Planar Graph Homologies and the 4-Color Theorem.

J.I. Pillay[1]
Institute of Actuaries Scotland.

Abstract
We introduce a Planar-Graph homology with the aim of simplifying the algebra associated with the establishment of minimal coloring associated with planar-graphs. We then use the theory introduced to propose a simple proof of the 4-color theorem. Finally we introduce a proposed relation between information exposed via a chosen homology and others forming a basis for a new platform.

Introduction.
Planar graph theory needs vary little in the way of introduction. The importance of a planar graph representation stems from its ability to simplify complex information. In a sence, by virtue of the nature of the representation, these dispose of unnecessary information and expose only the bare necessary on a platform capable of allowing one to make observations or inferences.
Many algebras have been developed via matrix algebra, thus providing a platform for use in investigations surrounding chromatic numbers[B]. Similar representations such as the Kneser Graph are further investigated for studying Fractional-Chromatic numbers[S-U], here the study extends to general (non-planar) graphs and hyper Graphs. Recent trends in Graph Theory[1][V] extend into areas as diverse as Engineering, Electronics and even Management Sciences where the study of work delegation is of interest. At this point, an impression is that all available algebra has been exhausted on this one homology. We believe that the time is ripe for the study into the relation (which we believe exists) between a homology and the platform it generates via existing algebra for use in investigations. Our investigations have led us to believe that the information exposed for use in deductions; seems neatly to be tied in with the nature of the homology itself. We will effect change in the information exposed via certain proposed alterations to the existing planar graph representation. This new representation offers an informative structure that enables one to see symmetry on a grand scale via the recursive progression of adjunctions of planar graphs on an alternate platform. The representation we propose, deforms vertices to blocks and reduces multiply extended lines from a point to a single horizontal line (see Fig.1) on planar graphs.
Each adjunction of a set of simply connected elements can be seen to be offered only a limited set of exposed or, available for adjunction elements. These, for the present, block point representations subsequently enable us to see pattern in the grand structure, which will be shown to be, in a sense, hieratical recursive in its entirety. Before making precise what it is we mean by hierarchies, these can be thought of as elements restricting placement-values available to other

[1]Dedicated to my asawa C.Wacal

adjacent elements by virtue of their dominance, expressed by being adjacent to two or more adjoined elements. See (Fig.1)
The key aspect in which this new representation differs from a planar graph is that vertices are deformed into blocks and lines are shrunk to points on a single line where appropriate. These deformations render the two representations topologically equivalent.

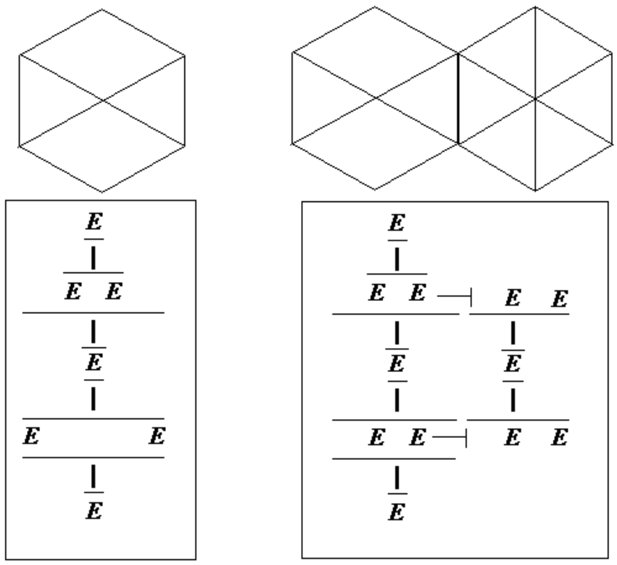

Figure 1: The representation to the right depicting lines between all vertices of the associated graph via a single bar-line extending from the bottom most element thereby also expressing its dominance over other vertically adjacent elements.

Each adjunction tends to nest selectively or otherwise, preceding elements. Reducing the initial set of elements to what is referred to as an elemental block, see (Fig.2), will highlight the emergence of a particular pattern.
Since arguments surrounding placement values for vertices are predicate on adjancies between elements, we need to be cautious with how we choose to have elements (as we refer to vertices), select placement-values. Continuous adjunctions of certain types of systems that have a nesting effect on others can again only 'connect' to other such systems via the same principle connections upon which the initial systems of elements are themselves built. As such these can be dealt with similarly via the reduction of such systems to elements and solving them in a similar manner to ordinary systems. This is the principle arguement around which our theory is built. We now make precise our previous arguements.

0. Preliminaries.

Notation

$C(n)$ Chromatic number n.

$A(\Gamma)$ Adjacency matrix.

$\overrightarrow{G}(S,[T],C(n))$ Go-To strategy with parameters (on set S,[strategy type],chromatic-number(n)).

$\{\xi| < H >\}$ A set of elements ξ of hierarchy $< H >$.

$< H|i >$ enumeration of hierarchal elements with respect to a V-system.

$V - system$ is a system with adjacency matrix of type (Expression 5).

$< H|I > V_\Sigma$ V-system V_Σ formed by the union of all associated elements $\xi\cup < H|j >$.

V_{ξ_i} elements of a V-system.

Definition 0.0

A representation that deforms vertices of a planar graph to blocks and reduces multiple lines extending from a single vertex to a single horizontal line (or H-line) forming a homology, is defined to be a canonical form.

Definition 0.1

Elements multiple connected to elements singly adjoined with ξ are said to have hierarchy $< h + 1 >$ over ξ.

$$\left[\frac{\xi_1 \quad \xi_2 \quad \xi_3}{\xi}\right] \mapsto A(\Gamma) := \begin{array}{c|ccc} & \xi_1 & \xi_2 & \xi_3 \\ \hline & 0 & 0 & 0 \\ \xi_{<H+1>} & 1 & 1 & 1 \\ & 0 & 0 & 0 \end{array} \tag{1}$$

Figure 1.1.1 The denominator like element has hierarchy $< H + 1 >$ over the adjoining elements, by the nature of the adjunction.

Definition 0.2

An element singly connected to ξ which is adjacent only to elements of $< H+1 >$ over ξ are said to have hierarchy $< H >$ with ξ. Written $\{\xi_j| < H >\}$

$$\left[\ \xi_1 \quad \xi_2 \quad \xi_3\ \right] \mapsto A(\Gamma) := \begin{array}{c|ccc} & \xi_1 & \xi_2 & \xi_3 \\ \hline \xi_1 & 0 & 1 & 0 \\ \xi_2 & 0 & 0 & 1 \\ \xi_3 & 0 & 0 & 0 \end{array} \tag{2}$$

Figure 1.1.2 $< H >$ connected elements.

Definition 0.3

Block representations of vertices on planar-graphs within a canonical form are referred to as elements.

Definition 0.4
A system is said to be solvable if its Planar-Graphs homology has chromatic number C(4).

1. Planar-Graphs and Grid-Forms

The construction of planar graphs can be described as a set $\mathcal{F}$ of n vertices between which another set $\mathcal{L}$ of non-overlapping lines can be drawn. Varying the order of vertices between which lines are drawn, along with number of such lines, one can construct any conceivable graph with n vertices. Since n is arbitrary, any planar graph can be constructed in this manner. To achieve this via grid forms, we begin with a set $C_1 := \{\xi_j| < H >\}$ of elements which is multiple adjoined to $C_2 := \{\xi_k| < H >\}$ such that elements of C_2 are of hierarchy $< H + 1 >$ over elements of C_1 and along a single face.
To this we may choose between the same face or its opposite and continue the above process to a necessary extent to form what we call a vertical system or simply V-system. Varying the number of elements in each row along with the reach of H-lines along each row, enables us to construct any possible V-system.

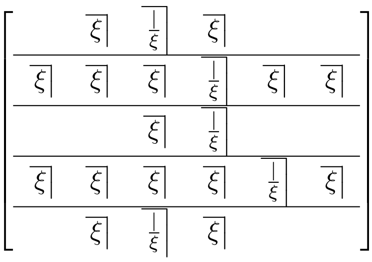

Figure 1.2.1 The figure describes the formation of a Vertical-system.

Such systems are associated with adjacency matrices of the form :

$$A(\Gamma)_{Part1} := \begin{array}{c|cccc cccc}
 & \xi_{j1} & \xi_{j2} & .. & \xi_{jp} & \xi_{h1} & \xi_{h2} & .. & \xi_{hq} \\
\hline
\xi_{1<H+1>} & [\,1 & 1 & .. & 1\,] & & & & \\
. & & & & & . & & & \\
\xi_{m<H+1>} & & & & & [\,1 & 1 & .. & 1\,]
\end{array} \tag{3}$$

$$A(\Gamma)_{Part2} := \begin{array}{c|cccc} & \xi_{X1} & \xi_{X2} & . & \xi_{X(pq)} \\ \hline \xi_{X1} & & 1 & & \\ \xi_{X2} & & & 1 & \\ . & & & & \searrow \\ \xi_{Xl} & & & & \end{array} \tag{4}$$

Figure 1.2.2 Note that such adjacencies are arbitrary when viewed within the framework of either planar graphs or adjacency matrices.

Moving now, along either one of the unused faces of a V-System, one may again repeat this process of choosing a face forming V-systems along and either switching to its opposite before continuation or resuming thereof, again to a necessary extent. Mixing the orientations of V-systems as such, to form a system, we will call a mixed or M-System. See(Fig.2).

Definition

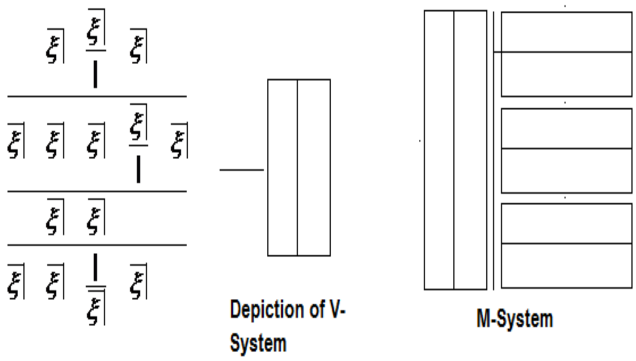

Figure 2: The representation to the right depicting a V-System where the bar indicates the orientation. To the left, is a typical set of adjunctions of V-systems by a change in orientation thus forming a M-System. One can continue such adjunctions to form more complex systems.

There exists a structural entity common to all V-systems. Along any one face of a V-system, there exists a set $\mathcal{L} := \{\xi_k| < H >\}$ of elements at odd intervals which are of hierarchy $< H + 1 >$ to elements of an adjoining vertical face, which follow through to the opposite face. We enumerate such elements as having hierarchy $< H|i >$, where $\{i\}$ is an enumeration of every such occurrence with respect to some initial V-system.
We bare in mind that every planar graph has an associated M-structure which can be formed via the principles described herein. The reason for which again, is simply that the deformations required of an associated canonical form does not violate the topological integrity of its associated planar graph. (See Expression. 5).

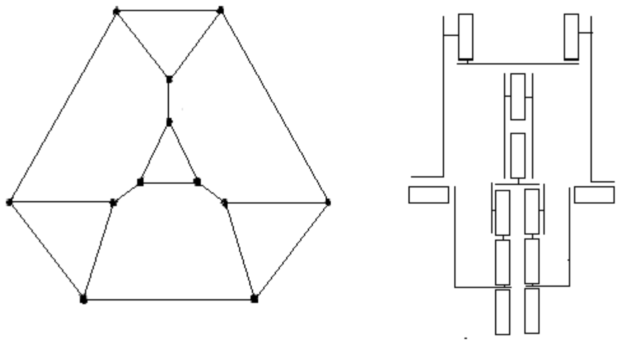

Figure 3: An M-system representation of a Caley graph to the left.

$$
\begin{array}{r|cccc c cccc}
 & \xi_{j1} & \xi_{j2} & .. & \xi_{jp} & & \xi_{h1} & \xi_{h2} & .. & \xi_{hp} \\
\hline
<H|1> \xi_{1<H+1>} & [\,1 & 1 & .. & 1\,] & & & & & \\
. & & & & & . & & & & \\
<H|m> \xi_{m<H+1>} & & & & & & [\,1 & 1 & .. & 1\,]
\end{array}
\qquad (5)
$$

Figure Each i in $< H|i >$ enumerates the occurrence of such a hierarchal adjunction with respect to some starting row of elements.

1.1. Reduction

1.1.1 Reduction Example

Given an $< H > -grid$, a $\overrightarrow{G}(< H >, [L, T], 3)$ or $\overrightarrow{G}(< H >, [R, T], 3)$ always exists, that solves the grid set.

Proof

Any one element can be affected up to three times. In addition, each element given the strategy, $\overrightarrow{G}(< H >, [L, T], 3)$ is at most subject to three faces of affecting elements to which the element in concern is absorbent. As such, the grid formation is solvable.

From hereon, the reduced grid set along with two $< H + 1 > -faceelements$ can be represented as :

$$
\begin{pmatrix} \sigma \\ \Sigma(L,T)| \\ \sigma \end{pmatrix} or \begin{pmatrix} \sigma \\ |\Sigma(L,T) \\ \sigma \end{pmatrix}
$$

The vertical line at the end of the central element signifies that the entire right/left face, aside from the σ-elements is open to any right/left face affector

element.
Since $|\Sigma(L,T)$ or $\Sigma(L,T)|$ is plausible given only the $< H >$-grid, we may make the reduction :

$$\begin{pmatrix} \sigma \\ < H > \Sigma \\ \sigma \end{pmatrix} \rightarrow \begin{pmatrix} \sigma \\ |\Sigma \\ \sigma \end{pmatrix} or \begin{pmatrix} \sigma \\ \Sigma| \\ \sigma \end{pmatrix}$$

where Σ denotes the top left/right most element.

The reduction above is a classical example of how one can reduce a system to a simple element, as we have demonstrated that under conditions of H-lines spanning only the circumference of such systems, both elements and such systems behave in much the same way as described by the properties of the representations above. Specifically, if surrounded by affecting H-lines, the reduced system is capable of directing an affector along any one of its faces, as the other faces have the remarkable property of being absorbent.

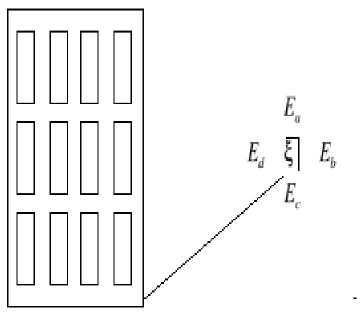

Figure 4:

This property is not restricted to $< H >$-grid s alone, in fact we will show shortly that this property extends to V-systems and as a consequence, to M-systems which will imply in a sense, that if V-systems are solvable, then so are generalized M-systems.

2. Application to the 4-Colour Theorem.

Proposition 2.1.
Any given V-system V_Σ can be solved via any of the go-to strategies $\{\vec{G}([L,T], \vec{G}([L,B], \vec{G}([R,T], \vec{G}([R,B]\}$.

Proof
Given a V-system :
in the bottom most row with set $\{\xi_k| < H >\}$, follow $\vec{G}(\xi_k, [X], 3)$ where the three placed at the end, indicates the number of influencing faces of the row that we appropriate. The final element ξ_e of the row is given hierarchy $< H+1 >$ over one of its vertically adjacent elements and follow $\vec{G}(\xi_k \cup \xi_e, [X], 3)$. Continuing this process through to the end of the V-system is possible due to the fact that the structure allows for this. Again, the final element $\{\xi_u\}$ of the V-system is again given $< H+1 >$ hierarchy over one of its horizontally adjacent elements. Since $[X]$ and the choice of the vertical/horizontal elements over which all ξ_e, ξ_u are given hierarchy is arbitrary, we can conclude that strategies toward any one corner (or even any element of a face) of a V-system is possible.
The above derives from the observation that, all elements of a V-System have one degree of freedom except for a single element t_ξ. Thus the entire system is solvable so long as we account for t_ξ by increasing its hierarchal status.

We express this strategy via the expression $\overrightarrow{G}(V_\Sigma, [R,T])$.

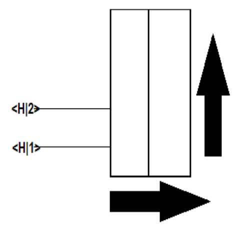

Figure 5: The horizontal arrow indicates that the progression is from left to right. The left being absorbent and the right being affecting. The vertical arrow does the same with exception to the direction moving from bottom to top.

Reduction Property 2.1.1
V_Σ can be treated as a singular element if none of the faces have splitting H-line adjancies.

Proof
We take note $\vec{G}(X)$ strategies associated with V-systems are structured in such a way that all faces of the system are absorbent aside from a single face associated the corner most or change-element ξ_σ associated with the system. We will

represent the corner along with the non-absorbent face in the following manner:

$$\left(\frac{\overset{o}{\boxed{\xi_\Sigma}}}{\Sigma}\right), \left(\frac{\overset{o}{\boxed{\xi_\Sigma}}}{\Sigma}\right), \left(\frac{o\,|\xi_\Sigma}{\Sigma}\right), \left(\frac{\xi_\Sigma|\,o}{\Sigma}\right)$$

The lines around the corner element depict the corner associated with the strategy. The (o) symbol is placed alongside a non-absorbent face. Thus any H-line associated with this element will have hierarchy $< H+1 >$ over elements adjacent along the H-line. These properties enable V_Σ to be treated as an element $V-\xi$, simply because, much like elements, such systems have three absorbent faces and a single non-absorbent one as is described via our preceding arguments.

Reduction Property 2.1.2

Case A) V_Σ with splitting H-line adjancies can be treated as two or more V_Σ-systems with each subsection belonging to the covering system associated wit the H-extension.

Case B) Is the case where multiple H-extensions form adjancies with an element ξ_p of V_Σ. Clearly here this case can be transposed by a single H-extension from the element covering the extensions from other systems.

Outline

All such H-lines will naturally emanate from some set $\mathcal{V} := \{V_\Sigma^k\}$ of V-systems. These will move along the associated faces to ξ_p. The reverse is naturally possible, that is for an H-Line to traverse along the appropriate faces to $\mathcal{V}$.

Proof

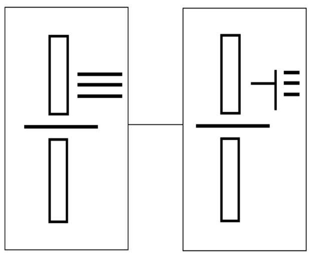

Figure 6:

The adjacency matrix associated with such adjacencies follows the schematic:

$$
A(\Gamma) = \begin{array}{c|ccccc}
 & J_1 & J_2 & J_3 & J_4 & . \\
\hline
. & 0 & 0 & 0 & 0 & . \\
\xi & 1 & 1 & 1 & 1 & . \\
. & 0 & 0 & 0 & 0 & . \\
. & . & . & . & . & . \\
. & . & . & . & . & .
\end{array} \tag{6}
$$

The transpose of which results in:

$$
A^T(\Gamma) = \begin{array}{c|ccccc}
 & . & \xi & . & . & . \\
\hline
J_1 & 0 & 1 & 0 & 0 & . \\
J_2 & 0 & 1 & 0 & 0 & . \\
J_3 & 0 & 1 & 0 & 0 & . \\
J_4 & 0 & 1 & 0 & 0 & . \\
. & . & . & . & . & .
\end{array} \tag{7}
$$

As such, it is easy to see that $A^T(\Gamma)$ always exists for adjacencies of type: $A(\Gamma)$ and such a transpose will denote a switch in hierarchy of the associated elements. Furthermore, $\forall i$, J_i may themselves be multiple H-connected with other elements and so on. As such, by switching the hierarchies of all $j_k \in \{J_i - \xi\}\forall i$ and subsequently following the same of elements of j_i and so on, until the very last of such connections, will ensure that $\forall \xi_i \in V(S)$ is associated with a single H-line extension.

Case C) In addition if it were mechanically required that the H-extensions be present for the case where certain elements are to be covered from certain extensions and other extensions are to be exposed to others, then each such subsystem with H-extension is to be considered a separate V_Σ-system.

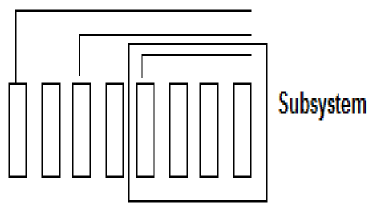

Figure 7:

All of the above reductions surmount to one thing, upon our canonical constructions, we need only consider cases where all faces of a V_Σ-system are enclosed by single H-extensions per face. Will emphasize later on why the case of such lines extending to multiple faces of a single V-system is trivial.

To better understand the following proof we take note of the proceeding interesting properties of V-systems. To recap, from our previous reductions, $\vec{G}(X)$ exists to any desired corner or face of a V-system. The adjunctions forming V-systems are very specific and do not include H-lines perpendicular to those within the system. Much like elements of a system, elements of hierarchy $< H + l >$ separate one series of such systems from another as described previously in our description of the formation of M-systems. As such, the connections surrounding the exterior of such systems comprise of the same possibilities as those possible between simple elements of a system. Thus the proof that follows takes advantage of this property by reducing a V-system with nested V-system as elements in exactly the same way that V-systems with simple elements are reduced.

Proposition 2.3.
M-systems have $\vec{G}(X)$ strategies that solve them.

Proof
From the preceeding Propositions, all $< H|1 > (V_\Sigma) :\to V_{\xi_i}$. Thus the V-system to which V_{ξ_i} belongs, (Regardless of its orientation), is solvable and subsequently reducible to an element of $< H|2 > (V_\Sigma)$ of $P(3)$.[2] This procedure can again be repeated $\forall (V_\Sigma) \in < H|2 > (V_\Sigma)$ thus forming a V-system to which the elements V_{ξ_i} of $P(3)$ belong, for which the process can again be repeated. Since all V-systems are of $P(3)$ just like elements belonging to such systems, we can follow this procedure for all $< H|i > (V_\Sigma)$ formed subsequent to aforementioned reductions. Finally, the V-system formed by the union $W := \{< H|I > V_\Sigma\}$, is again reducible to an element, as all $V_{\xi_i} \in W$ have been reduced to elements of W. This can again be repeated to the V-system to which W belongs and so on. (See Fig.8).

[2] P(n) denotes the number of effectible faces associated with the element.

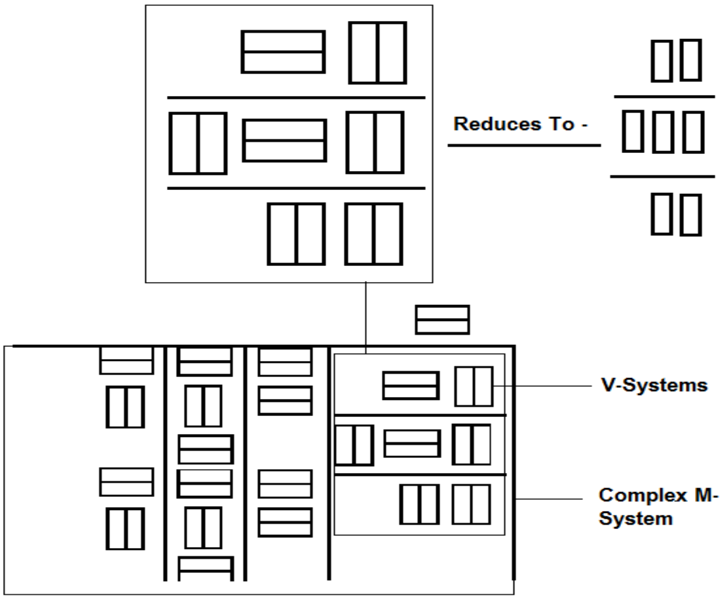

Figure 8: Note that post reduction, the surrounding connections are of the types associated with V-systems.

3. Topological Information.

As was pointed out in the initial portion of our article, a choice of representation exposes selective information, and It is this exposed information that interests us, as it is the manipulation of such information that has enabled us to prove a once difficult problem simplistically.
One could infer from the previous that the platform is responsible for the information it exposes. For instance given the platform exposed by planar graphs, one could make inferences on the information to which structures are mapped, via associated Propositions and associations involving number of vertices and lines. If instead our revised platform were used, the inferences would be transcribed via V-systems, M-systems and their formations. Though the same might be possible vis planar graphs, the platform is not conducive enough for exposing such structure formations clearly enough, as many structure types are entangled with others and as such make separation thereof difficult, if not impossible. A clear example of this is Apple and Hakken's proof of the 4-colour problem which involved the introduction of roughly a thousand plus forms for consideration [K-W]. As compared with our three, it seems very suggestive of entanglement of sort.

In a followup article, we will be investigating how such information comes to be and how one can possibly choose an appropriate representation for inference purposes. More specifically. let $(\mathcal{R}, \mathcal{P}(I))$ be the set respectively of representations and the set $\bigcup_{\forall i} X_i$ of inferences possible of the platform. We hope to investigate how transforming $T(\mathcal{R}, \mathcal{P}(I)) \mapsto (\mathcal{R}', \mathcal{P}'(I'))$ affects $\bigcup_{\forall i} X_i$.

References

[B] Norman Biggs, (1974). Algebraic Graph Theory: Cambridge Texts in Mathematics. Great Britan: Cambridge University Press.

[Bo] B.Bollobas, (1978). Advances in Graph Theory. US:Netherlands.

[K-W] Appel K, Hakken W, A976) Every Planar Map Is Four Colorable. Bull. Amer. Math. Soc. 82:711-712

[M] M.De Wilde, (1974).Closed graph theorems and webbed spaces. London:Pitman

[M-R] M.Molloy B.Reed, (2002). Graph Colouring and the Probabilistic Method. NY:Springer.

[M-U] R. Daniel Mauldin and Mariusz Urbanski, (2003). Graph Directed Markov Systems:Geometry and dynamics of limit sets. US:Cambridge University Press.

[S-U] Edward.R.Schcinerman, Daniel.H.Ullman, (1997). Fractional Graph Theory: A Rational Approach to the Theory of Graphs. US:J Wiley & Sons.

[V] R.Venkat Rao, (2007). Decision Making in the Manufacturing Environment:Using Graph Theory and Fuzzy Multiple AttributeDecision Making Methods. London:Springer.

[1] Lecture Notes in Mathematics: A collection of informal reports and seminars. NY: Springer-Verlag.

Acknowledgements

In a Professional circle.
Prof. E.E. Rosinger, Dr. W.E.Meyer, Dr. C.Zander, M.sc S.Swart

Personally.
M.sc Devaraj.N, U.Devaraj, R.Pillay, C.Wacal and Family My sincerest of gratitude extend to them.

Dr.P.Lingham For the many years of help extended by him to my family and me; without which many things would not have been possible, my sincerest of gratitude extends to him.

Dedicated with much love to my sister and Family in India.