_n are the cyclically ordered sets of flags attached to each of the connected component of the boundary of the spanning graph (V, A).

We recover $Z_G(\beta_e, 1, w)$ by setting $w_I = w$ and $\alpha_e = 1$, but the information pertaining to q is lost except for planar graphs. Indeed, in this case the genus of any subgraph is still 0 so that $|V| - |A| + b(A) = 2k(A)$ and thus $Z_G(\beta_e, q, w) = q^{|V|/2} Z_G(q^{-\frac{1}{2}}\beta_e, q^{\frac{1}{2}}w)$.

The polynomial $\Xi_G(\alpha_e, \beta_e, w_I)$ obeys the contraction/deletion rules for any semi-regular edges (i.e. all types of edges except self-loops). The structure of the flags of $G-e$ is left unchanged, but less variables w_I enter the polynomial since the number of boundaries decreases. For G/e, the flags attached to the vertex resulting from the contraction are merged respecting the cyclic order of flags and half-edges attached to the boundary of the subgraph made of the contracted edge only.

Proposition E.5.1 *The polynomial $\Xi_G(\alpha_e, \beta_e, w_I)$ obeys the contraction/deletion rule for a semi-regular edge,*

$$\Xi_G(\alpha_e, \beta_e, w_I) = \alpha_e \, \Xi_{G-e}(\alpha_{e' \neq e}, \beta_{e' \neq e}, w_I) + \beta_e \, \Xi_{G/e}(\alpha_{e' \neq e}, \beta_{e' \neq e}, w_I). \tag{57}$$

This follows from gathering in the expansion of $\Xi_G(\alpha_e, \beta_e, w_I)$ the terms that contain e and those that do not. The contraction/deletion rule may be extended to any edge provided we introduce vertices that are surfaces with boundaries as in [156].

The second interesting property of $\Xi_G(\alpha_e, \beta_e, w_I)$ lies in its invariance under duality. Recall that for a connected ribbon graph G with flags, its dual G^* is defined by taking as vertices the boundaries of G, with flags and half-edges attached in the cyclic order following the orientation of the boundary induced by that of G.

Proposition E.5.2 *For a connected graph G with dual G^*,*

$$\Xi_G(\alpha_e, \beta_e, w_I) = \Xi_{G^*}(\beta_e, \alpha_e, w_I). \tag{58}$$

Proof : First, recall that there is a natural bijection between the edges of G and those of G^*. Thus, to a subset A of edges of G we associate a subset A^* of edges of G^* which

is the image under the previous bijection of the complementary $E - A$. Then, the term corresponding to A in $\Xi_G(\alpha_e, \beta_e, w_I)$ equals that corresponding to A^* in $\Xi_{G^*}(\beta_e, \alpha_e, w_I)$. The only non trivial part in the last statement is the equality of the boundary terms in G and G^*, which is best understood by embedding G in a surface Σ. Then, the spanning graph (V^*, A^*), viewed as discs joined by ribbons, is homeomorphic to $\Sigma - (V, A)$, with the orientation reversed. Accordingly, they have the same boundary.

This relation may also be extended to non connected graphs at the price of introducing again vertices that are surfaces with holes. For example, the dual of a disjoint union of n vertices is the vertex made of a sphere with n holes. For a regular edge, the duality exchanges contraction (resp. deletion) in G with deletion (resp. contraction) in G^*. In the case of the deletion of a bridge in G, we have to contract a self-loop in G^*, thus leading to vertices that are surfaces with holes. Note that this implies a duality for the multivariate Bollobás-Riordan polynomial only at the special point $q = 1$, in agreement with the fact that the duality for the Bollobás-Riordan polynomial only holds when its arguments lies on a hypersurface [157].

Finally, let us come to the relation with the second Symanzik polynomial in NCQFT. For a given connected graph with momenta p_i such that $\sum_i p_i = 0$ attached to the flags, we decompose the latter polynomial into real and imaginary part,

$$V^\star_G(\alpha_e, \theta, p_i) = \mathcal{X}^\star_G(\alpha_e, \theta, p_i) + \mathrm{i}\, \mathcal{Y}^\star_G(\alpha_e, \theta, p_i). \tag{59}$$

Consider real variables w_i and define $w_I = \sum_i w_i$ for any cyclically oriented subset of flags, Then, expand $(\theta/2)^{|E|-|V|}\, \Xi_G(2\alpha_e/\theta, \theta w_I/2)$ to the first two orders at $w_i = 0$,

$$(\theta/2)^{|E|-|V|}\, \Xi_G(2\alpha_e/\theta, \theta w_I/2) = A\Big(\sum_i w_i\Big) + \sum_{i\neq j} B_{ij} w_i w_j + O\Big(w^3\Big). \tag{60}$$

The first order term reproduces the first Symanzik polynomial

$$U^\star_G(\alpha_e, \theta) = A, \tag{61}$$

whereas the second order terms yields the real part of the second Symanzik polynomial,

$$\mathcal{X}^\star_G(\alpha_e, \theta, p_i) = -\tfrac{1}{2} \sum_{i\neq j} A_{ij}\, p_i \cdot p_j. \tag{62}$$

To obtain the imaginary part, consider the variables

$$w_I = \tfrac{1}{2} \sum_{i<j} p_i \cdot \Theta p_j \tag{63}$$

if I contain all the flags and $w_I = 0$ otherwise. The previous definition involves a choice of a total order on I compatible with its cyclic structure, but momentum conservation $\sum_i p_i = 0$ implies that w_I does not depend on this choice. Then,

$$\mathcal{Y}^\star_G(\alpha_e, \theta, p_i) = (\theta/2)^{|E|-|V|}\, \Xi_G(2\alpha_e/\theta, w_I). \tag{64}$$

As a consequence of their expressions in terms of $\Xi_G(\alpha_e, \beta_e, w_I)$, the noncommutative Symanzik polynomials obey contraction/deletion rules for regular edges and duality relations. For example, the duality for the first Symanzik polynomial reads

$$(\theta/2)^{|V|}\, U^\star_G(\alpha_e, \theta) = (\theta/2)^{|V^*|} \Big(\prod_{e\in E} \frac{2\alpha_e}{\theta}\Big)\, U^\star_{G^*}\Big(\theta^2/\alpha_e, \theta\Big). \tag{65}$$

Beware that G^* is the dual graph whereas the star on polynomials such as $U^\star$ and $V^\star$ refer to the Moyal product. Analogous relations, though slightly more cumbersome, can be written for the second Symanzik polynomial.

Still an other way to categorify and regularize in the infrared is to introduce harmonic potentials on the edges rather than the vertices, leading to propagators based on the Mehler rather than the heat kernel. This is the so-called *vulcanization*. An extensive study of the corresponding commutative and noncommutative polynomials is under way as a companion paper [158].

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F

Paper 6

Quantum Field Theory on quantized Bergman domain

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Abstract. We present an oscillator realization of discrete series representations of group $SU(2,2)$. We give formulas for the coherent state star-product quantization of a Bergman domain D. A formulation of a (regularized) non-commutative scalar field on a quantized D is given.

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F.1 Introduction

The fundamental role of conformal group $SO(4,2)$ for Minkowski space-time was first stressed by Dirac, [72]. Its covering group $G = SU(2,2)$ describes conformal properties of spinning particles, see [73], where one can found a systematic introduction to the subject.

The group G and its orbits are fundamental for the twistor theory, [74]. It is also of essential importance for the ADS-CFT theory[75].

All unitary irreducible representations of the group G were classified by [77] and [76]. More general case of $SU(m,n)$ is treated in [78]. The discrete series $SU(2,2)$ representations were used by [79] and [81] for the investigation of conformal properties of fields on Minkowski space. The highest/lowest weight of the discrete series of representations has been studied by [79] and [80].

The importance of the deformations theory for quantum systems was first stressed by [82]. The deformation method was generalized to linear Poisson structures (related to Lie algebras) in [83] and to general Poisson structures in[84]. The relation between both approaches was described in [87].

The star-product formula represents an approach going beyond deformation theory. A general star-product approach, based on coherent states on co-adjoint orbits [88], was proposed in [89], for $SU(2)$ case the star-product formula was found in [90], see also [91] for $SU(n)$ orbits (the deformed algebra can be represented as a matrix algebra). A general formula for compact Lie groups was derived in [92].

For non-compact Lie groups the situation is more complicated. The case $SU(1,1)$ was briefly sketched in [89]. Similar approach was applied in [93] to a particular $SU(2,2)$ orbit - the complex Minkowski space. The corresponding deformed noncommutative algebra was represented in terms of 4 bosonic oscillator pairs.

A noncommutative field theory may be defined provided the noncommutative algebra of functions, with some additional structures, is specified on a configuration space(time). For $SU(2)$ case this was done in [94] and followed by various other papers. For the noncommutative Heisenberg group, the formulation of a noncommutative quantum field theory on a Moyal space, was given in [95]. Much more work has been done for models defined over the Euclidean deformed space-time. This culminated in [28] and [29], where a renormalizable nontrivial 4D model was found and studied. For a recent review see [24].

The noncommutative space-time model proposed by H. S. Snyder and C. N. Yang (see [146]), based on non-compact groups $SO(4,1)$ and $SO(5,1)$ has not been developed much further, mainly due to the success of renormalization theory approach to quantum field theory. In our opinion it could be a right time to return back to those old ideas.

The paper is organized as follows. In Section 1 we describe the Lie group $G = SU(2,2)$ and its Lie algebra $\mathbf{g} = su(2,2)$. The relevant mathematical background can be found, e.g., in [97] and [98]. In Section 3 we present a simple oscillator realization (in terms of 8 bosonic oscillator pairs) of most degenerate discrete series of representations which generalizes more common (Schwinger-Jordan) oscillator realizations used in the case of compact groups. In Section 4 we construct the system of coherent states for the representation in question and we give a corresponding star-product formula for the algebra of functions on a Bergman domain D. Finally, in Section 5 we construct a quantum field theory model on the quantized Bergman domain D.

F.2 The group $SU(2,2)$ and its Lie algebra

F.2.1 The definition of $SU(2,2)$

The group $G = SU(2,2)$ is the subgroup of $SL(4,\mathbf{C})$ matrices satisfying

$$g = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in G \Rightarrow g^\dagger \Gamma g = \Gamma, \ \Gamma = \begin{pmatrix} E & 0 \\ 0 & -E \end{pmatrix}. \tag{1}$$

Here all the entries a, b, c, d are 2×2- matrices and the symbols 0 and E denote the 2×2 zero matrix and the unit matrix, respectively. Inserting this into (1) one obtains two sets of equivalent constraints

$$a^\dagger a = E + c^\dagger c, \ \ c^\dagger d = E + b^\dagger b, \ \ a^\dagger b = c^\dagger d, \tag{2}$$

or,

$$a a^\dagger = E + b b^\dagger, \ \ d d^\dagger = E + c c^\dagger, \ \ a c^\dagger = b d^\dagger. \tag{3}$$

F.2.2 Maximal compact subgroup and Bergman domain

The maximal compact subgroup of $SU(2,2)$ is $K = S(U(2)\times U(2))$ which consists of the matrices

$$k = \begin{pmatrix} k_1 & 0 \\ 0 & k_2 \end{pmatrix}, \ k_{1,2} \in U(2), \quad \det(k_1)\det(k_2) = 1. \tag{4}$$

The corresponding Bergman domain D is a kind of Type 1 Cartan domain which defined as the group coset space :

$$D = G/K. \tag{5}$$

It can be represented as the set of all complex 2×2 matrix

$$Z = \begin{pmatrix} z_{11} & z_{12} \\ z_{21} & z_{22} \end{pmatrix} \tag{6}$$

with $Z^\dagger Z < E$. The group action on D is given by :

$$Z' = gZ = (aZ+b)(cZ+d)^{-1} \tag{7}$$

The Bergman domain D is a pseudo-convex domain where we could define Hilbert space $L^2(D, d\mu_N)$ (see (2)) of holomorphic functions with reproducing kernel $K(Z,W)$, where $Z, W \in D$. This reproducing kernel is also called Bergman kernel and it is well known that

$$K(Z,W) = \det(E - ZW^\dagger)^{-N}. \tag{8}$$

The Bergman domain D is an 8 dimensional rank 2 Hermitian symmetric space. It is also a Kähler manifold of 4 complex dimensions, with the metric given by

$$g_{i\bar{j}} = \frac{\partial^2}{\partial_{z_i}\partial_{\bar{z_j}}} \log(K(Z,\bar{Z})) \tag{9}$$

The topology of Bergman domain D is nontrivial. It not simply connected but has genus 4. One way to calculate the genus is by studying the corresponding complex Jordan triple. The interested reader could find more details in ([99]).

F.2.3 The Lie algebra g and the Haar measure

Let $\mathbf{g} = su(2,2)$ be the Lie algebra of G, so it is real and semisimple. It is formed by matrices satisfying

$$X^\dagger \Gamma + \Gamma X = 0 \tag{10}$$

Consider the Cartan decomposition of $\mathbf{g}$(see [98], [103], [104]) :

$$\mathbf{g} = \mathbf{k} + \mathbf{p}, \tag{11}$$

where $\mathbf{k}$ is the subset of all anti-hermitian matrices in $\mathbf{g}$

$$\mathbf{k} = \Big\{ \begin{pmatrix} A & 0 \\ 0 & D \end{pmatrix} : A^\dagger = -A,\ D^\dagger = -D,\ \mathrm{tr}(A+D) = 0,\ A, D \in M_2(C) \Big\}. \tag{12}$$

The set $\mathbf{k}$ is the Lie algebra of the maximal compact subgroup K in G. The subset $\mathbf{p}$ of all hermitian matrices in $\mathbf{g}$

$$\mathbf{p} = \Big\{ \begin{pmatrix} 0 & B \\ B^\dagger & 0 \end{pmatrix} : B \in M_2(C) \Big\} \tag{13}$$

is just a linear space and not a Lie algebra.

Let $\mathbf{a}$ be a maximal Abelian subalgebra in $\mathbf{p}$. We may choose for $\mathbf{a}$ the set of all matrices of the form

$$H_\Lambda = \begin{pmatrix} 0 & \Lambda \\ \Lambda & 0 \end{pmatrix} \tag{14}$$

where 0 means the 2×2 matrix with entries zeros and $\Lambda = \mathrm{diag}(\lambda_1, \lambda_2)$ is diagonal 2×2 with λ_1, λ_2 real. The corresponding subgroup consists of all matrices of the type :

$$\delta_\Lambda = \begin{pmatrix} C & S \\ S & C \end{pmatrix}, \qquad \begin{matrix} C = \mathrm{diag}(\mathrm{ch}\lambda_1, \mathrm{ch}\lambda_2), \\ S = \mathrm{diag}(\mathrm{sh}\lambda_1, \mathrm{sh}\lambda_2). \end{matrix} \tag{15}$$

Define the dual space $\mathbf{a}^\star$ spanned by the elements α_i satisfying $\alpha_i(H_\Lambda) = \lambda_i$.

Then the roots of $(\mathbf{g}, \mathbf{a})$ are given by

$$\pm 2\alpha_1,\ \pm 2\alpha_2,\ \pm(\alpha_1 - \alpha_2) \tag{16}$$

with multiplicities $m_{2\alpha_1} = m_{2\alpha_2} = 1$ and $m_{\alpha_1 \pm \alpha_2} = 2$.

On the root system we choose that the positive Weyl chamber given by $C^+ = \{\lambda_1, \lambda_2\}$ with $\lambda_1 > \lambda_2 > 0$. Then the positive roots are $2\alpha_1$, $2\alpha_2$ and $(\alpha_1 \pm \alpha_2)$. We use Σ donate the set of all roots and Σ^+ the set of positive roots.

Define

$$\rho = \frac{1}{2} \sum_{\alpha \in \Sigma^+} m_\alpha \alpha \tag{17}$$

So we have

$$\rho = \alpha_1 + \alpha_2 + (\alpha_1 + \alpha_2) + (\alpha_1 - \alpha_2) = 3\alpha_1 + \alpha_2 \tag{18}$$

Let $\mathbf{a}_c$ be the complex extension of $\mathbf{a}$. Follow the same procedure as shown above we could define the complex roots $\alpha_c \in \mathbf{a}_c^*$ as

$$\alpha_c^i(\mathbf{a}_c) = \tau_i, \quad i = 1, 2. \tag{19}$$

Here τ_i are complex numbers. The formula of ρ and τ_i will be used for constructing the eigenfunction of the radial part of the invariant Laplacian. See section F.5.

Now we have some physical interpretation of the Lie algebra : the first summand in formula (11) represents compact generators of *rotations*, whereas the second one represents non-compact generators *boosts*. Since the Lie algebras $su(2,2)$ and $so(4,2)$ are isomorphic we shall label rotations as X_{05} and x_{ab}, $a,b = 1,2,3,4$, and boosts as X_{0a} and X_{a5} considering them as generators $X_{AB} = -X_{BA}$, $A,B = 0,1,\ldots,5$, satisfying $so(4,2)$ commutation relations

$$[X_{AB}, X_{CD}] = \eta^{AC} X_{BD} - \eta^{BC} X_{AD} + \eta^{BD} X_{AC} - \eta^{AD} X_{BC}\,, \tag{20}$$

with the metric tensor $\eta^{AB} = \mathrm{diag}(+1,-1,-1,-1,-1,+1)$. Explicitly, the compact Lie algebra $\mathbf{k}$ is spanned by 7 anti-hermitian matrices and the basis of $\mathbf{p}$ is formed by 8 hermitian matrices given below :

$$S_{05} = \frac{i}{2}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix},\ S_{j4} = \frac{i}{2}\begin{pmatrix} \sigma_j & 0 \\ 0 & -\sigma_j \end{pmatrix},\ S_{ij} = \frac{i}{2}\varepsilon_{ijk}\begin{pmatrix} \sigma_k & 0 \\ 0 & \sigma_k \end{pmatrix},$$
$$S_{k5} = \frac{i}{2}\begin{pmatrix} 0 & \sigma_k \\ -\sigma_k & 0 \end{pmatrix},\ S_{0k} = \frac{1}{2}\begin{pmatrix} 0 & \sigma_k \\ \sigma_k & 0 \end{pmatrix},$$
$$S_{45} = \frac{1}{2}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix},\ S_{04} = \frac{1}{2}\begin{pmatrix} 0 & i \\ -i & 0 \end{pmatrix}, \tag{21}$$

where $i,j,k = 1,2,3$ and σ_1, σ_2, σ_3 are usual Pauli matrices.

The principal Cartan subalgebra

$$\mathbf{h} = \mathbf{a} \oplus \mathbf{u} \tag{22}$$

of G is spanned by three commuting generators : two noncompact X_{45} and X_{03} span $\mathbf{a}$, and the additional compact one X_{12} spans $\mathbf{u}$. The corresponding subgroups we shall denote as H, A and U : $H = A \times U$.

Any element of G possesses a unique Cartan decompositions

$$g = k\,\delta\,\tilde{q} = \tilde{k}\,\delta\,q\,, \tag{23}$$

where δ is some pure positive non-compact element of H with positive λ_1 and λ_2 given by formula (15). Further, $k = \tilde{k}\,u$, and $q = u\,\tilde{q}$ are elements of K, and u is the element of a compact subgroup U in H.

The Haar measure dg on $G = SU(2,2)$ is, in the parametrization (23), given as

$$dg \equiv dg(\lambda,k,q) = \rho(\lambda_1,\lambda_2)\,d\lambda_1\,d\lambda_2\,d\tilde{k}\,d\tilde{q}\,du\,, \tag{24}$$

where $d\tilde{k}$ and $d\tilde{q}$ denotes the normalized measures on K/U, and du is the usual measure on U (see [81]). The explicit form of $\rho(\lambda_1,\lambda_2)$ is constructed from the positive roots (see Section F.2.3) :

$$\rho(\Lambda) = \prod_{\alpha\in\Sigma^+} |\sinh\alpha(T)|^{m_\alpha} \tag{25}$$

where m_α is the multiplicity of the positive roots. So we have :

$$\rho(\lambda_1,\lambda_2) = \mathrm{sh}^2(\lambda_1+\lambda_2)\,\mathrm{sh}^2(\lambda_1-\lambda_2)\,\mathrm{sh}(2\lambda_1)\,\mathrm{sh}(2\lambda_2)\,. \tag{26}$$

F.3 Discrete series representation of $SU(2,2)$

The group $G = SU(2,2)$ possesses principal, supplementary and discrete series of unitary irreducible representations, see e.g., [76], [78]. The discrete series of representations is given by :

$$T_g f(Z) = [\det(CZ+D)]^{-N} f(Z'),\ N = 4,5,6\cdots \tag{1}$$

where Z' is given by formula (7) and $f(Z) \in L^2(D, d\mu_N)$ with the measure

$$d\mu_N(\bar{Z}, Z) \ = \ c_N\, d\bar{Z}\, dZ \det{}^{}(E - Z^\dagger Z)^{N-4}. \tag{2}$$

The normalization constant $c_N = \pi^{-4}(N-1)(N-2)^2(N-3)$ guarantees that the function $f_0(Z) \equiv 1$ has a unit norm, see [81].

We introduce a 4×2 matrix $\hat{Z} = (\hat{z}_{a\alpha})$, $a = 1, \ldots, 4$, $\alpha = 1,2$, of bosonic oscillators acting in Fock space and satisfying commutation relations

$$[\hat{z}_{a\alpha}, \hat{z}^\dagger_{b\beta}] \ = \ -\Gamma_{ab}\,\delta_{\alpha\beta}\,,$$

$$[\hat{z}_{a\alpha}, \hat{z}_{b\beta}] \ = \ [\hat{z}^\dagger_{a\alpha}, \hat{z}^\dagger_{b\beta}] \ = \ 0\,, \tag{3}$$

where Γ is a 4×4 matrix defined in (1). It can be easily seen that for all $g \in SU(2,2)$ these commutation relations are invariant under transformations :

$$\hat{Z} \ \mapsto \ g\,\hat{Z}, \quad \hat{Z}^\dagger \ \mapsto \ \hat{Z}^\dagger\, g^\dagger. \tag{4}$$

Since, $\Gamma = \mathrm{diag}(+1,+1,-1,-1)$ the upper two rows in $\hat{Z}$ corresponds to creation operators whereas the lower ones to annihilation operators :

$$\hat{Z} \ = \ \begin{pmatrix} \hat{a}^\dagger \\ \hat{b} \end{pmatrix} : \ \ [\hat{a}_{\alpha\beta}, \hat{a}^\dagger_{\gamma\delta}] \ = \ [\hat{b}_{\alpha\beta}, \hat{b}^\dagger_{\gamma\delta}] \ = \ \delta_{\alpha\beta}\,\delta_{\gamma\delta}, \quad \alpha,\beta,\gamma,\delta \ = \ 1,2. \tag{5}$$

and all other commutation relations among oscillator operators vanish. The Fock space $\mathcal{F}$ in question is generated from a normalized vacuum state $|0\rangle$, satisfying $\hat{a}_{\alpha\beta}\,|0\rangle \ = \ \hat{b}_{\alpha\beta}\,|0\rangle \ = \ 0$, by repeated actions of creation operators :

$$|m_{\alpha\beta},\, n_{\alpha\beta}\rangle \ = \ \prod_{\alpha\beta} \frac{(\hat{a}^\dagger_{\alpha\beta})^{m_{\alpha\beta}}\,(\hat{b}^\dagger_{\alpha\beta})^{n_{\alpha\beta}}}{\sqrt{m_{\alpha\beta}!\,n_{\alpha\beta}!}}\,|0\rangle\,. \tag{6}$$

We shall use the terminology that the state $|m_{\alpha\beta},\, n_{\alpha\beta}\rangle$ contains $m = \sum m_{\alpha\beta}$ particles a and $n = \sum n_{\alpha\beta}$ particles b.

The Lie algebra $su(2,2) = so(4,2)$ acting in Fock space can be realized in terms of oscillators. To any 4×4 matrix $X = (X_{ab})$ we assign the operator

$$\hat{X} \ = \ -\mathrm{tr}(\hat{Z}^\dagger \Gamma X \hat{Z}) \ = \ -\hat{z}^\dagger_{a\alpha}\,\Gamma_{ab}\,X_{bc}\,\hat{z}_{c\alpha}\,, \tag{7}$$

with $\hat{Z}^\dagger$ and $\hat{Z}$ given in (3) and (5) in terms of oscillators. Using commutation relations for annihilation and creation operators the commutator of operators $\hat{X} = -\mathrm{tr}(\hat{Z}^\dagger\Gamma X\hat{Z})$ and $\hat{Y} = -\,\mathrm{tr}(\hat{Z}^\dagger\Gamma Y\hat{Z})$ can be easily calculated :

$$[\hat{X},\hat{Y}] \ = \ [\mathrm{tr}(\hat{Z}^\dagger\Gamma X\hat{Z}), \mathrm{tr}(\hat{Z}^\dagger\Gamma Y\hat{Z})] \ = \ -\mathrm{tr}[\hat{Z}^\dagger\,\Gamma\,(XY - YX)\,\hat{Z}]\,. \tag{8}$$

It can be easily seen that the *anti-hermitian operators*

$$\hat{X}_{AB} \;=\; -\mathrm{tr}(\hat{Z}^{\dagger}\Gamma X_{AB}\hat{Z}), \quad A,B=0,1,\ldots,5, \tag{9}$$

with X_{AB} given in (21), satisfy in Fock space the $su(2,2)\cong so(4,2)$ commutation relations (20). The assignment

$$g \;=\; e^{\xi^{AB}X_{AB}} \;\in\; SU(2,2) \Rightarrow\; \hat{T}(g) \;=\; e^{\xi^{AB}\hat{X}_{AB}} \tag{10}$$

then defines a *unitary* $SU(2,2)$ *representation* in Fock space.

The adjoint action of $\hat{T}(g)$ on operators reproduces (4). In terms of a and b-oscillators in block-matrix notation this can be rewritten as

$$g=\begin{pmatrix} a & b \\ c & d \end{pmatrix}: \quad \begin{array}{ll} \hat{T}(g)\,\hat{a}^{\dagger}\,\hat{T}^{\dagger}(g) \;=\; a\,\hat{a}^{\dagger}+b\,\hat{b}, & \hat{T}(g)\,\hat{a}\hat{T}^{\dagger}(g) \;=\; \hat{a}\,a^{\dagger}+\hat{b}^{\dagger}\,b^{\dagger}, \\ \hat{T}(g)\,\hat{b}\,\hat{T}^{\dagger}(g) \;=\; d\,\hat{b}+c\,\hat{a}^{\dagger}, & \hat{T}(g)\,\hat{b}^{\dagger}\,\hat{T}^{\dagger}(g) \;=\; \hat{b}^{\dagger}\,d^{\dagger}+\hat{a}\,c^{\dagger}. \end{array} \tag{11}$$

Since any $g\in SU(2,2)$ possesses Cartan decomposition (23) we shall discuss separately rotations and special boosts given in (9). For rotations we obtain a mixing of annihilation and creation operators of a same type :

$$k=\begin{pmatrix} k' & 0 \\ 0 & k'' \end{pmatrix}: \quad \begin{array}{ll} \hat{T}(k)\,\hat{a}^{\dagger}\,\hat{T}^{\dagger}(k) \;=\; k'\,\hat{a}^{\dagger}, & \hat{T}(k)\,\hat{a}\hat{T}^{\dagger}(g) \;=\; \hat{a}\,k'^{\dagger}, \\ \hat{T}(g)\,\hat{b}\,\hat{T}^{\dagger}(k) \;=\; k''\,\hat{b}, & \hat{T}(k)\,\hat{b}^{\dagger}\,\hat{T}^{\dagger}(k) \;=\; \hat{b}^{\dagger}\,k''^{\dagger}. \end{array} \tag{12}$$

However, for special boosts from **a** we obtain Bogolyubov transformations :

$$\delta=\begin{pmatrix} C & S \\ S & C \end{pmatrix}: \quad \begin{array}{ll} \hat{T}(\delta)\,\hat{a}^{\dagger}\,\hat{T}^{\dagger}(g) \;=\; C\,\hat{a}^{\dagger}+S\,\hat{b}, & \hat{T}(\delta)\,\hat{a}\hat{T}^{\dagger}(\delta) \;=\; \hat{a}\,C+\hat{b}\,S, \\ \hat{T}(\delta)\,\hat{b}\,\hat{T}^{\dagger}(\delta) \;=\; C\,\hat{b}+S\,\hat{a}^{\dagger}, & \hat{T}(\delta)\,\hat{b}^{\dagger}\,\hat{T}^{\dagger}(\delta) \;=\; \hat{b}^{\dagger}\,C+\hat{a}\,S, \end{array} \tag{13}$$

with C and S determined in (15).

Using the explicit form of matrices X_{AB}, following from (10), the action of generators can be described in terms creation and annihilation of a- and b-particles :

(i) The action of *rotation* generators results in a replacement of some a-particle by an other a-particle and by replacement of b-particle by other (ab)-particle.

(ii) The action of *boost* generators results in creation of a pair (ab) of particles or in a destruction of ab pair.

In this context it is useful to consider lowering and rising operators labeled by arbitrary 2×2 complex matrix $B \;=\; (B_{\beta\gamma})$ entering (13), that annihilate and create ab pairs :

$$\hat{T}^{B}_{-} \;=\; \hat{a}_{\alpha\beta}\,B_{\beta\gamma}\,\hat{b}_{\gamma\alpha}, \quad \hat{T}^{B}_{+} \;=\; (\hat{T}^{B}_{+})^{\dagger} \;=\; \hat{a}^{\dagger}_{\alpha\beta}\,B^{*}_{\beta\gamma}\,\hat{b}^{\dagger}_{\gamma\alpha}. \tag{14}$$

More specifically, we can consider $B=E_{\beta\gamma}$ (the matrix with 1 in the intersection of β-th row and γ-th column and 0's otherwise). The corresponding lowering and rising operators are :

$$\hat{T}^{\beta\gamma}_{-} \;=\; \hat{a}_{\alpha\beta}\,\hat{b}_{\gamma\alpha}, \quad \hat{T}^{\beta\gamma}_{+} \;=\; (\hat{T}^{\beta\gamma}_{+})^{\dagger} \;=\; \hat{a}^{\dagger}_{\alpha\beta}\,\hat{b}^{\dagger}_{\gamma\alpha}. \tag{15}$$

In (14) and (15) the summation over α is understood. Any boost can be uniquely expressed as complex combinations of operators $T^{\beta\gamma}_{-}$ and $T^{\beta\gamma}_{+}$. It follows from (14) that the operator

$$\hat{N}\equiv\frac{1}{2}(\hat{N}_{\hat{b}}-\hat{N}_{\hat{b}}) \;=\; \frac{1}{2}(\hat{b}^{\dagger}_{\alpha\beta}\hat{b}_{\alpha\beta} \;-\; \hat{a}^{\dagger}_{\alpha\beta}\hat{a}_{\alpha\beta}) \;=\; \frac{1}{2}Tr(\hat{Z}^{\dagger}\Gamma\hat{Z}) \;-\; 2 \tag{16}$$

commutes with all generators $\hat{X}_{AB}$, $A, B = 0, 1, \ldots, 5$.

Below, we shall restrict ourselves to most degenerate discrete series representations which are specified by the eigenvalue of the operator $\hat{N}$ in the representation subspace. We start to construct the representation space $\mathcal{F}_N$ from a distinguished normalized state containing lowest number of particles :

$$\begin{aligned} |x_0\rangle &= \frac{\det(\hat{b}^\dagger)^{N-1}}{(N-1)!\,\sqrt{N}}\,|0\rangle,\ N = 1, 2, \cdots \\ &= \frac{1}{\sqrt{N}}\sum_{n=0}^{N-1}(-1)^n\,|\,0,0,0,0\,;\,N-1-n,n,n,N-1-n\rangle\,. \end{aligned} \tag{17}$$

Here, $\hat{b}^\dagger = (\hat{b}^\dagger_{\alpha\beta})$ is 2×2 matrix of b-particle creation operators, and N is a natural number that specifies the representation : $\hat{N}\,|x_0\rangle = (N-1)\,|x_0\rangle$. All other states in the representation space are obtained by the action of rising operators given in (14) : such states contain besides $(2N-2)$ b-particles a finite number of ab pairs.

The maximal compact subgroup $K = S(U(2)\times U(2))$ is the stability group of the state $|x_0\rangle$. The group action for $k = \mathrm{diag}(k', k'')$ reduces just to the phase transformation (see (12)) :

$$\hat{T}(k)\,|x_0\rangle \;=\; \det(k''^\dagger)^{N-1}\det(k')\,|x_0\rangle \;=\; \det(k'')^{-N}\,|x_0\rangle\,. \tag{18}$$

The first factor comes from $\hat{b}^\dagger \;\mapsto\; \hat{b}^\dagger\,k''^\dagger$ (see (12)), whereas the second factor comes from $\hat{T}(k)\,|0\rangle = \det(k')\,|0\rangle$ (due to the anti-normal ordering of the compact generators containing $\hat{a}$ and $\hat{a}^\dagger$).

Let us calculate the mean values of the operator $\hat{T}(g)$ in the state $|x_0\rangle$: $\omega_0(g) = \langle x_0|\,\hat{T}(g)\,|x_0\rangle$. Using Cartan decomposition $g = k\,\delta\,q$ and the action(17) of rotations we obtain :

$$\begin{aligned} \omega_0(g) \;&=\; \langle x_0|\,\hat{T}(g)\,|x_0\rangle \;=\; \langle x_0|\,\hat{T}(k)\,\hat{T}(\delta)\,\hat{T}(q)\,|x_0\rangle \\ &=\; \det(k'')^{-N}\det(q'')^{-N}\,\langle x_0|\,\hat{T}(\delta)\,|x_0\rangle. \end{aligned} \tag{19}$$

Thus it is enough to calculate the mean value for the special boost :

$$\delta = \begin{pmatrix} C & S \\ S & C \end{pmatrix} = \begin{pmatrix} E & 0 \\ T & E \end{pmatrix}\begin{pmatrix} C & 0 \\ 0 & C^{-1} \end{pmatrix}\begin{pmatrix} E & T \\ 0 & E \end{pmatrix} = t_+\,t_0\,t_-. \tag{20}$$

Here $C = \mathrm{diag}(\mathrm{ch}\,\lambda_1, \mathrm{ch}\,\lambda_2)$, $S = \mathrm{diag}(\mathrm{sh}\,\lambda_1, \mathrm{sh}\,\lambda_2)$ and $T = \mathrm{diag}(\mathrm{th}\,\lambda_1, \mathrm{th}\,\lambda_2)$. In the representation in question, the matrices t_+ and t_- are exponents of rising and lowering operators respectively :

$$\hat{T}(t_+) \;=\; e^{\mathrm{tr}(\hat{b}^\dagger\,T\,\hat{a}^\dagger)},\quad \hat{T}(t_-) \;=\; e^{-\mathrm{tr}(a\,T\,b)}.$$

Since $\hat{T}(t_-)$ contains $\hat{a}$ its action does not affect $|x_0\rangle$, and similarly $\hat{T}(t_+)$ containing $\hat{a}^\dagger$ does not affect $\langle x_0|$. The only non-trivial action comes from

$$\hat{T}(t_0) \;=\; e^{-\,\mathrm{tr}(\hat{a}\,\Lambda\,\hat{a}^\dagger)\,-\,\mathrm{tr}(\hat{b}^\dagger\,\Lambda\,\hat{b})},\quad \Lambda \;=\; \ln C\,. \tag{21}$$

Consequently,

$$\hat{T}(t_0)|x_0\rangle \;=\; \det(C)^{-N}\,|x_0\rangle\,. \tag{22}$$

The last equality follows from the identity

$$e^{-\mathrm{tr}(\hat{b}^\dagger\,\Lambda\,b)}\det(\hat{b}^\dagger)^N \;=\; e^{-N\,\mathrm{tr}(\Lambda)}\det(\hat{b}^\dagger)^N\,e^{-\mathrm{tr}(\hat{b}^\dagger\,\Lambda\,b)}$$

(which can be proven, e.g. by induction in N). From equations (18) and (21) we obtain remarkably simple results :

$$\omega_0(g) \;=\; \langle x_0|\,\hat{T}(g)\,|x_0\rangle \;=\; \det(k'')^{-N}\det(C)^{-N}\det(q'')^{-N} \;=\; \det(d)^{-N}. \tag{23}$$

Here $d \;=\; k''\,C\,q''$ is the lower-right block of matrix g (see the Cartan decomposition in (19)). We recovered the results valid in the holomorphic representation (1).

F.4 The star product

Starting from the normalized state $|x_0\rangle \;\in\; \mathcal{F}_N$ we shall construct the Perelomov's system of coherent states for the representation in question. We choose a set of boosts of the form :

$$g_x = k\,\delta\,k^\dagger \;=\; \begin{pmatrix} k'\,C\,k'^\dagger & k'\,S\,k''^\dagger \\ k''\,S\,k' & k''\,C\,k''^\dagger \end{pmatrix} \equiv \begin{pmatrix} C' & \tilde{S} \\ \tilde{S}^\dagger & C'' \end{pmatrix} \in G/K, \tag{1}$$

where $k \;\equiv\; \tilde{k} \;=\; \mathrm{diag}(k', k'')$ is an element of K with eliminated Cartan $U(1)$ factor, $C = \mathrm{diag}(\mathrm{ch}\,\lambda_1, \mathrm{ch}\,\lambda_2)$ and $S = \mathrm{diag}(\mathrm{sh}\,\lambda_1, \mathrm{sh}\,\lambda_2)$. Thus $x = x(k,\delta)$ depends on 8 parameters : 6 of them, hidden in k, are compact, and there are 2 non-compact ones λ_1 and λ_2 representing special boosts. To any boost g_x, given in (1), we assign coherent coherent state (see, [88])

$$|x\rangle \;=\; \hat{T}(g_x)\,|x_0\rangle \;=\; \hat{T}(k\,\delta\,k^\dagger)\,|x_0\rangle, \quad x = x(k,\delta). \tag{2}$$

Note : Any $x = x(k,\delta)$, can be uniquely assigned to the 2×2 complex matrix $z = k'\,T\,k''^\dagger$, $T = \mathrm{diag}(\mathrm{th}\,\lambda_1, \mathrm{th}\,\lambda_2)$, forming the bounded Bergman domain $D \cong G/K$.

Let us consider operators in the representation space of the form

$$\hat{F} \;=\; \int_G dg\,\tilde{F}(g)\,\hat{T}(g), \tag{3}$$

where $\tilde{F}(g)$ is a distribution on a group G with compact support $\mathrm{supp}\,\tilde{F}$. To any such operator we assign function on G/K by the prescription

$$F(x) \;=\; \langle x|\,\hat{F}\,|x\rangle \;=\; \int_G dg\,\tilde{F}(g)\,\omega(g,x), \tag{4}$$

where

$$\omega(g,x) \;\equiv\; \langle x|\,\hat{T}(g)\,|x\rangle \;=\; \omega_0(g_x^{-1}\,g\,g_x). \tag{5}$$

This equation combined with (23) offers an explicit form of $\omega(g,x)$ and is well suited for calculations.

The star-product of two functions $F(x) = \langle x|\,\hat{F}\,|x\rangle$ and $G(x) = \langle x|\,\hat{G}\,|x\rangle$ was defined in [89] :

$$(F \star G)(x) \;=\; \langle x|\,\hat{F}\hat{G}\,|x\rangle \;=\; \int_{G\times G} dg_1 dg_2\, \tilde{F}(g_1)\,\tilde{G}(g_2)\,\omega(g_1 g_2, x)$$

$$= \int_G dg\,(\tilde{F}\circ\tilde{G})(g)\,\omega(g,x). \tag{6}$$

In (6) the symbol $\tilde{F}\circ\tilde{G}$ denotes the convolution in the group algebra $\tilde{\mathcal{A}}_G$ of compact distributions :

$$(\tilde{F}\circ\tilde{G})(g) \;=\; \int_G dh\,\tilde{F}(gh^{-1})\,\tilde{G})(h). \tag{7}$$

Obviously, the mapping $\tilde{F} \mapsto F$ given in (4) is a homomorphism of the group algebra $\tilde{\mathcal{A}}_G$ into the star-algebra $\mathcal{A}^\star_G$ of functions (4) on $D = G/K$. The quantized Bergman domain $D_\star$ we identify with the noncommutative algebra of functions $\mathcal{A}^\star_G$ on D.

Note : We point out that as in the case of usual distributions, the convolution product may exist even for distribution with non-compact support provided there are satisfied specific restrictions at infinity.

It can be easily seen that

$$\mathrm{supp}\,(\tilde{F}\circ\tilde{G}) \;\subset\; (\mathrm{supp}\,\tilde{F})\,(\mathrm{supp}\,\tilde{G}) \;\equiv\; \{g = g_1 g_2 \,|\, g_1 \in \mathrm{supp}\,\tilde{F},\; g_2 \in \mathrm{supp}\,\tilde{G}\}.$$

Consequently, for a non-compact group there are two classes of group algebras :

(i) The first one is generated by distributions with a general compact support and the corresponding group algebra is simply the full algebra $\tilde{\mathcal{A}}_G$ defined in (6).

(ii) The second one is formed by distributions $\tilde{F}$ with $\mathrm{supp}\,\tilde{F}$ subset of a subgroup $H \subset K$, form a sub-algebra $\tilde{\mathcal{A}}_H$ of the group algebra $\tilde{\mathcal{A}}_G$.

In the second class there are are two interesting extremal cases :

(a) $\tilde{\mathcal{A}}_{\{e\}}$ corresponding to the trivial subgroup $H = \{e\}$ in $G = SU(2,2)$ ($\tilde{\mathcal{A}}_{\{e\}}$ is isomorphic to the enveloping algebra $\mathcal{U}(su(2,2))$, see e.g., [97] or [98]).

(b) $\tilde{\mathcal{A}}_K$ corresponding to the maximal compact subgroup K in G.

The deformation quantization on Lie group co-orbits in terms of the Lie group convolution algebra $\tilde{\mathcal{A}}_{\{e\}}$ was introduced by [83]. Here we follow the related coherent state construction of the star-star on Lie group co-orbits proposed in [89].

Any distribution $\tilde{F}$ can be given as a linear combination of finite derivatives of the group δ-function, i.e., as a linear combination of distributions

$$\tilde{F}_{A_1B_1\ldots A_nB_n}(g) \;=\; (\mathcal{X}_{A_1B_1}\,\ldots\,\mathcal{X}_{A_nB_n}\delta)(g), \tag{8}$$

where $\mathcal{X}_{AB}$ is the left-invariant vector field representing the generator X_{AB}. The explicit form of $\mathcal{X}_{AB}$ in terms of the coordinates of the Bergman domain has been given by [85] and [86].

Inserting this into (4) we obtain the corresponding function from $\mathcal{A}^\star_{\{e\}}$

$$F_{A_1B_1\ldots A_nB_n}(x) \;=\; (-1)^n(\mathcal{X}_{A_nB_n}\,\ldots\,\mathcal{X}_{A_1B_1}\omega)(g,x)|_{g=e}. \tag{9}$$

Here we used the fact that the operators $\mathcal{X}_{AB}$ are anti-hermitian differential operators with respect to the group measure dg. From (4) it follows directly that

$$(F_{A_1B_1\ldots A_nB_n} \star F_{C_1D_1\ldots C_mD_m})(x)$$

$$= (-1)^{n+m}(\mathcal{X}_{A_nB_n} \ldots \mathcal{X}_{A_1B_1}\, \mathcal{X}_{C_mD_m} \ldots \mathcal{X}_{C_1D_1}\omega)(g,x)|_{g=e}. \tag{10}$$

Equations (9) and (10) describe explicitly the homomorphism $\mathcal{U}(su(2,2)) \to \mathcal{A}^{\star}_{\{e\}}$.

Using exponential parametrization of the group element $g = e^{\xi^{AB}X_{AB}}$ formula for the symmetrized function (9) takes simple form :

$$F_{\{A_1B_1\ldots A_nB_n\}}(x) = (-1)^n(\partial_{\xi_{A_1B_1}} \ldots \partial_{\xi_{A_nB_n}}\omega)(e^{\xi^{AB}X_{AB}}, x)|_{\xi=0}$$

$$= (-1)^n \langle x|\, \hat{X}_{\{A_1B_1} \ldots \hat{X}_{A_nB_n\}}|x\rangle, \tag{11}$$

where $\{\ldots\}$ means symmetrization of double indexes and $\xi = 0$ means the evaluation at $\xi_{AB} = 0$ for $A, B = 0, 1, \ldots, 5$. Symmetrized form a basis of the algebra in question and symmetrized elements from the center of algebra correspond to Casimir operators. In the series of representation in question all Casimir operators are given in terms of a single operator $\hat{N}$ given in (16) which is represented by a constant function $N(x) = \langle x|\hat{N}|x\rangle = N$.

Example 1. : The function $\omega(g,x)$. Let us calculate the function $\omega(g,x) = \omega_0(g_x^{-1}\, g\, g_x)$ explicitly. Taking g and g_x as in (1) and (1) we have to calculate the product of three matrices :

$$g_x^{-1}\, g\, g_x = \begin{pmatrix} C' & -\tilde{S} \\ -\tilde{S}^{\dagger} & C'' \end{pmatrix} \begin{pmatrix} a & b \\ c & d \end{pmatrix} \begin{pmatrix} C' & \tilde{S} \\ \tilde{S}^{\dagger} & C'' \end{pmatrix} \equiv \begin{pmatrix} a_x & b_x \\ c_x & d_x \end{pmatrix},$$

where $d_x = C''\, d\, C'' + C''\, c\, \tilde{S}^{\dagger} - \tilde{S}\, c\, C'' - \tilde{S}^{\dagger}\, a\, \tilde{S}$. Using equation (23) for $\omega_0(g)$ we obtain

$$\begin{aligned} &\omega(g,x) = \det(d_x)^{-N} \\ &= \det(E - z_x^{\dagger}z_x)^N \det(d + cz_x^{\dagger} - z_x b - z_x a z_x^{\dagger})^{-N} \end{aligned} \tag{12}$$

Here $z_x = C'^{-1}\tilde{S} = k'Tk''^{\dagger}$ is 2×2 complex matrix from the Bergman domain. In this form the expression is convenient for calculations.

Example 2. : Fock space realization of the co-adjoint orbit : Our aim is to calculate the coordinates

$$\xi_{AB}(x) = \frac{1}{N}\langle x|\hat{X}_{AB}|x\rangle = \frac{1}{N}\langle x_0|\hat{T}^{\dagger}(g_x)\, \hat{X}_{AB}\hat{T}(g_x)|x_0\rangle, \tag{13}$$

for $A, B = 0, 1, \ldots, 5$. Taking into account (11) we see that $\xi_{AB}(x) = D^{CD}_{AB}(g_x)\, \xi_{AB}(x_0)$, where $(D^{CD}_{AB}(g)) = Ad^*_g$ is the matrix corresponding to the group action in co-adjoint algebra. Therefore it is sufficient to evaluate the coordinates at x_0 : $\xi_{AB}(x_0) = \frac{1}{N}\langle x_0|\hat{X}_{AB}|x_0\rangle$. A simple calculation gives : $\xi_{45}(x_0) = 1$ with all other $\xi_{AB}(x_0) = 0$. We see that $\xi_{AB}(x)$ just forms the co-adjoint orbit generated from $\xi_{AB}(x_0)$.

Example 3. The star-product of coordinates $\xi_{AB}(x)$: We have

$$(\xi_{AB} \star \xi_{CD})(x) \;=\; \frac{1}{2N^2}\,\langle x|\{\hat{X}_{AB}, \hat{X}_{CD}\}|x\rangle \;+\; \frac{1}{2N^2}\,\langle x|[\hat{X}_{AB}, \hat{X}_{CD}]|x\rangle,$$

where $\{\dots\}$ denotes anti-commutator and $[\dots]$ is commutator. Therefore, the second term is

$$\frac{1}{2N^2}\,\langle x|[\hat{X}_{AB}, \hat{X}_{CD}]|x\rangle \;=\; \frac{1}{2N}\,f^{EF}_{AB,CD}\,\xi_{EF}(x),$$

where we used the definition of $\hat{X}_{AB}$ and the short-hand notation for the commutator (20) : $[X_{AB}, X_{CD}] = f^{EF}_{AB,CD} X_{EF}$. The first term is proportional to the symmetrized function $F_{\{AB,CD\}}$ and we can use (11) :

$$\frac{1}{2N^2}\,\langle x|\{\hat{X}_{AB}, \hat{X}_{CD}\}|x\rangle \;=\; (1 + A_N)\,\xi_{AB}(x)\,\xi_{CD}(x) \;+\; B_N\,\delta_{AB,CD},$$

where we have a usual point-wise product of functions in the first term and $\delta_{AB,CD} = (1/2)(\delta_{AC}\delta_{BD} - \delta_{AD}\delta_{BC})$ in the second one. The coefficients A_N and B_N are of order $1/N$. Last two equations give

$$(\xi_{AB} \star \xi_{CD})(x) \;=\; (1 + A_N)\,\xi_{AB}(x)\,\xi_{CD}(x) \;+\; \frac{1}{2N}\,f^{EF}_{AB,CD}\,\xi_{EF}(x) \;+\; B_N\,\delta_{AB,CD}. \quad (14)$$

We see that the parameter of the non-commutativity is $\lambda_N = 1/N$. For $N \to \infty$ we recover the commutative product.

Théorème F.4.1 *The star product (6, 14) is associative and invariant under the transformation of* $SU(2,2)$ *group.*

The proof of this theorem follows directly the definition of the star product.

F.5 Quantum field on a Bergman domain D

F.5.1 The invariant Laplacian on D

The invariant Laplacian Δ_N is defined by :

$$\Delta_N \hat{T}_g = \hat{T}_g \Delta_N \quad (1)$$

where $\hat{T}_g$ is the representation operator given by (10). We have (see [100], [103]) :

$$\begin{aligned} &\Delta_N = tr[(E - ZZ^\dagger)\bar{\partial}_Z \cdot (E - Z^\dagger Z) \cdot \partial'_Z] \quad (2)\\ +\;&\det(E - Z^\dagger Z)^{-N} tr[(E - ZZ^\dagger)\bar{\partial}_Z \cdot (E - Z^\dagger Z) \cdot \partial'_Z(\det(E - Z^\dagger Z)^N)] \end{aligned}$$

where $\partial_Z = (\partial_{z_{ij}})$ is the 2×2 matrix of differential operators in the variables $Z = (z_{ij}) \in D$, $\bar{\partial}_Z$ and ∂'_Z denote, respectively the complex conjugate and the transpose of the matrix operator ∂_Z. It is understood that the operators $\bar{\partial}_Z$ and ∂'_Z do not differentiate the matrices $E - ZZ^\dagger$ and $E - Z^\dagger Z$.

The Laplacian Δ_N is self-adjoint on $L^2(D, d\mu_N)$ with respect to the measure given by (2).

In what follows we consider only the radial part of the invariant Laplacian Δ_N, as this part that contains information about most interesting physical quantities, e.g., the energy levels. The radial part of the invariant Laplacian could be constructed from the roots system introduced in section F.2.3, (see ([103], [104]) for more details). The radial part of the invariant Laplacian reads :

$$\Delta_N^r = \omega^{-1}(\sum_{i=1}^{2} \frac{1}{4} L_i \; - \; \frac{N}{2} \mathrm{th}\, \lambda_i \, \partial_{\lambda_i})\omega \tag{3}$$

where

$$\omega = 2(\mathrm{ch}\, 2\lambda_1 \; - \; \mathrm{ch}\, 2\lambda_2) \tag{4}$$

and

$$L_i = \frac{\partial^2}{\partial {\lambda_i}^2} + 2\mathrm{cth}\, 2\lambda_i \, \partial_{\lambda_i} \tag{5}$$

Let $\Phi(N, \tau_1, \tau_2)$ be the eigenfunction of Δ_N^r, we have (see[103]) :

$$\Delta_N^r \Phi(N, \tau_1, \tau_2) = -\frac{1}{4}[2(N-1)^2 + \tau_1^2 + \tau_2^2]\Phi(N, \tau_1, \tau_2) \tag{6}$$

where τ_1 and τ_2 are given by (19).

The operator $-\Delta_N$ is positive : it has the continuous spectrum

$$\big[-\frac{1}{2}(N-1)^2, +\infty\big) \tag{7}$$

for arbitrary τ_i, and the discrete finite spectrum

$$(N-1)(l_1 + l_2) \; + \; l_1^2 + l_2^2 \tag{8}$$

for $\tau_j, \;\; j = 1, 2$, imaginary :

$$\tau_j = -i(N - 1 - 2l_j), \;\; l_j = 0, 1, \cdots, \big[\frac{N-1}{2}\big]. \tag{9}$$

Here $\big[\frac{N-1}{2}\big]$ means the integer part of $\frac{N-1}{2}$. The discrete spectrum consists of $\frac{1}{2}\; k(k-1)$ points, where $k = \big[\frac{N-1}{2}\big]$.

F.5.2 A quantum field theory model

We shall present a construction of a real scalar (Euclidean) field theory on a quantized Bergman domain. The action of the model in question reads :

$$S[\Phi] \;\; = \;\; \int d\mu_N(\xi) \; [-\frac{1}{2}\Phi \star \Delta_N \Phi + \frac{1}{2} m^2 \Phi \star \Phi + V_\star(\Phi)] \, , \tag{10}$$

where $\Phi = \Phi(\xi)$ is a real scalar field depending on the noncommutative coordinates ξ and $d\mu_N(\xi)$ is the measure (2) expressed in terms of ξ. We suppose that V is a polynomial of Φ bounded from below. Then we expand the scalar field Φ in terms of the eigenfunction of the invariant Laplacian :

$$\Phi = \int d\tau_1 d\tau_2 \, C_N(\tau_1, \tau_2) \Phi(N, \tau_1, \tau_2) + \sum_{l_1, l_2} C_{N, l_1, l_2} \Phi(N, l_1, l_2), \tag{11}$$

where we integrated over the continuous part of spectrum and summed up over the discrete part of the spectrum. The coefficients $C_N(\tau_1, \tau_2)$ and C_{N,l_1,l_2} are arbitrary real numbers.

The quantum mean value of some polynomial field functional $F[\Phi]$ is defined as the functional integral over fields Φ :

$$\langle F[\Phi] \rangle \;=\; \frac{\int D\Phi\, e^{-S[\Phi]}\, F[\Phi]}{\int D\Phi\, e^{-S[\Phi]}}, \tag{12}$$

where $D\Phi = D_x d\Phi(x) \cong \prod_{\tau_j} dC_N(\tau_j)$ and $S[\Phi]$ denotes the corresponding action (10).

For the free field propagator we recover the quantum field theory results on a commutative Bergman domain D

$$< \Phi, \Phi >= \frac{1}{m^2 + \frac{1}{4}[2(N-1)^2 + \tau_1^2 + \tau_2^2]} \tag{13}$$

which is valid for arbitrary τ_j :

- For the discrete part of the spectrum where $\tau_j = -i(N - 1 - 2l_j)$, the quantum field theory is finite, it possesses a cutoff at the maximal energy level N.
- For the continuous part of the spectrum where the parameters τ_j are arbitrary, the theory is divergent. But it could be made finite after proper renormalization. This point will be studied in more detail in future publications.
- When N = finite, we have $\frac{1}{N}$ corrections for the vertices coming from the lowest order of the star product. The divergent behavior is similar to the semiclassical case.

F.6 Concluding remarks

In this paper we introduced an oscillator realization of the discrete series of $SU(2,2)$ representations. We performed a deformation quantization over the corresponding coset space $D = SU(2,2)/S(U(2) \times U(2))$. We presented an explicit expression of the star-product over D. Using this star product we constructed a QFT model over this noncommutative Bergman domain $D_\star$. This method can be applied to other $SU(m,n)$ type I Cartan domains, see [105], where $SU(2,1)$ is discussed in detail. Such results are of interest for both physics and mathematics. From the physical point of view, $SU(2,2)$ is the maximal symmetry group of the (compactified) Minkowski space. It is also of interest for the ADS-CFT correspondence, as $SU(2,2)$ is the double cover of $SO(4,2)$ conformal group. As the $SU(2,2)$ module, the Bergman domain D is a Kähler manifold which is important in supersymmetric Quantum field theory and string theory. In addition the Bergman domain D has nontrivial Shilov boundary, and the quantization of D could help us to understand this boundary problem in the framework of noncommutative geometry. These aspects are under study and will be discussed in a forthcoming paper.

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