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RANDOM UNCERTAINTIES MODEL IN DYNAMIC SUBSTRUCTURING USING A NONPARAMETRIC PROBABILISTIC MODEL

by

C. Soize¹ and **H. Chebli**²

¹ University of Marne-la-Vallée, Bat Copernic, 5 boulevard Descartes,
77454 Marne-la-Vallée, France - E-mail: soize@univ-mlv.fr

² Office National d'Etudes et de Recherches Aérospatiales, Chatillon, France
E-mail: chebli@onera.fr

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Abstract: This paper presents a new approach, called a nonparametric approach, for constructing a model of random uncertainties in dynamic substructuring in order to predict the matrix-valued frequency response functions of complex structures. Such an approach allows nonhomogeneous uncertainties to be modeled with the nonparametric approach. The Craig-Bampton dynamic substructuring method is used. For each substructure, a nonparametric model of random uncertainties is introduced. Such a nonparametric model does not require identifying the uncertain parameters in the reduced matrix model of each substructure as usually done for the parametric approach. This nonparametric model of random uncertainties is based on the use of a probability model for symmetric positive-definite real random matrices using the entropy optimization principle. The theory and a numerical example are presented in the context of the finite element method. The numerical results obtained show the efficiency of the model proposed.

INTRODUCTION

Here, we are interested in the low-frequency range where dynamical responses of structures modeled by finite element method (Zienkiewicz & Taylor, 1989) can be predicted by using

reduced matrix models constructed with the generalized coordinates of the mode-superposition method associated with the structural modes corresponding to the lowest eigenfrequencies of structures (Argyris & Mlejnek 1991, Bathe & Wilson 1976, Cheung & Leung 1992, Clough & Penzien 1975, G eradin & Rixen 1994, Meirovitch 1980, Ohayon & Soize 1998). In addition, dynamic substructuring methods are very efficient tools for dynamic analysis of complex structures (Benfield & Hruda 1971, Hintz 1975, Hurty 1965, MacNeal 1971, Meirovitch & Hale 1981, Rubin 1975). The Craig-Bampton method (Craig & Bampton 1968 for discrete case; Morand & Ohayon 1979 for continuous case) is very popular and efficient and is based on the use of the structural modes of each substructure with fixed coupling interface and the static boundary functions related to the coupling interface. Concerning the role played by uncertainties in structural dynamics modeling, it is known that the effects of uncertainties increase with the frequency. Consequently, in linear structural dynamics, numerical predictions with finite element models can be improved by introducing a model of random uncertainties. Random uncertainties in linear structural dynamics are usually modeled using parametric models. This means that 1) the uncertain parameters (scalars, vectors or fields) occurring in the boundary value problem (geometrical parameters; boundary conditions; mass density; mechanical parameters of constitutive equations; structural complexity, interface and junction modeling, etc.) have to be identified; 2) appropriate probabilistic models of these uncertain parameters have to be constructed, and 3) functions mapping the domains of uncertain parameters into the mass, damping and stiffness operators have to be constructed. Concerning details related to such a parametric approach, we refer the reader to (Haug et al 1986, Ibrahim 1987, Iwan & Jensen 1993, Lee & Singh 1994, Lin & Cai 1995, Soong 1973, Spanos & Zeldin 1994) for general developments, to (Ghanem & Spanos 1991, Kleiber et al 1992, Liu et al 1986, Shinozuka & Deodatis 1988, Spanos & Ghanem 1989, Vanmarcke & Grigoriu 1983) for general aspects related to stochastic finite elements and to (Ibrahim 1985, Kree & Soize 1986, Roberts & Spanos 1990, Soize 1994) for other aspects related to this kind of parametric models of random uncertainties in the context of developments written in stochastic dynamics and parametric stochastic excitations.

In this paper we present a new approach, that we will call a nonparametric approach, for constructing a model of random uncertainties in dynamic substructuring in order to predict the matrix-valued frequency response functions of complex structures. Such an approach allows

nonhomogeneous uncertainties to be modeled with the nonparametric approach. The use of dynamic substructuring allows a nonparametric approach of random uncertainties to be used when uncertainties are not homogeneous through the dynamic systems, that is to say, when the level of uncertainties differs from a part of the structure to another one. This nonparametric model of random uncertainties does not require identifying the uncertain local parameters in the reduced matrix model of each substructure as described above for the parametric approach but is based on the use of recent research (Soize 1999, 2000, 2001) in which the construction of a probability model for symmetric positive-definite real random matrices using the entropy optimization principle (Jaynes 1957, Kapur & Kesavan 1992) has been introduced and developed. These results allow the direct construction of a probabilistic model of the reduced matrix model of each substructure, for which the only information used in this construction is the available information constituted of the mean reduced matrix model of the substructure, the existence of second-order moments of inverses of certain random matrices and some algebraic properties relative to the positive-definiteness of these random matrices. It should be noted that these properties have to be taken into account in order to obtain a mechanical system with random uncertainties, which models a dynamic system. For instance if there are uncertainties on the reduced mass matrix, the probability distribution has to be such that this random reduced mass matrix be positive definite. If not, the probability model would be wrong because the reduced mass matrix of any substructure has to be positive definite.

We then propose an extension of the nonparametric model of random uncertainties to the Craig-Bampton substructuring method. The method presented could be used if the Craig-Bampton method was replaced by another substructuring techniques. Such an approach allows nonhomogeneous random uncertainties in a structure to be modeled by using the nonparametric approach. In a first section, we introduce the construction of the mean reduced dynamic stiffness matrix for a substructure using Craig & Bampton dynamic substructuring. The second section is devoted to the construction of the random reduced dynamic stiffness matrix for a substructure using the nonparametric model of random uncertainties. In the next section, we give a summary of the main results established in (Soize 2000, 2001) concerning the probability model for symmetric positive-definite real random matrices. The two next sections deal with the nonparametric model of random uncertainties of the reduced matrix model of each substructure

and the random reduced matrix model for the complete structure. Finally, a numerical example is presented.

CONSTRUCTION OF THE MEAN REDUCED DYNAMIC STIFFNESS MATRIX FOR A SUBSTRUCTURE USING DYNAMIC SUBSTRUCTURING

Mean finite element model for a substructure

Let us consider linear vibrations of a damped fixed structure \mathbb{S} submitted to external forces. We are interested in predicting the frequency response functions of structure \mathbb{S} in the frequency band of analysis $B = [\omega_{\min}, \omega_{\max}]$. In order to simplify the presentation, we limit the number of substructures to 2 and consequently, we assume that structure \mathbb{S} is decomposed into 2 substructures \mathbb{S}^r with $r = 1$ or 2 (the extension to a number of substructures greater than 2 is straightforward). Let Σ be the coupling interface between substructures \mathbb{S}^1 and \mathbb{S}^2 . Let Γ^r be the boundary of substructure \mathbb{S}^r (we then have $\Sigma \subset \Gamma^r$). Each substructure \mathbb{S}^r is assumed to be free on $\Gamma^r \setminus \Sigma$ (case of a *free substructure*) or fixed on a part of $\Gamma^r \setminus \Sigma$ (case of a *fixed substructure*). In this paper, the basic finite element model of structure \mathbb{S} is identified as the mean finite element model. In order to simplify the mathematical notations, exponent r related to substructure \mathbb{S}^r is canceled when no confusion is possible. For all ω in band B and for substructure \mathbb{S}^r , the mean matrix equation is written as

$$[\underline{\mathbb{A}}(\omega)] \underline{\mathbf{U}}(\omega) = \underline{\mathbf{F}}(\omega) \quad , \quad (1)$$

in which $\underline{\mathbf{U}}(\omega)$ is the \mathbb{C}^μ -valued vector constituted of the μ DOFs, $\underline{\mathbf{F}}(\omega)$ is the \mathbb{C}^μ -valued external force vector (due to load vector and coupling force vector) and $[\underline{\mathbb{A}}(\omega)]$ is the mean dynamic stiffness matrix which is written as

$$[\underline{\mathbb{A}}(\omega)] = -\omega^2[\underline{\mathbb{M}}] + i\omega[\underline{\mathbb{D}}] + [\underline{\mathbb{K}}] \quad , \quad (2)$$

where $[\underline{\mathbb{M}}]$, $[\underline{\mathbb{D}}]$ and $[\underline{\mathbb{K}}]$ are the mean mass, damping and stiffness matrices. Mean mass matrix $[\underline{\mathbb{M}}]$ is symmetric and positive definite. (1) For a fixed substructure, mean damping and stiffness matrices $[\underline{\mathbb{D}}]$ and $[\underline{\mathbb{K}}]$ are symmetric and positive definite (the substructure with free coupling interface Σ does not have rigid body modes). (2) For a free substructure, mean damping and stiffness matrices $[\underline{\mathbb{D}}]$ and $[\underline{\mathbb{K}}]$ are symmetric but are only semi-positive definite due to the

presence of rigid body modes (the substructure with free coupling interface Σ has rigid body modes); in this case, it is assumed that $[\underline{D}]$ and $[\underline{K}]$ have the same null space spanned by the μ_{rig} rigid body modes with $1 \leq \mu_{\text{rig}} \leq 6$. We introduce the usual decomposition of $\underline{\mathbf{U}}(\omega)$ and $\underline{\mathbf{F}}(\omega)$ with respect to the μ_i internal DOFs and the $\mu_{\Sigma} = \mu - \mu_i$ coupling DOFs. We then write

$$\underline{\mathbf{U}}(\omega) = \begin{bmatrix} \underline{\mathbf{U}}_i(\omega) \\ \underline{\mathbf{U}}_{\Sigma}(\omega) \end{bmatrix}, \quad \underline{\mathbf{F}}(\omega) = \begin{bmatrix} \underline{\mathbf{F}}_i(\omega) \\ \underline{\mathbf{F}}_{\Sigma}(\omega) + \underline{\mathbf{F}}_{\Sigma}^{\text{coup}}(\omega) \end{bmatrix}. \quad (3)$$

In Eq. (3), $\underline{\mathbf{F}}_{\Sigma}(\omega)$ is induced by the external prescribed load vector of the mean model on coupling interface Σ (for instance, due to the finite element discretization of an external prescribed body force field applied to the 3D domain of substructure \mathcal{S}^r) and $\underline{\mathbf{F}}_{\Sigma}^{\text{coup}}(\omega)$ is the coupling force vector of the mean model on coupling interface Σ . The corresponding block decomposition of the matrices are

$$[\underline{M}] = \begin{bmatrix} [\underline{M}_i] & [\underline{M}_c] \\ [\underline{M}_c]^T & [\underline{M}_{\Sigma}] \end{bmatrix}, \quad [\underline{D}] = \begin{bmatrix} [\underline{D}_i] & [\underline{D}_c] \\ [\underline{D}_c]^T & [\underline{D}_{\Sigma}] \end{bmatrix}, \quad [\underline{K}] = \begin{bmatrix} [\underline{K}_i] & [\underline{K}_c] \\ [\underline{K}_c]^T & [\underline{K}_{\Sigma}] \end{bmatrix}, \quad (4)$$

in which $[B]^T$ denotes the transpose of matrix $[B]$.

Mean reduced dynamic stiffness matrix for a substructure using Craig-Bampton method

The Craig-Bampton method (Craig & Bampton 1968) is used with N structural modes with fixed coupling interface Σ in order to construct the mean reduced dynamic stiffness matrix for substructure \mathcal{S}^r . We then obtain the following mean reduced matrix model for substructure \mathcal{S}^r ,

$$\begin{bmatrix} \underline{\mathbf{U}}_i(\omega) \\ \underline{\mathbf{U}}_{\Sigma}(\omega) \end{bmatrix} = [\underline{H}] \begin{bmatrix} \underline{\mathbf{q}}(\omega) \\ \underline{\mathbf{U}}_{\Sigma}(\omega) \end{bmatrix} \quad \text{with} \quad [\underline{H}] = \begin{bmatrix} [\underline{\Phi}] & [\underline{S}] \\ [0] & [I] \end{bmatrix}, \quad (5)$$

in which $\underline{\mathbf{q}}(\omega)$ is the \mathbb{C}^N -valued vector of the generalized coordinates, $[\underline{\Phi}]$ is the $(\mu_i \times N)$ real matrix whose columns are the structural modes of the mean model with fixed coupling interface, $[\underline{S}]$ is the $(\mu_i \times \mu_{\Sigma})$ real matrix defined by $[\underline{S}] = -[\underline{K}_i]^{-1}[\underline{K}_c]$, related to the static boundary functions and $[I]$ is the $(\mu_{\Sigma} \times \mu_{\Sigma})$ unity matrix. From Eqs. (1), (3) and (5), we deduced that

$$[\underline{A}(\omega)] \begin{bmatrix} \underline{\mathbf{q}}(\omega) \\ \underline{\mathbf{U}}_{\Sigma}(\omega) \end{bmatrix} = \begin{bmatrix} \underline{\mathcal{F}}(\omega) \\ \underline{\mathbf{F}}_{\Sigma}(\omega) + \underline{\mathbf{F}}_{\Sigma}^{\text{coup}}(\omega) \end{bmatrix}, \quad (6)$$

in which $[\underline{A}(\omega)]$ is the mean reduced dynamic stiffness matrix such that

$$[\underline{A}(\omega)] = -\omega^2[\underline{M}] + i\omega[\underline{D}] + [\underline{K}], \quad (7)$$

where reduced mass, damping and stiffness matrices are $(m \times m)$ real symmetric matrices, with $m = N + \mu_\Sigma$, such that

$$[\underline{M}] = [\underline{H}]^T [\underline{M}] [\underline{H}] , \quad [\underline{D}] = [\underline{H}]^T [\underline{D}] [\underline{H}] , \quad [\underline{K}] = [\underline{H}]^T [\underline{K}] [\underline{H}] \quad . \quad (8)$$

Since for a free or a fixed substructure, the null space of matrix $[\underline{H}]$ is reduced to $\{0\}$, from the previous properties, it can be deduced that reduced mass matrix $[\underline{M}]$ is symmetric and positive definite and, (1) for a fixed substructure, reduced damping and stiffness matrices $[\underline{D}]$ and $[\underline{K}]$ are symmetric and positive definite, (2) for a free substructure, reduced damping and stiffness matrices $[\underline{D}]$ and $[\underline{K}]$ are symmetric but are only semi-positive definite and $[\underline{D}]$ and $[\underline{K}]$ have the same null space spanned by the μ_{rig} rigid body modes with $1 \leq \mu_{\text{rig}} \leq 6$. In Eq. (6), the \mathbb{C}^N -valued vector $\underline{\mathcal{F}}(\omega)$ and the \mathbb{C}^{μ_Σ} -valued vector $\underline{\mathbb{F}}_\Sigma(\omega)$ are defined by

$$\underline{\mathcal{F}}(\omega) = [\underline{\Phi}]^T \underline{\mathbf{F}}_i(\omega) \quad , \quad (9)$$

$$\underline{\mathbb{F}}_\Sigma(\omega) = [\underline{S}]^T \underline{\mathbf{F}}_i(\omega) + \underline{\mathbf{F}}_\Sigma(\omega) \quad . \quad (10)$$

CONSTRUCTION OF THE RANDOM REDUCED DYNAMIC STIFFNESS MATRIX FOR A SUBSTRUCTURE USING A NONPARAMETRIC PROBABILISTIC MODEL OF RANDOM UNCERTAINTIES

In this section, we consider substructure \mathcal{S}^r with random uncertainties. We construct the probability model for the reduced dynamic stiffness matrix of this substructure using the principle introduced in the second section. This construction is fundamentally based on the use of the mean reduced dynamic stiffness matrix of substructure \mathcal{S}^r introduced in the previous section. In the following, we need notation relative to the following sets of matrices. Let $\mathbb{M}_n(\mathbb{R})$, $\mathbb{M}_n^S(\mathbb{R})$, $\mathbb{M}_n^{+0}(\mathbb{R})$ and $\mathbb{M}_n^+(\mathbb{R})$ be the set of all the $(n \times n)$ real, real symmetric, real symmetric semi-positive definite and real symmetric positive-definite matrices, respectively. We then have

$$\mathbb{M}_n^+(\mathbb{R}) \subset \mathbb{M}_n^{+0}(\mathbb{R}) \subset \mathbb{M}_n^S(\mathbb{R}) \subset \mathbb{M}_n(\mathbb{R}) \quad . \quad (11)$$

Let $\mathbb{M}_{\mu,n}(\mathbb{R})$ be the set of all the $(\mu \times n)$ real rectangular matrices. We then have $\mathbb{M}_n(\mathbb{R}) = \mathbb{M}_{n,n}(\mathbb{R})$.

Factorization of the mean reduced matrices for a substructure

For the construction of the nonparametric probabilistic model of random uncertainties, we need the factorization of the mean reduced mass, damping and stiffness matrices for each substructure \mathbb{S}^r . Since $[\underline{M}]$ belongs to $\mathbb{M}_m^+(\mathbb{R})$, then the Cholesky factorization of matrix $[\underline{M}]$ yields

$$[\underline{M}] = [\underline{L}_M]^T [\underline{L}_M] \quad , \quad (12)$$

in which $[\underline{L}_M]$ is an upper triangular sparse matrix in $\mathbb{M}_{m,m}(\mathbb{R})$.

(1) For a fixed substructure, since reduced damping and stiffness matrices $[\underline{D}]$ and $[\underline{K}]$ belong to $\mathbb{M}_m^+(\mathbb{R})$, then the Cholesky factorization of matrices $[\underline{D}]$ and $[\underline{K}]$ yield

$$[\underline{D}] = [\underline{L}_D]^T [\underline{L}_D] \quad , \quad (13)$$

$$[\underline{K}] = [\underline{L}_K]^T [\underline{L}_K] \quad , \quad (14)$$

in which $[\underline{L}_D]$ and $[\underline{L}_K]$ are upper triangular sparse matrices in $\mathbb{M}_{n,m}(\mathbb{R})$ with

$$n = m \quad . \quad (15)$$

(2) For a free substructure, reduced damping and stiffness matrices $[\underline{D}]$ and $[\underline{K}]$ belong to $\mathbb{M}_m^{+0}(\mathbb{R})$ and have the same null space spanned by μ_{rig} vectors deduced from the μ_{rig} rigid body modes of substructure \mathbb{S}^r with free coupling interface Σ . It can easily be proved that Eqs. (13)-(14) hold but $[\underline{L}_D]$ and $[\underline{L}_K]$ are rectangular ($n \times m$) real sparse matrices (which are almost upper triangular) in which

$$n = m - \mu_{\text{rig}} \quad . \quad (16)$$

The computation of such a factorization is usual and will not be explained here.

Construction of a nonparametric model of random uncertainties for the reduced dynamic stiffness matrix of a substructure

Let us assume that random uncertainties exist in substructure \mathbb{S}^r . Then its reduced dynamic stiffness matrix is a random matrix. As explained in the second section, the nonparametric probabilistic model of substructure \mathbb{S}^r consists in writing the random reduced dynamic stiffness matrix (see Eq. (7)) as

$$[\mathbf{A}(\omega)] = -\omega^2[\mathbf{M}] + i\omega[\mathbf{D}] + [\mathbf{K}] \quad , \quad (17)$$

in which $[\mathbf{M}]$, $[\mathbf{D}]$ and $[\mathbf{K}]$ are $(m \times m)$ real random matrices which are defined below. Consequently, the mean reduced matrix equations for substructure \mathcal{S}^r , defined by Eqs. (5)-(10), are replaced by the following random equations. Equation (5) is replaced by

$$\begin{bmatrix} \mathbf{U}_i(\omega) \\ \mathbf{U}_\Sigma(\omega) \end{bmatrix} = [\underline{H}] \begin{bmatrix} \mathbf{q}(\omega) \\ \mathbf{U}_\Sigma(\omega) \end{bmatrix} \quad , \quad (18)$$

in which $\mathbf{U}_i(\omega)$ and $\mathbf{U}_\Sigma(\omega)$ are the \mathbb{C}^{μ_i} – and \mathbb{C}^{μ_Σ} – random vectors constituted of the μ_i internal DOFs and μ_Σ coupling DOFs, where $[\underline{H}]$ is the matrix defined by Eq. (5) and where $\mathbf{q}(\omega)$ is the \mathbb{C}^N -valued random vector constituted of the N generalized coordinates. Equation (6) is replaced by the following. For all ω fixed in B , random vectors $\mathbf{q}(\omega)$ and $\mathbf{U}_\Sigma(\omega)$ verify the random reduced matrix equation

$$[\mathbf{A}(\omega)] \begin{bmatrix} \mathbf{q}(\omega) \\ \mathbf{U}_\Sigma(\omega) \end{bmatrix} = \begin{bmatrix} \underline{\mathcal{F}}(\omega) \\ \underline{\mathbb{F}}_\Sigma(\omega) + \mathbf{F}_\Sigma^{\text{coup}}(\omega) \end{bmatrix} \quad , \quad (19)$$

in which $\underline{\mathcal{F}}(\omega)$ is defined by Eq. (9) and vector $\underline{\mathbb{F}}_\Sigma(\omega)$ by Eq. (10), that is to say,

$$\underline{\mathbb{F}}_\Sigma(\omega) = [\underline{S}]^T \underline{\mathbf{F}}_i(\omega) + \underline{\mathbf{F}}_\Sigma(\omega) \quad , \quad (20)$$

where $\mathbf{F}_\Sigma^{\text{coup}}(\omega)$ is the random coupling force vector on coupling interface Σ . Finally, we have to define the available information concerning the random reduced mass, damping and stiffness matrices appearing in Eq. (17):

(a) Reduced mass matrix $[\mathbf{M}]$ is a random matrix with values in $\mathbb{M}_m^+(\mathbb{R})$ and, (1) for a fixed substructure, $[\mathbf{D}]$ and $[\mathbf{K}]$ are random matrices with values in $\mathbb{M}_m^+(\mathbb{R})$, (2) for a free substructure, $[\mathbf{D}]$ and $[\mathbf{K}]$ are random matrices with values in $\mathbb{M}_m^{+0}(\mathbb{R})$ and it is assumed that random matrices $[\mathbf{D}]$ and $[\mathbf{K}]$ have the same deterministic null space of dimension μ_{rig} , spanned by the μ_{rig} constant vectors deduced from the μ_{rig} rigid body modes of the mean finite element model; this assumption is automatically satisfied when the finite element model under consideration corresponds to the finite element discretization of a boundary value problem related to elastodynamics of a bounded continuum.

(b) The mean values of random reduced matrices $[\mathbf{M}]$, $[\mathbf{D}]$ and $[\mathbf{K}]$ are given by the mean reduced matrix model,

$$E\{[\mathbf{M}]\} = [\underline{M}] \quad , \quad E\{[\mathbf{D}]\} = [\underline{D}] \quad , \quad E\{[\mathbf{K}]\} = [\underline{K}] \quad , \quad (21)$$

in which E is the mathematical expectation.

(c) An additional available information will be introduced hereinafter concerning the existence of certain second-order moments of the random matrices introduced in the probabilistic model.

From Eqs. (12)-(16) and (21), we deduce that random reduced matrices $[\mathbf{M}]$, $[\mathbf{D}]$ and $[\mathbf{K}]$ can be written as

$$[\mathbf{M}] = [\underline{\mathbf{L}}_M]^T [\mathbf{G}_M] [\underline{\mathbf{L}}_M] \quad , \quad (22)$$

$$[\mathbf{D}] = [\underline{\mathbf{L}}_D]^T [\mathbf{G}_D] [\underline{\mathbf{L}}_D] \quad , \quad (23)$$

$$[\mathbf{K}] = [\underline{\mathbf{L}}_K]^T [\mathbf{G}_K] [\underline{\mathbf{L}}_K] \quad , \quad (24)$$

in which the available information for the random matrices $[\mathbf{G}_M]$, $[\mathbf{G}_D]$ and $[\mathbf{G}_K]$ is the following:

(a) Matrix $[\mathbf{G}_M]$ is a random matrix with values in $\mathbb{M}_m^+(\mathbb{R})$, matrices $[\mathbf{G}_D]$ and $[\mathbf{G}_K]$ are random matrices with values in $\mathbb{M}_n^+(\mathbb{R})$ with $n = m$ for a fixed substructure and $n = m - \mu_{\text{rig}}$ for a free substructure.

(b) The mean values of random matrices $[\mathbf{G}_M]$, $[\mathbf{G}_D]$ and $[\mathbf{G}_K]$ are

$$E\{[\mathbf{G}_M]\} = [I_m] \quad , \quad E\{[\mathbf{G}_D]\} = [I_n] \quad , \quad E\{[\mathbf{G}_K]\} = [I_n] \quad , \quad (25)$$

in which $[I_m]$ and $[I_n]$ are the $(m \times m)$ and $(n \times n)$ unity matrices respectively.

(c) Since $[\mathbf{G}_M]$ is a random matrix with values in $\mathbb{M}_m^+(\mathbb{R})$, $[\mathbf{G}_D]$ and $[\mathbf{G}_K]$ are random matrices with values in $\mathbb{M}_n^+(\mathbb{R})$, these matrices are invertible almost surely. As explained in (Soize 2000), this property does not imply that the second-order moment of their inverse exist and this kind of property is required. Consequently, we introduce the following constraint,

$$E\{\|[\mathbf{G}_M]^{-1}\|_F^2\} < +\infty \quad , \quad E\{\|[\mathbf{G}_D]^{-1}\|_F^2\} < +\infty \quad , \quad E\{\|[\mathbf{G}_K]^{-1}\|_F^2\} < +\infty \quad , \quad (26)$$

in which $\| [B] \|_F = (\text{tr}\{[B] [B]^T\})^{1/2}$ is the Frobenius norm of matrix $[B]$ where tr is the trace of the matrices. The last step concerns the construction of the probability model for symmetric positive-definite real random matrices $[\mathbf{G}_M]$, $[\mathbf{G}_D]$ and $[\mathbf{G}_K]$ with the available information defined by Eqs. (25) and (26).

PROBABILITY MODEL FOR SYMMETRIC POSITIVE-DEFINITE REAL RANDOM MATRICES

We then need to construct a probability model for symmetric positive-definite real random matrices $[\mathbf{G}_M]$, $[\mathbf{G}_D]$ and $[\mathbf{G}_K]$ using the available information defined by Eqs. (25) and (26). Let $[\mathbf{G}]$ be any one of these three matrices. In this section, we recall the main results established in (Soize 2000,2001) concerning the construction of such a probability model for a random matrix $[\mathbf{G}]$ with values in $\mathbb{M}_n^+(\mathbb{R})$ using the entropy optimization principle which allows the available information to be only used. It should be noted that the results obtained and presented below differ from the known results concerning the Gaussian Orthogonal Ensemble (GOE) which has been extensively studied in the literature (see for instance (Mehta 1991)).

Probability density function on the space of positive-definite symmetric real matrices and characteristic function

Let $[\mathbf{G}]$ be a random matrix with values in $\mathbb{M}_n^+(\mathbb{R})$ whose probability distribution $P_{[\mathbf{G}]} = p_{[\mathbf{G}]}([G]) \tilde{d}G$ is defined by a probability density function $[G] \mapsto p_{[\mathbf{G}]}([G])$ from $\mathbb{M}_n^+(\mathbb{R})$ into $\mathbb{R}^+ = [0, +\infty[$ with respect to the measure (volume element) $\tilde{d}G$ on $\mathbb{M}_n^+(\mathbb{R})$ defined by $\tilde{d}G = 2^{n(n-1)/4} \prod_{1 \leq i < j \leq n} dG_{ij}$. This probability density function is such that $\int_{\mathbb{M}_n^+(\mathbb{R})} p_{[\mathbf{G}]}([G]) \tilde{d}G = 1$. For all $[\Theta]$ in $\mathbb{M}_n^+(\mathbb{R})$, the characteristic function of random matrix $[\mathbf{G}]$ is defined by $\Phi_{[\mathbf{G}]}([\Theta]) = E\{\exp(i \ll [\Theta], [\mathbf{G}] \gg)\} = \int_{\mathbb{M}_n^+(\mathbb{R})} \exp(i \ll [\Theta], [G] \gg) p_{[\mathbf{G}]}([G]) \tilde{d}G$ in which $\ll [\Theta], [\mathbf{G}] \gg = \text{tr}\{[\Theta] [\mathbf{G}]^T\} = \text{tr}\{[\Theta] [\mathbf{G}]\}$.

Available information for construction of the probability model

We are interested in the construction of the probability distribution of a second-order random variable $[\mathbf{G}]$ with values in $\mathbb{M}_n^+(\mathbb{R})$ for which the available information is the mean value $[\underline{G}] = E\{[\mathbf{G}]\} = [I_n]$ (in which $[I_n]$ is the unity matrix in $\mathbb{M}_n^+(\mathbb{R})$) and the constraint defined by $E\{\ln(\det[\mathbf{G}])\} = v$ with $|v| < +\infty$. It is proved (Soize 2000 and 2001) that this constraint yields $E\{\|[\mathbf{G}]^{-1}\|_F^2\} < +\infty$. Consequently, the available information which is used to construct the probability model of random matrix $[\mathbf{G}]$ with values in $\mathbb{M}_n^+(\mathbb{R})$ is defined by the following constraints

$$\int_{\mathbb{M}_n^+(\mathbb{R})} p_{[\mathbf{G}]}([G]) \tilde{d}G = 1 \quad , \quad (27)$$

$$\int_{\mathbb{M}_n^+(\mathbb{R})} [G] p_{[\mathbf{G}]}([G]) \tilde{d}G = [I_n] \in \mathbb{M}_n^+(\mathbb{R}) \quad , \quad (28)$$

$$\int_{\mathbb{M}_n^+(\mathbb{R})} \ln(\det[\mathbf{G}]) p_{[\mathbf{G}]}([G]) \tilde{d}G = v \quad , \quad |v| < +\infty \quad . \quad (29)$$

It should be noted that, in the proposed theory, the covariance tensor of random matrix $[\mathbf{G}]$ is not considered as an available information.

Probability model using the maximum entropy principle

Introducing the measure of entropy (Shannon 1948) and the maximum entropy principle (Jaynes 1957) to construct the probability model of random matrix $[\mathbf{G}]$ with values in $\mathbb{M}_n^+(\mathbb{R})$ based only on the use of the available information defined by Eqs. (27)-(29), it is proved (Soize 2000) that, for $[\Theta] \in \mathbb{M}_n^S(\mathbb{R})$, probability density function $p_{[\mathbf{G}]}([G])$ and characteristic function $\Phi_{[\mathbf{G}]}([\Theta])$ of positive-definite random matrix $[\mathbf{G}]$ are written as

$$p_{[\mathbf{G}]}([G]) = \mathbb{1}_{\mathbb{M}_n^+(\mathbb{R})}([G]) \times c_G \times (\det[G])^{\lambda_G - 1} \times \exp\left(-\frac{(n-1+2\lambda_G)}{2} \text{tr}\{[G]\}\right) \quad , \quad (30)$$

$$\Phi_{[\mathbf{G}]}([\Theta]) = \left\{ \det\left([I_n] - \frac{2i}{(n-1+2\lambda_G)} [\Theta]\right) \right\}^{-(n-1+2\lambda_G)/2} \quad , \quad (31)$$

in which $\lambda_G > 0$ is a parameter depending on n and defined below, where \det denotes the determinant of matrices and where $\mathbb{1}_{\mathbb{M}_n^+(\mathbb{R})}([G])$ is equal to 1 if $[G] \in \mathbb{M}_n^+(\mathbb{R})$ and is equal to zero if $[G] \notin \mathbb{M}_n^+(\mathbb{R})$. When λ_G is an integer, the probability distribution defined by Eq. (30) or (31) coincides with a Wishart distribution (Anderson 1958). If λ_G is not an integer, then the probability distribution defined by Eq. (30) or (31) is not a Wishart distribution. In Eq. (30), positive constant c_G is written as

$$c_G = \frac{(2\pi)^{-n(n-1)/4} \left(\frac{n-1+2\lambda_G}{2}\right)^{n(n-1+2\lambda_G)/2}}{\left\{ \prod_{\ell=1}^n \Gamma\left(\frac{n-\ell+2\lambda_G}{2}\right) \right\}} \quad , \quad (32)$$

where, for $\Re z > 0$, $\Gamma(z)$ is the gamma function defined by $\Gamma(z) = \int_0^{+\infty} t^{z-1} e^{-t} dt$. Let δ_G be the positive real number defined by

$$\delta_G = \left\{ \frac{E\{\|[\mathbf{G}] - [\underline{G}]\|_F^2\}}{\|[\underline{G}]\|_F^2} \right\}^{1/2} \quad , \quad (33)$$

in which $[\underline{G}] = [I_n]$ and $\|[\underline{G}]\|_F^2 = n$. Parameter $\lambda_G(n)$ is defined by

$$\lambda_G(n) = \frac{1 - \delta_G^2}{2\delta_G^2} n + \frac{1 + \delta_G^2}{2\delta_G^2} \quad , \quad (34)$$

in which δ_G has to be chosen independent of n and such that

$$0 < \delta_G < \sqrt{\frac{n_0 + 1}{n_0 + 5}} \quad , \quad (35)$$

where n_0 is a fixed positive integer such that $n_0 \geq 1$. Equations (33) to (35) are used as follows. The lower bound n_0 of positive integer n is fixed. Then, the dispersion of the probability model is fixed by giving parameter δ_G , independent of n , a value such that Eq. (35) is satisfied. For each value of integer $n \geq n_0$, parameter $\lambda_G(n)$ is then calculated by using Eq. (34). It can then be proved (Soize 2000 and 2001) that

$$E\{\|[\mathbf{G}]^{-1}\|_F^2\} < +\infty \quad , \quad E\{\|[\mathbf{G}]\|_F^\eta\} < +\infty, \quad \forall \eta > 0 \quad , \quad (36)$$

and that the covariance $C_{jk,j'k'}^G = E\{([\mathbf{G}]_{jk} - [\underline{G}]_{jk})([\mathbf{G}]_{j'k'} - [\underline{G}]_{j'k'})\}$ of random variables $[\mathbf{G}]_{jk}$ and $[\mathbf{G}]_{j'k'}$ is written as

$$C_{jk,j'k'}^G = \frac{\delta_G^2}{n+1} \{ \delta_{j'k} \delta_{jk'} + \delta_{jj'} \delta_{kk'} \} \quad , \quad (37)$$

where $\delta_{jk} = 0$ if $j \neq k$ and $\delta_{jj} = 1$. The variance $V_{jk}^G = C_{jk,jk}^G$ of random variable $[\mathbf{G}]_{jk}$ is written as $V_{jk}^G = \frac{\delta_G^2}{n+1}(1 + \delta_{jk})$.

Representation of random matrix $[\mathbf{G}]$

Generally, $\lambda_G(n)$ defined by Eq. (34) is a positive real number. The following algebraic representation of positive-definite real random matrix $[\mathbf{G}]$ allows a procedure for the Monte Carlo numerical simulation of random matrix $[\mathbf{G}]$ to be defined. Random matrix $[\mathbf{G}]$ can be written as

$$[\mathbf{G}] = [\mathbf{L}_G]^T [\mathbf{L}_G] \quad , \quad (38)$$

in which $[\mathbf{L}_G]$ is an upper triangular random matrix with values in $\mathbb{M}_n(\mathbb{R})$ such that:

- (1) random variables $\{[\mathbf{L}_G]_{jj'}, j \leq j'\}$ are independent;
- (2) for $j < j'$, real-valued random variable $[\mathbf{L}_G]_{jj'}$ can be written as $[\mathbf{L}_G]_{jj'} = \sigma_n U_{jj'}$ in which $\sigma_n = \delta_G(n+1)^{-1/2}$ and where $U_{jj'}$ is a real-valued Gaussian random variable with zero mean and variance equal to 1;

(3) for $j = j'$, positive-valued random variable $[\mathbf{L}_G]_{jj}$ can be written as $[\mathbf{L}_G]_{jj} = \sigma_n \sqrt{2V_j}$ in which σ_n is defined above and where V_j is a positive-valued gamma random variable whose probability density function $p_{V_j}(v)$ with respect to dv is written as

$$p_{V_j}(v) = \mathbb{1}_{\mathbb{R}^+}(v) \frac{1}{\Gamma\left(\frac{n+1}{2\delta_G^2} + \frac{1-j}{2}\right)} v^{\frac{n+1}{2\delta_G^2} - \frac{1+j}{2}} e^{-v} \quad . \quad (39)$$

Probability model of a set of positive-definite symmetric real random matrices

Let us consider ν random matrices $[\mathbf{G}_1], \dots, [\mathbf{G}_\nu]$ with values in $\mathbb{M}_n^+(\mathbb{R})$ such that for each j in $\{1, \dots, \nu\}$, the probability density function of random matrix $[\mathbf{G}_j]$ satisfies Eqs. (27)-(29). Applying the maximum entropy principle, it is proved (Soize 2000) that the probability density function $([G_1], \dots, [G_\nu]) \mapsto p_{[\mathbf{G}_1], \dots, [\mathbf{G}_\nu]}([G_1], \dots, [G_\nu])$ from $\mathbb{M}_n^+(\mathbb{R}) \times \dots \times \mathbb{M}_n^+(\mathbb{R})$ into \mathbb{R}^+ with respect to the measure (volume element) $\tilde{d}G_1 \times \dots \times \tilde{d}G_\nu$ on $\mathbb{M}_n^S(\mathbb{R}) \times \dots \times \mathbb{M}_n^S(\mathbb{R})$ is written as

$$p_{[\mathbf{G}_1], \dots, [\mathbf{G}_\nu]}([G_1], \dots, [G_\nu]) = p_{[\mathbf{G}_1]}([G_1]) \times \dots \times p_{[\mathbf{G}_\nu]}([G_\nu]) \quad , \quad (40)$$

which means that $[\mathbf{G}_1], \dots, [\mathbf{G}_\nu]$ are independent random matrices.

NONPARAMETRIC MODEL OF RANDOM UNCERTAINTIES

We apply the results of the previous section to random matrices $[\mathbf{M}], [\mathbf{D}]$ and $[\mathbf{K}]$ defined by Eqs. (22)-(24), for which the available information concerning matrices $[\mathbf{G}_M], [\mathbf{G}_D]$ and $[\mathbf{G}_K]$ is defined by Eqs. (25)-(26). From Eq. (40), we deduce that these random matrices are independent. The dispersion of each random matrix $[\mathbf{M}], [\mathbf{D}]$ or $[\mathbf{K}]$ is controlled by parameter δ_M, δ_D or δ_K (see Eq. (33)), independent of m and n , chosen such that Eq. (35) holds,

$$0 < \delta_M, \delta_D, \delta_K < \sqrt{\frac{n_0 + 1}{n_0 + 5}} \quad ,$$

and defined by

$$\delta_B = \left\{ \frac{E\{\|[\mathbf{G}_B] - [\underline{\mathbf{G}}_B]\|_F^2\}}{\|[\underline{\mathbf{G}}_B]\|_F^2} \right\}^{1/2} \quad , \quad (41)$$

in which B is M, D or K . With this nonparametric model, the probability distribution of each random reduced matrix $[\mathbf{M}], [\mathbf{D}]$ or $[\mathbf{K}]$ of substructure \mathbb{S}^r , depends only on two parameters:

the mean reduced matrix $[\underline{M}]$, $[\underline{D}]$ or $[\underline{K}]$ associated with the mean mechanical model and corresponding to the design model, and a scalar parameter δ_M , δ_D or δ_K whose values have to be fixed by the designer in the interval $[0, 1[$ in order to give the dispersion level related to the random reduced matrix of the substructure. Parameter δ_M , δ_D or δ_K can be viewed as a global parameter. Because the model uncertainties which can be taken into account with the nonparametric model, cannot be directly quantified in terms of correlation between random variables, indirect methods have to be imagined in order to estimate parameter δ_M , δ_D or δ_K . *A priori*, such a parameter can be estimated (1) by using theoretical considerations and numerical experiments, (2) by using measurements of the frequency response functions for a given class of dynamic systems (that is to say, estimating δ parameter for that the measured experimental responses belong to the confidence region constructed with the nonparametric approach) and (3) from expertise. Concerning the last one, if there is no uncertainty for the reduced stiffness matrix of a substructure, then $\delta_K = 0$. On the other hand, if it is assumed that the global uncertainty for the reduced stiffness matrix of a substructure is 10%, then δ_K has to be 0.1.

In order to carry out the Monte Carlo numerical simulation (Cochran 1977, Kalos & Whitlock 1986, Rubinstein 1981) of random dynamical responses of the coupled system \mathbb{S} , we use the adapted algebraic representation of random matrices $[\mathbf{G}_M]$, $[\mathbf{G}_D]$ and $[\mathbf{G}_K]$ defined by Eqs. (38)-(39). We then have,

$$[\mathbf{M}] = [\underline{L}_M]^T [\mathbf{G}_M] [\underline{L}_M] \quad , \quad [\mathbf{G}_M] = [\mathbf{L}_{G_M}]^T [\mathbf{L}_{G_M}] \quad , \quad (42)$$

$$[\mathbf{D}] = [\underline{L}_D]^T [\mathbf{G}_D] [\underline{L}_D] \quad , \quad [\mathbf{G}_D] = [\mathbf{L}_{G_D}]^T [\mathbf{L}_{G_D}] \quad , \quad (43)$$

$$[\mathbf{K}] = [\underline{L}_K]^T [\mathbf{G}_K] [\underline{L}_K] \quad , \quad [\mathbf{G}_K] = [\mathbf{L}_{G_K}]^T [\mathbf{L}_{G_K}] \quad . \quad (44)$$

RANDOM REDUCED MATRIX MODEL FOR THE COMPLETE STRUCTURE

In order to distinguish substructure \mathbb{S}^1 ($r = 1$) from substructure \mathbb{S}^2 ($r = 2$), exponent r (with $r = 1$ or $= 2$) is reintroduced. The coupling conditions on coupling interface Σ consist in writing the continuity of the random displacement field which leads to write $\mathbf{U}_\Sigma^1 = \mathbf{U}_\Sigma^2 = \mathbf{U}_\Sigma$ and the equilibrium of coupling random force vectors leading to write $\mathbf{F}_\Sigma^{1, \text{coup}} + \mathbf{F}_\Sigma^{2, \text{coup}} = 0$. The block decomposition of random reduced dynamic stiffness matrix $[\mathbf{A}^r(\omega)]$ defined by Eq. (17)

corresponding to Eq. (19) is written as

$$[\mathbf{A}^r(\omega)] = \begin{bmatrix} [\mathcal{A}^r(\omega)] & [\mathbf{A}_c^r(\omega)] \\ [\mathbf{A}_c^r(\omega)]^T & [\mathbf{A}_\Sigma^r(\omega)] \end{bmatrix} . \quad (45)$$

Using the coupling conditions on interface Σ and Eq. (45) for $r = 1$ and $r = 2$, the random reduced matrix equation for structure \mathbb{S} is written as

$$\begin{bmatrix} [\mathcal{A}^1(\omega)] & [0] & [\mathbf{A}_c^1(\omega)] \\ [0] & [\mathcal{A}^2(\omega)] & [\mathbf{A}_c^2(\omega)] \\ [\mathbf{A}_c^1(\omega)]^T & [\mathbf{A}_c^2(\omega)]^T & [\mathbf{A}_\Sigma^1(\omega)] + [\mathbf{A}_\Sigma^2(\omega)] \end{bmatrix} \begin{bmatrix} \mathbf{q}^1(\omega) \\ \mathbf{q}^2(\omega) \\ \mathbf{U}_\Sigma(\omega) \end{bmatrix} = \begin{bmatrix} \underline{\mathcal{F}}^1(\omega) \\ \underline{\mathcal{F}}^2(\omega) \\ \underline{\mathbb{F}}_\Sigma(\omega) \end{bmatrix} , \quad (46)$$

in which $\underline{\mathcal{F}}^1(\omega)$ and $\underline{\mathcal{F}}^2(\omega)$ are defined by Eq. (9) for $r = 1$ and $r = 2$, and where

$$\underline{\mathbb{F}}_\Sigma(\omega) = [\underline{\mathcal{S}}^1]^T \underline{\mathbf{F}}_i^1(\omega) + [\underline{\mathcal{S}}^2]^T \underline{\mathbf{F}}_i^2(\omega) + \underline{\mathbf{F}}_\Sigma^1(\omega) + \underline{\mathbf{F}}_\Sigma^2(\omega) , \quad (47)$$

in which $\underline{\mathbf{F}}_i^r(\omega)$ and $\underline{\mathbf{F}}_\Sigma^r(\omega)$ are defined by Eqs. (3) for $r = 1$ and $r = 2$, and where $[\underline{\mathcal{S}}^r]$ is defined in Eq. (5) for $r = 1$ and $r = 2$. The random response $\mathbf{V}(\omega)$, constituted of the $\nu = \mu_i^1 + \mu_i^2 + \mu_\Sigma$ DOFs of structure \mathbb{S} , is then calculated by the following matrix equation

$$\mathbf{V}(\omega) = \begin{bmatrix} \mathbf{U}^1(\omega) \\ \mathbf{U}^2(\omega) \\ \mathbf{U}_\Sigma(\omega) \end{bmatrix} = \begin{bmatrix} [\underline{\Phi}^1] & [0] & [\underline{\mathcal{S}}^1] \\ [0] & [\underline{\Phi}^2] & [\underline{\mathcal{S}}^2] \\ [0] & [0] & [I] \end{bmatrix} \begin{bmatrix} \mathbf{q}^1(\omega) \\ \mathbf{q}^2(\omega) \\ \mathbf{U}_\Sigma(\omega) \end{bmatrix} . \quad (48)$$

NUMERICAL EXAMPLE

Definition of the mean model

We consider a mean model constituted of a rectangular, homogeneous, isotropic thin plate, simply supported, with a constant thickness $0.4 \times 10^{-3} \text{ m}$, width 0.5 m , length 1.0 m , mass density 7800 kg/m^3 , Young's modulus $2.1 \times 10^{11} \text{ N/m}^2$, Poisson's ratio 0.29. Two point masses of 3 kg and 4 kg are located at points $(0.4, 0.2)$ and $(0.75, 0.35)$, and three springs having the same stiffness coefficient $2.388 \times 10^7 \text{ N/m}$ are attached normally to the plate and located at points $(0.28, 0.22)$, $(0.54, 0.33)$ and $(0.83, 0.44)$. Consequently, the master structure defined above is not homogeneous. This master structure is decomposed into two substructures

\mathbb{S}^1 and \mathbb{S}^2 . The first one has a length $0.6 m$ and the second one, a length $0.4 m$ (see Figure 1). The frequency band of analysis is $B = 2\pi \times [1, 100] rad/s$. The mean finite element model of the master structure is constructed using 4-nodes bending plate elements. The mesh size is $0.01 m \times 0.01 m$. The master structure has $\nu = 14849$ DOFs corresponding to $\mu_i^1 = 8840$, $\mu_i^2 = 5860$ and $\mu_\Sigma = 149$. The mean damping matrix of the master structure is constructed using a Rayleigh model corresponding to a mean damping rate $\underline{\xi} = 0.03$ for eigenfrequencies $f_1 = 2.6Hz$ and $f_{35} = 106.38Hz$ of the mean master structure. The master structure is subjected to an external load vector $\underline{\eta}(\omega)\underline{\mathbf{g}}$ in which spatial part $\underline{\mathbf{g}} = (g_1, \dots, g_n) \in \mathbb{R}^n$ is such that $g_j = 0$ for all $j \in \{1, \dots, n\}$ excepted for DOF dof_1 corresponding to the node whose (x, y) coordinates are $(0.24, 0.24)$. Function $\omega \mapsto \underline{\eta}(\omega)$ is defined by $\underline{\eta}(\omega) = \mathbb{1}_B(\omega)$ in which $\omega \mapsto \mathbb{1}_B(\omega)$ is such that $\mathbb{1}_B(\omega) = 1$ if $\omega \in B$ and $\mathbb{1}_B(\omega) = 0$ if $\omega \notin B$. We introduce dof_2 and dof_3 as the observed DOFs corresponding to the nodes whose (x, y) coordinates are $(0.39, 0.31)$ and $(0.79, 0.24)$ respectively.

Dynamic substructuring model with random uncertainties

Concerning the dynamic substructuring model with random uncertainties, the calculations are carried out with $N = N^1 = N^2$. The dispersion parameters are defined for each substructure \mathbb{S}^r ($r = 1, 2$) by $n_0 = 4$ and $\delta_M = \delta_D = \delta_K = 0.1$. Random responses of the master structure modeled by substructuring are obtained by solving Eqs. (46)-(48). A Monte Carlo numerical simulation is carried out with $n_s = 1000$ samples denoted as $\{\theta_j\}_{j=1}^{n_s}$ and for each θ_j , $\|\mathbf{V}(\theta_j)\|_B^2 = \int_B \|\mathbf{V}(\omega; \theta_j)\|^2 d\omega$ is calculated. The mean value $\|\mathbf{V}\|_B^2 = E\{\|\mathbf{V}\|_B^2\}$ of random variable $\|\mathbf{V}\|_B^2$ is then estimated by $\|\mathbf{V}\|_B^2 \simeq (1/n_s) \sum_{j=1}^{n_s} \|\mathbf{V}(\theta_j)\|_B^2$.

Convergence analysis

Figure 2 shows the graph of $n_s \mapsto 10 \log_{10}(\|\mathbf{V}\|_B^2)$ for $N = 10, 20, 30$ and 100 . A good convergence is obtained for $n_s = 500$. Figure 3 shows the graph of $N \mapsto 10 \log_{10}(\|\mathbf{V}\|_B^2)$ for a number of samples fixed to $n_s = 500$. A good convergence is obtained for $N = 20$. From Figures 2 and 3, it can be deduced that $n_s = 500$ and $N = 20$ correspond to a good approximation. For ω fixed in B , let $dB_k(\omega)$ be the random variable defined by $dB_k(\omega) = 10 \log_{10}(|V_k(\omega)|^2)$ with $\mathbf{V}(\omega) = (V_1(\omega), \dots, V_\nu(\omega))$.

Confidence region associated with a given probability level

The construction of the confidence region related to random variable $\text{dB}_k(\omega)$ is carried out as explained in (Soize & Bjaoui 2000). This confidence region is defined by the upper and lower envelopes $\omega \mapsto \text{dB}_k^+(\omega)$ and $\omega \mapsto \text{dB}_k^-(\omega)$ of the frequency-response-function modulus corresponding to a given probability level P_c and is such that the probability $\mathcal{P}\{\text{dB}_k^-(\omega) < \text{dB}_k(\omega) \leq \text{dB}_k^+(\omega)\} = P_c$. By construction, only $\text{dB}_k^+(\omega)$ is unknown and the lower envelope is given by $\text{dB}_k^-(\omega) = 2 \text{dB}_k^0(\omega) - \text{dB}_k^+(\omega)$ with $\text{dB}_k^0(\omega) = 10 \log_{10}(|E\{V_k(\omega)\}|^2)$. The upper envelope $\text{dB}_k^+(\omega)$ is constructed using the Chebychev's inequality. We can then write $\mathcal{P}\{|V_k(\omega) - E\{V_k(\omega)\}| \geq a_k(\omega)\} \leq \text{Var}\{V_k(\omega)\}/a_k^2(\omega)$ in which $\text{Var}\{V_k(\omega)\}$ is the variance of random variable $V_k(\omega)$. We deduce that $\mathcal{P}\{\text{dB}_k^-(\omega) < \text{dB}_k(\omega) < \text{dB}_k^+(\omega)\} \geq P_c$ with $\text{dB}_k^+(\omega) = 20 \log_{10}(|E\{V_k(\omega)\}| + a_k(\omega))$ and $P_c = 1 - \text{Var}\{V_k(\omega)\}/a_k^2(\omega)$.

Figures 4, 5 and 6 show the confidence region defined by the upper and lower envelopes (thick solid lines) constructed with $P_c = 0.95$ for excited DOF $k = \text{dof}_1$ in plate 1 and for observed DOFs $k = \text{dof}_2$ and $k = \text{dof}_3$ in plates 1 and 2 respectively. The graph of $f \mapsto \text{dB}_k^0(f)$ is represented by the thin solid line with f -axis in Hertz. These figures show that the size of the confidence region increases in the frequency band when frequency is increasing. This phenomenon is due to the fact that the sensitivity of an eigenmode to random uncertainties increases with its rank.

Extreme value statistics associated with samples

In addition, we introduce functions $\omega \mapsto dB_k^{\max}(\omega; \boldsymbol{\theta}) = \max_{j=1, \dots, n_s} \text{dB}_k(\omega; \theta_j)$ and $\omega \mapsto dB_k^{\min}(\omega; \boldsymbol{\theta}) = \min_{j=1, \dots, n_s} \text{dB}_k(\omega; \theta_j)$ in which $\boldsymbol{\theta} = (\theta_1, \dots, \theta_{n_s})$. Figures 7, 8 and 9 are related to excited DOF $k = \text{dof}_1$ in plate 1, to observed DOF $k = \text{dof}_2$ in plate 1 and to observed DOF $k = \text{dof}_3$ in plate 2. Each figure shows the comparison between $f \mapsto dB_k^{\max}(f; \boldsymbol{\theta})$ (upper thin solid line) and $f \mapsto dB_k^{\min}(f; \boldsymbol{\theta})$ (lower thin solid line) with the confidence region defined by its upper envelope $f \mapsto \text{dB}_k^+(f)$ (upper thick solid line) and its lower envelope $f \mapsto \text{dB}_k^-(f)$ (lower thick solid line) corresponding to $P_c = 0.95$, in which f is in Hertz. These three figures allow us to conclude that the confidence region approach yields a very good approximation of the extreme value statistics.

CONCLUSION

We have presented an approach allowing the random uncertainties to be modeled by a nonparametric model for prediction of frequency response functions in linear structural dynamics using the Craig-Bampton dynamic substructuring method in the low-frequency range. The parametric approaches existing in literature are very useful when the number of uncertain parameters is small and when the probabilistic model can be constructed for the set of parameters considered. The nonparametric approach presented is useful when the number of uncertain parameters is high or when the probabilistic model is difficult to construct for the set of parameters considered. In addition, the parametric approaches do not allow the model uncertainties to be taken into account (because a parametric approach is associated with a fixed model exhibiting some parameters), whereas the nonparametric approach proposed allows to take into account the model uncertainties. The main interest of using such a nonparametric approach for random uncertainties modeling in dynamic substructuring is its capability to model nonhomogeneous random uncertainties in the global structure. This means that a substructure can be taken without uncertainties and another one with uncertainties. If random uncertainties is taken into account in a substructure, the dynamic part (structural modes of the substructure with fixed interface) and the static part (static boundary functions of the coupling interface) is taken into account by the nonparametric probabilistic approach. It can be seen that such an approach is perfectly adapted for modeling random uncertainties in a complex mechanical junction (a substructure with random uncertainties) realizing the attachment of two main substructures without significant uncertainties.

An explicit construction and representation of the probability model have been obtained and are very well suited to algebraic calculus and to Monte Carlo numerical simulation. The fundamental properties related to the convergence of the stochastic solution with respect to the dimension of the random reduced matrix model of each substructure has been analyzed. This convergence analysis carried out has allowed the consistency of the theory proposed to be proved and the parameters of the probability distribution of the random reduced matrices to be clearly defined.

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LEGENDS ACCOMPANYING EACH FIGURE

FIG. 1. Geometry of the master structure: \bullet point masses, $+$ springs, \odot input force, \circ output normal displacements.

FIG. 2. Graph of $n_s \mapsto 10 \log_{10}(\|\mathbf{V}\|_B^2)$ for $N = 10, 20, 30$ and 100 .

FIG. 3. Graph of $N \mapsto 10 \log_{10}(\|\mathbf{V}\|_B^2)$ for $n_s = 500$.

FIG. 4. For excited DOF $k = \text{dof}_1$ in plate 1, graph of function $f \mapsto \text{dB}_k^0(f)$ (thin solid line) with f in Hertz and confidence region corresponding to $P_c = 0.95$, defined by its upper envelope $f \mapsto \text{dB}_k^+(f)$ (upper thick solid line) and its lower envelope $f \mapsto \text{dB}_k^-(f)$ (lower thick solid line).

FIG. 5. For observed DOF $k = \text{dof}_2$ in plate 1, graph of function $f \mapsto \text{dB}_k^0(f)$ (thin solid line) with f in Hertz and confidence region corresponding to $P_c = 0.95$, defined by its upper envelope $f \mapsto \text{dB}_k^+(f)$ (upper thick solid line) and its lower envelope $f \mapsto \text{dB}_k^-(f)$ (lower thick solid line).

FIG. 6. For observed DOF $k = \text{dof}_3$ in plate 2, graph of function $f \mapsto \text{dB}_k^0(f)$ (thin solid line) with f in Hertz and confidence region corresponding to $P_c = 0.95$, defined by its upper envelope $f \mapsto \text{dB}_k^+(f)$ (upper thick solid line) and its lower envelope $f \mapsto \text{dB}_k^-(f)$ (lower thick solid line).

FIG. 7. For excited DOF $k = \text{dof}_1$ in plate 1, comparison between $f \mapsto dB_k^{\max}(f; \theta)$ (upper thin solid line) and $f \mapsto dB_k^{\min}(f; \theta)$ (lower thin solid line) with the confidence region defined by its upper envelope $f \mapsto \text{dB}_k^+(f)$ (upper thick solid line) and its lower envelope $f \mapsto \text{dB}_k^-(f)$ (lower thick solid line) corresponding to $P_c = 0.95$, in which f is in Hertz.

FIG. 8. For observed DOF $k = \text{dof}_2$ in plate 1, comparison between $f \mapsto dB_k^{\max}(f; \theta)$ (upper thin solid line) and $f \mapsto dB_k^{\min}(f; \theta)$ (lower thin solid line) with the confidence region defined by its upper envelope $f \mapsto \text{dB}_k^+(f)$ (upper thick solid line) and its lower envelope $f \mapsto \text{dB}_k^-(f)$ (lower thick solid line) corresponding to $P_c = 0.95$, in which f is in Hertz.

FIG. 9. For observed DOF $k = \text{dof}_3$ in plate 2, comparison between $f \mapsto dB_k^{\max}(f; \theta)$ (upper thin solid line) and $f \mapsto dB_k^{\min}(f; \theta)$ (lower thin solid line) with the confidence region defined by its upper envelope $f \mapsto \text{dB}_k^+(f)$ (upper thick solid line) and its lower envelope $f \mapsto \text{dB}_k^-(f)$ (lower thick solid line) corresponding to $P_c = 0.95$, in which f is in Hertz.