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Eulerian and Hamiltonian Directed Hypergraphs.

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Eulerian and Hamiltonian Directed Hypergraphs.

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Abstract: Let $H = (\mathcal{V}, \mathcal{E})$ be a directed hypergraph, also called a dihypergraph. Each vertex $v \in \mathcal{V}$ is incident to some hyperarcs in \mathcal{E} . Conversely, each hyperarc $E \in \mathcal{E}$ is incident to some vertices in \mathcal{V} . H is Eulerian if there is a dicycle C such that each hyperarc $E \in \mathcal{E}$ appears exactly once in C . Similarly, H is Hamiltonian if there is a dicycle C' such that every vertex $v \in \mathcal{V}$ appears exactly once in C' .

We show that both problems are NP-complete. Some necessary conditions for a dihypergraph to be Eulerian are presented. We exhibit some families of hypergraphs for which those are sufficient conditions. We also generalize a part of the properties of the Eulerian digraphs to the uniform and regular directed hypergraphs. Stronger generalizations of *Eulerianity* to dihypergraphs are also studied.

Finally, we show that the generalized de Bruijn and Kautz dihypergraphs are Eulerian and Hamiltonian in most cases. We also study some properties of their bipartite representation digraph.

Key-words: directed hypergraphs, Eulerian and Hamiltonian dicycles, de Bruijn and Kautz dihypergraphs

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Hypergraphes Orientés Eulériens et Hamiltoniens.

Résumé : Soit un Hypergraphe orienté $H = (\mathcal{V}, \mathcal{E})$, qu'on appelle aussi un dihypergraphe. Chaque sommet $v \in \mathcal{V}$ est relié à des hyperarcs dans \mathcal{E} . Réciproquement, chaque hyperarc $E \in \mathcal{E}$ est relié à des sommets dans \mathcal{V} . H est Eulérien s'il existe un dicycle C tel que chaque hyperarc $E \in \mathcal{E}$ n'apparaisse qu'une et une seule fois dans C . De même H est Hamiltonien s'il existe un dicycle C' tel que chaque sommet $v \in \mathcal{V}$ n'apparaisse qu'une et une seule fois dans C' .

Nous montrons que ces deux problèmes sont NP-complets. Des conditions nécessaires pour qu'un hypergraphe orienté soit Eulérien sont présentées. Nous exhibons des familles d'hypergraphes pour lesquelles elles sont suffisantes. Nous généralisons aussi certaines propriétés des graphes orientés Eulériens aux hypergraphes orientés uniformes et réguliers. D'autres généralisations plus fortes de l'*Eulerianité* aux dihypergraphes sont aussi étudiées.

Enfin nous montrons que les hypergraphes de de Bruijn et de Kautz sont Eulériens et Hamiltoniens dans la plupart des cas. Nous étudions également certaines propriétés de leur représentation bipartite.

Mots-clés : hypergraphes orientés, dicycles Eulériens et Hamiltoniens, hypergraphes de de Bruijn et de Kautz

1 Introduction

Two concepts that are well-known and studied in Graph Theory are: Eulerian and Hamiltonian (di)cycles. An *Eulerian* (di)cycle in a (di)graph G is a (di)cycle C such that each *edge* of G (resp. each *arc* of G) appears exactly once in C . A close notion is the *Hamiltonian* (di)cycle: where each *vertex* of G appears exactly once (see [8]).

A connected graph is Eulerian if and only if each vertex has an even degree, and a strongly connected digraph is Eulerian if and only if each vertex has equal indegree and outdegree. Therefore for a given (di)graph G , deciding whether G has an Eulerian (di)cycle can be done in polynomial time. On the other hand, deciding whether G has a Hamiltonian (di)cycle is an NP-complete problem [13].

A recent work generalizes the *graph*-theoretic concept of an Euler cycle to *undirected* hypergraphs [26]. We now generalize the *digraph*-theoretic concept of an Eulerian dicycle to *directed* hypergraphs. Basically, the difference between a usual digraph D and a dihypergraph H is that (hyper)arcs in H may have multiple heads and multiple tails. We say that an Eulerian dicycle in a dihypergraph is a dicycle C such that a *hyperarc* appears exactly once in C . We also define and study a generalization of Hamiltonian dicycles to directed hypergraphs. Similarly, a Hamiltonian dicycle in a dihypergraph is a dicycle C such that a *vertex* appears exactly once in C .

We show that deciding if a dihypergraph is Eulerian is an NP-complete problem. Actually, deciding whether a dihypergraph is Eulerian is as hard as deciding if a dihypergraph is Hamiltonian. However, we show that some results about the Eulerian digraphs can be generalized, in the case where the studied dihypergraphs are *uniform* and *regular*. Finally, we study a special family of regular, uniform dihypergraphs, including Kautz and de Bruijn directed hypergraphs [7].

Definitions and discussion about the proper generalization of an Eulerian dicycle to directed hypergraphs are given in Section 2. In Section 3, the mentioned-above equivalence between Eulerian and Hamiltonian dicycles are presented. Results about Eulerian, uniform, regular dihypergraphs are presented in Section 4. Section 5 deals with the particular subclass of the Consecutive- (d, s) dihypergraphs. The last section shows the existence of complete Berge dicycles (dicycles that are simultaneously Eulerian and Hamiltonian) in various cases of Consecutive dihypergraphs.

2 Definitions and Notations

2.1 Directed Hypergraphs

An *undirected hypergraph* H , simply called a hypergraph, is a pair $(\mathcal{V}(H), \mathcal{E}(H))$ where $\mathcal{V}(H)$ is a non-empty set of elements (called *vertices*) and $\mathcal{E}(H)$ is a collection of subsets (or *edges*) of $\mathcal{V}(H)$. If we replace $\mathcal{E}(H)$ by a set of ordered pairs of subsets of $\mathcal{V}(H)$ (called *hyperarcs*), H is a *directed hypergraph* (or *dihypergraph*).

Let $E = (E^-, E^+)$ be a hyperarc in $\mathcal{E}(H)$. Then the vertex sets E^- and E^+ are called the *in-set* and the *out-set* of the hyperarc E , respectively. The sets E^- and E^+ need not to be disjoint. Moreover, at most one of these sets may be empty. The hyperarc E is said to *join* the vertices of E^- to the vertices of E^+ . Furthermore, the vertices of E^- are said to be *incident to* the hyperarc E and the vertices of E^+ are said to be *incident from* E . The vertices of E^- are *adjacent to* the vertices of E^+ , and the vertices of E^+ are *adjacent from* the vertices of E^- .

If E is a hyperarc in a dihypergraph H , then $|E^-|$ is the *in-size* and $|E^+|$ is the *out-size* of E . The *maximum in-size* and the *maximum out-size* of H are respectively:

$$s^-(H) = \max_{E \in \mathcal{E}(H)} |E^-| \quad \text{and} \quad s^+(H) = \max_{E \in \mathcal{E}(H)} |E^+|.$$

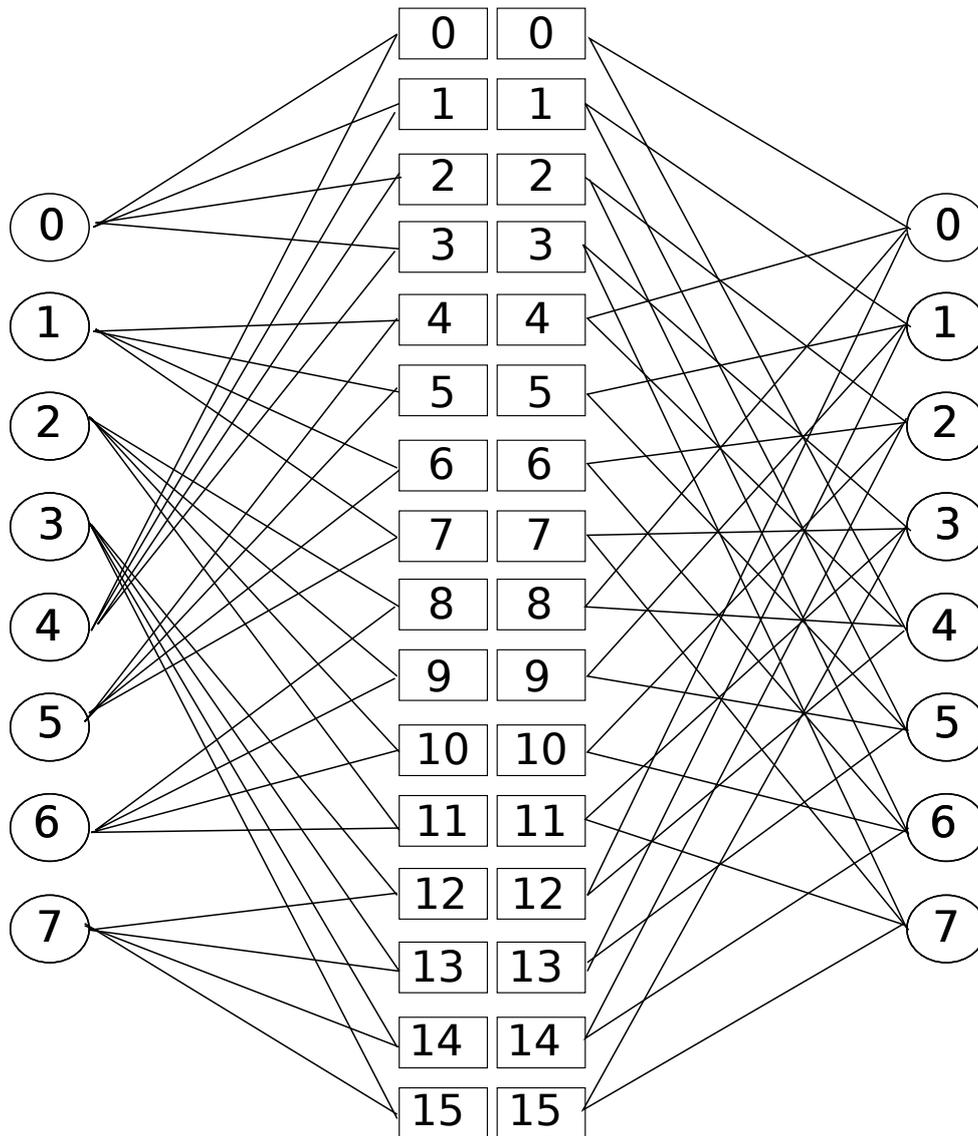


Figure 1: The bipartite representation digraph of the de Bruijn dihypergraph $GBH_1(4, 8, 2, 16)$

The *order* of H is the number of vertices in $\mathcal{V}(H)$ and is denoted by $n(H)$. The number of hyperarcs in H is denoted by $m(H)$. We note that a *digraph* is a directed hypergraph $D = (\mathcal{V}(D), \mathcal{E}(D))$ with $s^-(D) = s^+(D) = 1$.

Let v be a vertex in H . The *in-degree* of v is the number of hyperarcs that contain v in their out-set, and is denoted by $d_H^-(v)$. Similarly, the *out-degree* of vertex v is the number of hyperarcs that contain v in their in-set, and is denoted by $d_H^+(v)$.

To a directed hypergraph H , we associate a **bipartite representation digraph** $R(H) = (\mathcal{V}_1(R) \cup \mathcal{V}_2(R), \mathcal{E}(R))$, where $\mathcal{V}_1(R) = \mathcal{V}(H)$, $\mathcal{V}_2(R) = \mathcal{E}(H)$ and $\mathcal{E}(R) = \{v_i E_j \mid v_i \in E_j^-\} \cup \{E_j v_i \mid v_i \in E_j^+\}$.

This representation digraph is useful for drawing dihypergraphs. To make the figure more readable, we duplicate vertices and hyperarcs, and we put in the left part the arcs of $R(H)$ from vertices of H to hyperarcs of H , and in the right part those from $\mathcal{E}(H)$ to $\mathcal{V}(H)$. Remark that when you inverse the respective roles of $\mathcal{V}_1(R)$ and $\mathcal{V}_2(R)$ in $R(H)$, you intuitively exchange the role of the vertices with the role of the hyperarcs in H . This is an informal definition of the *dual dihypergraph* H^* . Formally:

$$\mathcal{V}(H^*) = \mathcal{E}(H)$$

$$\mathcal{E}(H^*) = \mathcal{V}(H)$$

and a vertex e is in V^- if and only if $v \in E^+$; similarly, e is in V^+ if and only if $v \in E^-$.

It is important to notice that a hyperarc $V \in \mathcal{E}(H^*)$ may have an empty in-set (if $d_H^-(v) = 0$) or an empty out-set (if $d_H^+(v) = 0$).

Other transformations are possible. For example, in the *mirror dihypergraph* \tilde{H} (sometimes called the symmetric image of H), the orientation of every hyperarc in H is inversed:

$$\mathcal{V}(\tilde{H}) = \mathcal{V}(H) \text{ and}$$

$$\mathcal{E}(\tilde{H}) = \{(E^+, E^-) \mid (E^-, E^+) \in \mathcal{E}(H)\}.$$

When we replace the hyperarcs by complete bipartite digraphs, we get the *underlying multidigraph* $U(H)$: This multidigraph has as vertex set $\mathcal{V}(U(H)) = \mathcal{V}(H)$, and as arc set $\mathcal{E}(U(H))$, that is the multiset of all ordered pairs (u, v) such that $u \in E^-$ and $v \in E^+$, for some hyperarc $E \in \mathcal{E}(H)$.

We emphasize that $U(H)$ needs not to be simple: the number of arcs from u to v in $U(H)$ is the number of hyperarcs $E = (E^-, E^+)$ in H such that $u \in E^-$ and $v \in E^+$. Thus, the in- and out-degrees of a vertex in $U(H)$ are

$$d_{U(H)}^-(u) = \sum_{\substack{E \in \mathcal{E}(H) \\ E^+ \ni u}} |E^-| \quad \text{and} \quad d_{U(H)}^+(u) = \sum_{\substack{E \in \mathcal{E}(H) \\ E^- \ni u}} |E^+|.$$

Observe that the underlying multidigraph of a given dihypergraph is unique. However, a given digraph D can be the underlying digraph of many dihypergraphs H . In fact, the hyperarcs in H form a *legitimate* partition of the arcs in D . A *legitimate* partition means that vertices in the in-set E^- of a hyperarc must be adjacent to every vertex in E^+ , in D .

Remark 1. If $s^- = s^+ = 1$ then $U(H) = H$.

Moreover, H can always be transformed into a BF-graph:

Definition 1 ([21]). A *Backward hyperarc*, or simply *B-arc*, is a hyperarc $E = (E^-, E^+)$ with $|E^+| = 1$. A *Forward hyperarc*, or simply *F-arc*, is a hyperarc $E = (E^-, E^+)$ with $|E^-| = 1$.

A *BF-graph* is a hypergraph H whose hyperarcs are either *B-arcs* or *F-arcs*. When all the hyperarcs are *B-arcs*, H is a *B-graph*. In the same way, when all the hyperarcs are *F-arcs*, H is a *F-graph*.

Let $E = (E^-, E^+)$ be a hyperarc of H that is neither a *B-arc* nor a *F-arc*. First we add a new vertex e in H . Then we replace E by the *B-arc* $E_1 = (E^-, \{e\})$ and the *F-arc* $E_2 = (\{e\}, E^+)$. When iterating this process we get a *BF-graph*, denoted $\mathbf{BF}(H)$.

By a dipath in H we mean a sequence $P = u_0, E_0, \dots, u_i, E_i, \dots, u_{p-1}, E_{p-1}, u_p$, such that, for all i, j , we have $u_i \in \mathcal{V}(H)$, $E_j \in \mathcal{E}(H)$, and $u_i \in E_{i-1}^+ \cap E_i^-$. We say that P is a dipath of length p .

One can build the *line directed hypergraph* $L(H)$ (*denoted line dihypergraph of H*). Vertices are labeled by the dipaths of length 1 in H , and hyperarcs are labeled by the dipaths of length 1 in H^* :

$$\begin{aligned}\mathcal{V}(L(H)) &= \bigcup_{E \in \mathcal{E}(H)} \{(uEv) \mid u \in E^-, v \in E^+\}, \\ \mathcal{E}(L(H)) &= \bigcup_{v \in \mathcal{V}(H)} \{(EvF) \mid v \in E^+ \cap F^-\};\end{aligned}$$

where the in-set and the out-set of hyperarc (EvF) are defined as :

$$\begin{aligned}(EvF)^- &= \{(uEv) \mid u \in E^-\}, \\ (EvF)^+ &= \{(vFw) \mid w \in F^+\}.\end{aligned}$$

Particularly when D is a digraph, $L(D)$ is called the *line digraph* of D .

The following results are used in the next sections:

Theorem 1 ([6]). *Let H be a directed hypergraph.*

1. *The digraphs $R(L(H))$ and $L^2(R(H))$ are isomorphic.*
2. *The digraphs $U(L(H))$ and $L(U(H))$ are isomorphic.*
3. *The digraphs $(L(H))^*$ and $L(H^*)$ are isomorphic.*

2.2 Eulerian and Hamiltonian Dicycles in Dihypergraphs

Let H be a directed hypergraph. A dipath $P = u_0, E_0, \dots, u_i, E_i, \dots, u_{p-1}, E_{p-1}, u_p$ is a dicycle in H if and only if we have $u_0 = u_p$.

Dicycles in a digraph are a special case of dicycles in a dihypergraph.

Remark 2. *A dicycle in the dihypergraph H can be seen as a dicycle in its bipartite representation $R(H)$.*

In the same way, we can extend the digraph-theoretic notions of *Eulerian* dicycles and *Hamiltonian* dicycles, to the dihypergraphs:

Definition 2. *Let H be a dihypergraph. We say that H is Eulerian (resp. H is Hamiltonian) if, and only if, there is a dicycle C in H , such that every hyperarc of H (resp. every vertex of H) appears in C exactly once. We call C an Eulerian dicycle (resp. a Hamiltonian dicycle).*

Remark 3. *A dihypergraph H is Eulerian if and only if there is a dipath $P = u_0, E_0, \dots, u_i, E_i, \dots, u_{m-1}, E_{m-1}, u_m$, such that every hyperarc of H appears exactly once in P , and u_m is in E_0^- .*

Indeed, when $u_0 \neq u_p$, we can always replace u_0 by u_p , and in so doing we have an Eulerian dicycle.

Our generalization of an Eulerian dicycle to dihypergraphs is close to the extension of an Euler tour to the *undirected* hypergraphs, that was proposed in [26].

Definition 3 ([26]). Let H_u be an undirected hypergraph. A tour is a sequence $T = v_0, E_0, v_1, \dots, v_{m-1}, E_{m-1}, v_0$ where for all i , $v_i \neq v_{i+1}$ and v_i, v_{i+1} are in hyperedge E_i . Particularly, T is called an Euler tour when every edge appears exactly once in T . H_u is an Eulerian hypergraph if there exists an Euler tour T in H_u .

Remark 4. An Eulerian dicycle in H (resp. a Hamiltonian dicycle in H) is a dicycle in $R(H)$, such that each vertex of $\mathcal{V}_2(R)$ (resp. of $\mathcal{V}_1(R)$) appears exactly once.

As a consequence, a necessary and sufficient condition for $R(H)$ to be Hamiltonian is that there is a dicycle C in H , such that C is simultaneously an Eulerian dicycle and a Hamiltonian dicycle in H . In reference to the undirected case [5], we call C a *complete Berge dicycle*:

Definition 4. Let H be a directed hypergraph. A complete Berge dicycle in H is a dicycle C in H , such that C is an Eulerian dicycle and a Hamiltonian dicycle in H .

In the following sections, we focus on Eulerian dihypergraphs. We assume that the studied dihypergraphs have no isolated vertex, without any loss of generality.

3 First Results

3.1 Some conditions

First we recall the well-known characterization of the Eulerian digraphs:

Theorem 2 ([4]). Let D be a digraph. The following statements are equivalent:

1. D is Eulerian.
2. D is (strongly) connected, and for all vertex $v \in \mathcal{V}(D)$ $d^-(v) = d^+(v)$.
3. D is (strongly) connected, and it has a cycle decomposition (that is its arcs can be partitioned into arc-disjoint dicycles).

A digraph D is weakly connected when its underlying multigraph is connected. It is a strongly connected digraph provided for each pair of vertices $u, v \in \mathcal{V}(D)$ there exists a dipath from u to v . These connectivity notions can be extended to dihypergraphs as well [6]. We say that H is strongly (resp. weakly) connected when its underlying multidigraph $U(H)$ is strongly (resp. weakly) connected. $U(H)$ is weakly connected when its *underlying* multigraph $G_{U(H)}$ is a connected multigraph. The digraph-theoretic notions of vertex-connectivity and arc-connectivity are also generalized by the dihypergraph-theoretic notions of vertex-connectivity and hyperarc-connectivity. Unlike 1-arc connected digraphs, 1-hyperarc connected dihypergraphs are not always 1-vertex connected.

Remark that unlike an Eulerian digraph, an Eulerian dihypergraph does not need to be strongly connected. Indeed, let H be an Eulerian dihypergraph. If we add a new vertex x in H , such that x is incident to only one hyperarc E of H , and $d^-(x) = 0$, then H is still Eulerian, but H cannot be strongly connected anymore.

On the other hand, we have the following necessary condition:

Proposition 3. Let H be a dihypergraph. If H is Eulerian, then H is weakly connected.

Proof. Let H be a dihypergraph. Let $G_{U(H)}$ be the *underlying* multigraph of $U(H)$. We want to prove that $G_{U(H)}$ is connected.

Remark first that for all hyperarc $E \in \mathcal{E}(H)$, vertices in the subset $E^- \cup E^+$ are in the same connected component in $G_{U(H)}$, by the definition of $U(H)$.

Moreover, let E, F be two distinct hyperarcs in $\mathcal{E}(H)$. Since there is an Eulerian dicycle in H , therefore, $\exists u \in E^+, \exists v \in F^-$, such that there is a dipath in H from u to v . Since there is a dipath from u to v in H , therefore

there is a dipath P from u to v in $U(H)$. Let P_u be the *undirected* version of P in $G_{U(H)}$. P_u is a path from u to v (resp. from v to u) in $G_{U(H)}$. So subsets $E^- \cup E^+$ and $F^- \cup F^+$ are in the same connected component in $G_{U(H)}$ too. Therefore, $G_{U(H)}$ is connected. \square

A necessary condition for a uniform, undirected hypergraph H_u to be Eulerian is presented in [26]. A k -uniform undirected hypergraph is a hypergraph H_u such that for all hyperedge E in $\mathcal{E}(H_u)$, E is a subset of $\mathcal{V}(H_u)$ with k elements. An extension of this property to dihypergraphs is presented and studied in the next section.

Theorem 4 ([26]). *Let H_u be a k -uniform, undirected hypergraph. If H_u is an Eulerian hypergraph, then*

$$|V_{\text{odd}}(H_u)| \leq (k-2)m(H_u)$$

where $V_{\text{odd}}(H_u)$ denotes the set of vertices in H_u that are incident to an odd number of edges, and $m(H_u)$ is the total number of edges in H_u .

Using the same idea we prove a necessary condition for a directed hypergraph H to be Eulerian.

Theorem 5. *Let H be a dihypergraph. If H is Eulerian then:*

$$\sum_{v \in \mathcal{V}(H)} |d^+(v) - d^-(v)| \leq \sum_{E \in \mathcal{E}(H)} [(|E^+| - 1) + (|E^-| - 1)].$$

Proof. Let $m = m(H)$ be the number of hyperarcs in H . Let $C = v_0, E_0, v_1, \dots, v_{m-1}, E_{m-1}, v_0$ be an Eulerian dicycle in H . By definition, a given vertex may appear many times in C , but every hyperarc appears exactly once in the dicycle C . Let us find the maximum number of occurrences of a given vertex v in C . $\forall i \neq j$ we may have $v_i = v_j$, but we are sure that $E_i \neq E_j$. So a vertex v can appear, at most, $\min(d^+(v), d^-(v))$ times in C and, as a consequence, we have the following inequality:

$$\sum_{v \in \mathcal{V}(H)} \min(d^+(v), d^-(v)) \geq m(H)$$

Moreover, we know that:

$$\min(d^+(v), d^-(v)) = \frac{1}{2}(d^+(v) + d^-(v) - |d^+(v) - d^-(v)|)$$

Since

$$\sum_{v \in \mathcal{V}(H)} d^+(v) = \sum_{E \in \mathcal{E}(H)} |E^-|$$

$$\sum_{v \in \mathcal{V}(H)} d^-(v) = \sum_{E \in \mathcal{E}(H)} |E^+|$$

therefore, the following inequalities are all equivalent to the previous one:

$$\begin{aligned} \frac{1}{2} \sum_{v \in \mathcal{V}(H)} (d^+(v) + d^-(v) - |d^+(v) - d^-(v)|) &\geq m(H) \\ \sum_{v \in \mathcal{V}(H)} (d^+(v) + d^-(v) - |d^+(v) - d^-(v)|) &\geq 2m(H) \\ \sum_{v \in \mathcal{V}(H)} d^+(v) + \sum_{v \in \mathcal{V}(H)} d^-(v) - \sum_{v \in \mathcal{V}(H)} |d^+(v) - d^-(v)| &\geq \sum_{E \in \mathcal{E}(H)} 2 \\ \sum_{E \in \mathcal{E}(H)} |E^-| + \sum_{E \in \mathcal{E}(H)} |E^+| - \sum_{v \in \mathcal{V}(H)} |d^+(v) - d^-(v)| &\geq \sum_{E \in \mathcal{E}(H)} 2 \\ \sum_{u \in \mathcal{V}(H)} |d^+(u) - d^-(u)| &\leq \sum_{E \in \mathcal{E}(H)} [(|E^+| - 1) + (|E^-| - 1)] \end{aligned}$$

\square

For a directed graph, Theorem 5 is equivalent to the Euler condition:

Corollary 1. *Let D be a digraph. If D is Eulerian, then for all $u \in \mathcal{V}(D)$, $d^+(u) = d^-(u)$.*

Theorem 5 is not a sufficient condition for a strongly connected dihypergraph H to be Eulerian: a counter-example is presented in Section 3.3, Figure 6.

Another necessary condition was proposed by N. Cohen (private communication), who transposed the search of an Eulerian dicycle to a *flow networks* problem.

Definition 5. *Let $N = (\mathcal{V}, \mathcal{E})$ be a directed graph, with two specific vertices s and t (the source and the sink, respectively). N is called a network. Each arc $uv \in \mathcal{E}$ has a capacity $c(uv) \in \mathbb{N}^*$. A flow is a mapping $f : \mathcal{E} \rightarrow \mathbb{N}$, such that:*

1. for all $uv \in \mathcal{E}$, $f(uv) \leq c(uv)$ (capacity constraint)
2. for all vertex $v \neq s, t$ $\sum_{uv \in \mathcal{E}} f(uv) = \sum_{vw \in \mathcal{E}} f(vw)$ (conservation of flows)

The value of the flow is $|f| = \sum_{v \in \mathcal{V}} f(sv)$.

Let H be a directed hypergraph. If there is a hyperarc E whose in-set (resp. whose out-set) is empty, then H cannot be Eulerian. Else, let φ be a function from $\mathcal{E}(H)$ to $\mathcal{V}(H)^2$, such that for all E , $\varphi(E) \in E^- \times E^+$. Actually we transform the hyperarcs into arcs. So we get a digraph, denoted $D_\varphi[H] = (\mathcal{V}(H), \varphi(\mathcal{E}(H)))$. Observe that $D_\varphi[H]$ is a subdigraph of $U(H)$. There are $\prod_{E \in \mathcal{E}(H)} |E^-||E^+|$ possible φ functions.

Remark 5. *A dihypergraph H is Eulerian if and only if there exists a function φ such that $D_\varphi[H]$ is an Eulerian digraph.*

By Theorem 2, a necessary and sufficient condition for a digraph D to be Eulerian is that for every vertex v , $d^-(v) = d^+(v)$ and D must be connected. When D satisfies only the first condition, we call it a *balanced* digraph.

Let $s, t \notin \mathcal{V}(H)$ be two vertices. Consider the network $N_f(H) = (\mathcal{V}_f(H), \mathcal{E}_f(H))$, such that:

$$\mathcal{V}_f(H) = \mathcal{V}(H) \cup \{E^- \mid E \in \mathcal{E}(H)\} \cup \{E^+ \mid E \in \mathcal{E}(H)\} \cup \{s, t\}$$

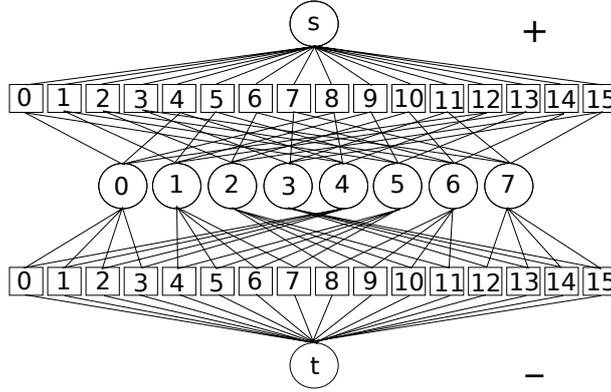
$$\mathcal{E}_f(H) = \{sE_j^+ : E_j \in \mathcal{E}(H)\} \cup \{E_j^-t : E_j \in \mathcal{E}(H)\} \cup \{E_j^+v : v \in E_j^+\} \cup \{vE_j^- : v \in E_j^-\}$$

Each arc $uv \in \mathcal{E}_f(H)$ has a capacity of 1. In those conditions, by the max-flow min-cut Theorem, the maximum value of an $s-t$ flow in $N_f(H)$ is the greatest number of pairwise arc-disjoint dipaths between s and t in $N_f(H)$. An upper bound of this value is $m(H)$.

Lemma 1. *Let H be a dihypergraph. If H is Eulerian, then the maximum value of an $s-t$ flow in $N_f(H)$ is $m(H)$. More precisely, there exists φ such that $D_\varphi[H]$ is a balanced digraph if and only if this condition is verified.*

Proof. If H is Eulerian, by Remark 5, there exists φ such that $D_\varphi[H]$ is a balanced digraph. As a consequence, we only need to show the second part of the lemma.

Let us assume the maximum value of the flow is $m(H)$, that is there are $m(H)$ pairwise arc-disjoint dipaths in $N_f(H)$ between s and t . Let $P_1, \dots, P_{m(H)}$ be $m(H)$ pairwise arc-disjoint dipaths in $N_f(H)$ between s and t . By construction those dipaths are of the form $s, E_{j_1}^+, v_i, E_{j_2}^-, t$, such that every $E_{j_1}^+$ (resp. every $E_{j_2}^-$) appears in only one

Figure 2: The flow networks $N_f(GBH_1(4, 8, 2, 16))$.

dipath P_k . We define the function φ as follows: if $s, E_{j_1}^+, v_i, E_{j_2}^-, t$ and $s, E_{j_2}^+, v_{i'}, E_{j_3}^-, t$ are among the $m(H)$ pairwise arc-disjoint dipaths between s and t , then $\varphi(E_{j_2}) = (v_i, v_{i'})$. Clearly, $D_\varphi[H]$ is a balanced digraph.

Now, let us suppose there exists φ such that $D_\varphi[H]$ is a balanced digraph. Let v be a vertex of $D_\varphi[H]$. Since $d_{D_\varphi[H]}^-(v) = d_{D_\varphi[H]}^+(v)$ one can define a bijection between the out-links and the in-links of v . Those arcs are all associated to a different hyperarc by φ . Therefore we can build $d_{D_\varphi[H]}^-(v)$ pairwise arc-disjoint dipaths in $N_f(H)$ between s and t . Since $\sum_{v \in \mathcal{V}(H)} d_{D_\varphi[H]}^-(v) = m(H)$ by construction of $D_\varphi[H]$, therefore there are $m(H)$ pairwise arc-disjoint dipaths in $N_f(H)$ between s and t . \square

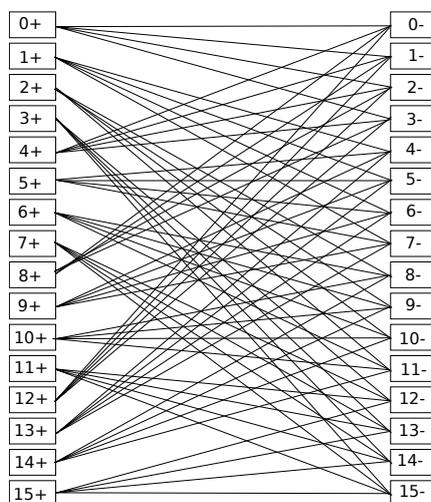
Another formulation of Lemma 1 is proposed. Actually, it is easier to show the characterization if we transpose the flow networks problem to a *perfect matching* problem. The relation between both of these problems is well known. Let $BP(H) = (\mathcal{V}_1(BP), \mathcal{V}_2(BP), \mathcal{E}(BP))$ be a bipartite graph, with:

$$\begin{aligned}\mathcal{V}_1(BP) &= \{E_j^+ : E_j \in \mathcal{E}(H)\} \\ \mathcal{V}_2(BP) &= \{E_j^- : E_j \in \mathcal{E}(H)\} \\ \mathcal{E}(BP) &= \{E_j^+ E_j^- : E_j^+ \cap E_j^- \neq \emptyset\}\end{aligned}$$

Lemma 2. *Let H be a directed hypergraph. There is a perfect matching in $BP(H)$ if and only if the maximum value of an $s-t$ flow in $N_f(H)$ is $m(H)$.*

Proof. Let e, f be two vertex-disjoint edges in $BP(H)$: $e = E_{i_1}^+ E_{i_2}^-$ and $f = E_{j_1}^+ E_{j_2}^-$. By construction of $BP(H)$ $E_{i_1}^+ \cap E_{i_2}^- \neq \emptyset$ and $E_{j_1}^+ \cap E_{j_2}^- \neq \emptyset$. Therefore we can select a vertex $u \in E_{i_1}^+ \cap E_{i_2}^-$ and a vertex $v \in E_{j_1}^+ \cap E_{j_2}^-$. Clearly, $P_1 = s, E_{i_1}^+, u, E_{i_2}^-, t$ and $P_2 = s, E_{j_1}^+, v, E_{j_2}^-, t$ are two dipaths in $N_f(H)$ between s and t . Furthermore, since e and f are vertex-disjoint, therefore $E_{i_1} \neq E_{j_1}$ and $E_{i_2} \neq E_{j_2}$. As a consequence, P_1 and P_2 are arc-disjoint.

Reciprocally, let $P_1 = s, E_{i_1}^+, u, E_{i_2}^-, t$ and $P_2 = s, E_{j_1}^+, v, E_{j_2}^-, t$ be two arc-disjoint dipaths in $N_f(H)$ between s and t . By construction of $N_f(H)$, u is in $E_{i_1}^+ \cap E_{i_2}^-$ and v is in $E_{j_1}^+ \cap E_{j_2}^-$. Therefore $e = E_{i_1}^+ E_{i_2}^-$ and $f = E_{j_1}^+ E_{j_2}^-$ are edges in $BP(H)$. Moreover, since P_1 and P_2 are arc-disjoint, therefore $E_{i_1} \neq E_{j_1}$ and $E_{i_2} \neq E_{j_2}$. As a consequence, e and f are vertex-disjoint.

Figure 3: The bipartite graph $BP(GBH_1(4, 8, 2, 16))$.

Let us assume there is a perfect matching in $BP(H)$. That means there are $m(H)$ pairwise vertex-disjoint edges in $BP(H)$. So there are $m(H)$ pairwise arc-disjoint dipaths in $N_f(H)$ between s and t . Reciprocally, if there are $m(H)$ pairwise arc-disjoint dipaths in $N_f(H)$ then there are $m(H)$ pairwise vertex-disjoint edges in $BP(H)$ - that is a perfect matching. \square

A well-known condition allowing a bipartite graph to have a perfect matching is the Hall's Theorem. We use the following result so that we can prove a necessary and sufficient condition for a dihypergraph to have a balanced digraph.

Theorem 6 (Hall, [5]). *Let $G = (\mathcal{V}_1 \cup \mathcal{V}_2, \mathcal{E})$ be a bipartite graph such that $|\mathcal{V}_1| = |\mathcal{V}_2|$. There is a perfect matching in B if and only if for every subset $S \subset \mathcal{V}_1$, $|Adj(S)| \geq |S|$, where $Adj(S)$ denotes the set of vertices adjacent to some vertex of S .*

Let X be a subset of $\mathcal{V}(H)$. Let $d^+(X)$ (also denoted $d_H^+(X)$) be the number of hyperarcs $E \in \mathcal{E}(H)$ such that $E^- \cap X \neq \emptyset$, and $d_s^-(X)$ (also denoted $d_{s,H}^-(X)$) be the number of hyperarcs E such that $E^+ \subseteq X$. We are now able to prove the following result:

Theorem 7. *Let H be a dihypergraph. There exists φ such that $D_\varphi[H]$ is a balanced digraph if and only if for every subset $X \subseteq \mathcal{V}(H)$ $d_s^-(X) \leq d^+(X)$.*

Proof. Let us assume there exists φ such that $D_\varphi[H]$ is a balanced digraph. For every subset $X \subseteq \mathcal{V}(H)$, for every hyperarc E such that $E^+ \subseteq X$, we necessarily have $(u, v) = \varphi(E) \in \mathcal{V}(H) \times X$. Since $D_\varphi[H]$ is balanced, hence there must be the image $\varphi(F)$ of some hyperarc F whose origin is v . Clearly, $F^- \cap X$ is not empty. Hence $d_s^-(X) \leq d^+(X)$.

Reciprocally, let us assume that for every subset $X \subseteq \mathcal{V}(H)$, $d_s^-(X) \leq d^+(X)$. Let $S = \{E_{j_1}^+, E_{j_2}^+, \dots, E_{j_{|S|}}^+\}$ be a subset of $\mathcal{V}_1(BP)$, and $X = \bigcup_{k=1}^{|S|} E_{j_k}^+$. Observe that $|Adj(S)| = d^+(X)$. Since $d_s^-(X) \geq |S|$, therefore $|Adj(S)| \geq |S|$.

Furthermore, $\mathcal{V}_1(BP)$ and $\mathcal{V}_2(BP)$ obviously have the same cardinality. By Theorem 6, there is a perfect matching in $BP(H)$. Therefore by Lemma 2, there are $m(H)$ pairwise arc-disjoint dipaths in $N_f(H)$ between s and t . So, by Lemma 1, there exists φ such that $D_\varphi[H]$ is a balanced digraph. \square

Observe that we may define $d^-(X)$ and $d_s^+(X)$ in the same way as $d^+(X)$ and $d_s^-(X)$. Another formulation of Theorem 7 is: there exists φ such that $D_\varphi[H]$ is a balanced digraph if and only if for every subset $X \subset \mathcal{V}(H)$ $d_s^+(X) \leq d^-(X)$.

Proposition 8. *Let H be a B-graph (resp. a F-graph). If for all subset $X \subset \mathcal{V}(H)$, $d_s^-(X) \leq d^+(X)$, then:*

$$\sum_{u \in \mathcal{V}(H)} |d^+(u) - d^-(u)| = \sum_{E \in \mathcal{E}(H)} [(|E^+| - 1) + (|E^-| - 1)].$$

Proof. By symmetry, if Proposition 8 is true for B-graphs, then it is true for F-graphs. Let H be a B-graph. Observe that, for every subset X , $d_s^-(X) = d^-(X) = \sum_{x \in X} d^-(x)$. If, for every subset X , $d^-(X) \leq d^+(X)$, then, for every vertex v , $d^-(v) \leq d^+(v)$. Therefore:

$$\sum_{v \in \mathcal{V}(H)} |d^+(v) - d^-(v)| = \sum_{v \in \mathcal{V}(H)} d^+(v) - d^-(v) = \sum_{v \in \mathcal{V}(H)} d^+(v) - \sum_{v \in \mathcal{V}(H)} d^-(v) = \sum_{E \in \mathcal{E}(H)} |E^-| - \sum_{E \in \mathcal{E}(H)} |E^+|$$

As for a B-graph $|E^+| = 1$ for every $E \in \mathcal{E}(H)$:

$$\sum_{E \in \mathcal{E}(H)} |E^-| - \sum_{E \in \mathcal{E}(H)} |E^+| = \sum_{E \in \mathcal{E}(H)} |E^-| - \sum_{E \in \mathcal{E}(H)} 1 = \sum_{E \in \mathcal{E}(H)} |E^-| - 1 = \sum_{E \in \mathcal{E}(H)} [|E^+| - 1 + |E^-| - 1]$$

\square

Consequently, Theorem 7 is stronger than Theorem 5 for B-graphs and F-graphs.

Remark 6. *Deciding whether there exists φ such that $D_\varphi[H]$ is a balanced digraph can be done in polynomial time. But the connectivity of $D_\varphi[H]$ cannot be guaranteed.*

Actually, deciding whether there exists φ such that $D_\varphi[H]$ is strongly connected is an NP-complete problem [25]. But the condition of Theorem 7 is sufficient if we impose that for every φ , $D_\varphi[H]$ is connected.

Proposition 9. *Let H be a dihypergraph. We denote $\lambda = \sum_{E \in \mathcal{E}(H)} |E^+||E^-| - 1$. Let $G_{U(H)}$ be the underlying multigraph of the underlying multidigraph $U(H)$. If $G_{U(H)}$ is $\lambda + 1$ -edge connected, then H is Eulerian if and only if for every subset X , $d_s^-(X) \leq d^+(X)$.*

Proof. Let E be a hyperarc in H . By construction of $U(H)$, there are $|E^-||E^+|$ arcs of $U(H)$ that are associated to E . Hence the association of E to a single arc, by some function φ , requires the deletion of $|E^-||E^+| - 1$ of these arcs. Therefore, if $G_{U(H)}$ is $\lambda + 1$ -edge connected, then every $D_\varphi[H]$ is weakly-connected. \square

This solution appears very restrictive. Another approach is the *skeleton-connectivity*. Let H be a BF-graph. We know that for all hyperarc $E \in \mathcal{E}(H)$, E^- or E^+ is a singleton, denoted $\{e\}$. The choice of e is not unique when E is an arc. Observe that for all φ , $\{e\}$ is in $\{\varphi(E)^-, \varphi(E)^+\}$. Consequently, we have a better control over the possible values of $\varphi(E)$ than in the general case.

We define a graph $S(H) = (\mathcal{V}(S), \mathcal{E}(S))$ such that $\mathcal{V}(S)$ is isomorphic to $\mathcal{E}(H)$, and if E, F are two hyperarcs in H , then EF is in $\mathcal{E}(S)$ if and only if they share a singleton. Formally, the edge exists if and only if there is $\{u\} \in \{E^+, E^-\}$, there is $\{v\} \in \{F^+, F^-\}$, such that $u = v$. We call H a skeleton-connected dihypergraph when S is a connected graph.

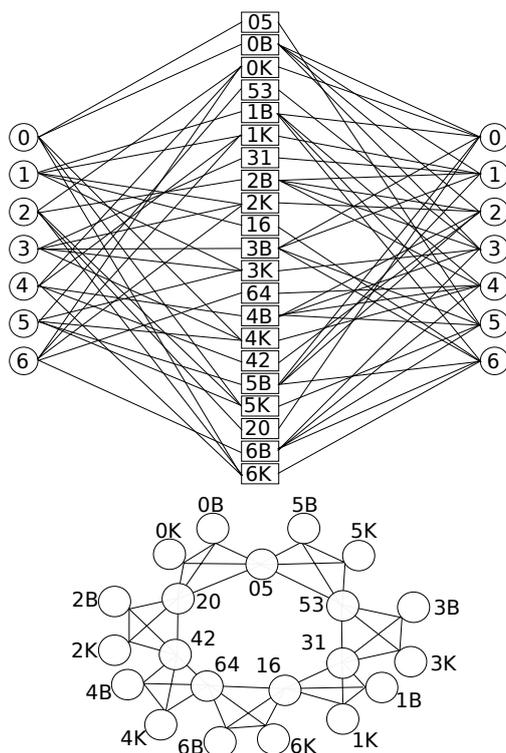


Figure 4: A skeleton-connected dihypergraph.

Lemma 3. *Let H be a BF-graph. If H is skeleton-connected then for every φ , $D_\varphi[H]$ is weakly connected.*

Proof. Let E, F be two hyperarcs in H , such that EF is an edge of $S(H)$. By definition, there exists $\{u\} \in \{E^+, E^-\}$, there exists $\{v\} \in \{F^+, F^-\}$, such that $u = v$. Furthermore since $|\{u\}| = 1$, therefore $\{u\}$ is in $\{\varphi(E)^+, \varphi(E)^-\}$ and in $\{\varphi(F)^+, \varphi(F)^-\}$. So $\varphi(E)\varphi(F)$ is still an edge of $S(D_\varphi[H])$.

Consequently, $S(H)$ connected implies $S(D_\varphi[H])$ connected. Clearly, skeleton-connectivity implies weakly-connectivity for a digraph. Therefore, $D_\varphi[H]$ is weakly connected. \square

We generalize the Euler’s Theorem to BF-graphs:

Theorem 10. *Let H be a skeleton-connected BF-graph. H is Eulerian if and only if for every subset X of $\mathcal{V}(H)$ $d_s^-(X) \leq d^+(X)$.*

Proof. That follows from Lemma 3 and Theorem 7. \square

In Section 2.1, we detail the method of construction of a BF-graph $BF(H)$ from any given dihypergraph H . So we may imagine using the method and combining it with Theorem 10, in order to get Eulerian dihypergraphs. Unfortunately, $BF(H)$ may be skeleton-connected if and only if H is already a BF-graph (hence $H = BF(H)$), or $m(H) = 1$.

From any dihypergraph H , such that for each hyperarc E , $E^- \neq \emptyset$ and $E^+ \neq \emptyset$, one can obtain a B-graph $B(H)$ in the following way: let $E \in \mathcal{E}(H)$ be a hyperarc that is not a B-arc. Then we order the elements of E^+ , denoted $e_1, e_2, \dots, e_{|E^+|}$, and we replace E by the $|E^+|$ B-arcs $Ee_1, Ee_2, \dots, Ee_{|E^+|}$, such that for all j , with $1 \leq j \leq |E^+|$:

$$(Ee_j)^- = E^-,$$

$$(Ee_j)^+ = \{e_j\}.$$

In the same way, H could be transformed into a F-graph, denoted $F(H)$. We present a sufficient condition for $B(H)$ and $F(H)$ to be Eulerian.

Proposition 11. *Let H be a directed hypergraph, having no hyperarc E such that $E^- = \emptyset$ or $E^+ = \emptyset$. If $R(H)$ is Eulerian, then both $B(H)$ and $F(H)$ are Eulerian.*

Proof. By construction of the bipartite representation digraph $R(H)$, $\mathcal{E}(R(H)) = \mathcal{E}_1 \cup \mathcal{E}_2$, where $\mathcal{E}_1 = \{u_i E_j : u_i \in E_j^-\}$ and $\mathcal{E}_2 = \{E_j u_i : u_i \in E_j^+\}$. One can easily check that there is a natural bijection f between \mathcal{E}_1 and the F-arcs in $F(H)$, plus another one b between \mathcal{E}_2 and the B-arcs of $B(H)$. Let us assume that $R(H)$ is Eulerian.

Let $C = u_0, u_0 E_0, E_0, E_0 u_1, u_1, \dots, u_k, u_k E_k, E_k, E_k u_0, u_0$ be an Eulerian dicycle in $R(H)$. Then $C_F = u_0, f(u_0 E_0), u_1, f(u_1 E_1), \dots, u_k, f(u_k E_k), u_0$ is an Eulerian dicycle in $F(H)$, and $C_B = u_0, b(E_0 u_1), u_1, b(E_1 u_2), \dots, u_k, b(E_k u_0), u_0$ is an Eulerian dicycle in $B(H)$. \square

3.2 Equivalences and Complexity

We know that from a dihypergraph H , we can build other dihypergraphs that are related to H (see Section 2.1). The search of an Eulerian dicycle in H is equivalent to the search of other dicycles in those directed hypergraphs:

Proposition 12. *Let H be a dihypergraph. The following statements are equivalent:*

1. H is Eulerian.
2. H^* is Hamiltonian.
3. \tilde{H} is Eulerian.
4. $BF(H)$ is Eulerian.

Proof. **Let us prove that H is Eulerian if, and only if, H^* is Hamiltonian.** To a dicycle $C = v_0, E_0, v_1, \dots, v_p, E_p, v_0$ of H we associate in H^* a dicycle $C^* = e_0, V_1, e_2, \dots, e_p, V_0, e_0$. If C is an Eulerian dicycle in H (i.e. C contains each edge exactly once) then C^* contains each vertex of H^* exactly once, and so C^* is a Hamiltonian dicycle of H^* . Conversely, if C^* is a Hamiltonian dicycle of H^* (each vertex appears exactly once), C is an Eulerian dicycle.

We prove now that H is Eulerian if, and only if, \tilde{H} is Eulerian. Since $\tilde{\tilde{H}} = H$ we only need to prove the implication. For a given hyperarc $E = (E^-, E^+) \in \mathcal{E}(H)$ we call $\tilde{E} = (E^+, E^-)$ the associated hyperarc in $\mathcal{E}(\tilde{H})$. Let $C = v_0, E_0, \dots, v_{m-1}, E_{m-1}, v_0$ be an Eulerian dicycle in H . Then, by definition of \tilde{H} , $\tilde{C} = v_0, \tilde{E}_{m-1}, \dots, v_1, \tilde{E}_0, v_0$ is an Eulerian dicycle in \tilde{H} .

Finally, let us show that H is Eulerian if and only if $BF(H)$ is Eulerian. Let $C = v_0, E_0, v_1, \dots, v_{m(H)-1}, E_{m(H)-1}, v_0$ be an Eulerian dicycle in H . For all j , if E_j is neither a B-arc nor a F-arc, E_j is replaced in $BF(H)$ by a B-arc $E_{j,1}$ and a F-arc $E_{j,2}$, that are linked by a new vertex e_j ; in order to obtain an Eulerian dicycle C_{BF} of $BF(H)$, we replace E_j by the sequence $E_{j,1}, e_j, E_{j,2}$ in C . Conversely, assume that C_{BF} is an Eulerian dicycle in $BF(H)$. By construction of $BF(H)$, $E_{j,1}$ only shares a common vertex with $E_{j,2}$. Consequently, $E_{j,1}, e_j, E_{j,2}$ is a subsequence of C_{BF} , where e_j is the vertex of $BF(H)$ by which $E_{j,1}$ and $E_{j,2}$ are linked. So we may replace this subsequence by E_j for every such E_j , and we get a dicycle C that is an Eulerian dicycle in H . \square

Remark 7. *Since $(H^*)^* = H$, H is Hamiltonian if, and only if, H^* is Eulerian.*

These equivalences may help to prove that some given dihypergraphs are Eulerian. Remark that because of Proposition 12, searching an Eulerian dicycle in a directed hypergraph H is as hard as searching a Hamiltonian dicycle in its dual. Since it is already the case when the studied dihypergraphs are digraphs [2], deciding whether a dihypergraph H is Hamiltonian is an NP-complete problem. So the following result is not that surprising:

Theorem 13. *Deciding if a dihypergraph H is Eulerian is NP-complete.*

Proof. Let C be a dipath. One can verify, in $O(|\mathcal{E}(H)|)$ operations, whether C is an Eulerian dicycle in H . Consequently, the problem is in NP.

Now, assume that H is an instance for the Hamiltonian problem: *Given a dihypergraph H . Is there a Hamiltonian dicycle in H ?* This problem is NP-complete. Moreover, the dual of H can be built in $O(|\mathcal{E}(H)| + |\mathcal{V}(H)|)$ operations. By Proposition 12, H is Hamiltonian if and only if H^* is Eulerian. Hence, the Eulerian problem is NP-complete too. \square

Finding Hamiltonian dihypergraphs is useful to build Eulerian dihypergraphs. The following lemma is used in the next sections:

Lemma 4. *Let H be a directed hypergraph. H is Hamiltonian if, and only if, its underlying multidigraph $U(H)$ is Hamiltonian.*

Proof. By definition of $U(H)$, any dicycle in H is a dicycle in $U(H)$, with the same vertices, and reciprocally. \square

This lemma, and its proof, have a direct application. Let \mathcal{G} be a family of Hamiltonian digraphs. Let D be a digraph in \mathcal{G} . Using a *legitimate* partition of the arcs in $\mathcal{E}(D)$, we can build a dihypergraph H , whose underlying digraph is D . Consequently, we build a set \mathcal{H} of Hamiltonian dihypergraphs. Let $\mathcal{H}^* = \{H : H^* \in \mathcal{H}\}$. Then \mathcal{H}^* is a set of Eulerian dihypergraphs, by Proposition 12.

Here is another application: let D be a digraph. From D we build a dihypergraph H_D such that:

$$\mathcal{V}(H_D) = \mathcal{V}(D)$$

and for every vertex v we associate a hyperarc V defined as follows:

$$\begin{aligned} V^- &= \{v\} \\ V^+ &= \{w \in \mathcal{V}(D) : vw \in \mathcal{E}(D)\} \end{aligned}$$

An Eulerian dicycle in H_D (resp. a Hamiltonian dicycle) is always a complete Berge dicycle. Therefore H_D is Eulerian if and only if D is Hamiltonian.

3.3 Line Dihypergraphs Properties

For a given digraph D , the line digraph $L(D)$ is Hamiltonian if, and only if, D is Eulerian (see [13]). First, we give a new proof of this result.

Lemma 5. *Let D be a digraph. Its line digraph $L(D)$ is isomorphic to $U(D^*)$.*

Proof. Let D be a digraph. By construction of its dual dihypergraph, $\mathcal{V}(U(D^*))$ is isomorphic to $\mathcal{E}(D)$. Since every arc in D is represented by a vertex in $L(D)$, therefore $\mathcal{V}(L(D))$ and $\mathcal{V}(U(D^*))$ are isomorphic.

Let v be a vertex in D . Let $E_v = (E_v^-, E_v^+)$ be the associated hyperarc in D^* . E_v^- is the set of the arcs $uv \in \mathcal{E}(D)$, and E_v^+ is the set of the arcs $vw \in \mathcal{E}(D)$. As a consequence, for each vertex uv in $U(D^*)$, uv is incident to the vertices $vw \in \mathcal{V}(U(D^*))$. Those are the relations of incidence in $L(D)$. Moreover, in D^* the vertex uv is only incident to the hyperarc E_v . Therefore, $U(D^*)$ is a simple digraph. \square

Corollary 2. *Let D be a digraph. D is Eulerian if and only if its line digraph $L(D)$ is Hamiltonian.*

Proof. Let D be a digraph.

$$\begin{aligned} D \text{ is Eulerian} &\iff \\ D^* \text{ is Hamiltonian, by Proposition 12} &\iff \\ U(D^*) \text{ is Hamiltonian, by Lemma 4} &\iff \\ L(D) \text{ is Hamiltonian, by Lemma 5} &\iff \end{aligned}$$

□

This property is useful for some special families of digraphs, such that Kautz and de Bruijn digraphs, that are stable by line digraph operation [13]. Using induction, one can prove that every digraph of the family is Hamiltonian. It was shown in [6] that de Bruijn and Kautz dihypergraphs are also stable by line dihypergraph operation. So it is natural to wonder whether Corollary 2 can be generalized to dihypergraphs. Unfortunately, that is not exactly the case.

Proposition 14. *Let H be a directed hypergraph.*

$$L(H) \text{ is Hamiltonian} \iff U(H) \text{ is Eulerian.}$$

Proof.

$$\begin{aligned} U(H) \text{ Eulerian} &\iff \\ L(U(H)) \text{ Hamiltonian by Corollary 2} &\iff \\ U(L(H)) \text{ Hamiltonian by Theorem 1} &\iff \\ L(H) \text{ Hamiltonian, by Lemma 4.} & \end{aligned}$$

□

We show with two counter-examples that both implications do not hold: there are dihypergraphs H Eulerian such that $L(H)$ is not Hamiltonian, and there are dihypergraphs H not Eulerian such that $L(H)$ is Hamiltonian. By Proposition 14, we only need to study the relation between H Eulerian and $U(H)$ Eulerian.

Consider the following dihypergraph $H_1 = (\mathcal{V}(H_1), \mathcal{E}(H_1))$:

$$\mathcal{V}(H_1) = \{1, 2, 3, 4, 5\} \text{ and } \mathcal{E}(H_1) = \{E, F\}$$

The bipartite representation digraph of H_1 is in Figure 5.

$1, E, 3, F, 1$ is an Eulerian dicycle in H_1 . But $d_{U(H_1)}^+(2) = \sum_{2 \in E^-} |E^+| = 4$, that is different than $d_{U(H_1)}^-(2) = \sum_{2 \in E^+} |E^-| = 3$. As a consequence, $U(H_1)$ cannot be Eulerian, by Corollary 1.

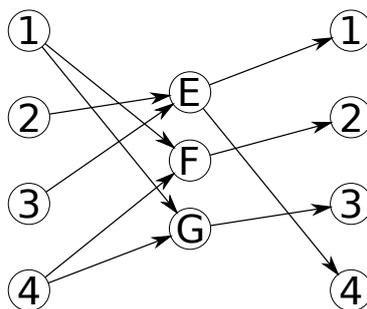


Figure 5: An example of H Eulerian and $U(H)$ not Eulerian.

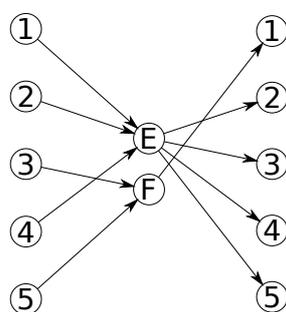


Figure 6: An example of H not Eulerian and $U(H)$ Eulerian.

On the other hand, there exist dihypergraphs H that are not Eulerian such that $U(H)$ is Eulerian and so $L(H)$ is Hamiltonian: let $H_2 = (\mathcal{V}(H_2), \mathcal{E}(H_2))$ be the following dihypergraph:

$$\mathcal{V}(H_2) = \{1, 2, 3, 4\} \text{ and } \mathcal{E}(H_2) = \{E, F, G\}$$

The bipartite representation of H_2 is in Figure 6.

Remark that H_2 verifies the necessary condition of Theorem 5. Furthermore, H_2 is strongly connected. One may observe that its underlying multidigraph $U(H_2)$ is Eulerian (it is even a 2-regular digraph). However, H_2 is not Eulerian because it does not verify the condition of Theorem 7. Indeed, $d_s^-(\{2, 3\}) = 2$, which is strictly greater than $d^+(\{2, 3\}) = 1$.

We show in the next sections there are dihypergraphs H , that are not digraphs, that are Eulerian and such that $U(H)$ is Eulerian.

Nonetheless, Corollary 2 could be generalized in this way:

Proposition 15. *Let H be a dihypergraph. If H is Eulerian, then there is a dicycle C in $L(H)$, whose length is $m(H)$, and such that a vertex in $\mathcal{V}(L(H))$ appears at most once in C .*

Proof. Let $u_0, E_0, u_1, \dots, u_{m(H)-1}, E_{m(H)-1}, u_0$ be an Eulerian dicycle in H . Let $C = (u_0 E_0 u_1), [E_0 u_1 E_1], (u_1 E_1 u_2), \dots, (u_{m(H)-1} E_{m(H)-1} u_0), [E_{m(H)-1} u_0 E_0], (u_0, E_0, u_1)$. C is a dicycle in $L(H)$ whose length is $m(H)$. Moreover, for all $j \neq j'$ vertex $u_j E_j u_{j+1}$ is different than vertex $u_{j'} E_{j'} u_{j'+1}$ because H is Eulerian and consequently $E_j \neq E_{j'}$. \square

Hence, if there are enough Eulerian dicycles in H , whose associated dicycles in $U(H)$ are arc-disjoint, then there may be a Hamiltonian dicycle in $L(H)$. Based on this intuition, we now present a stronger notion of Eulerian

dihypergraphs. Let c be a positive integer. We assume that H is a dihypergraph such that for every hyperarc E , $|E^-||E^+| = c$.

Definition 6. A digraph partition of H is a collection of functions $\varphi_1, \varphi_2, \dots, \varphi_c$ from $\mathcal{E}(H)$ to $\mathcal{V}(H)^2$, such that for all $i \neq j$ for every hyperarc E , $\varphi_i(E) \neq \varphi_j(E)$. Formally: $\mathcal{E}(D_{\varphi_1}[H]), \mathcal{E}(D_{\varphi_2}[H]), \dots, \mathcal{E}(D_{\varphi_c}[H])$ is a partition of $\mathcal{E}(U(H))$.

We now introduce the notion of *strongly Eulerian dihypergraphs*. A directed hypergraph H is strongly Eulerian if there is a digraph-partition $\varphi_1, \varphi_2, \dots, \varphi_c$, such that for every j , $D_{\varphi_j}[H]$ is an Eulerian digraph.

By Remark 5, a strongly Eulerian dihypergraph is an Eulerian dihypergraph. Furthermore, a digraph is strongly Eulerian if and only if it is an Eulerian digraph. Non-trivial examples of strongly Eulerian dihypergraphs are presented in the Appendix.

Lemma 6. Let H be a dihypergraph. If H is strongly Eulerian, then there is a cycle decomposition in $U(H)$.

Proof. Let $\varphi_1, \varphi_2, \dots, \varphi_c$ be a digraph-partition of H . Observe that an arc in $U(H)$ is also an arc in one and only one $D_{\varphi_j}[H]$, where $1 \leq j \leq c$.

Let us assume that for every $1 \leq j \leq c$ $D_{\varphi_j}[H]$ is an Eulerian digraph. Let C_j be an Eulerian dicycle in $D_{\varphi_j}[H]$. Then C_1, C_2, \dots, C_c is a cycle decomposition of $U(H)$. \square

We now show the following result:

Theorem 16. Let H be a dihypergraph. If H is strongly Eulerian, then $L(H)$ is Hamiltonian.

Proof. Since H is strongly Eulerian, therefore H is Eulerian. By Proposition 3, H is weakly connected. Consequently, $U(H)$ is also weakly connected. By Lemma 6, there is a cycle decomposition in $U(H)$. Therefore, by Theorem 2, $U(H)$ is Eulerian. If $U(H)$ is Eulerian, then $L(H)$ is Hamiltonian by Proposition 14. \square

3.4 Kronecker Product of Dihypergraphs

Eulerian and Hamiltonian properties of the digraph products have been intensively studied. We propose a brief study of a dihypergraph product. Let \mathcal{H} be a family of Eulerian (resp. Hamiltonian) dihypergraphs. We want to know whether the product of two elements of \mathcal{H} is still an Eulerian (resp. Hamiltonian) dihypergraph. The studied dihypergraph product is a generalization of a digraph product:

Definition 7. Let $G_1 = (\mathcal{V}_1, \mathcal{E}_1)$, $G_2 = (\mathcal{V}_2, \mathcal{E}_2)$, be two digraphs. Their Kronecker product $G_1 \otimes G_2$ is a digraph $G = (\mathcal{V}, \mathcal{E})$ such that:

1. $\mathcal{V} = \mathcal{V}_1 \times \mathcal{V}_2$
2. $\mathcal{E} = \{(\langle u_1, u_2 \rangle, \langle v_1, v_2 \rangle) : (u_1, v_1) \in \mathcal{E}_1 \text{ and } (u_2, v_2) \in \mathcal{E}_2\}$

Among the properties of Kronecker product, we are interested in its Hamiltonian properties:

Proposition 17 ([23]). Let C_1, C_2 be two dicycles whose respective sizes are n_1 and n_2 . Then $G = C_1 \otimes C_2$ is the union of $\gcd(n_1, n_2)$ disjoint dicycles, whose size is $\text{lcm}(n_1, n_2)$.

We define the Kronecker product of two dihypergraphs H_1, H_2 , denoted $H = H_1 \otimes H_2$, as follows:

$$\begin{aligned}\mathcal{V}(H) &= \mathcal{V}(H_1) \times \mathcal{V}(H_2) \\ \mathcal{E}(H) &\approx \mathcal{E}(H_1) \times \mathcal{E}(H_2)\end{aligned}$$

such that for each E in $\mathcal{E}(H_1)$ and for each F in $\mathcal{E}(H_2)$, we have a hyperarc of H defined as follows:

$$(E, F)^- = E^- \times F^-$$

$$(E, F)^+ = E^+ \times F^+$$

Lemma 7. *Let H_1, H_2 be two dihypergraphs.*

$U(H_1 \otimes H_2)$ and $U(H_1) \otimes U(H_2)$ are isomorphic.

Proof. One may easily check that $\mathcal{V}(U(H_1 \otimes H_2))$ and $\mathcal{V}(U(H_1) \otimes U(H_2))$ are isomorphic. Let u_1, v_1 be two vertices of H_1 and u_2, v_2 be two vertices of H_2 . Let us suppose there are k_1 arcs in $U(H_1)$ from u_1 to v_1 . They are associated to k_1 different hyperarcs of H_1 , denoted E_1, E_2, \dots, E_{k_1} . Similarly, let us suppose there are k_2 arcs in $U(H_2)$ from u_2 to v_2 , associated to hyperarcs F_1, F_2, \dots, F_{k_2} . Then there are $k_1 k_2$ arcs in $U(H_1 \otimes H_2)$ from (u_1, u_2) to (v_1, v_2) , that are associated to hyperarcs $(E_1, F_1), \dots, (E_1, F_{k_2}), \dots, (E_{k_1}, F_1), \dots, (E_{k_1}, F_{k_2})$. That is the same number of arcs from (u_1, u_2) to (v_1, v_2) in $U(H_1) \otimes U(H_2)$. \square

Lemma 8. *Let H_1, H_2 be two dihypergraphs.*

$(H_1 \otimes H_2)^*$ and $H_1^* \otimes H_2^*$ are isomorphic.

Proof. Vertices in $H_1^* \otimes H_2^*$ and in $(H_1 \otimes H_2)^*$ are labeled by $\mathcal{E}(H_1) \times \mathcal{E}(H_2)$. Hyperarcs in $H_1^* \otimes H_2^*$ are labeled by $\mathcal{V}(H_1) \times \mathcal{V}(H_2)$, such that, if v_1 is a vertex of H_1 and v_2 is a vertex of H_2 :

$$(V_1, V_2)^- = \{(e, f) : v_1 \in E^+ \text{ and } v_2 \in F^+\} = \{(e, f) : (v_1, v_2) \in (E, F)^+\}$$

$$(V_1, V_2)^+ = \{(e, f) : v_1 \in E^- \text{ and } v_2 \in F^-\} = \{(e, f) : (v_1, v_2) \in (E, F)^-\}$$

Those are the same relations of incidence as in $(H_1 \otimes H_2)^*$. Therefore $H_1^* \otimes H_2^*$ and $(H_1 \otimes H_2)^*$ are isomorphic. \square

We deduce from these lemmas the following theorem:

Theorem 18. *Let H_1, H_2 be two dihypergraphs.*

1. *If $n(H_1)$ is coprime to $n(H_2)$ and H_1, H_2 are Hamiltonian, then $H_1 \otimes H_2$ is Hamiltonian.*
2. *If $m(H_1)$ is coprime to $m(H_2)$ and H_1, H_2 are Eulerian, then $H_1 \otimes H_2$ is Eulerian.*

Proof. Let us assume that H_1 and H_2 are Hamiltonian. Then $U(H_1)$ and $U(H_2)$ are Hamiltonian, by Lemma 4. Furthermore, if $n(H_1)$ is coprime to $n(H_2)$, then $U(H_1) \otimes U(H_2)$ is Hamiltonian, by Proposition 17. $U(H_1) \otimes U(H_2)$ is isomorphic to $U(H_1 \otimes H_2)$, by Lemma 7. Therefore, $H_1 \otimes H_2$ is Hamiltonian, by Lemma 4.

Now let us assume that H_1 and H_2 are Eulerian. That means that H_1^* and H_2^* are Hamiltonian by Proposition 12. If $m(H_1) = n(H_1^*)$ is coprime to $m(H_2) = n(H_2^*)$, then $H_1^* \otimes H_2^*$ is Hamiltonian. $H_1^* \otimes H_2^*$ is isomorphic to $(H_1 \otimes H_2)^*$, by Lemma 8. Therefore, $(H_1 \otimes H_2)^*$ is Hamiltonian. By Proposition 12, $H_1 \otimes H_2$ is Eulerian. \square

4 The Properties of d -regular, s -uniform Dihypergraphs

Let (s^-, s^+) be a couple of positive integers. A (s^-, s^+) -uniform dihypergraph H is a dihypergraph such that the in-size (resp. the out-size) of every hyperarc in H equals s^- (resp. equals s^+). When $s^- = s^+ = s$ we also say that H is a s -uniform dihypergraph.

Regular dihypergraphs generalize regular digraphs. Let (d^-, d^+) be a couple of positive integers. A (d^-, d^+) -regular dihypergraph H is a dihypergraph such that the in-degree (resp. the out-degree) of every vertex in H equals d^- (resp. d^+). When $d^- = d^+ = d$ we also say that H is a d -regular dihypergraph. Remark that a dihypergraph H is (p, q) -uniform if, and only if, its dual dihypergraph H^* is (p, q) -regular, for some positive integers p, q .

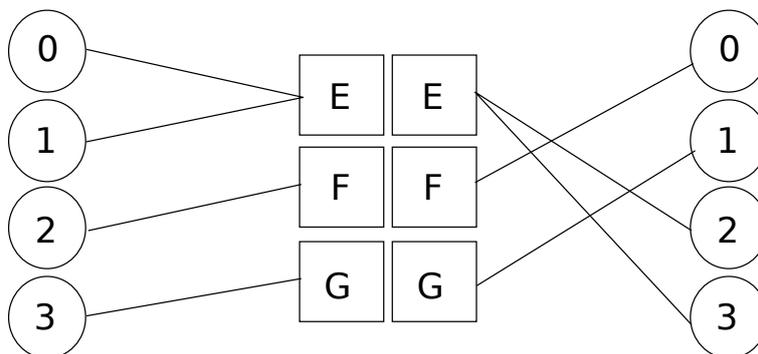


Figure 7: A regular dihypergraph that is not Eulerian.

When the studied dihypergraphs are uniform, Theorem 5 can be reformulated, in a very similar way to [26]:

Corollary 3. *Let H be an Eulerian dihypergraph. If H is (s^-, s^+) -uniform, then :*

$$\sum_{v \in \mathcal{V}(H)} |d^+(v) - d^-(v)| \leq (s^+ + s^- - 2)m(H)$$

Observe that even though d -regular dihypergraphs always verify the necessary condition of Theorem 5, they are not always Eulerian (see Figure 7).

The next result is a generalization of the Hamiltonian property of the disjoint unions of dicycles:

Proposition 19. *Let H be a 1-regular dihypergraph. H is Hamiltonian if and only if H is connected, and for all hyperarc $E \in \mathcal{E}(H)$, $|E^-| = |E^+|$. Consequently, deciding whether H is Hamiltonian can be done in polynomial time.*

Proof. Let H be a 1-regular dihypergraph. Remark that its dual H^* is a digraph. By Theorem 2, H^* is Eulerian if and only if it is connected and, for every vertex $e \in \mathcal{V}(H^*)$, $d^-(e) = d^+(e)$. Moreover, by Proposition 12, H is Hamiltonian if and only if H^* is Eulerian. \square

This does not mean that deciding whether a uniform dihypergraph is Eulerian is an easier task than in the general case. We recall the following result about regular digraphs:

Theorem 20 (see [2]). *Deciding whether a 2-regular digraph D is Hamiltonian is an NP-complete problem.*

In [26], the authors use a similar result about 3-regular graphs, for proving that deciding whether a k -uniform hypergraph, $k \geq 3$, is Eulerian is an NP-complete problem. We do the same for uniform dihypergraphs. First observe that if the dihypergraphs are 1-uniform, that is they are digraphs, we know that deciding whether a digraph is Eulerian can be done in polynomial time. [13]

Theorem 21. *Let (s^-, s^+) be a couple of positive integers. If $s^- \geq 2$, or $s^+ \geq 2$, then deciding whether a (s^-, s^+) -uniform dihypergraph is Eulerian is an NP-complete problem.*

Proof. By Proposition 12, we only need to prove the case when $s^+ \geq 2$; indeed, the mirror dihypergraph of a (s^-, s^+) -uniform dihypergraph is a (s^+, s^-) -uniform dihypergraph. Furthermore, by Theorem 13, we already know that the problem is in the NP-class. The idea consists in associating, in a polynomial time, to a 2-regular digraph D a dihypergraph H_D , such that H_D is Eulerian if and only if D is Hamiltonian; we add useless vertices in the process in H_D to have the desired uniformity.

Let $D = (\mathcal{V}(D), \mathcal{E}(D))$ be a 2-regular digraph. We define the dihypergraph H_D with the following rules:

1. $\mathcal{V}(H_D) = \mathcal{V}(D) \cup [A \times \mathcal{V}(D)] \cup [B \times \mathcal{V}(D)]$, where A and B are two sets satisfying $|A| = s^- - 1$ and $|B| = s^+ - 2$.
2. To each vertex $v \in \mathcal{V}(D)$ we associate a hyperarc $E_v \in \mathcal{E}(H_D)$, such that $E_v^- = \{v\} \cup [A \times \{v\}]$, and $E_v^+ = \{w_v, w'_v\} \cup [B \times \{v\}]$, where w_v and w'_v are the out-neighbors of v in D .

By construction, H_D is a (s^-, s^+) -uniform dihypergraph. Suppose that D is Hamiltonian. Let $C = v_0, v_0v_1, v_1, \dots, v_{n-1}, v_{n-1}v_0, v_0$ be a Hamiltonian dicycle in D . From C , we build a dicycle C_D in H_D , $C_D = v_0, E_{v_0}, v_1, E_{v_1}, \dots, v_{n-1}, E_{v_{n-1}}, v_0$, where E_{v_i} is the hyperarc that is induced by v_i . By definition of a Hamiltonian dicycle, $\forall v \in \mathcal{V}(D)$, v appears only once in C . Therefore, by construction of H_D , $\forall E_v \in \mathcal{E}(H_D)$, E_v appears exactly once in C_D . So C_D is an Eulerian dicycle in H_D .

Moreover, remark that $\forall E, F \in \mathcal{E}(H_D)$, by construction of H_D we have $E^+ \cap F^- \subset \mathcal{V}(D)$. Let $C_D = v_0, E_0, v_1, \dots, v_{m-1}, E_{m-1}, v_0$ be an Eulerian dicycle in H_D . Because of the previous remark, we know that $\forall i, v_i \in \mathcal{V}(D)$. But a vertex $v \in \mathcal{V}(D)$ is incident to only one hyperarc in H_D . As a consequence, for all i , E_i is the hyperarc that is induced by v_i . So $C = v_0, v_0v_1, v_1, \dots, v_{m-1}, v_{m-1}v_0, v_0$ is a Hamiltonian dicycle in D . \square

Nonetheless, we are able to prove more results than in the general case, when the studied dihypergraphs are uniform and also regular. When H is a digraph, we know that:

Theorem 22 (see [13]). *Let D be a weakly-connected digraph. If D is regular, then all its iterated line digraphs $L^n(D)$, $n \geq 1$, are Hamiltonian and Eulerian.*

We are able to prove that, more generally:

Theorem 23. *Let H be a weakly-connected, d -regular, s -uniform directed hypergraph. Then $\forall n \geq 1$, $L^n(H)$ is Eulerian and Hamiltonian.*

Proof. Since H is d -regular and s -uniform, then $U(H)$ is a ds -regular multidigraph. As a consequence, for all n $L^n(U(H))$ is also ds -regular. By Theorem 1, we show by induction that for all n , $U(L^n(H))$ is isomorphic to $L^n(U(H))$. So for all n , $U(L^n(H))$ is Eulerian (because it is a *regular* multidigraph), that is equivalent, by Proposition 14, to $L(L^n(H)) = L^{n+1}(H)$ Hamiltonian.

Moreover, H^* is s -regular, d -uniform. Therefore, $\forall n \geq 1$ $L^n(H^*)$ is also Hamiltonian. Again by Theorem 1, we prove by induction that for all n , $(L^n(H))^*$ is isomorphic to $L^n(H^*)$. Therefore, by Proposition 12, $\forall n \geq 1$ $L^n(H)$ is Eulerian. \square

Remark 8. *Theorem 23 holds when H is (d^-, d^+) -regular, H is (s^-, s^+) -uniform, if we add the extra-condition: $d^-s^- = d^+s^+$.*

In the case when $s = d$ we are able to prove slightly more than Theorem 23:

Proposition 24. *Let H be a d -regular, d -uniform dihypergraph. There is a complete Berge dicycle in $L(H)$. In other words, the bipartite representation digraph $R(L(H))$ is Hamiltonian.*

Proof. Because of the d -regularity, d -uniformity of H , its own bipartite representation digraph $R(H)$ is d -regular. Therefore, $\forall i \geq 1$ $L^i(R(H))$ is Hamiltonian. Particularly when $i = 2$. By Theorem 1, we know that $L^2(R(H))$ and $R(L(H))$ are isomorphic. Therefore, $R(L(H))$ is Hamiltonian. \square

Uniform B-graphs and F-graphs also have an interesting property. When their line dihypergraph $L(H)$ is Hamiltonian, they are not necessarily Eulerian, but they verify the necessary condition of Theorem 7:

Proposition 25. *Let H be a $(1, s^+)$ -uniform dihypergraph (resp. a $(s^-, 1)$ -uniform dihypergraph). If $L(H)$ is Hamiltonian, then for all subset $X \subset \mathcal{V}(H)$, $d_s^-(X) \leq d^+(X)$.*

Proof. By symmetry, we only show the property for B-graphs.

Let H be a $(s^-, 1)$ -uniform dihypergraph, such that its line dihypergraph $L(H)$ is Hamiltonian. By Proposition 14, its underlying multidigraph $U(H)$ is Eulerian. Therefore, for all subset $X \subset \mathcal{V}(H)$, we have

$$d_{U(H)}^-(X) = \sum_{x \in X} d_{U(H)}^-(x) = \sum_{x \in X} d_{U(H)}^+(x) = d_{U(H)}^+(X)$$

Since H is a $(s^-, 1)$ -uniform dihypergraph, therefore $d_{U(H)}^-(X) = s^- \cdot d_H^-(X)$. Furthermore:

$$d_{U(H)}^+(X) = \sum_{E \in \mathcal{E}(H) \text{ s.t. } E^- \cap X \neq \emptyset} |E^- \cap X| \leq \sum_{E \in \mathcal{E}(H) \text{ s.t. } E^- \cap X \neq \emptyset} |E^-| = s^- \cdot d_H^+(X)$$

Consequently:

$$d_{s,H}^-(X) = d_H^-(X) = \frac{d_{U(H)}^-(X)}{s^-} = \frac{d_{U(H)}^+(X)}{s^-} \leq d_H^+(X)$$

□

4.1 A Particular Subfamily

One can find a family of uniform dihypergraphs for which the condition of Theorem 5 is a sufficient condition. But that imposes very restrictive conditions over this family.

Let \mathcal{H} be a set of dihypergraphs, such that for all $H \in \mathcal{H}$:

1. H is (s^-, s^+) -uniform, for some couple of positive integers (s^-, s^+) .
2. for all hyperarc $E \in \mathcal{E}(H)$, $E^- \cap E^+ = \emptyset$.
3. for all hyperarcs $E, F \in \mathcal{E}(H)$, such that $E \neq F$, if $E \cap F \neq \emptyset$ then $E^+ \cap F^- \neq \emptyset$ and $F^+ \cap E^- \neq \emptyset$.

We associate to $H \in \mathcal{H}$ an *undirected* hypergraph H_u , whose set of vertices is $\mathcal{V}(H)$, and whose set of edges is $\mathcal{E}_u = \{E^- \cup E^+ : E \in \mathcal{E}(H)\}$. H_u is called the *underlying hypergraph* of H . By construction, H_u is a k -uniform hypergraph, where $k = s^- + s^+$. The Eulerian properties of the undirected uniform hypergraphs have been studied in [26]. Particularly, the authors of this paper define the *strongly connectivity* of an *undirected* uniform hypergraph:

Definition 8 ([26]). *Let H_u be a k -uniform, undirected hypergraph. Let G be a graph, whose vertices are the edges of H_u , such that an edge E is linked to another edge F if and only if $|E \cap F| = k - 1$. H_u is called a *strongly connected hypergraph* when G is a connected graph.*

Theorem 26 ([26]). *Let H_u be a k -uniform, undirected hypergraph. If H_u is strongly connected, then H_u is Eulerian if and only if*

$$|V_{\text{odd}}(H_u)| \leq (k-2)m(H_u)$$

This theorem generalizes the Euler condition for a graph G to be Eulerian: G is Eulerian if and only if it is connected and all its vertices have an even degree.

Theorem 27. *Let $H \in \mathcal{H}$ be a (s^-, s^+) -uniform dihypergraph, and H_u be the underlying hypergraph of H . If H_u is strongly connected, then H is Eulerian if and only if*

$$\sum_{v \in \mathcal{V}(H)} |d^+(v) - d^-(v)| \leq (s^- + s^+ - 2)m(H)$$

Proof. By Corollary 3, if H is Eulerian then $\sum_{v \in \mathcal{V}(H)} |d^+(v) - d^-(v)| \leq (s^- + s^+ - 2)m(H)$. Let us assume this condition is verified.

Let v be a vertex in H . Since for all hyperarc E , $E^- \cap E^+ = \emptyset$, therefore the degree of v in H_u , that is the number of edges that contain v , denoted $d(v)$, is equal to $d_H^-(v) + d_H^+(v)$. Consequently, when $d(v)$ is odd, $d_H^-(v) \neq d_H^+(v)$. Therefore $\sum_{v \in \mathcal{V}(H)} |d^+(v) - d^-(v)|$ is an upper bound of $|V_{\text{odd}}(H_u)|$. We recall that by construction,

H_u is a $s^- + s^+$ -uniform hypergraph. Moreover, H_u is strongly connected. Since $|V_{\text{odd}}(H_u)| \leq (s^- + s^+ - 2)m(H)$, and $m(H) = |\mathcal{E}_u|$, therefore H_u is Eulerian by Theorem 26.

We denote E_j the edge in H_u that is associated to hyperarc E_j . Let $T = u_0, E_0, u_1, \dots, u_{m-1}, E_{m-1}, u_0$ be an Euler tour in H_u . For all j , we have $u_{j+1} \in E_j \cap E_{j+1}$ by definition of an Euler tour. Consequently, $E_j \cap E_{j+1} \neq \emptyset$. By definition of \mathcal{H} , there exists a vertex $v_{j+1} \in E_j^+ \cap E_{j+1}^-$ because $E_j^+ \cap E_{j+1}^- \neq \emptyset$. As a consequence, $C = v_0, E_0, v_1, \dots, v_{m-1}, E_{m-1}, v_0$ is an Eulerian dicycle in H . \square

Remark 9. *Actually, the property " H is in $\mathcal{H} \implies H$ is uniform" could be replaced by the following one: if H is a dihypergraph in \mathcal{H} , then there exists an integer k such that for all hyperarc $E \in \mathcal{E}(H)$, $|E^-| + |E^+| = k$.*

4.2 Universal Cycles and Universally Eulerian Dihypergraphs

Finally, we present another kind of so-called Eulerian dicycles. Let \mathcal{F} be a set of N combinatorial objects, and A be an alphabet. An element of \mathcal{F} is associated to one or many words of length n over A .

Definition 9 ([27]). *A universal cycle (or ucycle) of \mathcal{F} is a word $w = a_0 a_1 a_2 \dots a_{N-1}$ over A , with each $F \in \mathcal{F}$ represented by exactly one subword $a_i a_{i+1} \dots a_{i+n}$ (index addition is interpreted modulo N).*

Observe that a same word of length n could represent many elements in \mathcal{F} . So there are possibly more than one bijection between \mathcal{F} and the subwords of length n in a ucycle.

It was suggested in [1] to use universal cycles as an extension of the Eulerian cycles to the uniform hypergraphs. This can be done with the uniform dihypergraphs too. Indeed, let $H = (\mathcal{V}(H), \mathcal{E}(H))$ be a (s^-, s^+) -uniform dihypergraph. A hyperarc $E \in \mathcal{E}(H)$ could be represented as a word of length $k = s^- + s^+$ over the alphabet $A = \mathcal{V}(H)$, denoted $w = v_0 v_1 \dots v_{k-1}$, where $\{v_0, v_1, \dots, v_{s^- - 1}\}$ and $\{v_{s^-}, v_{s^- + 1}, \dots, v_{k-1}\}$ are the in-set and the out-set of E respectively. In so doing, a hyperarc is associated to $s^- ! s^+ !$ words of length k over the alphabet A .

Definition 10. *Let H be a (s^-, s^+) -uniform dihypergraph. H is universally Eulerian if there is a ucycle of $\mathcal{E}(H)$ over the alphabet $\mathcal{V}(H)$.*

Observe that an Eulerian digraph D is a universally Eulerian digraph. Indeed, let $v_0, v_0 v_1, v_1, \dots, v_{m-1}, v_{m-1} v_0, v_0$ be an Eulerian dicycle in D . Then $v_0 v_1 v_2 \dots v_{m-1}$ is a ucycle of $\mathcal{E}(D)$ over the alphabet $\mathcal{V}(D)$.

Proposition 28. *A universally Eulerian dihypergraph is an Eulerian dihypergraph.*

Proof. Let H be a (s^-, s^+) -uniform dihypergraph. Let us assume that H is universally Eulerian, and let $C_U = v_0 v_1 \dots v_{m-1}$ be a universal cycle of $\mathcal{E}(H)$ over the alphabet $\mathcal{V}(H)$. We denote by E_i the hyperarc that is represented by $v_i v_{i+1} \dots v_{i+k-1}$, where $k = s^- + s^+$. Remark that v_{i+s^-} is in $E_i^+ \cap E_{i+1}^-$. As a consequence, $C = v_{m-1+s^-}, E_0, v_{s^-}, E_1, \dots, E_i, v_{i+s^-}, E_{i+1}, \dots, v_{m-2+s^-}, E_{m-1}, v_{m-1+s^-}$ is an Eulerian dicycle in H . \square

Proposition 28 shows that universal Eulerianicity is a stronger notion than simple Eulerianicity. Like strongly Eulerian dihypergraphs, universally Eulerian dihypergraphs verify the following property:

Theorem 29. *Let H be a (s^-, s^+) -uniform dihypergraph. if H is universally Eulerian, then $L(H)$ is Hamiltonian.*

We prove Theorem 29 with two Lemmas:

Lemma 9. *Let H be a universally Eulerian dihypergraph. Let C_U be a ucycle of $\mathcal{E}(H)$ over $\mathcal{V}(H)$. For all vertex v :*

$$d^-(v) = s^+ \cdot |C_U|_v$$

$$d^+(v) = s^- \cdot |C_U|_v$$

where $|C_U|_v$ denotes the number of occurrences of v in C_U .

Proof. By definition, s^- consecutive vertices in C_U are the in-set of a given hyperarc, and s^+ consecutive vertices are the out-set of a given hyperarc.

Let v be a vertex in H . When $d^-(v) = d^+(v) = 0$, Lemma 9 holds. Else v is necessarily in C_U , and so $d^-(v), d^+(v) \neq 0$. Let us assume that $C_U = v_0 v_1, \dots, v_{m-1}$, and $v = v_i$ for some $i \in \{0, \dots, m-1\}$. Then v is in the in-set of $E_i, E_{i-1}, \dots, E_{i-s^-+1}$ and in the out-set of $E_{i+1-k}, E_{i+2-k}, \dots, E_{i+s^+-k}$, where $k = s^- + s^+$. Furthermore, vertex v cannot be twice in the same in-set nor twice in the same out-set. Consequently, any occurrence of v in C_U increases the in-degree $d^-(v)$ of s^+ (resp. the out-degree $d^+(v)$ of s^-). \square

Lemma 10. *Let H be a (s^-, s^+) -uniform dihypergraph. If H is universally Eulerian, then $U(H)$ is Eulerian.*

Proof. Let v be a vertex in H . Observe that $d_{U(H)}^-(v) = s^- d_H^-(v)$ and $d_{U(H)}^+(v) = s^+ d_H^+(v)$. Therefore, by Lemma 9, for all vertex v , $d_{U(H)}^-(v) = d_{U(H)}^+(v)$. In other words, $U(H)$ is a balanced digraph.

Furthermore, since H is universally Eulerian, therefore H is Eulerian by Proposition 28. By Proposition 3, H , and so $U(H)$ is weakly connected. Therefore $U(H)$ is Eulerian by Theorem 2. \square

Theorem 29 is now a direct consequence of Lemma 10 and Proposition 14.

5 Consecutive Dihypergraphs

5.1 De Bruijn, Kautz and Consecutive digraphs

5.1.1 Definitions

Definition 11. *The generalized de Bruijn digraph $GB(d, n)$ (also called Reddy-Pradhan-Khul digraph), is the digraph whose vertices are labeled with the integers modulo n ; there is an arc from vertex i to vertex j if, and only if, $j \equiv di + \alpha \pmod{n}$, for some α with $0 \leq \alpha \leq d-1$.*

If $n = d^D$, $GB(d, n)$ is nothing else than the de Bruijn digraph $B(d, D)$. [18]

Definition 12. *The generalized Kautz digraph $GK(d, n)$ (also called Imase-Itoh digraph), is the digraph whose vertices are labeled with the integers modulo n ; there is an arc from vertex i to vertex j if, and only if, $j \equiv -di - d + \alpha \pmod{n}$, for some α with $0 \leq \alpha \leq d-1$.*

If $n = d^D + d^{D-1}$, $GK(d, n)$ is nothing else than the Kautz digraph $K(d, D)$. [3]

Both of those families of digraphs can be generalized in the following way:

Definition 13. The Consecutive- d digraph $G(d, n, q, r)$ (see [14]) is the digraph whose vertices are labeled modulo n , such that there is an arc from vertex i to vertex j if, and only if, $j \equiv qi + r + \alpha \pmod{n}$, for some α with $0 \leq \alpha \leq d - 1$. We assume that $1 \leq d, q \leq n - 1$, and $0 \leq r \leq n - 1$.

When $q = d$ and $r = 0$, $G(d, n, d, 0) = GB(d, n)$.

When $q = r = n - d$, $G(d, n, n - d, n - d) = GK(d, n)$.

Finally, we present a subfamily that was introduced in [18]: Let λ be a positive integer, with $1 \leq \lambda \leq d$. Then $GB_\lambda(d, n)$ is the subdigraph of $GB(d, n)$ such that there is a link from i to j if, and only if, $j \equiv di + \alpha \pmod{n}$, for some α with $0 \leq \alpha \leq \lambda - 1$. Actually, the digraph $GB_\lambda(d, n)$ is nothing else than the consecutive digraph $G(\lambda, n, d, 0)$. But the notation of $GB_\lambda(d, n)$ helps to understand that it is a subdigraph of $GB(d, n)$. More precisely, provided $\lambda = d$, $GB_d(d, n) = GB(d, n)$. We can define in a similar way $GK_\lambda(d, n)$.

5.1.2 Hamiltonian properties

Consecutive- d digraphs have been intensively studied, especially in [17], [14], [24], [10], [11], [20], [16], [19], and [9]. Particularly, the characterization of the Hamiltonian Consecutive- d digraphs is nearly complete:

Theorem 30 ([11], [17], [14], [24]). Let $G = G(d, n, q, r)$ be a Consecutive- d digraph.

If $d = 1$, then G is Hamiltonian if and only if all of the 4 following conditions hold:

1. $\gcd(n, q) = 1$.
2. For every prime number p such that $p|n$, p divides $q - 1$.
3. If n is a multiple of 4, then $q - 1$ is also divisible by 4.
4. $\gcd(n, q - 1, r) = 1$.

When $d = 2$, G is Hamiltonian if and only if one of the following conditions is verified:

1. $\gcd(n, q) = 2$
2. $\gcd(n, q) = 1$ and either $G(1, n, q, r)$ or $G(1, n, q, r + 1)$ is Hamiltonian.

When $d = 3$:

1. If $\gcd(n, q) \geq 2$ then G is Hamiltonian if and only if $3 \geq \gcd(n, q)$.
2. If $1 \leq |q| \leq 3$, and n is coprime to q , then G is always Hamiltonian.

If $d \geq 4$, then G is Hamiltonian if and only if $d \geq \gcd(n, q)$.

Corollary 4 ([14]). Let $G = G(d, n, q, r)$ be a Consecutive- d digraph. If $\lambda = \gcd(n, q) \geq 2$, then G is Hamiltonian if, and only if, $\lambda \leq d$.

The only remaining case is when $d = 3$, for which we only have a partial characterization.

In particular, the characterization of the Hamiltonian generalized de Bruijn digraphs (resp. the Hamiltonian generalized Kautz digraphs) is complete:

Theorem 31 ([18]). *If $\lambda = \gcd(n, d) \geq 2$ then $GB_\lambda(d, n)$ and $GK_\lambda(d, n)$ are Hamiltonian.*

Theorem 32 ([18], [15]). *$GB(d, n)$ is Hamiltonian if and only if one of the following conditions holds:*

1. $d \geq 3$.
2. $d = 2$ and n is even.

Similarly, $GK(d, n)$ is Hamiltonian if and only if one of the following conditions holds:

1. $d \geq 3$.
2. $d = 2$ and n is even.
3. $d = 2$ and n is a power of 3.

$GB(d, n)$ and $GK(d, n)$ are also Eulerian [18].

Finally, consecutive- d digraphs have interesting line digraph properties. We use the following relations [18]:

Proposition 33. *If $\gcd(n, d) = \lambda \geq 2$ then*

$$L(GB_\lambda(d, \frac{n}{\lambda})) = GB_\lambda(d, n)$$

$$L(GK_\lambda(d, \frac{n}{\lambda})) = GK_\lambda(d, n)$$

Particularly, we have for all n :

$$L(GB(d, n)) = GB(d, dn) \text{ and } L(GK(d, n)) = GK(d, dn).$$

Proof. A possible mapping from the arcs of $GB_\lambda(d, \frac{n}{\lambda})$ to the vertices in $GB_\lambda(d, n)$ is this one: let (i, j) be an arc of $GB_\lambda(d, \frac{n}{\lambda})$ such that $j \equiv di + \alpha \pmod{\frac{n}{\lambda}}$, then we associate the vertex of $GB_\lambda(d, n)$ $f(i, j) \equiv di + \alpha \pmod{n}$ to the arc (i, j) . Since $\gcd(n, d) = \lambda$ we can show that f is a one-to-one mapping. Furthermore, if $j \equiv di + \alpha \pmod{\frac{n}{\lambda}}$ then $\lambda j \equiv \lambda(di + \alpha) \pmod{n}$. Consequently:

$$d.f(i, j) + \beta \equiv d.(di + \alpha) + \beta \equiv \frac{d}{\lambda}\lambda(di + \alpha) + \beta \equiv \frac{d}{\lambda}\lambda.j + \beta \equiv dj + \beta \pmod{n}$$

So we have $\{d.f(i, j) + \beta : 0 \leq \beta \leq \lambda - 1\} = \{f(j, k) : (j, k) \text{ is an arc of } GB_\lambda(d, \frac{n}{\lambda})\}$ □

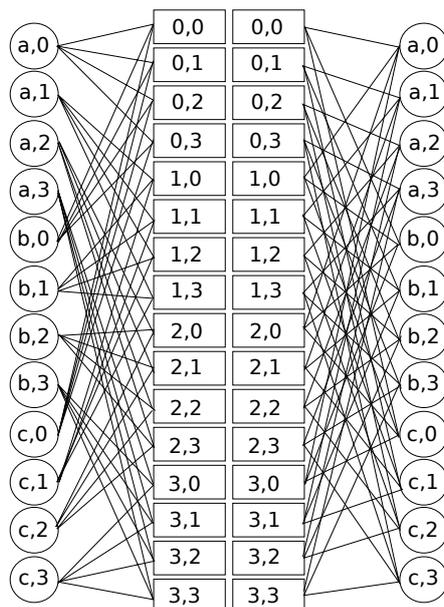
5.2 Definitions of de Bruijn and Kautz dihypergraphs

In [7] de Bruijn and Kautz digraphs have been generalized to dihypergraphs. Two distinct definitions were proposed for de Bruijn dihypergraphs. First scheme is a generalization of the alphabetical definition of a de Bruijn digraph. In the dihypergraph $B_1(d, D, s)$, d is the size of a first alphabet A , and s is the size of a second alphabet B . Vertex labels are of the form $[BA]^D$, and hyperarc labels are of the form $[A][BA]^{D-1}[A]$.

Vertex $(b_1, a_1, b_2, a_2, \dots, b_{D-1}, a_{D-1}, b_D, a_D)$ belongs to the in-set of hyperarcs $(a_1, b_2, a_2, \dots, a_{D-1}, b_D, a_D, \alpha)$, with $\alpha \in A$.

Hyperarc $(a_0, b_1, a_1, \dots, b_{D-1}, a_{D-1}, a_D)$ has as out-set all the vertices $(b_1, a_1, \dots, b_{D-1}, a_{D-1}, \beta, a_D)$, with $\beta \in B$.

An extension of the latest definition, denoted by $GBH_1(d, n, s, m)$, exists, when d, n, s, m verify some particular conditions detailed in [7]. With this definition, $GBH_1(d, (ds)^D, s, d^2(ds)^{D-1})$ and $B_1(d, D, s)$ are isomorphic. However there is another arithmetical definition for the de Bruijn dihypergraphs:

Figure 8: The de Bruijn dihypergraph $B_1(4, 1, 3)$.

Definition 14 ([7]). Let (d, n, s, m) be four positive integers, such that: $dn \equiv 0 \pmod{m}$ and $sm \equiv 0 \pmod{n}$. The generalized de Bruijn dihypergraph $GBH_2(d, n, s, m)$ can be defined as follows. Vertex labels (resp. hyperarc labels) are the integers modulo n (resp. modulo m). Any vertex i is incident to the hyperarcs $di + \alpha \pmod{m}$, for every $0 \leq \alpha \leq d - 1$. Any hyperarc j has for out-set the vertices $sj + \beta \pmod{n}$, for every $0 \leq \beta \leq s - 1$.

Note that the condition $dn \equiv 0 \pmod{m}$ follows from the fact that i and $i + n$ should be incident to the same hyperarcs $d(i + n) + \alpha = di + \alpha \pmod{m}$. Similarly $E_j^+ = E_{j+m}^+$ implies $sm \equiv 0 \pmod{n}$.

Particularly when $n = m$, it can be useful to remark that in the bipartite digraph $R(GBH_2(d, n, s, n))$, the incidence relations between the left side (from vertices to hyperarcs) are the same as in $GB(d, n)$, and the incidence relations between the right side (from hyperarcs to vertices) are the same as in $GB(s, n)$.

Furthermore, note that when $n = (ds)^D$, and $m = d^2(ds)^{D-1}$, those de Bruijn dihypergraphs are also isomorphic to a subfamily of alphabetically-defined dihypergraphs, noted $B_2(d, D, s)$.

In the same way, we can define the Kautz dihypergraph with an arithmetical definition:

Definition 15 ([7]). Let (d, n, s, m) be four positive integers, such that: $dn \equiv 0 \pmod{m}$ and $sm \equiv 0 \pmod{n}$. The generalized Kautz dihypergraph, denoted by $GKH(d, n, s, m)$, is the dihypergraph whose vertices (resp. hyperarcs) are labeled by the integers modulo n (resp. modulo m), such that a vertex i is incident to hyperarcs $di + \alpha \pmod{m}$, for every $0 \leq \alpha \leq d - 1$, and hyperarc j has for out-set the vertices $-sj - s + \beta \pmod{n}$, for every $0 \leq \beta \leq s - 1$.

Inverting the labeling of the hyperarcs, it has been proposed in [7] an equivalent definition for Kautz dihypergraphs:

Definition 16 ([7]). Let (d, n, s, m) be four positive integers, such that: $dn \equiv 0 \pmod{m}$ and $sm \equiv 0 \pmod{n}$. The generalized Kautz dihypergraph, denoted by $GKH(d, n, s, m)$, is the dihypergraph whose vertices (resp. hyperarcs) are labeled by the integers modulo n (resp. modulo m), such that a vertex i is incident to hyperarcs $-di - d + \alpha \pmod{m}$, for every $0 \leq \alpha \leq d - 1$, and hyperarc j has for out-set the vertices $sj + \beta \pmod{n}$, for every $0 \leq \beta \leq s - 1$.

When $GBH_1(d, n, s, m)$ and $GBH_2(d, n, s, m)$ have a property in common, we refer to them by $GBH(d, n, s, m)$.

Theorem 34 ([7]). *The underlying multidigraph of $GBH(d, n, s, m)$ (resp. $GKH(d, n, s, m)$) is $GB(ds, n)$ (resp. $GK(d, n)$).*

We recall that de Bruijn and Kautz dihypergraphs are stable by line dihypergraph operation:

Theorem 35 ([6]). *The line dihypergraph of $GBH_2(d, n, s, m)$ (resp. of $GKH(d, n, s, m)$) is $GBH_2(d, dsn, s, dsm)$ (resp. is $GKH(d, dsn, s, dsm)$).*

Furthermore, the de Bruijn dihypergraphs $B_1(d, D, s)$ have similar properties:

Theorem 36 ([6]). *The line dihypergraph of $B_1(d, D, s)$ is isomorphic to $B_1(d, D + 1, s)$.*

Remark that $B_1(d, D, s)$ is d -regular and s -uniform [7].

5.3 Definition and general properties of Consecutive dihypergraphs

5.3.1 Definition

As in the digraph case, Kautz and de Bruijn dihypergraphs are particular cases of a larger class of dihypergraphs. For preventing the lector from confusing vertices with hyperarcs, we denote by i the vertex i , but by E_j the hyperarc labeled by j .

Let (d, n, q, r, s, m, p, t) be eight positive integers, such that $qn \equiv 0 \pmod{m}$ and $pm \equiv 0 \pmod{n}$. We call *Consecutive- (d, s)* dihypergraph a dihypergraph $H([d, n, q, r], [s, m, p, t])$, whose vertices (resp. whose hyperarcs) are labeled by the integers modulo n (resp. modulo m). Vertex i belongs to the in-set of hyperarc E_j if, and only if, $j \equiv qi + r + \alpha \pmod{m}$, for some α with $0 \leq \alpha \leq d - 1$. In the same way, the out-set of hyperarc E_j contains vertex k if, and only if, $k \equiv pj + t + \beta \pmod{n}$, for some β with $0 \leq \beta \leq s - 1$.

Remark 10. *This definition allows empty in-sets. Clearly, a hyperarc whose in-set is empty is useless. So we may imagine a way to modify Consecutive- (d, s) dihypergraphs such that there is no hyperarc whose in-set is empty.*

This idea is not experimented in this paper.

When $q = d$, $p = s$ and $r = t = 0$, $H([d, n, d, 0], [s, m, s, 0]) = GBH_2(d, n, s, m)$.

When $q = r = n - d$ and $p = t = m - s$, $H([d, n, n - d, n - d], [s, m, m - s, m - s]) = GBH_2(d, n, s, m)$ too.

When $q = d$, $r = 0$ and $p = t = m - s$, $H([d, n, d, 0], [s, m, m - s, m - s]) = GKH(d, n, s, m)$.

Finally, when $q = r = n - d$, $p = s$ and $t = 0$, $H([d, n, n - d, n - d], [s, m, s, 0]) = GKH(d, n, s, m)$.

5.3.2 Properties

We generalize to the consecutive- (d, s) dihypergraphs some useful properties of the Kautz and de Bruijn dihypergraphs. An obvious property is the stability by the dual operation:

$$(H([d, n, q, r], [s, m, p, t]))^* = H([s, m, p, t], [d, n, q, r])$$

Particularly:

$$(GBH_2(d, n, s, m))^* = GBH_2(s, m, d, n) \text{ and } (GKH(d, n, s, m))^* = GKH(s, m, d, n) \text{ [7].}$$

First, we generalize Theorem 34 to consecutive- (d, s) dihypergraphs as follows:

Proposition 37. *Let $H([d, n, q, r], [s, m, p, t])$ be a Consecutive- (d, s) dihypergraph. Its underlying multidigraph is a union of Consecutive- s digraphs. More precisely:*

$$U(H([d, n, q, r], [s, m, p, t])) = \bigcup_{\alpha=0}^{d-1} G(s, n, pq, p(r + \alpha) + t) \subset G_r(p(d - 1) + s, n, pq, pr + t)$$

where $G_r(p(d-1) + s, n, pq, pr + t)$ is a digraph based on $G(p(d-1) + s, n, pq, pr + t)$ such that some of the arcs may be repeated.

Proof. Let U be the underlying multidigraph of $H([d, n, q, r], [s, m, p, t])$. A vertex i is adjacent to vertices $pqi + pr + t + p\alpha + \beta$, for every $0 \leq \alpha \leq d-1$ and $0 \leq \beta \leq s-1$. So the inclusion of U into $G_r(p(d-1) + s, n, pq, pr + t)$ is trivial.

Moreover, let U_α be the subdigraph generated by the arcs uv such that $v \in E_{qu+\alpha}^+$, with fixed $\alpha \in \{0, \dots, d-1\}$. It is isomorphic to $G(s, n, pq, p(r+\alpha) + t)$. Finally, it remains to observe that $U = \bigcup_{\alpha=0}^{d-1} U_\alpha$. \square

Particularly, $U(H([d, n, q, r], [s, m, p, t])) = G_r(p(d-1) + s, n, pq, pr + t)$ when $1 \leq p \leq s$, or $-s \leq p \leq -1$.

We can also generalize Theorem 35 to consecutive dihypergraphs, but we need extra conditions. For example we get the following generalization:

Proposition 38. *If $q = \lambda d$, $p = \mu s$, n and m are prime with $\lambda\mu$, λ is prime with s and μ is prime with d then $L(H([d, n, q, 0], [s, m, p, 0])) = H([d, nds, q, 0], [s, mds, p, 0])$*

Proof. In $L(H([d, n, q, 0], [s, m, p, 0]))$ vertex i, E_j, i' is joined to the arc $E_j, i', E_{j'}$ where: $j \equiv qi + \alpha \pmod{m}$ for some $0 \leq \alpha \leq d-1$, $i' \equiv pj + \beta \pmod{n}$ for some $0 \leq \beta \leq s-1$, and $j' \equiv qi' + \alpha' \pmod{m}$ for some $0 \leq \alpha' \leq d-1$.

Consider the one-to-one mapping of the vertices of $L(H([d, n, q, 0], [s, m, p, 0]))$ in the set of integers modulo dsn : to the vertex i, E_j, i' , where $j \equiv qi + \alpha \pmod{m}$ and $i' \equiv pj + \beta \equiv pqi + p\alpha + \beta \pmod{n}$, we associate the integer $x = pqi + p\alpha + \beta$ modulo dsn .

Now consider the one-to-one mapping of the hyperarcs of $L(H([d, n, q, 0], [s, m, p, 0]))$ in the set of integers modulo dsm : to the arc $E_j, i', E_{j'}$ where $i' \equiv pj + \beta \pmod{n}$ and $j' \equiv qi' + \alpha' \pmod{m}$, we associate the integer $y = pqj + q\beta + \alpha'$ modulo dsm .

Let us prove that these mappings are one-to-one mappings due to the hypothesis. By symmetry of the proof, we only need to show that the mapping of the vertices is a one-to-one mapping.

Let (i_1, α_1, β_1) , (i_2, α_2, β_2) be two triples such that $pqi_1 + p\alpha_1 + \beta_1 \equiv pqi_2 + p\alpha_2 + \beta_2 \pmod{dsn}$. This is equivalent to the equation $pq(i_1 - i_2) \equiv p(\alpha_2 - \alpha_1) + (\beta_2 - \beta_1) \pmod{dsn}$. Since s divides p and s divides dsn therefore s divides $\beta_2 - \beta_1$, and so $\beta_2 - \beta_1 = 0$. We now get the equation $pq(i_1 - i_2) \equiv p(\alpha_2 - \alpha_1) \pmod{dsn}$, which is equivalent to the other equation $\lambda\mu d(i_1 - i_2) \equiv \mu(\alpha_2 - \alpha_1) \pmod{dn}$. Again, since d divides $\lambda\mu d$ and d divides dn , therefore d divides $\mu(\alpha_2 - \alpha_1)$. Furthermore, since μ is coprime to d , d divides $\alpha_2 - \alpha_1$, and so $\alpha_2 - \alpha_1 = 0$ too. Finally, we get the equation $\lambda\mu d(i_1 - i_2) \equiv 0 \pmod{dn}$, that is $\lambda\mu(i_1 - i_2) \equiv 0 \pmod{n}$. Since $\lambda\mu$ is prime with n , therefore $i_1 \equiv i_2 \pmod{n}$. Consequently, the mapping of the vertices is a one-to-one mapping.

The adjacency relation are kept as vertex $x = pqi + p\alpha + \beta$ representing the vertex i, E_j, i' is joined in $H([d, nds, q, 0], [s, mds, p, 0])$ to the hyperarcs $y = qx + \alpha' = pq^2i + pq\alpha + q\beta + \alpha' = pqj + q\beta + \alpha'$ that are the images of the hyperarcs $E_j, i', E_{j'}$ to which x is adjacent. \square

Note that when $\lambda = \mu = 1$ we get Theorem 35 in the case of the generalized de Bruijn dihypergraphs.

5.4 Eulerian and Hamiltonian properties

Theorem 39. *Let $H = GBH(d, n, s, m)$ be a de Bruijn dihypergraph. H is Hamiltonian if and only if one of the following cases is verified:*

1. $ds \geq 3$
2. $ds = 2$ and n is an even number.
3. $ds \in \{1, 2\}$, and $n = 1$.

Proof. Regardless of the scheme that is used for defining H , $U(H) = GB(ds, n)$, by Theorem 34. By Theorem 32, we know that the de Bruijn digraph $GB(ds, n)$ is Hamiltonian if and only if $ds \geq 3$ or $ds = 2$ and n is even (or $n = 1$). Therefore, by Lemma 4, Theorem 39 follows. \square

First scheme for the de Bruijn dihypergraphs is apart. So we deal with it first. In the case when the studied de Bruijn dihypergraphs are alphabetically-defined ($B_1(d, D, s)$) we are able to prove they are also Eulerian:

Theorem 40. $B_1(d, D, s)$ is Eulerian and Hamiltonian.

Proof. Since $B_1(d, D, s)$ and $GBH_1(d, (ds)^D, s, d^2(ds)^{D-1})$ are isomorphic, therefore we already know, by Theorem 39, every $B_1(d, D, s)$ is Hamiltonian. But there is another way for proving it: using Theorem 36, we show by induction that $L^{D-1}(B_1(d, 1, s)) = B_1(d, D, s)$. $B_1(d, 1, s)$ is d -regular, and s -uniform, and weakly-connected. Therefore, by Theorem 23, $\forall D \geq 2$ $B_1(d, D, s)$ is Eulerian and Hamiltonian.

Finally, let us prove that $B_1(d, 1, s)$ is Eulerian:

Let A, B be the two alphabets that are used for representing $B_1(d, 1, s)$. We recall that $|A| = d$, and $|B| = s$. In this special case: $\mathcal{V}(B_1(d, 1, s)) = B \times A$ and $\mathcal{E}(B_1(d, 1, s)) = A \times A$. Remark that hyperarcs (a_i, a_j) , (a_j, a_k) can always be linked, for example by vertex $(b_0, a_j) \in (a_i, a_j)^+ \cap (a_j, a_k)^-$. So we omit in the following dipaths the vertex between two such hyperarcs.

For every $0 \leq i \leq d-1$ we construct a dipath :

$$P_0 = (a_0, a_0), (a_0, a_2), (a_2, a_0), \dots, (a_0, a_j), (a_j, a_0), \dots, (a_0, a_{d-1}), (a_{d-1}, a_0); 2 \leq j \leq d-1$$

$$P_{d-1} = (a_{d-2}, a_{d-1}), (a_{d-1}, a_{d-1}), (a_{d-1}, a_{d-2}), \dots, (a_{d-j}, a_{d-j-1}), \dots, (a_2, a_1), (a_1, a_0); 1 \leq j \leq d-1$$

Otherwise $P_i = (a_{i-1}, a_i), (a_i, a_i), (a_i, a_{i+2}), (a_{i+2}, a_i), \dots, (a_i, a_j), (a_j, a_i), \dots, (a_i, a_{d-1}), (a_{d-1}, a_i)$, where $i+2 \leq j \leq d-1$.

Remark that the concatenation $P_0 P_1 \dots P_{d-1}$ is an Eulerian dicycle in $B_1(d, 1, s)$. So $B_1(d, 1, s)$ is Eulerian. \square

Because of Theorem 32, giving a necessary and sufficient condition for a de Bruijn dihypergraph to be Hamiltonian is the easy part. We now prove that Theorem 39 implies a necessary and sufficient condition for $GBH_2(d, n, s, m)$ to be Eulerian:

Theorem 41. Let $H = GBH_2(d, n, s, m)$ be a de Bruijn dihypergraph. H is Eulerian if, and only if, one of the following cases is verified:

1. $ds \geq 3$
2. $ds = 2$ and m is even.
3. $ds \in \{1, 2\}$, and $m = 1$

Proof. $H^* = GBH_2(s, m, d, n)$. So Theorem 39 gives a necessary and sufficient condition for H^* to be Hamiltonian. By Proposition 12, this is also a necessary and sufficient condition for H to be Eulerian. \square

Corollary 5. $B_2(d, D, s)$ is Eulerian and Hamiltonian.

Proof. Since $B_2(d, D, s)$ and $GBH_2(d, (ds)^D, s, d^2(ds)^{D-1})$ are isomorphic, therefore it is a direct application of Theorems 39 and 41. Indeed if $ds = 2$, then n and m are even when $d = 2$ or $D \geq 2$, and n is even and $m = 1$ when $d = D = 1$. \square

The method that is used for deciding whether $GBH_2(d, n, s, m)$ is Eulerian or Hamiltonian can be applied to Kautz dihypergraphs in the same way. By Theorem 32, we have necessary and sufficient conditions for a generalized Kautz digraph to be Hamiltonian. Consequently:

Theorem 42. *Let $H = GKH(d, n, s, m)$ be a Kautz dihypergraph.*

1. *If $ds \geq 3$ then H is Eulerian and Hamiltonian.*
2. *If $ds = 2$ then H is Eulerian (resp. Hamiltonian) if, and only if m (resp. n) is even or a power of 3.*
3. *If $d = s = 1$, then H is Eulerian (resp. Hamiltonian) if, and only if, $m \in \{1, 2\}$ (resp. $n \in \{1, 2\}$).*

ds	n	m	$GBH_2(d, n, s, m)$	$GKH(d, n, s, m)$
1			Eulerian $\iff m = 1$	Eulerian $\iff m \in \{1, 2\}$
1			Hamiltonian $\iff n = 1$	Hamiltonian $\iff n \in \{1, 2\}$
2	even		Hamiltonian	Hamiltonian
2		even	Eulerian	Eulerian
2	odd		Hamiltonian $\iff n = 1$	Hamiltonian $\iff n$ is a power of 3
2		odd	Eulerian $\iff m = 1$	Eulerian $\iff m$ is a power of 3
≥ 3			Eulerian and Hamiltonian	Eulerian and Hamiltonian

Table 1: Cases when Kautz and de Bruijn dihypergraphs are Eulerian or Hamiltonian

Those methods can be generalized to the Consecutive- (d, s) dihypergraphs too. But they are not sufficient for completing the characterization of their Eulerian and Hamiltonian properties. By the dual operation, a condition for a Consecutive- (d, s) dihypergraph (not) to be Hamiltonian is always associated to a condition (not) to be Eulerian. So we detail some conditions for a Consecutive- (d, s) dihypergraph (not) to be Hamiltonian, and we present the associated conditions (not) to be Eulerian as corollaries.

d	Eulerian Properties	s	Hamiltonian Properties
1	$G(1, m, pq, 1)$ is Hamiltonian and $\gcd(m, pq - 1, q(\beta + t) + r) = 1$, for some $0 \leq \beta \leq s - 1$	1	$G(1, n, pq, 1)$ is Hamiltonian and $\gcd(n, pq - 1, p(\alpha + r) + t) = 1$, for some $0 \leq \alpha \leq d - 1$
2	1) $\gcd(m, pq) = 2$ 2) $\gcd(m, pq) = 1$ and $G(1, m, pq, q(t + \beta) + r)$ is Hamiltonian or $G(1, m, pq, q(t + \beta) + r + 1)$ is Hamiltonian for some $0 \leq \beta \leq s - 1$	2	1) $\gcd(n, pq) = 2$ 2) $\gcd(n, pq) = 1$ and $G(1, n, pq, p(r + \alpha) + t)$ is Hamiltonian or $G(1, n, pq, p(r + \alpha) + t + 1)$ is Hamiltonian for some $0 \leq \alpha \leq d - 1$
3	1) $2 \leq \gcd(m, pq) \leq 3$ 2) $\gcd(m, pq) = 1$, with $1 \leq pq \leq 3$	3	1) $2 \leq \gcd(n, pq) \leq 3$ 2) $\gcd(n, pq) = 1$, with $1 \leq pq \leq 3$
≥ 4	$d \geq \gcd(m, pq)$	≥ 4	$s \geq \gcd(n, pq)$

Table 2: Sufficient conditions for a Consecutive- (d, s) dihypergraph $H([d, n, q, r], [s, m, p, t])$ to be Eulerian or Hamiltonian.

Proposition 43. *Let $H = H([d, n, q, r], [s, m, p, t])$ be a Consecutive- (d, s) dihypergraph. If the consecutive digraph $G(s, n, pq, p(r + \alpha) + t)$ is Hamiltonian for some $0 \leq \alpha \leq d - 1$, then H is Hamiltonian.*

Proof. By Proposition 37, $G(s, n, pq, p(r + \alpha) + t)$ is a subdigraph of $U(H)$. Therefore, if $G(s, n, pq, p(r + \alpha) + t)$ is Hamiltonian, then $U(H)$ is Hamiltonian, and so H is Hamiltonian by Lemma 4. \square

Corollary 6. *Let $H = H([d, n, q, r], [s, m, p, t])$ be a Consecutive- (d, s) dihypergraph. If the consecutive digraph $G(d, m, pq, q(t + \beta) + r)$ is Hamiltonian for some $0 \leq \beta \leq s - 1$, then H is Eulerian.*

In the same way, we have a sufficient condition for a Consecutive- (d, s) dihypergraph not to be Hamiltonian (resp. Eulerian).

Proposition 44. *Let $H = H([d, n, q, r], [s, m, p, t])$ be a Consecutive- (d, s) dihypergraph. If the Consecutive digraph $G(p(d - 1) + s, n, pq, pr + t)$ is not Hamiltonian, then H is not Hamiltonian.*

Proof. By Proposition 37, $U(H)$ is a subdigraph of G_r , where G_r is a version of $G(p(d - 1) + s, n, pq, pr + t)$ with possibly repeated arcs. Clearly, repeating some arcs does not change the Hamiltonicity. Therefore, if $G(p(d - 1) + s, n, pq, pr + t)$ is not Hamiltonian, then G_r is not Hamiltonian either. As a consequence, $U(H)$ is not Hamiltonian, and so H is not Hamiltonian, by Lemma 4. \square

Corollary 7. *Let $H = H([d, n, q, r], [s, m, p, t])$ be a Consecutive- (d, s) dihypergraph. If the Consecutive digraph $G(q(s - 1) + d, m, pq, qt + r)$ is not Hamiltonian, then H is not Eulerian.*

Particularly when $U(H) = G_r(p(d - 1) + s, n, pq, pr + t)$ (resp. when $U(H^*) = G_r(q(s - 1) + d, m, pq, qt + r)$), Proposition 44 (resp. Corollary 7) becomes a necessary and sufficient condition for H to be Hamiltonian (resp. Eulerian).

Corollary 8. *Let $H = H([d, n, q, r], [s, m, p, t])$ be a Consecutive- (d, s) dihypergraph.*

1. *When $1 \leq |p| \leq s$, H is Hamiltonian if and only if the Consecutive digraph $G(p(d - 1) + s, n, pq, pr + t)$ is Hamiltonian.*
2. *Similarly, when $1 \leq |q| \leq d$, H is Eulerian if and only if the Consecutive digraph $G(q(s - 1) + d, m, pq, qt + r)$ is Hamiltonian.*

6 Existence of Complete Berge Dicycles

In this section, we show the existence of Complete Berge Dicycles in some consecutive dihypergraphs. More precisely, we try to decide whether there is a complete Berge dicycle in $GBH_2(d, n, s, m)$, i.e there is a Hamiltonian dicycle in its bipartite representation digraph.

A necessary condition for a dihypergraph H to have a complete Berge dicycle is that $n(H) = m(H)$. Otherwise, there are more hyperarcs than vertices, or more vertices than hyperarcs and, as a consequence, $R(H)$ cannot be Hamiltonian.

6.1 The Main Result

We now study the particular case when $d = s$, even though some lemmas apply for $d \neq s$. Actually, we are always able to decide whether there is a complete Berge dicycle in $GBH_2(d, n, d, n)$:

Theorem 45. *There is a complete Berge dicycle in $GBH_2(d, n, d, n)$ if and only if one of the following conditions is verified:*

1. $d \geq 3$
2. $d = 2$ and n is an even number

3. $d = 2$ and n is a power of 3

4. $d = n = 1$

Recall that for $d \geq 2$, those are exactly the cases when $GK(d, n)$ is Hamiltonian, as presented in Theorem 32. The entire demonstration of Theorem 45 is decomposed in Lemmas 11, 12, 15, 16, 14, 17, 13, and 18.

Lemma 11. *If $d = 1$, then there is a complete Berge dicycle in $GBH_2(1, n, s, n)$ if and only if the de Bruijn digraph $GB(s, n)$ is Hamiltonian.*

Proof. If $d = 1$, then every vertex i is only incident to hyperarc E_i . So we may not distinguish the vertices from the hyperarcs, and we get a digraph, the relations of incidence of which are the relations of incidence between hyperarcs and vertices in the original dihypergraph. Therefore, Lemma 11 follows. \square

The case when $d = 1$ is entirely solved by Lemma 11.

For the next proofs, we see a Hamiltonian dicycle in the digraph $GB(d, n)$ or $GK(d, n)$ as a circular permutation σ in Z_n . If j is the vertex that follows i in the Hamiltonian dicycle, then $\sigma(i) = j$; if k is the vertex that follows j in the dicycle, then $\sigma^2(i) = k$, and so on.

Concatenation of digraph dicycles

Most of the remaining proofs use the notion of *dual counterpart*.

Definition 17. *Let $H = (\mathcal{V}, \mathcal{E})$ be a dihypergraph, and $P = v_0, E_0, v_1, \dots, E_{p-2}, v_{p-1}$ be a dipath in H . The dual counterpart of P , denoted by P^* , is the representation of P in the dual dihypergraph H^* . In other words, $P^* = V_0, e_0, V_1, \dots, e_{p-2}, V_{p-1}$.*

Observe that the dual counterpart of a dipath is not necessarily a dipath itself, unless P is a dicycle. Moreover, as it was remarked in [12], the relations of incidence in $GBH_2(d, n, s, m)$ can be simultaneously *reversed*. Indeed, we can transform the labels of each hyperarc, labeling hyperarc i with label $m - 1 - i$, and if we keep the previous relations of incidence we get the following definition:

Definition 18 ([12]). *Let (d, n, s, m) be four positive integers, such that: $dn \equiv 0 \pmod{m}$ and $sm \equiv 0 \pmod{n}$. The de Bruijn dihypergraph $GBH_2(d, n, s, m)$ can be defined as follows. Vertex labels (resp. hyperarc labels) are the integers modulo n (resp. modulo m). Any vertex i is incident to hyperarcs $-di - d + \alpha \pmod{m}$, for every $0 \leq \alpha \leq d - 1$. Any hyperarc j is has for out-set the vertices $-sj - s + \beta \pmod{n}$, for every $0 \leq \beta \leq s - 1$.*

As a result, when n is an odd number, Theorem 45 can be partially proven with a concatenation of dicycles:

Lemma 12. *If $GK(d, n)$ is Hamiltonian, and n is odd, then there is a complete Berge dicycle in $GBH_2(d, n, d, n)$.*

Proof. We use Definition 18 for the proof. Let $0, \sigma(0), \sigma^2(0), \dots, \sigma^{n-1}(0)$ be a Hamiltonian dicycle in $GK(d, n)$. Observe that $P_1 = 0, E_{\sigma(0)}, \sigma^2(0), \dots, \sigma^{2j}(0), E_{\sigma^{2j+1}(0)}, \dots, \sigma^{n-1}(0)$, for $0 \leq j \leq \frac{n-1}{2}$, is a dipath in $GBH_2(d, n, d, n)$. Let $P_2 = P_1^* = E_0, \sigma(0), E_{\sigma^2(0)}, \dots, E_{\sigma^{n-1}(0)}$ be its *dual counterpart*. Since n is odd, then P_1 and P_2 can be linked together, and $C = P_1 P_2 = 0, E_{\sigma(0)}, \sigma^2(0), \dots, \sigma^{n-1}(0), E_0, \sigma(0), E_{\sigma^2(0)}, \dots, E_{\sigma^{n-1}(0)}$ is a complete Berge dicycle in $GBH_2(d, n, d, n)$. \square

In the same way, if $GB(d, n)$ is Hamiltonian, and n is odd, then there is a complete Berge dicycle in $GBH_2(d, n, d, n)$. However, by Theorem 32, if $GB(d, n)$ is Hamiltonian then $GK(d, n)$ is also Hamiltonian, but the converse is not true. That is why we use $GK(d, n)$, hence Definition 18, for the proof. The case for which the Hamiltonian properties of $GB(d, n)$ are not enough to conclude is this one:

Lemma 13. *If n is a power of 3, then there is a complete Berge dicycle C in $GBH_2(2, n, 2, n)$. For all i , the next vertex after i in C is $4i + 1 \pmod{n}$.*

Proof. It can be seen as a particular case of Lemma 12. Indeed, we know by [15] that $i \rightarrow -2i - 1$ is a Hamiltonian dicycle in $GK(2, n)$. Using Definition 18, it gives us a complete Berge dicycle in $GBH_2(2, n, 2, n)$. Remark that if C is a complete Berge dicycle in $GBH_2(d, n, d, n)$ the order in which we visit the vertices in C does not depend on the chosen definition for the de Bruijn dihypergraphs. Since $-2(-2i - 1) - 1 = 4i + 2 - 1 = 4i + 1$, therefore $i \rightarrow 4i + 1$ is a complete Berge dicycle in $GBH_2(2, n, 2, n)$. \square

This dicycle can be found with the following algorithm:

1. Start from vertex 0,
2. From vertex i , go to the hyperarc E_{2i} ,
3. From hyperarc E_j , go to vertex $2j + 1$,
4. Stop when $2j + 1 = 0$.

We show, in the following way, that Lemma 12 is a consequence of Proposition 17 (Kronecker product):

Second Proof of Lemma 12. Let n be an odd number. One may consider the bipartite representation digraph $R(GBH_2(d, n, d, n))$ as the Kronecker product $GK(d, n) \otimes C_2$, where C_2 is a canonical dicycle of size 2. Let C_n be a Hamiltonian dicycle in $GK(d, n)$. Its length is obviously n . Since $\gcd(n, 2) = 1$, therefore, by Corollary 17, $C_n \otimes C_2$ is a dicycle of length $2n$. Therefore, $C_n \otimes C_2$ is a Hamiltonian dicycle of $R(GBH_2(d, n, d, n))$. \square

Actually, Lemmas 12 and 13 are nearly enough for solving the case when n is odd. Indeed, they show that when n is odd, then there is a complete Berge dicycle in $GBH_2(d, n, d, n)$ when $d \geq 3$, or $d = 2$ and n is a power of 3. However, it does not prove that $GBH_2(2, n, 2, n)$ has no complete Berge dicycle when n is odd and n is not a power of 3. This remaining proof will be given later, using another method.

When $d \geq 4$, and n is coprime to d , we can prove Theorem 45 with the Hamiltonian properties of Consecutive- d digraphs, and a refinement of the previous method:

Lemma 14. *If $d \geq 4$, and n is coprime to d , then there is a complete Berge dicycle in $GBH_2(d, n, d, n)$.*

Proof. We need $d \geq 4$ because the Hamiltonian properties of the Consecutive-3 digraphs have yet to be completely solved. We arbitrarily choose to associate hyperarc E_{di} to vertex i for all i . Since $\gcd(n, d) = 1$, therefore d is invertible in Z_n , and $i \rightarrow di$ is a bijection between vertices and hyperarcs. If, for all i , we associate vertex i to the hyperarc E_{di} , we get a digraph G , whose vertices are the integers modulo n ; there is an arc in G from i to j if, and only if, hyperarc E_{di} is incident to vertex j in $GBH_2(d, n, d, n)$, i.e. $j \equiv d^2i + \alpha \pmod{n}$, for some α with $0 \leq \alpha \leq d - 1$.

G is the Consecutive- d digraph $G(d, n, d^2, 0)$. By Theorem 30, we know that G is Hamiltonian. Let $0, \sigma(0), \sigma^2(0), \dots, \sigma^{n-1}(0)$ be a Hamiltonian dicycle in G . Then, by construction, $0, E_0, \sigma(0), E_{d\sigma(0)}, \dots, \sigma^{n-1}(0), E_{d\sigma^{n-1}(0)}, 0$ is a complete Berge dicycle in $GBH_2(d, n, d, n)$. \square

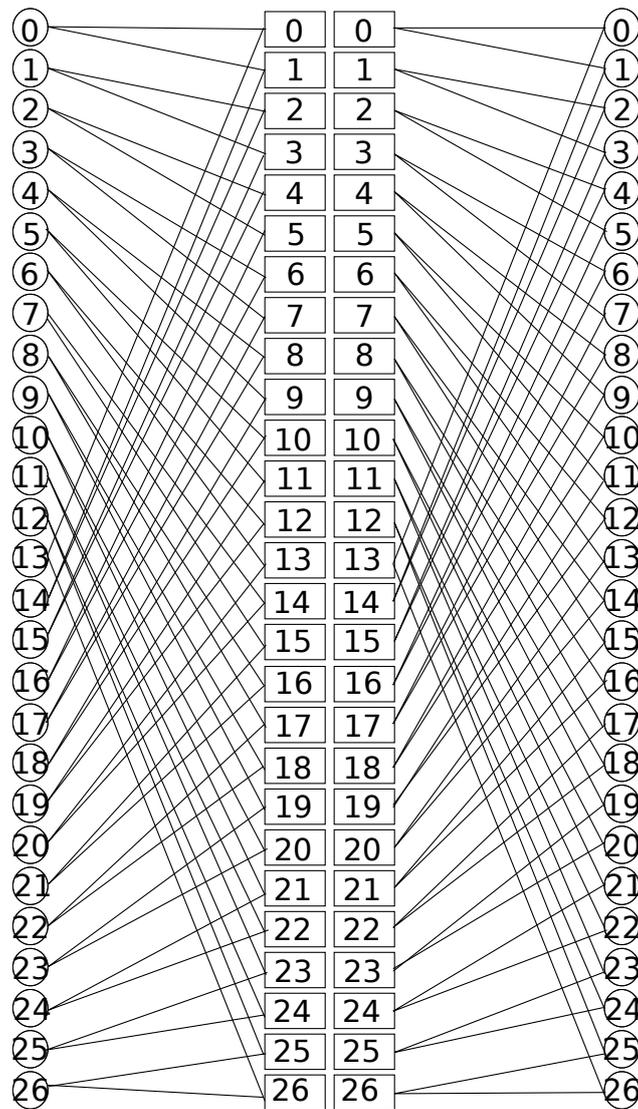


Figure 9: The de Bruijn dihypergraph $GBH_2(2, 27, 2, 27)$.

Such kind of concatenation does not work when n is an even number. However, there is a way to adapt the proof of Lemma 12 when d divides n .

Lemma 15. *If $d|n$, then there is a complete Berge dicycle in $GBH_2(d, n, d, n)$.*

Proof. Since $d|n$, therefore $GB(d, n)$ is Hamiltonian by Theorem 32. When n is odd, we can use Lemma 12. Therefore we assume that n is even. Let $0, \sigma(0), \sigma^2(0), \dots, \sigma^{n-1}(0)$ be a Hamiltonian dicycle in $GB(d, n)$.

Let $P_1 = 0, E_{\sigma(0)}, \sigma^2(0), \dots, E_{\sigma^{n-1}(0)}$ be the dipath in $GBH_2(d, n, d, n)$ built with half of indices for the vertices, and the other half for the hyperarcs. Let $P_2 = E_0, \sigma(0), E_{\sigma^2(0)}, \dots, \sigma^{n-1}(0)$ be its dual counterpart. Since $d|n$ we have the following property: *two vertices that are incident to the same hyperarc are incident to the same hyperarcs.*

There cannot be two vertices that have between 1 and $d-1$ hyperarcs in common. Remark that 0 and $\sigma^{n-1}(0)$ are incident to the same hyperarc E_0 . So $\sigma^{n-1}(0)$ is also incident to hyperarc $E_{\sigma(0)}$, and $E_{\sigma^{n-1}(0)}$ is also incident to vertex $\sigma(0)$. Therefore we can insert P_2 into P_1 between 0 and $E_{\sigma(0)}$. In so doing we get the dicycle $0, E_0, \sigma(0), E_{\sigma^2(0)}, \dots, \sigma^{n-1}(0), E_{\sigma(0)}, \sigma^2(0), \dots, E_{\sigma^{n-1}(0)}, 0$ which is a complete Berge dicycle in $GBH_2(d, n, d, n)$. \square

Using Line Digraphs and Line Dihypergraphs

One can also prove that if d^2 divides n , then there is a complete Berge dicycle in $GBH_2(d, n, d, n)$:

Proof. Let us write $n = d^2m$. The bipartite representation digraph $D = R(GBH_2(d, m, d, m))$ is d -regular. Therefore, by Theorem 22, for all $i \geq 1$, $L^i(D)$ is Hamiltonian. Particularly when $i = 2$. By Theorem 1, $L^2(D)$ and $R(L(GBH_2(d, m, d, m)))$ are isomorphic. Moreover, by Theorem 35, $L(GBH_2(d, m, d, m)) = GBH_2(d, n, d, n)$. Therefore, there is a complete Berge dicycle in $GBH_2(d, n, d, n)$, by Proposition 24. \square

More generally, if $ds|n$ then $R(GBH_2(d, n, s, n))$ is isomorphic to the line digraph of $D = L(R(GBH_2(d, \frac{n}{ds}, s, \frac{n}{ds})))$, by Theorems 1 and 35. One can easily check that D is an Eulerian digraph. Consequently, $R(GBH_2(d, n, s, n))$ is Hamiltonian.

Furthermore, Lemma 15 is itself a particular case of the following lemma:

Lemma 16. *If $\gcd(d, n) \geq 2$ then there is a complete Berge dicycle in $GBH_2(d, n, d, n)$.*

Proof. We note $\lambda = \gcd(n, d)$. If there is a complete Berge dicycle in the subdihypergraph $GBH_{(\lambda, \lambda)}(d, n, d, n)$, then there is a complete Berge dicycle in $GBH_2(d, n, d, n)$. By hypothesis, $\exists n' \geq 1$ such that $n = n'\lambda$. Then remark the following relation between the bipartite representation digraphs:

$$L(R(GBH_{(\lambda, \lambda)}(d, n', d, n'))) \text{ and } R(GBH_{(\lambda, \lambda)}(d, n, d, n)) \text{ are isomorphic.} \quad (1)$$

Indeed, we know that $L(GB_{\lambda}(d, n')) = GB_{\lambda}(d, n)$ by Proposition 33. Considering the incidence relations between hyperarcs and vertices (resp. between vertices and hyperarcs), previous remark follows.

Clearly, $R(GBH_{(\lambda, \lambda)}(d, n', d, n'))$ is λ -regular. Moreover, $GB_{\lambda}(d, n')$ is Eulerian, and so strongly connected, because its line digraph $GB_{\lambda}(d, n')$ is Hamiltonian, by Theorem 32. Let i, j be two vertices in $GB_{\lambda}(d, n')$. Since the loop 00 is an arc in $GB_{\lambda}(d, n')$, therefore one can always find a dipath of even length between i and j . Indeed, there is a dipath from i to 0 , there is a dipath from 0 to j , and we can insert a loop between both of those dipaths if needed. Consequently $GBH_{(\lambda, \lambda)}(d, n', d, n')$ is connected. So its bipartite representation digraph is also connected. Therefore, $R(GBH_{(\lambda, \lambda)}(d, n', d, n'))$ is Eulerian. So its line digraph is Hamiltonian, by Theorem 22. \square

Like in Lemmas 12 and 15, Lemma 16 can also be proven with a concatenation of dicycles:

Proof. Since $GBH_\lambda(d, n)$ is Hamiltonian, by Theorem 32, we can choose $0, \sigma(0), \sigma^2(0), \dots, \sigma^{n-1}(0)$, a Hamiltonian dicycle of $GB_\lambda(d, n)$. The property: *two vertices are incident to the same hyperarcs, or have no hyperarc in common*, used for Lemma 15, is still true for $GBH_{(\lambda, \lambda)}(d, n, d, n)$. Therefore we can adapt the proof of Lemma 15 and we show that $0, E_0, \sigma(0), E_{\sigma^2(0)}, \dots, \sigma^{n-1}(0), E_{\sigma(0)}, \sigma^2(0), \dots, E_{\sigma^{n-1}(0)}, 0$ is a complete Berge dicycle of $GBH_{(\lambda, \lambda)}(d, n, d, n)$ when n is even. \square

A link-method interchange

The only remaining cases are $d = 2$ and $d = 3$. First we deal with the case when $d = 3$. Here is a useful definition for the following results:

Definition 19. Let C_1, C_2 be two dicycles, that are subdigraphs of the same digraph D . A pair of interchange is a couple of vertices (i, j) , such that i is in C_1 whereas j is in C_2 . Those vertices must verify the following property : *the predecessor of i in C_1 is incident to j in D , and the predecessor of j in C_2 is incident to i in D too.*

Those pairs of interchange are a precious tool for merging disjoint dicycles together, using a method that is sometimes called the *link-method interchange*. That method is at the basis of many proofs concerning the Hamiltonian Properties of Consecutive- d digraphs (see [17], [14], [10], [11], and [15]). Many examples are given in the rest of the paper.

Lemma 17. *For every positive integer n , there is a complete Berge dicycle in $GBH_2(3, n, 3, n)$.*

Proof. If n is odd, that is a direct consequence of Lemma 12 and Theorem 32. Furthermore, when n is a multiple of 3, that is a consequence of Lemma 15. So we assume that n is even, and that n is coprime to 3.

Let R be the bipartite representation digraph of $GBH_2(3, n, 3, n)$. To every vertex i we associate the hyperarc E_{3i+1} . Similarly, to every hyperarc E_j we associate the vertex $3j + 1$. Since $\gcd(n, 3) = 1$, we partition the digraph R into pairwise vertex-disjoint dicycles, denoted C_1, C_2, \dots, C_k . If there is only one dicycle we are done as it is Hamiltonian. Otherwise we will use interchange pairs to merge successively the dicycles till we have only one. But we have to be careful to do independent interchanges.

Let us assume that $k \geq 2$. We first claim that if i and $i + 1$ are on two disjoint dicycles C_x and C_y then $(i, i + 1)$ is an interchange pair. Indeed, let $E_j, E_{j'}$ be the predecessor of i in C_x and the predecessor of $i + 1$ in C_y , respectively. By construction: $3j + 1 = i$, and $3j' + 1 = i + 1$. Consequently:

$$3j + 2 = i + 1 \text{ and } 3j' = i$$

So we can merge C_x and C_y into a new dicycle $C_{x \cup y}$, such that $E_{j'}$ is the predecessor of i and E_j is the predecessor of $i + 1$ in $C_{x \cup y}$. Similarly we have that if E_j and E_{j+1} are on two disjoint dicycles then (E_j, E_{j+1}) is an interchange pair. We have to be careful not to use twice the same vertex in an interchange pair, as the predecessor have changed when doing the first merging. For example, when vertices $i, i + 1, i + 2$ are in three distinct dicycles C_x, C_y, C_z , then we can merge C_x and C_y with the pair $(i, i + 1)$, we can merge C_y and C_z with the pair $(i + 1, i + 2)$, but we cannot merge $C_{x \cup y}$ and C_z with $(i + 1, i + 2)$, nor C_x and $C_{y \cup z}$ with $(i, i + 1)$. Consequently, we restrain ourselves to the pairs of interchange $g(i) = (2i, 2i + 1)$, with $0 \leq i \leq n/2 - 1$. In so doing, all the possible exchanges are pairwise independent (that is they are disjoint). Remark that we can also define those pairs of interchange by the pairs $g(i) = (2i \pmod n, 2i + 1 \pmod n)$, with $i \in \mathbb{Z}_n$. In that case $g(i) = g(i + \frac{n}{2})$.

In the same way, we cannot use, for the hyperarcs, two consecutive pairs of interchange (E_j, E_{j+1}) and (E_{j+1}, E_{j+2}) . This time we restrain ourselves to the pairs of interchange $f(j) = (E_{2j+1 \pmod n}, E_{2j+2 \pmod n})$, with $j \in \mathbb{Z}_n$. Those pairs are pairwise independent.

Let UC_1, UC_2, \dots, UC_k be the undirected versions of the dicycles C_1, C_2, \dots, C_k , and L be the union of those cycles (therefore L is an undirected graph). For all i, j we associate in L an edge $(2i \pmod n, 2i+1 \pmod n)$ to the pair $g(i)$, and an edge $(E_{2j+1 \pmod n}, E_{2j+2 \pmod n})$ to the pair $f(j)$. We get a graph G . Remark that if G is connected, then we can merge all the dicycles C_x , with $1 \leq x \leq k$, into one dicycle C . Indeed, when UC_x and UC_y are linked by a pair $g(i)$ or a pair $f(j)$ in G , that means that C_x and C_y can be merged into one dicycle $C_{x \cup y}$. Moreover, all the possible interchange pairs are pairwise independent. Observe that if all the pairs $g(i)$, with $i \in \mathbb{Z}_n$, can be linked with some edges in G , then G is connected.

Let i be a integer modulo n . The vertex $2i \pmod n$ is adjacent to the hyperarc $E_{6i+1 \pmod n}$ in G , and the vertex $2i+1 \pmod n$ is adjacent to the hyperarc $E_{6i+4 \pmod n}$ in G . Moreover, since n is even, and $6i+3$ is odd, therefore $6i+3 \pmod n$ is odd. So $(E_{6i+3 \pmod n}, E_{6i+4 \pmod n})$ is an interchange pair $f(j)$ for some j . Actually $6i+3 = 6i+2+1 \equiv 6i+2*3*3^{-1}+1 \equiv 6i+6*3^{-1}+1 \equiv 3*(2*(i+3^{-1}))+1 \pmod n$. As a consequence, $g(i)$ and $g(i+3^{-1})$ are connected in G . Since 3^{-1} is a generator element in \mathbb{Z}_n , therefore all the $g(i)$ can be linked together. \square

Figure 10 shows the case $n = 8$. Here $3 = 3^{-1}$. Note that if we used the pairs $g(i) = (2i, 2i+1)$ and $f'(j) = (E_{2j}, E_{2j+1})$ we would get a non connected graph.

Arithmetical Results

Finally, we deal with the case when $d = 2$. Lemmas 13 and 15 show that there is a complete Berge dicycle in $GBH_2(2, n, 2, n)$ when n is even, or n is odd and n is a power of 3. We still have to prove there is no complete Berge dicycle in the remaining cases. But we need to introduce the Euler function first:

Definition 20. *The Euler function, denoted by φ , associates to a positive integer n , the number $\varphi(n)$ of positive integers, that are lower than n and coprime to n . This is also the number $\varphi(n)$ of elements $i \in \mathbb{Z}_n$ such that $x \rightarrow x+i$ is a circular permutation in \mathbb{Z}_n . In other words, $\varphi(n)$ is the number of generator elements in \mathbb{Z}_n .*

The Euler function can be defined inductively with the following rules:

1. $\varphi(1) = 1$;
2. If p is a prime number and $m \geq 1$, then $\varphi(p^m) = (p-1)p^{m-1}$;
3. If a is coprime to b , then $\varphi(ab) = \varphi(a)\varphi(b)$.

Let a, b be two positive integers. If a is coprime to b , then a is invertible in \mathbb{Z}_b , and the powers of a are a multiplicative group in \mathbb{Z}_b . The order of a in \mathbb{Z}_b is the minimum $i \geq 1$ such that $a^i \equiv 1 \pmod b$.

Theorem 46 (Euler). *If a is coprime to b then $a^{\varphi(b)} \equiv 1 \pmod b$.*

So we know the order of a divides $\varphi(b)$.

Lemma 18. *If n is an odd number, then there is a complete Berge dicycle in $GBH_2(2, n, 2, n)$ if, and only if, n is a power of 3.*

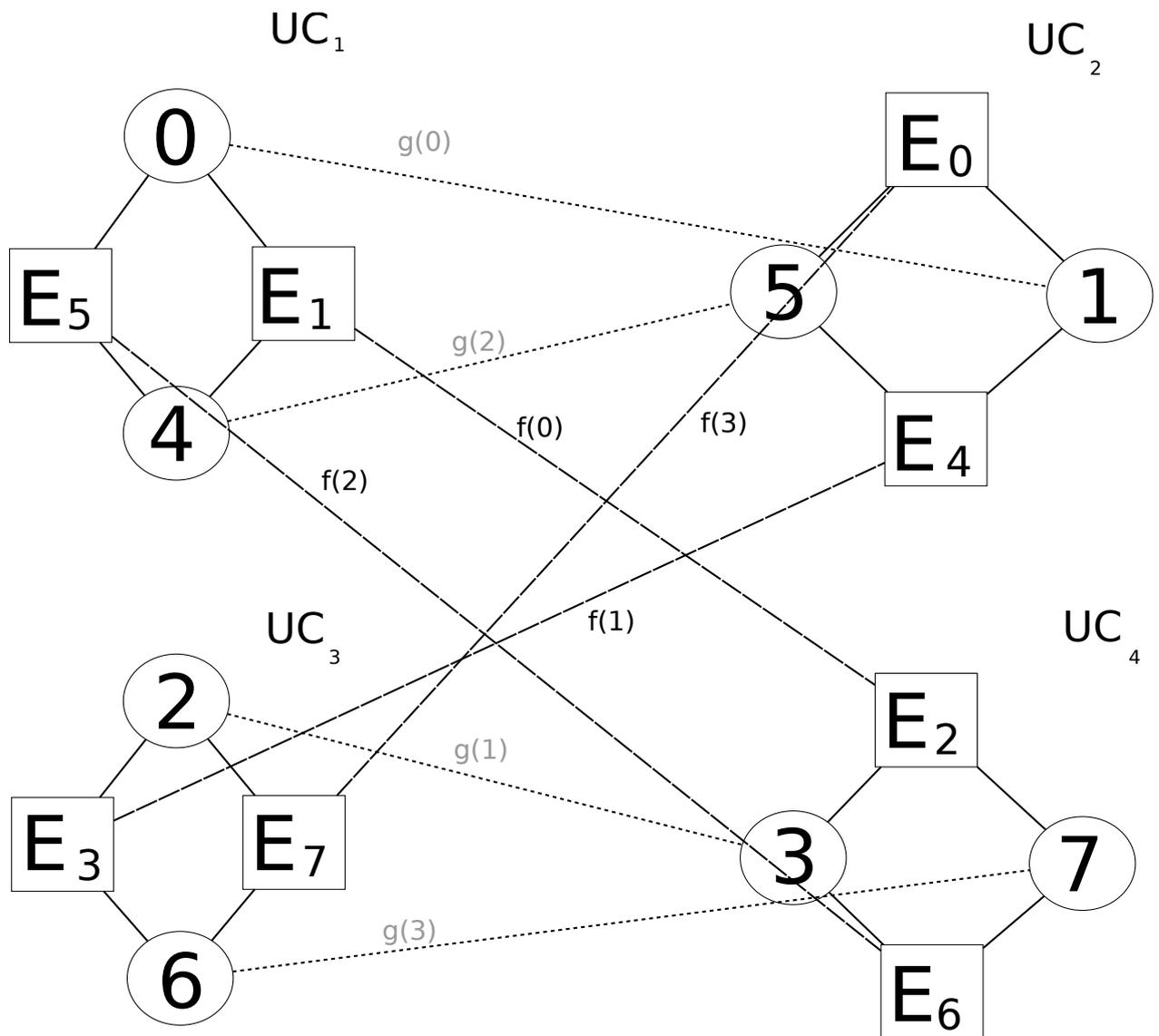


Figure 10: An application of the link-method interchange to the de Bruijn dihypergraph $GBH_2(3, 8, 3, 8)$.

Proof. Let us suppose there is a complete Berge dicycle C in $GBH_2(2, n, 2, n)$. We distinguish two cases: either there exists a vertex i such that i precedes in C the hyperarc E_{2i} , and we will show this holds for all the vertices; or any vertex i precedes in C the hyperarc E_{2i+1} . To prove this fact, consider that vertex i precedes hyperarc $E_{2i \pmod n}$ in C . Since $\gcd(2, n) = 1$, therefore 2 is inversible in \mathbb{Z}_n . Thus, for $i' = i - 2^{-1} \pmod n$, since $2i' + 1 = 2i \pmod n$, then vertex i' precedes in C hyperarc $E_{2i' \pmod n}$ too, and so on. Consequently, since 2^{-1} is a generator of \mathbb{Z}_n , then every vertex i precedes in C hyperarc E_{2i} . Similarly, we can prove that either every hyperarc E_j in C precedes the vertex $2j$, or every hyperarc E_j in C precedes the vertex $2j + 1$.

Let us call σ the application of \mathbb{Z}_n that associates any vertex i to its successor $\sigma(i)$, in a given complete Berge dicycle. There are only four possibilities for σ :

1. $e : i \rightarrow 4i \pmod n$
2. $f : i \rightarrow 4i + 1 \pmod n$
3. $g : i \rightarrow 4i + 2 \pmod n$
4. $h : i \rightarrow 4i + 3 \pmod n$

Since $4 \cdot 0 = 0$, solution 1 does not generate a complete Berge dicycle. Besides, if $\gcd(n, 3) = 1$ then none of the other solutions work. Indeed, the equation $\sigma(i) = i \iff 4i + k = i \pmod n \iff 3i = -k \pmod n$, for some k with $1 \leq k \leq 3$, will always have a solution.

It remains to consider the case $n = c3^p$, $p \geq 1$ and $\gcd(3, c) = 1$.
By induction, we prove that :

1. $f^m(0) = \frac{4^m - 1}{3} \pmod n$
2. $g^m(0) = 2f^m(0) = \frac{2(4^m - 1)}{3} \pmod n$
3. $h^m(0) = 3f^m(0) = 4^m - 1 \pmod n$.

Let φ be the Euler function. We know that $\varphi(n) = \varphi(c)\varphi(3^p) = 2\varphi(c)3^{p-1}$. Since $\gcd(n, 2) = 1$, then, by the Euler's Theorem (Theorem 46), $2^{\varphi(n)} = 1 \pmod n$. Therefore, $4^{\varphi(c)3^{p-1}} = 1 \pmod n$ and, since $\varphi(c)3^{p-1} < n$, h never generates a complete Berge dicycle either.

Moreover, since 2 is inversible in \mathbb{Z}_n , then we also know that f generates a complete Berge dicycle if, and only if, this is also the case for solution g . Actually, when we choose f , we choose g in the dual, and reciprocally. Let us concentrate now on f .

The equation $f^m(0) = f^{m'}(0)$ is equivalent to $\frac{4^m - 1}{3} = \frac{4^{m'} - 1}{3} \pmod n$, which implies that $4^m = 4^{m'} \pmod{3n}$. Again because of the Euler's Theorem, we know that $4^{\varphi(c)3^p} = 1 \pmod{3n}$. And the only solution for having $\varphi(c) = c$ is $c = 1$; otherwise $\varphi(c) < c$. Therefore, if n is not a power of 3, f and g do not generate a complete Berge dicycle.

By Lemma 13, $i \rightarrow 4i + 1$ gives a complete Berge dicycle when n is a power of 3. Therefore there is a complete Berge dicycle in $GBH_2(2, n, 2, n)$ if and only if n is a power of 3. \square

Remark 11. This proof for $d = 2$ and n odd could be shortened using the characterization of the Hamiltonian Consecutive-1 digraphs. Indeed, we prove there are only four possibilities for having a complete Berge dicycle in $GBH_2(2, n, 2, n)$. In the original proof, we deal with them as applications σ of \mathbb{Z}_n , namely: e, f, g and h . But these four solutions are also equivalent to some Consecutive-1 digraphs. Indeed, they correspond, respectively, to the relations of incidence in $G(1, n, 4, 0)$, $G(1, n, 4, 1)$, $G(1, n, 4, 2)$ and $G(1, n, 4, 3)$. Then deciding whether one of these four solutions generate a complete Berge dicycle is the same thing as deciding whether one of these four Consecutive-1 digraphs is a Hamiltonian digraph. Furthermore, by Theorem 30, we know whether one of those digraphs is Hamiltonian, depending on the value of n . For all n , $G(1, n, 4, 0)$ and $G(1, n, 4, 3)$ are never Hamiltonian. Moreover, $G(1, n, 4, 1)$ and $G(1, n, 4, 2)$ are Hamiltonian if and only if n is a power of 3. Since, the Hamiltonicity of, at least, one of these digraphs, is a necessary and sufficient condition for $H = GBH_2(2, n, 2, n)$ to have a complete Berge dicycle, therefore there is a complete Berge dicycle in H if and only if n is a power of 3.

$n \setminus d$	1	2	3	≥ 4
even	Lemma 11	Lemma 15 or 16	Lemmas 15, 17	Lemmas 16, 14
odd	Lemma 11	Lemmas 13, 18	Lemma 12	Lemmas 16, 14

6.2 Generalization

The methods applied in the previous section can be extended to a larger family of digraphs, as it is detailed in the Appendix. Some similar results can be found for the Consecutive- (d, s) dihypergraphs $H([d, n, q, r], [s, n, p, t])$ too:

Lemma 19. Let $H = H([1, n, q, r], [s, n, p, t])$ be a Consecutive- $(1, s)$ dihypergraph. There is a complete Berge dicycle in H if and only if q is coprime to n and $G(s, n, pq, pr + t)$ is Hamiltonian.

Proof. Each vertex in H is incident to only one hyperarc. Therefore a necessary condition for having a complete Berge dicycle in H is that for all $i \neq i'$, $qi + r \not\equiv qi' + r \pmod{n}$. That is equivalent to $\gcd(n, q) = 1$.

Conversely, if n is coprime to q , then we define a digraph G such that $\mathcal{V}(G) = \mathbb{Z}_n$, and there is an arc from vertex i to vertex i' if and only if E_{qi+r} is incident to i' that is $i' = pqi + pr + t + \beta$, $0 \leq \beta \leq s - 1$. So G is isomorphic to $G(s, n, pq, pr + t)$. The lemma follows from the fact that there is a complete Berge dicycle in H if and only if G is Hamiltonian. \square

Furthermore, we can partially solve the case when $\gcd(n, q) \geq 2$. In that case, we always know whether there is a complete Berge dicycle in $H([d, n, q, r], [d, n, q, r])$. We show this characterization with the following two lemmas, the first of which deals with a more general case.

Lemma 20. Let $H = H([d, n, q, r], [s, n, p, t])$ be a consecutive- (d, s) dihypergraph. We denote $\lambda = \gcd(n, q)$. If $\lambda \geq \max(2, d + 1)$, then there is no complete Berge dicycle in H .

Proof. Let $n' = \frac{n}{\lambda}$. Observe that for all k , with $0 \leq k \leq \lambda - 1$, vertices i and $i + kn'$ are incident to the same hyperarcs. Since $n'd < n$, therefore there are empty in-sets. When there are empty in-sets, there is no Eulerian dicycle, hence there is no complete Berge dicycle either. \square

Observe that it is the same result when $\mu = \gcd(n, p) \geq \max(2, s + 1)$.

If d is greater than λ , then we can adapt the proof of Lemma 16.

Lemma 21. Let $H = H([d, n, q, r], [d, n, q, r])$ be a consecutive- (d, d) dihypergraph. We denote $\lambda = \gcd(n, q)$. If $2 \leq \lambda \leq d$, then there is a complete Berge dicycle in H .

Proof. Let $n' = \frac{n}{\lambda}$. Since $L(G(\lambda, n', q, r)) = G(\lambda, n, q, r)$ ([14]), therefore $L(R(H([\lambda, n', q, r], [\lambda, n', q, r])))$ is isomorphic to $R(H([\lambda, n, q, r], [\lambda, n, q, r]))$. Observe that $G(\lambda, n', q, r)$ is Eulerian and so connected [14].

First, let us assume that n' is coprime to q . Let R_u be the undirected version of $R(H([\lambda, n', q, r], [\lambda, n', q, r]))$. Since $qi + r = q(i - q^{-1}) + r + 1$, therefore there is a path of length 2 between vertices i and $i - q^{-1}$ in R_u . Consequently, all the vertices are linked together in R_u , because q^{-1} is a generator element of \mathbb{Z}_n . So $R(H([\lambda, n', q, r], [\lambda, n', q, r]))$ is weakly connected.

Else, let $\lambda' = \gcd(n', q)$. Clearly, λ' is lower than λ . Therefore, the dihypergraph $H([\lambda, n', q, r], [\lambda, n', q, r])$ satisfies the conditions of Lemma 21, and we show by induction that its bipartite representation digraph is always a weakly connected digraph.

$R(H([\lambda, n', q, r], [\lambda, n', q, r]))$ is a connected regular digraph, i.e. it is an Eulerian digraph. Therefore, its line digraph $R(H([\lambda, n, q, r], [\lambda, n, q, r]))$, hence $R(H([d, n, q, r], [d, n, q, r]))$, is Hamiltonian. \square

Remark 12. The last two lemmas show that when $\lambda \geq 2$, then there is a complete Berge dicycle in $H([d, n, q, r], [d, n, q, r])$ if and only if $\lambda \leq d$, that is $G(d, n, q, r)$ is Hamiltonian.

Other extensions of the results about $GBH_2(d, n, d, n)$ are proposed. For example, when n is odd, we easily generalize Lemma 12:

Lemma 22. Let $H = H([d, n, q, r], [d, n, q, r])$ be a Consecutive- (d, d) dihypergraph. If n is odd and the Consecutive- d digraph $G(d, n, q, r)$ is Hamiltonian, then there is a complete Berge dicycle in H .

Proof. Let C_n be a Hamiltonian dicycle in $G(d, n, q, r)$, and C_2 be a canonical dicycle of length 2. By Corollary 17, $C_n \otimes C_2$ is a dicycle of length $2n$. Moreover, observe that $R(H)$ is isomorphic to $G(d, n, q, r) \otimes C_2$. Therefore $C_n \otimes C_2$ is a Hamiltonian dicycle in $R(H)$. \square

If n is coprime to q , then we can generalize Lemma 14:

Lemma 23. Let $H = H([d, n, q, r], [s, n, p, t])$ be a consecutive- (d, s) dihypergraph. If q is coprime to n , and $G(s, n, pq, r)$ is Hamiltonian then there is a complete Berge dicycle in H .

Proof. To every vertex i we associate the hyperarc E_{qi} . That is a bijection between vertices and hyperarcs because $\gcd(n, q) = 1$. In so doing, we get a digraph G , whose set of vertices is labeled by \mathbb{Z}_n , such that there is an arc from vertex i to vertex j if and only if E_{qi} is incident to j . Actually, G is isomorphic to the Consecutive- s digraph $G(s, n, pq, r)$. Moreover, $G(s, n, pq, r)$ is Hamiltonian. Let $0, \sigma(0), \dots, \sigma^{n-1}(0), 0$ be a Hamiltonian dicycle in $G(s, n, pq, r)$. Then $0, E_0, \sigma(0), E_{q\sigma(0)}, \dots, \sigma^i(0), E_{q\sigma^i(0)}, \dots, \sigma^{n-1}(0), E_{q\sigma^{n-1}(0)}, 0$ is a complete Berge dicycle in H . \square

We can also extend Lemma 17 to the Consecutive dihypergraphs when n is even:

Lemma 24. Let $H = H([d, n, q, r], [s, n, p, t])$ be a Consecutive- (d, s) dihypergraph. If n is even, n is coprime to pq , $\gcd(n, \frac{q-1}{2}) = 1$, r is even and $d, s \geq 3$, then there is a complete Berge dicycle in H .

Proof. To every vertex i , we associate the hyperarc E_{qi+r+1} . Similarly, to every hyperarc E_j , we associate the vertex $pj + t + 1$. Since $\gcd(n, p) = \gcd(n, q) = 1$, therefore we get pairwise vertex-disjoint dicycles in $R(H)$, denoted C_1, C_2, \dots, C_k . If $k = 1$, then C_1 is a Hamiltonian dicycle in $R(H)$.

Let us assume that $k \geq 2$. We use the pairs of interchange $g(i) = (2i \pmod n, 2i + 1 \pmod n)$, and $f(j) = (E_{2j+1 \pmod n}, E_{2j+2 \pmod n})$, with $i, j \in \mathbb{Z}_n$. Those pairs are disjoint.

Let UC_1, UC_2, \dots, UC_k be the undirected versions of the dicycles C_1, C_2, \dots, C_k , and L be the union of those cycles. For all i, j we associate in L an edge $(2i \pmod n, 2i + 1 \pmod n)$ to the pair $g(i)$, and an edge $(E_{2j+1 \pmod n}, E_{2j+2 \pmod n})$ to the pair $f(j)$. We get a graph G . Remark that if G is connected, then we can merge all the dicycles C_x , with

$1 \leq x \leq k$, into one dicycle C . Moreover, if all the pairs $g(i)$, with $i \in \mathbb{Z}_n$, can be linked with some edges in G , then G is connected.

Let i be an integer modulo n . Vertex $2i$ is adjacent to hyperarc $E_{2qi+r+1}$, and vertex $2i+1$ is adjacent to hyperarc $E_{2qi+r+q+1}$. Since q is coprime to n , and n is even, therefore q is odd. So $(E_{2qi+r+q}, E_{2qi+r+q+1})$ is a pair of interchange. Furthermore $2qi+q = 2qi + (q-1) + 1 = 2qi + 2q' + 1 = 2q(i + q^{-1}q') + 1$, where $q' = (q-1)/2$. Since $\gcd(n, q') = 1$ therefore $q^{-1}q'$ is a generator element in \mathbb{Z}_n , and so all the pairs $g(i)$ are linked together. \square

Furthermore, when n is coprime to q and $d = 2$ (resp. when n is coprime to p and $s = 2$), we always know whether there is a complete Berge dicycle in $H([2, n, q, r], [s, n, p, t])$ (resp. in $H([d, n, q, r], [2, n, p, t])$).

Lemma 25. *Let $H = H([2, n, q, r], [s, n, p, t])$ be a Consecutive dihypergraph. If n is coprime to q , then there is a complete Berge dicycle in H if and only if either $G(s, n, pq, pr+t)$ or $G(s, n, pq, pr+t+p)$ is a Hamiltonian Consecutive- s digraph.*

Proof. Suppose that there exists a vertex i incident to the hyperarc E_{qi+r} . Then the vertex $j = i - q^{-1}$ is incident to the hyperarc E_{qj+r} , because $qj+r+1 = qi+r$. Consequently, as q^{-1} is a generator of \mathbb{Z}_n , all the vertices j in a complete Berge dicycle are incident to the hyperarc E_{qj+r} . Therefore the vertices are ordered in such a way after vertex i appears vertex $p(qi+r) + t + \beta$ for some β , $0 \leq \beta \leq s-1$, and that corresponds to a Hamiltonian dicycle in $G(s, n, pq, pr+t)$. Similarly, if some vertex i is joined to the hyperarc E_{qi+r+1} , then all the vertices j are also joined to E_{qj+r+1} , and a Berge dicycle will correspond to a Hamiltonian dicycle in $G(s, n, pq, pr+t+p)$. \square

$n \setminus d$	1	2	3	≥ 4
even	Lemma 19	Lemmas 20, 21	Lemma 24 (partially)	Lemmas 20, 21, 23
odd	Lemma 19	Lemma 25	Lemma 22 (partially)	Lemmas 20, 21, 23

Table 3: Necessary and sufficient conditions for $H([d, n, q, r], [d, n, q, r])$ to have a complete Berge dicycle

Thank to these new lemmas, we nearly always know whether there is a complete Berge dicycle in $H([d, n, q, r], [d, n, q, r])$. The only remaining case is when $d = 3$.

6.3 The particular case of the Kautz Dihypergraphs

The Kautz dihypergraph $GKH(d, n, d, n)$ is close to the dihypergraph $GBH_2(d, n, d, n)$, but the existence of complete Berge dicycles in it is much harder to prove due to its asymmetry. Indeed, the relations of incidence from its vertices to the hyperarcs are not the same as the relations of incidence from its hyperarcs to the vertices.

Nonetheless, we are able to show the existence of complete Berge dicycles in $GKH(d, n, d, n)$ for some particular values of (d, n) . Remark that $R(GKH(d, n, d, n))$ is isomorphic to the bipartite digraph $BD(d, n)$ (see [22]).

Theorem 47. *Let $H = GKH(d, n, d, n)$ be a Kautz dihypergraph. There is a complete Berge dicycle in H if one of the following conditions is verified:*

1. $d \geq 4$
2. $d = 3$ and n is even
3. $d = 2$, and n is even or n is a power of 5
4. $d = 1$ and $n \in \{1, 2\}$

Note that $GKH(d, n, d, n) = H([d, n, d, 0], [d, n, n-d, n-d])$. Therefore Conditions 2 and 4 are corollaries of Lemmas 24 and 19, respectively. When d is coprime to n , Condition 1 is a direct consequence of Lemma 23 and Theorem 30. We prove the other cases with the following Lemmas:

Lemma 26. *Let $H = GKH(2, n, 2, n)$ be a Kautz dihypergraph. If n is odd, then there is a complete Berge dicycle in H if and only if n is a power of 5.*

Proof. H is isomorphic to $H([2, n, 2, 0], [2, n, n - 2, n - 2])$. By Lemma 25, there is a complete Berge dicycle in H if and only if either $G(2, n, n - 4, n - 2)$ or $G(2, n, n - 4, n - 4)$ is Hamiltonian. By Theorem 30, one of those digraphs is Hamiltonian if and only if either $G(1, n, n - 4, n - 1)$, $G(1, n, n - 4, n - 2)$, $G(1, n, n - 4, n - 3)$ or $G(1, n, n - 4, n - 4)$ is Hamiltonian. Again by Theorem 30, a necessary and sufficient condition for one of those digraphs to be Hamiltonian is that n is a power of 5. \square

Remark that neither $GK(2, n)$, nor $GB(2, n)$ is Hamiltonian when n is a power of 5.

It remains the case when $\gcd(d, n) \geq 2$.

Lemma 27. *If $\lambda = \gcd(d, n) \geq 2$, then there is a complete Berge dicycle in $GKH(d, n, d, n)$.*

Proof. Let us denote $n' = \frac{n}{\lambda}$. First observe that $L(R(GKH_{(\lambda, \lambda)}(d, n', d, n')))$ and $R(GKH_{(\lambda, \lambda)}(d, n, d, n))$ are isomorphic. Indeed, $L(R(GKH_{(\lambda, \lambda)}(d, n', d, n')))$ is a bipartite digraph $(\mathcal{V}_1, \mathcal{V}_2, \mathcal{E})$. Vertices in \mathcal{V}_1 are labeled by the arcs in $GBH_{\lambda}(d, n')$, and vertices in \mathcal{V}_2 are labeled by the arcs in $GK_{\lambda}(d, n')$. To an arc $i_1 j_1$ in $GB_{\lambda}(d, n')$, we associate the label $f(i_1, j_1)$, where $f(i_1, j_1) = di_1 + \alpha \pmod{n}$ if $j_1 = di_1 + \alpha \pmod{n'}$. Similarly, to an arc $i_2 j_2$ in $GK_{\lambda}(d, n')$, we associate the label $f(n' - 1 - i_2, j_2)$. This label exists, because $d(n' - 1 - i_2) + \beta = j_2 \pmod{n'}$ for some β with $0 \leq \beta \leq \lambda - 1$, and so $(n' - 1 - i_2)j_2$ is an arc in $GB_{\lambda}(d, n')$. Observe that those mappings are one to one. Furthermore:

$$\{d(n - 1 - f(i_1, j_1)) + \alpha : 0 \leq \alpha \leq \lambda - 1\} = \{d(n - 1 - j_1) + \alpha : 0 \leq \alpha \leq \lambda - 1\} = \{f(n' - 1 - j_1, k_1) : j_1 k_1 \in \mathcal{E}(GK_{\lambda}(d, n'))\}$$

$$\{d(f(n' - 1 - i_2, j_2)) + \beta : 0 \leq \beta \leq \lambda - 1\} = \{d j_2 + \beta : 0 \leq \beta \leq \lambda - 1\} = \{f(j_2, k_2) : j_2 k_2 \in \mathcal{E}(GB_{\lambda}(d, n'))\}$$

The relations of incidence between \mathcal{V}_1 and \mathcal{V}_2 are the relations of incidence in $GK_{\lambda}(d, n)$. In the same way, the relations of incidence between \mathcal{V}_2 and \mathcal{V}_1 are the relations of incidence in the digraph $GB_{\lambda}(d, n)$. By Definition 16, this bipartite digraph is isomorphic to $R(GKH_{(\lambda, \lambda)}(d, n, d, n))$.

Let us show that $R(GKH_{(\lambda, \lambda)}(d, n', d, n'))$ is connected. Actually, the proof is similar to the one presented in Lemma 21. First, let us assume that $\gcd(n', d) = 1$. Let R_u be the undirected version of $R(GKH_{(\lambda, \lambda)}(d, n', d, n'))$. For all i , $i, di, i - d^{-1}$ is a path of R_u between vertices i and $i - d^{-1}$. Since d^{-1} is a generator element in \mathbb{Z}_n , therefore all the vertices are linked together. Else, let us denote $\lambda' = \gcd(n', d)$. Obviously, $\lambda' \leq \lambda$. By induction, if there is a complete Berge dicycle in $GKH_{(\lambda', \lambda')}(d, n', d, n')$, therefore its bipartite representation digraph is connected. Hence $R(GKH_{(\lambda, \lambda)}(d, n', d, n'))$ is connected too.

Moreover, $R(GKH_{(\lambda, \lambda)}(d, n', d, n'))$ is regular. Therefore it is an Eulerian digraph. So its line digraph $R(GKH_{(\lambda, \lambda)}(d, n, d, n))$, hence $R(GKH(d, n, d, n))$, is Hamiltonian. \square

We conjecture the following characterization of the complete Berge dicycles in $GKH(d, n, d, n)$:

Conjecture 48. *Let $H = GKH(d, n, d, n)$ be a Kautz dihypergraph. If $d \geq 3$ then there is a complete Berge dicycle in H .*

7 Conclusion

We define an Eulerian dicycle in a dihypergraph as a dicycle C such that every hyperarc appears exactly once in C , and a Hamiltonian dicycle as a dicycle C' such that every vertex appears exactly once in C' . We show that the Eulerian dihypergraphs are the dual dihypergraphs of the Hamiltonian dihypergraphs. Consequently, both of the problems are NP-complete. Nonetheless, we are able to provide some necessary conditions for a dihypergraph to

be Eulerian, hence necessary conditions for its dual to be Hamiltonian. We prove those conditions are sufficient for some particular subclasses of dihypergraphs, that are detailed in this paper. Furthermore, we also show that the study of Eulerian dihypergraphs could be restrained to the study of Eulerian BF-graphs, without any loss of generality.

Partial extensions of some known results about the Eulerian digraphs are found for the regular, uniform dihypergraphs. Particularly, the iterated line dihypergraphs of a uniform, regular dihypergraph are always Eulerian and Hamiltonian. We also propose two stronger definitions of an Eulerian dihypergraph, namely the strongly Eulerian and universally Eulerian dihypergraphs. We show that those stronger notions imply the simple Eulerianity. Moreover, stronger results about the Eulerian digraphs can be extended to strongly Eulerian and universally Eulerian dihypergraphs. Especially, the line dihypergraph of a strongly (resp. universally) Eulerian dihypergraph is Hamiltonian.

Finally, we generalize the Kautz and the de Bruijn dihypergraphs by the Consecutive- (d, s) dihypergraphs. We present some basic properties of this new family of dihypergraphs. We also completely solve the characterization of the Eulerian and Hamiltonian de Bruijn and Kautz dihypergraphs. In a lot of cases, we give some necessary and sufficient conditions for a Consecutive- (d, s) dihypergraph to be Eulerian or Hamiltonian.

The existence of complete Berge dicycles in $H([d, n, q, r], [s, n, p, t])$, is investigated too. Since they correspond to Hamiltonian dicycles in the bipartite representation digraphs $R(H([d, n, q, r], [s, n, p, t]))$, we were able to use in this part the Hamiltonian properties of the Kronecker product, of the Consecutive digraphs, and the link-method interchange that is at the basis of many proofs of Hamiltonicity for the Consecutive digraphs. Although we were able to build a complete Berge dicycle in $GBH_2(d, n, d, n)$ from a Hamiltonian dicycle in $GK(d, n)$ in many cases, we were unable to do the inverse operation.

Since BF-graphs are amongst the simplest dihypergraphs, we think the study of their Eulerian properties could be useful, in order to find new Eulerian dihypergraphs. Moreover, although the definition of a strongly Eulerian dihypergraph is a bit artificial, we think that strongly and universally Eulerian dihypergraphs should be further studied.

Observe that our definition of a dicycle, presented in Section 2.2, is called "simple dicycle" in the literature. Other definitions of dicycles can be found in [21] and [28], namely: B-, F-, BF- and L-dicycles. One may consider using them for defining new kinds of Eulerian dicycles, hence new definitions of an Eulerian dihypergraph.

We mention in the paper the properties of the Consecutive dihypergraphs may be improved using some modified Consecutive- d digraphs, as the ones presented in [20]. Let $D(d, n, q, r), D(s, m, p, t)$ be two such modified Consecutive digraphs. Then we could define the modified Consecutive- (d, s) dihypergraph $D([d, n, q, r], [s, m, p, t])$, such that the relations of incidence between the n vertices and the m hyperarcs (resp. between the m hyperarcs and the n vertices) are the relations of incidence of $D(d, n, q, r)$ (resp. of $D(s, m, p, t)$). They may have more interesting properties than the original Consecutive dihypergraphs.

Finally, we remark that it is easier to study the existence of complete Berge dicycles in $H([d, n, q, r], [s, n, p, t])$ when $d = s, q = p$, and $r = t$. Actually, that is because $R(H([d, n, q, r], [d, n, q, r]))$ can be *factorized* using the Kronecker product; indeed: $R(H([d, n, q, r], [d, n, q, r])) = G(d, n, q, r) \otimes C_2$. More generally, $R(H([d, n, q, r], [s, n, p, t]))$ can always be factorized, using a digraph product presented in [3]. Using this product, denoted \otimes_{ϕ_n} , $R(H([d, n, q, r], [s, n, p, t]))$ is isomorphic to $G(d, n, q, r) \otimes_{\phi_n} G(s, n, p, t)$. A study of the Hamiltonian properties of this product may help finding new results about the complete Berge dicycles in the Consecutive- (d, s) dihypergraphs. A complete characterization of the Hamiltonian Consecutive digraphs may help too. Both of these ideas should be explored further.

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8 Appendix

8.1 On Strongly Eulerian Dihypergraphs

We are unable to give a full characterization of the strongly Eulerian dihypergraphs. However, we present some basic results about them.

Proposition 49. *Let D_1, D_2, \dots, D_c be Eulerian digraphs, such that there is no isolated vertex, $\mathcal{V}(D_1) = \mathcal{V}(D_2) = \dots = \mathcal{V}(D_c) = \mathcal{V}$, and $\mathcal{E}(D_1), \mathcal{E}(D_2), \dots, \mathcal{E}(D_c)$ are pairwise disjoint sets. Let us assume that all these digraphs are d -regular. Then one can build a strongly Eulerian dihypergraph, whose underlying multidigraph is $\bigcup_{i=1}^c D_i$.*

Proof. Let H be a dihypergraph, such that $\mathcal{V}(H) = \mathcal{V}$. We define $\mathcal{E}(H)$ as follows: let v be a vertex. For all i , with $1 \leq i \leq c$, we order the in-links of v in D_i , denoted $u_1^i v, u_2^i v, \dots, u_d^i v$. Then we define d B-arcs V_1, V_2, \dots, V_d , such that for all j with $1 \leq j \leq d$,

$$V_j^- = \bigcup_{i=1}^c \{u_j^i\}$$

$$V_j^+ = \{v\}$$

By construction, $m(H) = d \cdot n(H)$, and H is a $(c, 1)$ -uniform dihypergraph. Let φ_i be a function such that for all hyperarc V_j , we have $\varphi_i(V_j) = (u_{i,j}, v)$. Clearly, $\varphi_1, \varphi_2, \dots, \varphi_c$ is a digraph-partition of H . Furthermore, for all i we have $D_{\varphi_i}[H] = D_i$. Therefore H is strongly Eulerian. \square

Observe that H may be a F-graph too, using the same kind of construction. We may also define a strongly Eulerian dihypergraph, using only one d -regular digraph.

Proposition 50. *Let D be a simple, connected, d -regular digraph. For all vertex v we define a bijection between the out-neighbours of v and $\{0, 1, \dots, d-1\}$. Let $\sigma_1, \sigma_2, \dots, \sigma_c$ be permutations of $\{0, 1, \dots, d-1\}$, such that for all i for every $j \neq j'$, $\sigma_j(i) \neq \sigma_{j'}(i)$. Then one can define a strongly Eulerian dihypergraph from D and these permutations, denoted $H = H_D[\sigma_1, \dots, \sigma_c]$.*

Proof. Let H be a dihypergraph such that $\mathcal{V}(H) = \mathcal{V}(D)$. We define $\mathcal{E}(H)$ as follows: let v be a non-isolated vertex. We define d hyperarcs V_1, V_2, \dots, V_d that are associated to v , such that:

$$V_j^- = \{v\}$$

$$V_j^+ = \bigcup_{i=1}^c \{\sigma_i(u_j)\}$$

where u_j is the j th out-neighbour of v in D . By construction, H is a $(1, c)$ -uniform dihypergraph. Let φ_i be a function such that for all hyperarc V_j we have $\varphi_i(V_j) = (v, \sigma_i(u_j))$. Clearly, $\varphi_1, \varphi_2, \dots, \varphi_c$ is a digraph-partition of H . Furthermore, for every i , $D_{\varphi_i}[H]$ is isomorphic to D . Therefore H is strongly Eulerian. \square

Propositions 49 and 50 can be combined for creating more complicated strongly Eulerian dihypergraphs.

Finally, we present a non-trivial example of strongly Eulerian dihypergraphs, using the Consecutive- (d, s) dihypergraphs.

Proposition 51. *The dihypergraph $H = H([d, n, q, r], [s, n, p, t])$ is strongly Eulerian if it satisfies the following conditions:*

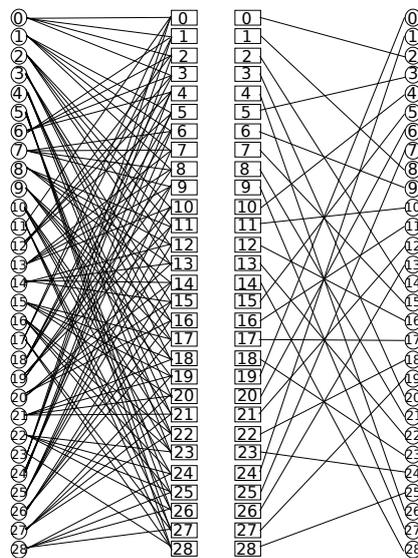


Figure 11: The Consecutive- (d, s) dihypergraph $H([4, 29, 5, 0], [1, 29, 6, 2])$ is strongly Eulerian.

1. $pq - 1 \neq 2$ is prime,
2. n is a power of $pq - 1$
3. $0 < p(d - 1 + r) + (t + s - 1) < pq - 1$

Proof. First observe that $\gcd(n, p) = \gcd(n, q) = 1$. Therefore H can be partitioned into the digraphs $G(1, n, pq, p(\alpha + r) + \beta + t)$, with $0 \leq \alpha \leq d - 1$ and $0 \leq \beta \leq s - 1$. By Theorem 30, a necessary and sufficient condition for $G(1, n, pq, p(\alpha + r) + \beta + t)$ to be Hamiltonian (hence a dicycle) is that:

1. n is coprime to pq ,
2. every prime number k , such that k divides n , divides $pq - 1$,
3. $\gcd(n, pq - 1, p(\alpha + r) + \beta + t) = 1$.

Conditions 1 and 2 are automatically verified. Moreover, $p(\alpha + r) + \beta + t \leq p(d - 1 + r) + s - 1 + t < pq - 1$. Since $pq - 1$ is prime, therefore $pq - 1$ is coprime to $p(\alpha + r) + \beta + t$, and so Condition 3 is also verified. Consequently, all the $G(1, n, pq, p(\alpha + r) + \beta + t)$ are dicycles, that is they are Eulerian. \square

8.2 The Hamiltonian Property of Large Generalized Cycles

The digraph $BGC(p, d, n)$ is defined in [22] as the Kronecker product of the de Bruijn digraph $GB(d, n)$ with a canonical dicycle of length p , that is $BGC(p, d, n) = GB(d, n) \otimes C_p$. Using some methods presented in Section 6, we give necessary and sufficient conditions for $BGC(p, d, n)$ to be Hamiltonian in all the cases.

If $p = 1$, then $BGC(1, d, n)$ is isomorphic to the generalized de Bruijn digraph $GB(d, n)$. Moreover, if $p = 2$, then $BGC(2, d, n)$ is isomorphic to the bipartite representation digraph $R(GBH_2(d, n, d, n))$. We recall the Hamiltonian properties of the generalized de Bruijn digraph in Theorem 32 and we deal with the existence of complete Berge dicycle in $GBH_2(d, n, d, n)$ in Section 6. Consequently, we consider that p is greater than 2.

Theorem 52. For $p \geq 3$ the digraph $BGC(p, d, n)$ is Hamiltonian if and only if one of the following conditions is verified:

1. $d \geq 3$,
2. $d = 2$ and n is even,
3. $d = 2$, and for every prime number q such that q divides n , q divides $2^p - 1$,
4. $d = n = 1$.

If $d = 1$, then we trivially verify that $BGC(p, 1, n)$ is Hamiltonian if and only if $GB(1, n)$ is Hamiltonian. So the only solution is the degenerate case when $n = 1$.

Let us assume that $\gcd(n, d) \geq 2$:

Lemma 28. If $\lambda = \gcd(n, d) \geq 2$, then $BGC(p, d, n)$ is Hamiltonian.

Proof. Let $n' = \frac{n}{\lambda}$. We denote by $BGC_\lambda(p, d, n)$ the digraph $GB_\lambda(d, n) \otimes C_p$. Observe that $L(GB_\lambda(d, n')) = GB_\lambda(d, n) \implies L(BGC_\lambda(p, d, n')) = BGC_\lambda(p, d, n)$. Moreover, $GB_\lambda(d, n')$ is Eulerian, and so strongly connected. Since there is a loop in $GB_\lambda(d, n')$, namely 00 , therefore there is always a dipath of length l , such that $l = 0 \pmod{p}$, between two of its vertices. Consequently, $BGC_\lambda(p, d, n')$ is also strongly connected. Furthermore, it is a λ -regular digraph. Therefore $BGC_\lambda(p, d, n')$ is Eulerian, and so its line digraph $BGC_\lambda(p, d, n)$ is Hamiltonian. As a consequence, $BGC(p, d, n)$ is Hamiltonian too. \square

If $d \geq 4$, and n is coprime to d , then we can use the Hamiltonian Properties of the Consecutive- d digraphs:

Lemma 29. If $d \geq 4$ and n is coprime to d , then $BGC(p, d, n)$ is Hamiltonian.

Proof. Consider the consecutive- d digraph $G = G(d, n, d^p, 0)$. Since $\gcd(n, d^p) = 1$ and $d \geq 4$, we know that G is a Hamiltonian digraph by Theorem 30. Let $0, \sigma(0), \sigma^2(0), \dots, \sigma^{n-1}(0), 0$ be a Hamiltonian dicycle in G .

We recall that vertices in $BGC(p, d, n)$ are labeled by $\mathbb{Z}_n \times \mathbb{Z}_p$, then we consider the following dicycle $C = (0, 0), (0, 1), \dots, (0, p-1), (\sigma(0), 0), (d\sigma(0), 1), \dots, (d^{p-1}\sigma(0), p-1), (\sigma^2(0), 0), \dots, (\sigma^i(0), 0), (d\sigma^i(0), 1), \dots, (d^{p-1}\sigma^i(0), p-1), \dots, (\sigma^{n-1}(0), 0), (d\sigma^{n-1}(0), 1), \dots, (d^{p-1}\sigma^{n-1}(0), p-1), (0, 0)$. It is effectively a dicycle as, by definition, there is an arc from vertex (i, k) to vertex $(di, k+1)$, with $0 \leq k \leq p-2$; furthermore, there is also an arc from vertex $(d^{p-1}\sigma^i(0), p-1)$ to vertex $(\sigma^{i+1}(0), 0)$, as by the adjacency relations of G there exists α , $0 \leq \alpha \leq d-1$ such that $\sigma^{i+1}(0) = d^p \sigma^i(0) + \alpha = d(d^{p-1}\sigma^i(0)) + \alpha$.

Let us note that since $\gcd(n, d) = 1$, the mapping (i, k) to its successor $(di, k+1)$ is one-to-one. So it suffices to verify that all the vertices with a given k , for example $k = 0$, are different, which follows from the fact the $\sigma^i(0)$, $0 \leq i \leq n-1$ are all different in the Hamiltonian dicycle of G . \square

The only remaining cases are when $d = 2$ and $d = 3$. If $d = 2$ and n is even, then there is a Hamiltonian dicycle on $BGC(p, 2, n)$ by Lemma 28. Else:

Lemma 30. If n is odd, then $BGC(p, 2, n)$ is Hamiltonian if and only if for all prime number q such that $q|n$, q divides $2^p - 1$.

Proof. Let us assume that there is a Hamiltonian dicycle C in $BGC(p, d, n)$. Let k_0 be a fixed integer modulo p . Suppose that there exists some i such that the vertex (i, k_0) precedes the vertex $(2i, k_0 + 1)$ in C . Then the vertex (j, k_0) must precede the vertex $(2j, k_0 + 1)$ in C , with $j = i - 2^{-1}$. Furthermore, since 2^{-1} is a generator element in \mathbb{Z}_n , therefore each vertex (i, k_0) has to precede the vertex $(2i, k_0 + 1)$ in C . Otherwise, each vertex (i, k_0) has to precede the vertex $(2i + 1, k_0 + 1)$ in C , and so there are only two possibilities for a given k_0 . At the end, there are only 2^p possible Hamiltonian dicycles, namely : $C_0, C_1, \dots, C_{2^p-1}$, such that after a vertex $(i, 0)$, the next vertex

in C_r whose label is also in $\mathbb{Z}_n \times \{0\}$ is $(2^p i + r, 0)$. Then observe that C_r is a Hamiltonian dicycle if and only if $i \rightarrow 2^p i + r$ is a circular permutation. Clearly n is coprime to 2^p . By Theorem 30, a necessary condition for $i \rightarrow 2^p i + r$ to be a circular permutation, that is $G(1, n, 2^p, r)$ is Hamiltonian, is that for all prime number q such that q divides n , q also divides $2^p - 1$. This condition is also sufficient, because we can always choose $r = 1$, and in so doing $\gcd(n, 2^p - 1, r) = 1$. \square

Remark 13. Especially when $M_p = 2^p - 1$ is a Mersenne prime number, $BGC(p, 2, n)$ is Hamiltonian if and only if n is a power of M_p , and there are always $\varphi(M_p) = M_p - 1$ possible Hamiltonian dicycles, where φ denotes the Euler's function.

If $d = 3$, we give a complete characterization of the Hamiltonian digraphs $BGC(p, 3, n)$, using two methods of link-interchanges.

Lemma 31. *If n is even, then $BGC(p, 3, n)$ is Hamiltonian.*

Proof. When n is a multiple of 3, that is a direct consequence of Lemma 29. Consequently, we assume that n is coprime to 3. To a vertex (i, k) we associate the vertex $(3i + 1, k + 1)$. In so doing, since 3 is coprime to n , therefore we partition $BGC(p, 3, n)$ into mutually vertex-disjoint dicycles. Let us denote those dicycles by C_1, C_2, \dots, C_m . If $m = 1$, then C_1 is a Hamiltonian dicycle.

Like in Lemma 17, we use the following pairs $g(i) = [(2i, 0), (2i + 1, 0)]$ and $f(j) = [(2j + 1, 1), (2j + 2, 1)]$, with $i, j \in \mathbb{Z}_n$. Those pairs are mutually disjoint. Let UC_1, UC_2, \dots, UC_m be the undirected versions of the dicycles C_1, C_2, \dots, C_m , and L be the union of those cycles. For all i, j we associate in L an edge $[(2i \pmod n, 0), (2i + 1 \pmod n, 0)]$ to the pair $g(i)$, and an edge $[(2j + 1 \pmod n, 1), (2j + 2 \pmod n, 1)]$ to the pair $f(j)$. We get a graph G . Remark that if G is connected, then we can merge all the dicycles C_x , with $1 \leq x \leq m$, into one dicycle C . Moreover, if all the pairs $g(i)$, with $i \in \mathbb{Z}_n$, can be linked with some edges in G , then G is connected.

Let i be a integer modulo n . Vertex $(2i, 0)$ is incident to vertex $(6i + 1, 1)$ and vertex $(2i + 1, 0)$ is incident to vertex $(6i + 4, 1)$. Since n is even and $6i + 3$ is odd, therefore $6i + 3 \pmod n$ is odd. Consequently, $[(6i + 3, 1), (6i + 4, 1)]$ is a pair of interchange $f(j)$ for some j . Furthermore $6i + 3 = 6(i + 3^{-1}) + 1$, and so $g(i)$ and $g(i + 3^{-1})$ are linked together. Since 3^{-1} is a generator element in \mathbb{Z}_n , therefore all the pairs $g(i)$ are connected. \square

It is interesting to notice that no extra-pairs of interchange $[(i, k), (i + 1, k)]$ with $k \neq \{0, 1\}$ are needed. The demonstration of Lemma 31 is similar to the proof of Lemma 17.

We propose another set of interchanges when n is odd.

Lemma 32. *If n is odd, then $BGC(p, 3, n)$ is Hamiltonian.*

Proof. The proof is quite the same as for Lemma 31, except that we choose the set of interchange $P = \{[(2i + 1, k), (2i + 2, k)] : 2i + 1 < n\}$. Clearly, all the pairs in P are disjoint. Let UC_1, UC_2, \dots, UC_m be the undirected versions of the dicycles C_1, C_2, \dots, C_m , and L be the union of those cycles. For all pair $[(2i + 1, k), (2i + 2, k)] \in P$, we add an edge $[(2i + 1, k), (2i + 2, k)]$ in L . We get a graph G , and we want to show that G is connected for completing the proof.

By [18], G is connected when $p = 1$. Consequently, for every $i, j \in \mathbb{Z}_n$, for all $k_1 \in \mathbb{Z}_p$, there exists $k_2 \in \mathbb{Z}_p$ such that there is a path in G from (i, k_1) to (j, k_2) . Particularly, for every $i \in \mathbb{Z}_n$, for all $k_1 \in \mathbb{Z}_p$, there exists $k_2 \in \mathbb{Z}_p$ such that there is a path in G from (i, k_1) to $(-2^{-1}, k_2)$. Furthermore, $3 * (-2^{-1}) + 1 = (1 + 2) * (-2^{-1}) + 1 = -2^{-1} - 1 + 1 = -2^{-1}$. Therefore, for all $k_1, k_2 \in \mathbb{Z}_p$, there is a path in G from $(-2^{-1}, k_1)$ to $(-2^{-1}, k_2)$. So G is connected. \square

In the same way, the digraph $KGC(p, d, n)$ is defined as follows in [22]. Its set of vertices is labeled by $\mathbb{Z}_n \times \mathbb{Z}_p$. For all k , with $0 \leq k \leq p - 2$, a vertex (i, k) is incident to the vertices $(di + \alpha, k + 1)$, with $0 \leq \alpha \leq d - 1$. Furthermore, a vertex $(i, p - 1)$ is incident to the vertices $(-di - d + \beta, 0)$, with $0 \leq \beta \leq d - 1$. Observe that $KGC(2, d, n)$ is isomorphic to the bipartite digraph $BD(d, n)$, that is the representation digraph $R(GKH(d, n, d, n))$

(see Section 6.3). Moreover, the digraph $GK(d, n) \otimes C_p$ is isomorphic to $KGC(p, d, n)$ if p is odd, and is isomorphic to $BGC(p, d, n)$ otherwise. Thus using the same methods, we can prove the following Theorem 52:

Theorem 53. *The digraph $KGC(p, d, n)$ is Hamiltonian if one of the following conditions is verified:*

1. $d \geq 4$,
2. $d = 3$, and n is even, or p is odd, or 3 divides n ,
3. $d = 2$ and n is even,
4. $d = 2$, and for every prime number q such that q divides n , q divides $2^p + 1$,
5. $d = 1$, and $n \in \{1, 2\}$.

Furthermore, if $d = 1$ or $d = 2$, the last three conditions are necessary and sufficient conditions.

We also propose the following conjecture:

Conjecture 54. *If $d \geq 3$ then $KGC(p, d, n)$ is Hamiltonian.*

The conjecture is true for d greater than 3.



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