Numerical null controllability of the 1D heat equation: primal methods

Enrique Fernandez-Cara, Arnaud Munch

Numerical null controllability of the 1D heat equation: primal methods

Enrique Fernández-Cara* and Arnaud Münch†

Abstract

This paper deals with the numerical computation of distributed null controls for the 1D heat equation, with Dirichlet boundary conditions. The goal is to compute a control that drives (a numerical approximation of) the solution from a prescribed initial state at $t = 0$ to zero at $t = T$. Using ideas from Fursikov and Imanuvilov [18], we consider the control that minimizes over the class of admissible null controls a functional that involves weighted integrals of the state and the control, with weights that blow up near T. The optimality system is equivalent to a differential problem that is fourth-order in space and second-order in time. We first address the numerical solution of the corresponding variational formulation by introducing a space-time finite element that is C^1 in space and C^0 in time. We prove a strong convergence result for the approximate controls and then we present some numerical experiments. IWe also introduce a mixed variational formulation and we prove well-posedness through a suitable *inf-sup* condition. We introduce a (non-conformal) C^0 finite element approximation and we provide new numerical results. In both cases, thanks to an appropriate change of variable, we observe a polynomial dependance of the condition number with respect to the discretization parameter. Furthermore, with this second method, the initial and final conditions are satisfied exactly.

Keywords: One-dimensional heat equation, null controllability, finite element methods, mixed finite elements, Carleman inequalities.

Mathematics Subject Classification (2010)- 35K35, 65M12, 93B40.

1 Introduction. The null controllability problem

We are mainly concerned in this work with the null controllability problem for the 1D heat PDE. The state equation is the following:

$$\begin{cases} y_t - (a(x)y_x)_x + A(x,t)\, y = v 1_\omega, & (x,t) \in (0,1) \times (0,T) \\ y(x,t) = 0, & (x,t) \in \{0,1\} \times (0,T) \\ y(x,0) = y_0(x), & x \in (0,1). \end{cases} \tag{1}$$

Here, $\omega \subset\subset (0,1)$ is a (small) non-empty open interval, 1_ω is the associated characteristic function, $T > 0$, $a \in L^\infty(0,1)$ with $a(x) \geq a_0 > 0$ a.e., $A \in L^\infty((0,1) \times (0,T))$ and $y_0 \in L^2(0,1)$. In (1), $v \in L^2(\omega \times (0,T))$ is the *control* and $y = y(x,t)$ is the associated *state.*

*Dpto. EDAN, University of Sevilla, Aptdo. 1160, 41080 Sevilla, Spain. E-mail: `cara@us.es`. Partially supported by grants MTM2006-07932 and MTM2010-15592 (DGES, Spain).

†Laboratoire de Mathématiques, Université Blaise Pascal (Clermont-Ferrand 2), UMR CNRS 6620, Campus des Cézeaux, 63177 Aubière, France. E-mail: `arnaud.munch@math.univ-bpclermont.fr`. Partially supported by grant MTM2010-15592 (DGES, Spain).

In the sequel, for any $\tau > 0$, we will denote by Q_τ, Σ_τ and q_τ the sets $(0,1)\times(0,\tau)$, $\{0,1\}\times(0,\tau)$ and $\omega \times (0,\tau)$, respectively. We will also use the following notation:

$$Ly := y_t - (a(x)y_x)_x + A(x,t)\, y, \quad L^* z := -z_t - (a(x)z_x)_x + A(x,t)\, z.$$

For any $y_0 \in L^2(0,1)$ and $v \in L^2(q_T)$, it is well-known that there exists exactly one solution y to (1), with

$$y \in C^0([0,T]; L^2(0,1)) \cap L^2(0,T; H_0^1(0,1)).$$

Accordingly, for any final time $T > 0$, the associated null controllability problem (at T) is the following: for each $y_0 \in L^2(0,1)$, find $v \in L^2(q_T)$ such that the associated solution to (1) satisfies

$$y(x,T) = 0, \quad x \in (0,1). \tag{2}$$

The controllability of PDEs is an important area of research and has been the subject of many papers in recent years. Some relevant references are [38, 39, 31] and [11]. For linear heat equations, see [30, 23, 1, 27, 3] and [4]; for similar semilinear systems, the first contributions have been given in [41, 26, 13, 23] and [18].

This paper is devoted to design and analyze efficient numerical methods for the previous null controllability problem.

The numerical approximation of null controls for (1) is a difficult issue. As shown below, this is mainly due to the strong regularization property of the heat kernel, that renders the numerical problem severely ill-posed.

So far, the approximation of the control of minimal L^2 norm has focused most of the attention. The first contribution was due to Carthel, Glowinski and Lions in [8], who made use of duality arguments. However, the resulting problems involve some dual spaces which are very difficult (if not impossible) to approximate numerically.

More precisely, the null control of minimal norm in $L^2(q_T)$ is given by $v = \phi\, 1_\omega$, where ϕ solves the backward heat equation

$$\begin{cases} -\phi_t - (a(x)\phi_x)_x + A(x,t)\phi = 0, & (x,t) \in (0,1)\times(0,T) \\ \phi(x,t) = 0, & (x,t) \in \{0,1\}\times(0,T) \\ \phi(x,T) = \phi_T(x), & x \in (0,1) \end{cases} \tag{3}$$

and ϕ_T minimizes the strictly convex and coercive functional

$$\mathcal{I}(\phi_T) = \frac{1}{2}\|\phi\|^2_{L^2(q_T)} - (\phi(\cdot\,,0), y_0)_{L^2(0,1)} \tag{4}$$

over the Hilbert space $\mathcal{H}$ defined by the *completion* of $L^2(0,1)$ with respect to the norm $\|\phi\|_{L^2(q_T)}$.

The coercivity of $\mathcal{I}$ in $\mathcal{H}$ is a consequence of the so-called *observability inequality*

$$\|\phi(\cdot\,,0)\|^2_{L^2(0,1)} \le C \iint_{q_T} |\phi|^2\, dx\, dt \qquad \forall \phi_T \in L^2(0,1), \tag{5}$$

that holds for for some constant $C = C(\omega, T)$ and, in turn, this is a consequence of some appropriate *global Carleman* inequalities; see [18] and [14]. But, as discussed in length in [36] (see also [24, 33]), the minimization of $\mathcal{I}$ is numerically ill-posed, essentially because of the hugeness of $\mathcal{H}$. Notice that, in particular, $H^{-s}(0,1) \subset \mathcal{H}$ for any $s > 0$; see also [2], where the degree of ill-posedness is investigated in the boundary situation.

All this explains why in [8] the *approximate controllability* problem is considered and $\mathcal{I}$ is replaced by $\mathcal{I}_\epsilon$, where

$$\mathcal{I}_\epsilon(\phi_T) := \mathcal{I}(\phi_T) + \epsilon\|\phi_T\|_{L^2(0,1)}$$

for any $\epsilon > 0$. Now, the minimizer $\phi_{T,\epsilon}$ belongs to $L^2(0,1)$ and the corresponding control v_ϵ produces a state y_ϵ with $\|y_\epsilon(\cdot\,,T)\|_{L^2(0,1)} \leq \epsilon$. But, as $\epsilon \to 0^+$, high oscillations are observed for the controls v_ϵ near the controllability time T, see [36].

In this paper, we will consider the following extremal problem, introduced by Fursikov and Imanuvilov in [18]:

$$\begin{cases} \text{Minimize} \;\; J(y,v) = \dfrac{1}{2}\displaystyle\iint_{Q_T} \rho^2|y|^2\,dx\,dt + \dfrac{1}{2}\iint_{q_T} \rho_0^2|v|^2\,dx\,dt \\ \text{Subject to} \;\; (y,v) \in \mathcal{C}(y_0,T). \end{cases} \tag{6}$$

Here, we denote by $\mathcal{C}(y_0,T)$ the linear manifold

$$\mathcal{C}(y_0,T) = \{\, (y,v) : v \in L^2(q_T),\; y \text{ solves (1) and satisfies (2)} \,\}$$

and we assume (at least) that

$$\begin{cases} \rho = \rho(x,t),\, \rho_0 = \rho_0(x,t) \text{ are continuous and } \geq \rho_* > 0 \text{ in } Q_T \text{ and} \\ \rho, \rho_0 \in L^\infty(Q_{T-\delta}) \quad \forall \delta > 0 \end{cases} \tag{7}$$

(hence, they can blow up as $t \to T^-$).

This paper is organized as follows.

In Section 2, we recall some results from [18] and we present the details of the variational approach to the null controllability problem. The optimal pair y and v are written in terms of a new function p, the unique solution to (15). In Section 3, we analyze the numerical approximation of the variational formulation (22), that is obtained from (15) after a change of variables, see (18). The main advantage of (22) is that it involves no weight growing exponentially explicitly. The approximation makes use of a finite element that is C^1 in space and C^0 in time. We prove a convergence result as the discretization parameters go to zero and then we present some numerical experiments. In order to avoid C^1 in space finite elements, we introduce in Section 4 the mixed variational formulation (57), which is equivalent to (22) and we prove well-posedness, see theorem 4.1. Some numerical experiments, based on a non conformal C^0 finite element, are discussed in Section 4.2 and highlight once again a polynomial dependance of the condition number. Finally, some further comments, additional results and concluding remarks are given in Section 5.

2 A variational approach to the null controllability problem

In the sequel, unless otherwise specified, it will be assumed that

$$a \in C^1([0,1]), \quad a(x) \geq a_0 > 0 \quad \forall x \in [0,1]. \tag{8}$$

Under this assumption, for any $A \in L^\infty(Q_T)$, the linear system (1) is null-controllable.

Let ρ and ρ_0 be functions satisfying (7) and let us consider the extremal problem (6). Then we have the following:

Theorem 2.1 *For any $y_0 \in L^2(0,1)$ and $T > 0$, there exists exactly one solution to* (6).

The proof is simple. Indeed, from the null controllability of (1), $\mathcal{C}(y_0,T)$ is non-empty. Furthermore, it is a closed convex set of $L^2(Q_T) \times L^2(q_T)$; in fact, it is a closed linear manifold, whose supporting space is the set of all (z,w) such that $w \in L^2(q_T)$,

$$\begin{cases} z_t - (a(x)z_x)_x + A(x,t)\,z = w1_\omega, & (x,t) \in (0,1)\times(0,T) \\ z(x,t) = 0, & (x,t) \in \{0,1\}\times(0,T) \\ z(x,0) = 0, & x \in (0,1) \end{cases} \tag{9}$$

and

$$z(x,T) = 0, \quad x \in (0,1).$$

On the other hand, $(y,v) \mapsto J(y,v)$ is strictly convex, proper and lower semi-continuous on the space $L^2(Q_T) \times L^2(q_T)$ and $J(y,v) \to +\infty$ as $\|(y,v)\|_{L^2(Q_T)\times L^2(q_T)} \to +\infty$. Hence, the extremal problem (6) certainly possesses a unique solution.

Since we are looking for controls such that the associated states satisfy (2), it is a good idea to choose weights ρ and ρ_0 blowing up to $+\infty$ as $t \to T^-$; this can be viewed as a reinforcement of the constraint (2)

When (8) holds, there exist "good" weight functions ρ and ρ_0 that blow up at $t = T$ and provide a very suitable solution to the original null controllability problem. They were determined and systematically used by Fursikov and Imanuvilov and are the following:

$$\left\{ \begin{array}{l} \rho(x,t) = \exp\left(\dfrac{\beta(x)}{T-t}\right), \quad \rho_0(x,t) = (T-t)^{3/2}\rho(x,t), \quad \beta(x) = K_1\left(e^{K_2} - e^{\beta_0(x)}\right) \\ \text{where the } K_i \text{ are sufficiently large positive constants (depending on } T,\, a_0 \text{ and } \|a\|_{C^1}) \\ \text{and } \beta_0 \in C^\infty([0,1]),\ \beta_0 > 0 \text{ in } (0,1),\ \beta_0(0) = \beta_0(1) = 0,\ |\beta_0'| > 0 \text{ outside } \omega. \end{array} \right. \tag{10}$$

The roles of ρ and ρ_0 are clarified by the following arguments and results, which are mainly due to Fursikov and Imanuvilov. First, let us set

$$P_0 = \{\, q \in C^\infty(\overline{Q}_T) : q = 0 \ \text{ on } \ \Sigma_T \,\}.$$

In this linear space, the bilinear form

$$(p,q)_P := \iint_{Q_T} \rho^{-2} L^*p\, L^*q \, dx\, dt + \iint_{q_T} \rho_0^{-2} p\, q\, dx\, dt$$

is a scalar product. Indeed, if we have $q \in P_0$, $L^*q = 0$ in Q_T and $q = 0$ in q_T, then, by the well known *unique continuation property,* we necessarily have $q \equiv 0$.

Let P be the completion of P_0 for this scalar product. Then P is a Hilbert space and the following results hold:

LEMMA 2.1 *Assume that a satisfies (8) and ρ and ρ_0 are given by (10). Let us also set*

$$\rho_1(x,t) = (T-t)^{1/2}\rho(x,t), \quad \rho_2(x,t) = (T-t)^{-1/2}\rho(x,t). \tag{11}$$

Then there exists $C > 0$ only depending on ω, T, a_0, $\|a\|_{C^1}$ and $\|A\|_{L^\infty}$, such that one has the following for all $q \in P$:

$$\begin{aligned} &\iint_{Q_T} \left[\rho_2^{-2}\left(|q_t|^2 + |q_{xx}|^2\right) + \rho_1^{-2}|q_x|^2 + \rho_0^{-2}|q|^2\right] dx\, dt \\ &\qquad \le C\left(\iint_{Q_T} \rho^{-2}|L^*q|^2\, dx\, dt + \iint_{q_T} \rho_0^{-2}|q|^2\, dx\, dt\right). \end{aligned} \tag{12}$$

The proof is given in [18]; see also [14].

LEMMA 2.2 *Let the assumptions of lemma 2.1 hold. Then, for any $\delta > 0$, one has*

$$P \hookrightarrow C^0([0, T-\delta]; H_0^1(0,1)),$$

where the embedding is continuous. In particular, there exists $C > 0$, only depending on ω, T, a_0, $\|a\|_{C^1}$ and $\|A\|_{L^\infty}$, such that

$$\|q(\cdot\,,0)\|^2_{H_0^1(0,1)} \le C\left(\iint_{Q_T} \rho^{-2}|L^*q|^2\, dx\, dt + \iint_{q_T} \rho_0^{-2}|q|^2\, dx\, dt\right) \tag{13}$$

for all $q \in P$.

PROOF: Let $\delta > 0$ be given and let us consider the Banach space $C^0([0, T-\delta]; L^2(0,1))$. Let q be given in P. Then, in view of lemma 2.1 and the fact that all the weights ρ_i are bounded from above in $Q_{T-\delta}$, we see that

$$q, q_t, q_x, q_{xx} \in L^2(Q_{T-\delta}),$$

with norms in this space bounded by a constant times $\|q\|_P$.

In particular, $t \mapsto q(\cdot\,, t)$ and $t \mapsto q_t(\cdot\,, t)$, respectively regarded as a $H^2(0,1)$-valued and a $L^2(0,T)$-valued function, are square-integrable. This implies that $t \mapsto q(\cdot\,, t)$, regarded as a $H^1_0(0,1)$-valued function, is continuous on $[0,T]$. □

PROPOSITION 2.1 *Assume that a satisfies (8) and let ρ and ρ_0 be given by (10). Let (y, v) be the corresponding optimal pair. Then there exists $p \in P$ such that*

$$y = \rho^{-2} L^* p, \quad v = -\rho_0^{-2} p|_{q_T}. \tag{14}$$

The function p is the unique solution to

$$\left\{ \begin{array}{l} \displaystyle\iint_{Q_T} \rho^{-2} L^* p \, L^* q \, dx \, dt + \iint_{q_T} \rho_0^{-2} p \, q \, dx \, dt = \int_0^1 y_0(x) \, q(x,0) \, dx \\ \forall q \in P; \quad p \in P. \end{array} \right. \tag{15}$$

PROOF: In view of lemma 2.2 and the well known *Lax-Milgram lemma,* there exists exactly one solution p to (15). Let us introduce y and v according to (14). We will check that (y, v) solves (6); this will prove the result.

First, notice that $y \in L^2(Q_T)$ and $v \in L^2(q_T)$. Also, in view of (15), we have

$$\left\{ \begin{array}{l} \displaystyle\iint_{Q_T} y \, L^* q \, dx \, dt = \iint_{q_T} v \, q \, dx \, dt + \int_0^1 y_0(x) \, q(x,0) \, dx \\ \forall q \in P; \quad y \in L^2(Q_T). \end{array} \right. \tag{16}$$

But this means that y is the solution to (1) in the transposition sense. Since $y_0 \in L^2(0,1)$ and $v \in L^2(q_T)$, y must coincide with the unique weak solution to (1). In particular, $y \in C^0([0,T]; L^2(0,1))$ and, taking into account (14), we find that (2) holds. In other words, $(y, v) \in \mathcal{C}(y_0, T)$.

Finally, let $(z, w) \in \mathcal{C}(y_0, T)$ be such that $J(z, w) < +\infty$. Then, it is immediate that

$$\begin{aligned} J(z,w) &\geq J(y,v) + \iint_{Q_T} \rho^2 y \, (z - y) \, dx \, dt + \iint_{q_T} \rho_0^2 v \, (w - v) \, dx \, dt \\ &= J(y,v) - \iint_{Q_T} L^* p \, (z - y) \, dx \, dt - \iint_{q_T} p \, (w - v) \, dx \, dt \\ &= J(y,v). \end{aligned}$$

Hence, (y, v) solves (6).

This ends the proof. □

Remark 1 In this proposition, the regularity assumption on the diffusion coefficient a can be relaxed. Indeed, when a is piecewise C^1 and satisfies the ellipticity hypothesis $a \geq a_0 > 0$, it is also possible to construct weights ρ and ρ_0 such that the previous results hold; see [3]. □

Remark 2 In view of (14) and (15), it is clear that the function p furnished by proposition 2.1 solves, at least in the distributional sense, the following differential problem, that is second-order in time and fourth-order in space:

$$\begin{cases} L(\rho^{-2}L^*p) + \rho_0^{-2}p\,1_\omega = 0, & (x,t) \in (0,1)\times(0,T) \\ p(x,t) = 0, \quad (-\rho^{-2}L^*p)(x,t) = 0 & (x,t) \in \{0,1\}\times(0,T) \\ (\rho^{-2}L^*p)(x,0) = y_0(x), \quad (\rho^{-2}L^*p)(x,T) = 0, & x \in (0,1). \end{cases} \tag{17}$$

Notice that, here, no information is obtained on $p(\cdot\,,T)$. □

3 A first method: solving a variational equality

In the sequel, we will take ρ and ρ_0 as in (10). In account of proposition 2.1, a strategy to find the solution (y,v) to (6) is to first solve (15) and then use (14).

We know that the solution p to (15) belongs to $C^0([0,T];H_0^1(0,1))$. However, for any $s \geq 0$, there is no reason to have $p(\cdot\,,t)$ bounded in $H^{-s}(0,1)$ as $t \to T^-$. This means that it can be difficult to approximate with robustness the variational equality (15). Hence, it will appear very efficient to perform a change of variable, so as to, somehow, we "normalize" the space P.

3.1 An equivalent variational reformulation

The idea is to rewrite the variational equality (15) in terms of a new variable z, given by

$$z(x,t) = (T-t)^{-\alpha}\rho_0^{-1}(x,t)p(x,t), \quad (x,t) \in Q_T \tag{18}$$

for some appropriate $\alpha \geq 0$. We define Z as the completion of P_0 for the scalar product

$$(z,\overline{z})_Z := \iint_{Q_T} \rho^{-2}L^*((T-t)^\alpha\rho_0 z)L^*((T-t)^\alpha\rho_0\overline{z})\,dx\,dt + \iint_{q_T}(T-t)^{2\alpha}z\overline{z}\,dx\,dt \tag{19}$$

or, equivalently, we set

$$Z = \{\,(T-t)^{-\alpha}\rho_0\,p : p \in P\,\}.$$

We see that

$$\rho^{-1}L^*((T-t)^\alpha\rho_0 z) = A_1 z + A_2 z_t + A_3 z_x + A_4 z_{xx}, \tag{20}$$

where the $A_i = A_i(x,t)$ satisfy:

$$\begin{cases} A_1 = \Big((\frac{3}{2}+\alpha)(T-t)^{\alpha+1/2} - \beta(T-t)^{\alpha-1/2}\Big) \\ \qquad\quad - c\Big((T-t)^{\alpha+1/2}\beta_{xx} + \beta_x^2(T-t)^{\alpha-1/2}\Big) + A(T-t)^{\alpha+3/2}, \\ A_2 = -(T-t)^{\alpha+3/2}, \quad A_3 = -2c\beta_x(T-t)^{\alpha+1/2}, \quad A_4 = -c(T-t)^{\alpha+3/2}. \end{cases} \tag{21}$$

Consequently, the variational equality (15) can be rewritten as follows:

$$\begin{cases} \displaystyle\iint_{Q_T}(A_1z + A_2z_t + A_3z_x + A_4z_{xx})(A_1\overline{z} + A_2\overline{z}_t + A_3\overline{z}_x + A_4\overline{z}_{xx})\,dx\,dt \\ \qquad\displaystyle + \iint_{q_T}(T-t)^{2\alpha}z\overline{z}\,dx\,dt = T^\alpha\int_0^1 y_0(x)\rho_0(x,0)\overline{z}(x,0)\,dx \\ \forall \overline{z} \in Z;\ z \in Z. \end{cases} \tag{22}$$

The well-posedness of this formulation is an obvious consequence of the well-posedness of (15).

PROPOSITION 3.1 *The variational equality* (22) *possesses exactly one solution* $z \in Z$. *Moreover, the unique solution* (y, v) *to* (6) *is given by*

$$y = \rho^{-1}(A_1 z + A_2 z_t + A_3 z_x + A_4 z_{xx}), \quad v = -(T-t)^\alpha \rho_0^{-1} z 1_\omega, \tag{23}$$

where $z \in Z$ *solves* (22).

In order to have all the coeffcicients A_i in $L^\infty(Q_T)$, it suffices to take $\alpha \geq 1/2$. Notice that, thanks to the previous change of variable, the functions ρ and ρ_0 in $\rho^{-1}L^* p = \rho^{-1}L^*((T-t)^\alpha \rho_0 z)$ compensate each other, so that no exponential function appears anymore in (22).

3.2 Numerical analysis of the variational equality

Let us introduce the bilinear form $m(\cdot\,,\cdot)$, with

$$\begin{aligned} m(z, \overline{z}) := & \iint_{Q_T} (A_1 z + A_2 z_t + A_3 z_x + A_4 z_{xx})(A_1 \overline{z} + A_2 \overline{z}_t + A_3 \overline{z}_x + A_4 \overline{z}_{xx})\, dx\, dt \\ & + \iint_{q_T} s^{2\alpha} z \overline{z}\, dx\, dt \end{aligned} \tag{24}$$

and the linear form ℓ, with

$$\langle \ell, \overline{z} \rangle := T^\alpha \int_0^1 y_0(x) \rho_0(x, 0) \overline{z}(x, 0)\, dx\, dt. \tag{25}$$

Then (22) reads as follows:

$$m(z, \overline{z}) = \langle \ell, \overline{z} \rangle \quad \forall \overline{z} \in Z; \quad z \in Z. \tag{26}$$

3.2.1 Finite dimensional approximation

For any finite dimensional space $Z_h \subset Z$, we can introduce the following problem:

$$m(z_h, \overline{z}_h) = \langle \ell, \overline{z}_h \rangle \quad \forall \overline{z}_h \in Z_h; \quad z_h \in Z_h. \tag{27}$$

Obviously, (27) is well-posed. Furthermore, we have the following classical result:

LEMMA 3.1 *Let* $z \in Z$ *be the unique solution to* (26) *and let* $z_h \in Z_h$ *be the unique solution to* (27). *We have*

$$\|z - z_h\|_Z \leq \inf_{\overline{z}_h \in Z_h} \|z - \overline{z}_h\|_Z. \tag{28}$$

PROOF: We write that

$$\|z_h - z\|_Z^2 = m(z_h - z, z_h - z) = m(z_h - z, z_h - \overline{z}_h) + m(z_h - z, \overline{z}_h - z).$$

The first term term vanishes for all $\overline{z}_h \in Z_h$. The second one is bounded by $\|z_h - z\|_Z \|\overline{z}_h - z\|_Z$. So, we get

$$\|z - z_h\|_Z \leq \|z - \overline{z}_h\|_Z \quad \forall \overline{z}_h \in Z_h$$

and the result follows. □

As usual, this result can be used to prove the convergence of z_h towards z as $h \to 0$ when the spaces Z_h are chosen appropriately.

More precisely, assume that $\mathcal{H} \subset \mathbf{R}^d$ is a *net* (i.e. a generalized sequence) that converges to zero and Z_h is as above for each $h \in \mathcal{H}$. Let us introduce the interpolation operators $\Pi_h : P_0 \to Z_h$ and let us assume that the finite dimensional spaces Z_h are chosen such that

$$\|\Pi_h z - z\|_Z \to 0 \text{ as } h \in \mathcal{H},\ h \to 0, \quad \forall z \in P_0. \tag{29}$$

We then have:

PROPOSITION 3.2 *Let $z \in Z$ be the solution to* (26) *and let $z_h \in Z_h$ be the solution to* (27) *for each $h \in \mathcal{H}$. Then*

$$\|z - z_h\|_Z \to 0 \quad \textit{as} \quad h \in \mathcal{H},\ h \to 0. \tag{30}$$

PROOF: Let us choose $\epsilon > 0$. From the density of P_0 in Z, there exists $z_\epsilon \in P_0$ such that $\|z - z_\epsilon\|_Z \leq \epsilon$. Therefore, from lemma 3.1, we find that

$$\begin{aligned} \|z - z_h\|_Z &\leq \|z - \Pi_h z_\epsilon\|_Z \\ &\leq \|z - z_\epsilon\|_Z + \|z_\epsilon - \Pi_h z_\epsilon\|_Z \\ &\leq \epsilon + \|z_\epsilon - \Pi_h z_\epsilon\|_Z. \end{aligned} \tag{31}$$

From (29), $\|z_\epsilon - \Pi_h z_\epsilon\|_Z$ goes to zero as $h \in \mathcal{H}$, $h \to 0$ and the result follows. □

3.2.2 The finite dimensional spaces Z_h

We will now indicate which are the good spaces Z_h.

The spaces Z_h have to be chosen so that $\rho^{-1} L^*((T-t)^\alpha \rho_0 z_h)$ belongs to $L^2(Q_T)$ for any $z_h \in Z_h$. This means that z_h must possess first-order time derivatives and up to second-order spatial derivatives in $L^2_{loc}(Q_T)$. Therefore, an approximation based on a standard triangulation of Q_T requires spaces of functions that must be C^0 in t and C^1 in x.

For large integers N_x and N_t, we set $\Delta x = 1/N_x$, $\Delta t = T/N_t$ and $h = (\Delta x, \Delta t)$. We introduce the associated uniform quadrangulations $\mathcal{Q}_h$, with $Q_T = \bigcup_{K \in \mathcal{Q}_h} K$ and we assume that $\{\mathcal{Q}_h\}_h$ is a regular family. Then, we introduce the space Z_h as follows:

$$Z_h = \{\, z_h \in C^{1,0}_{x,t}(\overline{Q}_T) : z_h|_K \in \mathbb{P}(K) \ \ \forall K \in \mathcal{Q}_h,\ z_h = 0 \ \text{ on } \Sigma_T \,\}. \tag{32}$$

Here, $C^{1,0}_{x,t}(\overline{Q}_T)$ is the space of the functions in $C^0(\overline{Q}_T)$ that are continuously differentiable with respect to x in $\overline{Q}_T$ and $\mathbb{P}(K)$ denotes the following space of polynomial functions in x and t:

$$\mathbb{P}(K) = (\mathbb{P}_{3,x} \otimes \mathbb{P}_{1,t})(K), \tag{33}$$

where $\mathbb{P}_{\ell,\xi}$ is the space of polynomial functions of order ℓ in the variable ξ.

Obviously, Z_h is finite dimensional subspace of Z.

According to the specific geometry of Q_T, we shall analyze the situation for a uniform quadrangulation $\mathcal{Q}_h$. Each element $K \in \mathcal{Q}_h$ is of the form

$$K_{kl} = (x_k, x_{k+1}) \times (t_l, t_{l+1}),$$

with

$$x_{k+1} = x_k + \Delta x,\ t_{l+1} = t_l + \Delta t, \quad \text{for} \ \ k = 1, ..., N_x,\ l = 1, ..., N_t.$$

It is then easy to see that a function $z_h \in \mathbb{P}(K_{kl})$ is uniquely determined by the real numbers $\{z_h(x_{k+m}, t_{l+n})\}$ and $\{(z_h)_x(x_{k+m}, t_{l+n})\}$, with $m, n = 0, 1$.

More precisely, let us introduce the functions

$$
\begin{aligned}
L_{0k}(x) &= \frac{(\Delta x + 2x - 2x_k)(\Delta x - x + x_k)^2}{(\Delta x)^3}, \quad L_{1k}(x) = \frac{(x - x_k)^2(-2x + 2x_k + 3\Delta x)}{(\Delta x)^3}, \\
L_{2k}(x) &= \frac{(x - x_k)(\Delta x - x + x_k)^2}{(\Delta x)^2}, \quad L_{3k}(x) = \frac{-(x - x_k)^2(h - x + x_k)}{(\Delta x)^2},
\end{aligned} \tag{34}
$$

and

$$
\mathcal{L}_{0l}(t) = \frac{t_l - t + \Delta t}{\Delta t}, \quad \mathcal{L}_{1l}(t) = \frac{t - t_l}{\Delta t}. \tag{35}
$$

Then, the following result is not difficult to prove:

LEMMA 3.2 *Let $u \in P_0$ and let us define the function $\Pi_h u$ as follows: on each $K_{kl} = (x_k, x_k + \Delta x) \times (t_l, t_l + \Delta t)$, we set*

$$
\Pi_h u(x,t) := \sum_{i,j=0}^{1} L_{ik}(x)\mathcal{L}_{jl}(t)u(x_{i+k}, t_{j+l}) + \sum_{i,j=0}^{1} L_{i+2,k}(x)\mathcal{L}_{jl}(t)u_x(x_{i+k}, t_{j+l}). \tag{36}
$$

Then $\Pi_h u$ is the unique function in Z_h that satisfies

$$
\Pi_h u(x_k, t_l) = u(x_k, t_l), \quad (\Pi_h u(x_k, t_l))_x = u_x(x_k, t_l), \quad \forall k, l. \tag{37}
$$

The linear mapping $\Pi_h : P_0 \mapsto Z_h$ is by definition the interpolation operator associated to Z_h.

In the next Section, we will use the following result:

LEMMA 3.3 *For any $u \in P_0$ and any $(x,t) \in K_{kl}$, one has:*

$$
u - \Pi_h u = \sum_{i,j=0}^{1} m_{ij} u_x(x_{i+k}, t_{j+l}) + \sum_{i,j=0}^{1} L_{ik}\mathcal{L}_{jl}\mathcal{R}[u; x_{i+k}, t_{j+l}], \tag{38}
$$

where

$$
m_{ij}(x,t) := (L_{ik}(x)(x - x_i) - L_{i+2,k}(x))\mathcal{L}_j(t)
$$

and

$$
\left\{
\begin{aligned}
\mathcal{R}[u; x_{i+k}, t_{j+l}](x,t) := &\int_{t_{j+l}}^{t} u_t(x_{i+k}, s)\, ds + (x - x_{i+k}) \int_{t_{j+l}}^{t} (t - s) u_{xt}(x_{i+k}, s)\, ds \\
&+ \int_{x_{i+k}}^{x} (x - s) u_{xx}(s,t)\, ds.
\end{aligned}
\right.
$$

PROOF: The equality (38) is a consequence of the following Taylor expansion with integral remainder :

$$
\begin{aligned}
u(x,t) = u(x_i, t_j) &+ (x - x_i)u_x(x_i, t_j) + \int_{t_j}^{t} u_t(x_i, s)\, ds \\
&+ (x - x_i) \int_{t_j}^{t} (t - s) u_{xt}(x_i, s)\, ds + \int_{x_i}^{x} (x - s) u_{xx}(s,t)\, ds
\end{aligned} \tag{39}
$$

and the fact that $\sum_{i,j=0}^{1} L_{ik}(x)\mathcal{L}_{jl}(t) = 1$. $\square$

3.2.3 An estimate of $\|z - \Pi_h z\|_Z$ and some consequences

We will now prove that (29) holds.

Thus, let us fix $z \in P_0$ and let us first see that

$$\iint_{q_T} (T-t)^{2\alpha} |z - \Pi_h z|^2 \, dx \, dt \to 0 \quad \text{as} \quad \Delta x, \Delta t \to 0^+. \tag{40}$$

For each $K_{kl} \in \mathcal{Q}_h$ (denoted by K in the sequel), we write:

$$\iint_K ((T-t)^\alpha)^2 |z - \Pi_h z|^2 \, dx \, dt \le T^{2\alpha} \iint_K |z - \Pi_h z|^2 \, dx \, dt. \tag{41}$$

Using lemma 3.3, we have:

$$\begin{aligned} \iint_K |z - \Pi_h z|^2 \, dx \, dt &= \iint_K \Big(\sum_{i,j} m_{ij} z_x(x_i, t_j) + \sum_{i,j} L_i \mathcal{L}_j \mathcal{R}[z; x_{i+k}, t_{j+l}] \Big)^2 \, dx \, dt \\ &\le 8 \|z_x\|^2_{L^\infty(K)} \sum_{i,j} \iint_K |m_{ij}|^2 \, dx \, dt + 8 \sum_{i,j} \iint_K |L_i \mathcal{L}_j \mathcal{R}[z; x_{i+k}, t_{j+l}]|^2 \, dx \, dt \end{aligned} \tag{42}$$

where we have omitted the indices k and l.

Moreover,

$$\begin{aligned} |\mathcal{R}[z; x_{i+k}, t_{j+l}]|^2 \le & 3 \|z_t(x_i, \cdot)\|^2_{L^2(t_1,t_2)} |t - t_j| + (x - x_i)^2 |t - t_j|^3 \|z_{xt}(x_i, \cdot)\|^2_{L^2(t_1,t_2)} \\ & + |x - x_i|^3 \|z_{xx}(\cdot\,, t)\|^2_{L^2(x_1,x_2)}. \end{aligned} \tag{43}$$

Consequently, we get:

$$\begin{aligned} & \sum_{i,j} \iint_K |L_i \mathcal{L}_j \mathcal{R}[z; x_{i+k}, t_{j+l}]|^2 \, dx \, dt \\ & \quad \le 3 \sup_{x \in (x_1,x_2)} \|z_t(x, \cdot)\|^2_{L^2(t_1,t_2)} \sum_{i,j} \iint_K |L_i(x) \mathcal{L}_j(t)|^2 |t - t_j| \, dx \, dt \\ & \quad + \sup_{x \in (x_1,x_2)} \|z_{tx}(x, \cdot)\|^2_{L^2(t_1,t_2)} \sum_{i,j} \iint_K |L_i(x) \mathcal{L}_j(t)|^2 |t - t_j|^3 (x - x_i)^2 \, dx \, dt \\ & \quad + \sup_{t \in (t_1,t_2)} \|z_{xx}(\cdot\,, t)\|^2_{L^2(x_1,x_2)} \sum_{i,j} \iint_K |L_i(x) \mathcal{L}_j(t)|^2 |x - x_i|^3 \, dx \, dt. \end{aligned} \tag{44}$$

After some tedious computations, one finds that

$$\sum_{i,j} \iint_K |m_{ij}|^2 \, dx \, dt = \frac{8}{945} (\Delta x)^3 \Delta t \sum_{i,j} \iint_K |L_i(x) \mathcal{L}_j(t)|^2 |t - t_j| \, dx \, dt = \frac{13}{105} \Delta x (\Delta t)^2, \tag{45}$$

$$\sum_{i,j} \iint_K |L_i(x) \mathcal{L}_j(t)|^2 |t - t_j|^3 (x - x_i)^2 \, dx \, dt = \frac{19}{9450} (\Delta x)^3 (\Delta t)^4, \tag{46}$$

$$\sum_{i,j} \iint_K |L_i(x) \mathcal{L}_j(t)|^2 |x - x_i|^3 \, dx \, dt = \frac{11}{630} (\Delta x)^4 \Delta t. \tag{47}$$

This leads to the estimate

$$\begin{aligned}\iint_K |z-\Pi_h z|^2\,dx\,dt \le\ & \frac{64}{945}(\Delta x)^3\Delta t\|z_x\|^2_{L^\infty(K)}\\
&+\frac{312}{105}\Delta x(\Delta t)^2 \sup_{x\in(x_1,x_2)}\|(r^{-1}p)_t(x,\cdot)\|^2_{L^2(t_1,t_2)}\\
&+\frac{152}{9450}(\Delta x)^3(\Delta t)^4 \sup_{x\in(x_1,x_2)}\|z_{tx}(x,\cdot)\|^2_{L^2(t_1,t_2)}\\
&+\frac{88}{630}(\Delta x)^4\Delta t \sup_{t\in(t_1,t_2)}\|z_{xx}(\cdot\,,t)\|^2_{L^2(x_1,x_2)}.\end{aligned}\tag{48}$$

We deduce that

$$\begin{aligned}\iint_{q_T}|z-\Pi_h z|^2\,dx\,dt \le\ & K_1|q_T|\|z_x\|^2_{L^\infty(q_T)}(\Delta x)^2+K_2|\omega|\|z_t\|^2_{L^2(0,T;L^\infty(\omega))}(\Delta t)^2\\
&+K_3|\omega|\|z_{tx}\|^2_{L^2(0,T;L^\infty(\omega))}(\Delta x)^2(\Delta t)^4\\
&+K_4T\|z_{xx}\|^2_{L^\infty(0,T;L^2(\omega))}(\Delta x)^4\end{aligned}\tag{49}$$

for some finite positive constants K_i. Hence, for any $z\in P_0$ one has

$$\iint_{q_T}(T-t)^{2\alpha}|z-\Pi_h(z)|^2\,dx\,dt\to 0 \quad\text{as}\quad \Delta x,\Delta t\to 0.\tag{50}$$

On the other hand, taking $\alpha = 1/2$ (in order to get bounded coefficients A_i), after similar computations, we get:

$$\begin{aligned}\iint_K |A_1(z-\Pi_h z)+&A_2(z-\Pi_h z)_t+A_3(z-\Pi_h z)_x+A_4(z-\Pi_h z)_{xx}|^2\,dx\,dt\\
&\le 4\|A_1\|^2_{L^\infty(Q_T)}\iint_K|z-\Pi_h z|^2\,dx\,dt\\
&+4\|A_2\|^2_{L^\infty(Q_T)}\iint_K|(z-\Pi_h z)_t|^2\,dx\,dt\\
&+4\|A_3\|^2_{L^\infty(Q_T)}\iint_K|(z-\Pi_h z)_x|^2\,dx\,dt\\
&+4\|A_4\|^2_{L^\infty(Q_T)}\iint_K|(z-\Pi_h z)_{xx}|^2\,dx\,dt\end{aligned}\tag{51}$$

and, proceeding as above, we see that all these quantities go to 0 as $h=(\Delta x,\Delta t)\to(0,0)$ (it suffices to differentiate (38) with respect to t and x; for more details, we refer the reader to [16]).

This proves the convergence of z_h towards z in the space Z, that is, (29).

Consequently, we have the following result:

PROPOSITION 3.3 *Let $z_h\in Z_h$ be the unique solution of (27) and let y_h, v_h be the functions defined by*

$$y_h=\rho^{-1}(A_1z_h+A_2z_{h,t}+A_3z_{h,x}+A_4z_{h,xx}),\quad v_h=-(T-t)^\alpha\rho_0^{-1}z_h\,1_\omega.$$

Then,

$$\|v-v_h\|_{L^2(q_T)}\to 0\ \ \textit{and}\ \ \|y-y_h\|_{L^2(Q_T)}\ \ \textit{as}\qquad h\to 0.$$

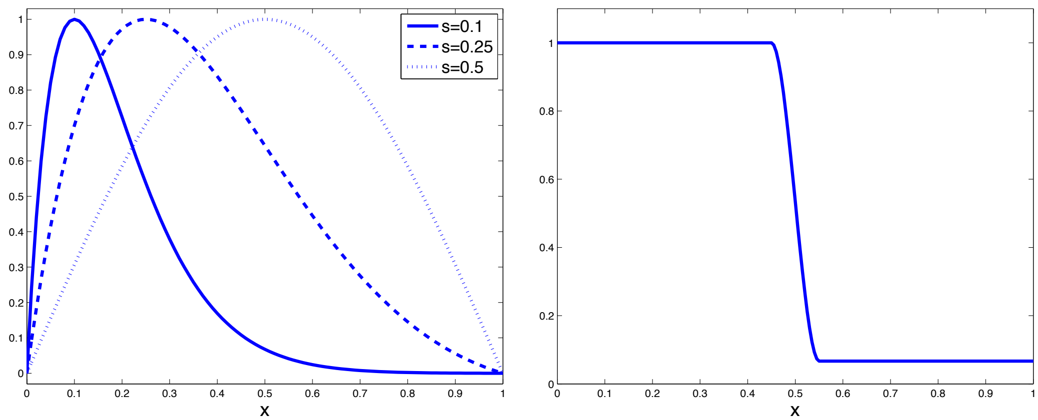

Figure 1: **Left**: The function $\beta_{0,s}$ on $(0,1)$ for $s = 0.1$, $s = 0.25$ and $s = 0.5$. **Right**: The non-constant C^1 diffusion coefficient a used in Section 4.2 (first case).

In order to take into account the numerical approximation of the weights and the data that we we necessarily have to perform in practice, we will consider in the next section a third problem:

$$m_h(\hat{z}_h, \overline{z}_h) = \langle \ell_h, \overline{z}_h \rangle \quad \forall \overline{z}_h \in Z_h; \quad \hat{z}_h \in Z_h, \tag{52}$$

where

$$\begin{aligned} m_h(z_h, \overline{z}_h) := \iint_{Q_T} &((\pi_h A_1) z_h + (\pi_h A_2) z_{h,t} + (\pi_h A_3) z_{h,x} + (\pi_h A_4) z_{h,xx}) \\ &((\pi_h A_1) \overline{z}_h + (\pi_h A_2) \overline{z}_{h,t} + (\pi_h A_3) \overline{z}_{h,x} + (\pi_h A_4) \overline{z}_{h,xx})\, dx\, dt \\ + \iint_{q_T} &\pi_{\Delta t}((T-t)^{2\alpha}) z_h \overline{z}_h \, dx\, dt \end{aligned} \tag{53}$$

and

$$\langle \ell_h, \overline{z}_h \rangle := T^\alpha \int_0^1 \pi_{\Delta x}(y_0 \rho_0(\cdot\,,0)) \overline{z}_h(x,0)\, dx. \tag{54}$$

Here, for any function $f \in L^\infty(Q_T)$, $\pi_h f$ denotes the piecewise linear function which coincides with f at all vertices of $\mathcal{Q}_h$. Similar (self-explanatory) meanings can be assigned to $\pi_{\Delta x} f$ and $\pi_{\Delta t} f$.

We refer the reader to [16], where the strong convergence of $\hat{z}_h$ towards z is proved, together with some *a priori* estimates, explicit in h and $\|z\|_Z$.

3.3 Numerical experiments (I)

We present now some numerical experiments concerning the solution of (52), which can in fact be viewed as a linear system involving a sparse, definite positive and symmetric matrix of order $2N_x N_t$. We denote by $\mathcal{M}_h$ this matrix, so that $(z_h, \overline{z}_h)_{Z_h} = (\mathcal{M}_h\{z_h\}, \{\overline{z}_h\})$. Once the variable z_h is known, the control v_h is given by $v_h = -\pi_h((T-t)^\alpha \rho_0^{-1}) z_h\, 1_\omega$. The corresponding controlled state may be first obtained from (23). Then, this approximation y_h satisfies the controllability requirement (2) (that is, $y_h(\cdot\,,T) = 0$), but not exactly the initial condition. Instead, in order to check the action of the control function v_h, the approximation y_h may be obtained by solving (1) using a finite element method in space and time in the standard way.

For any $s \in (0,1)$, we consider the function $\beta_{0,s}$:

$$\beta_{0,s}(x) = \frac{x(1-x)e^{-(x-c_s)^2}}{s(1-s)e^{-(s-c_s)^2}}, \quad c_s = s - \frac{1-2s}{2s(1-s)}. \tag{55}$$

If s belongs to ω, we easily check that $\beta_{0,s}$ satisfies the conditions in (10). In the numerical experiments, we will take ρ and ρ_0 as in (10) with $\beta_0 = \beta_{0,s}$, s being the middle point of ω, $K_1 = 1/10$ and $K_2 = 2\|\beta_0\|_{L^\infty(0,1)} = 2$.

The function $\beta_{0,s}$ is plotted in Figure 1-Left for $s = 1/10, 1/4$ and $s = 1/2$. The weights ρ^{-2} and ρ_0^{-2} corresponding to $s = 1/2$ are displayed in Figure 2.

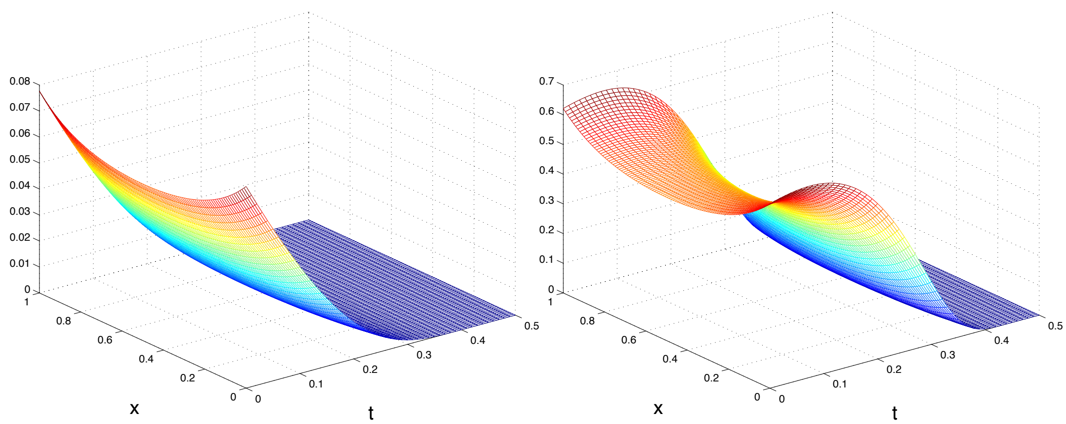

Figure 2: The weights ρ^{-2} and ρ_0^{-2} defined by (10), with $\beta_0 = \beta_{0,1/2}$ defined by (55), $K_1 = 0.1$ and $K_2 = 2\|\beta_0\|_{L^\infty(0,1)}$.

We use an exact integration method in order to compute the components of $\mathcal{M}_h$ and Gauss method to solve the corresponding linear system.

Le us consider a constant diffusion function $a \equiv a_0 = 10^{-1}$ in $(0,1)$. The initial state y_0 is the first eigenfunction of the Dirichlet-Laplacian, that is $y_0(x) \equiv \sin(\pi x)$ and $T = 1/2$. We also take $A \equiv 1$ and $\alpha = 1/2$, so that all the coefficients appearing in the formulation belong to $L^\infty(Q_T)$. Tables 1 and 2 collect relevant numerical values for $\omega = (0.2, 0.8)$ and $\omega = (0.3, 0.6)$ respectively. For $\omega = (0.2, 0.8)$, we take $\beta_0 = \beta_{0,1/2}$. For $\omega = (0.3, 0.6)$, we take $\beta_0 = \beta_{0,0.45}$. Moreover, for simplicity, we always set $\Delta x = \Delta t$.

These Tables clearly show that the solution z_h converges as $h \to 0$, as well as v_h and y_h. The influence of the size of ω on the norm of the control is also emphasized. The absolute errors, displayed in the last two rows, are computed assuming that $h = (1/320, 1/320)$ provides a reference solution.

For $\omega = (0.2, 0.8)$, we observe that $\|y - y_h\|_{L^2(Q_T)} \approx \mathcal{O}(h^{1.19})$ and $\|v - v_h\|_{L^2(q_T)} \approx \mathcal{O}(h^{1.25})$, while for $\omega = (0.3, 0.6)$ we observe a slightly slower convergence: $\|y - y_h\|_{L^2(Q_T)} \approx \mathcal{O}(h^{0.95})$ and $\|v - v_h\|_{L^2(q_T)} \approx \mathcal{O}(h^{0.85})$. We also check that the null controllability requirement (2) is very well satisfied: indeed, we observe that $\|y_h(\cdot, T) - y(\cdot, T)\|_{L^2(0,1)} \approx \mathcal{O}(h^{1.97})$ and $\|y_h(\cdot, T) - y(\cdot, T)\|_{L^2(0,1)} \approx \mathcal{O}(h^{1.65})$ for $\omega = (0.2, 0.8)$ and $\omega = (0.3, 0.6)$, respectively. Notice that, as a consequence of the change of variable (18), the L^2 norm of $z_h(\cdot, T)$ remains bounded with h.

In Tables 3 and 4, we consider the case $\alpha = 0$, once again with $\omega = (0.2, 0.8)$ and $\omega = (0.3, 0.6)$. In this case, the coefficient A_1 is weakly singular, like $(T-t)^{-1/2}$; see (21). This is simply handled by numerically replacing $(T-t)$ by $(T-t+10^{-10})$. The approximation z_h is different here, but

$\Delta x, \Delta t$	1/20	1/40	1/80	1/160	1/320
$\kappa(\mathcal{M}_h)$	1.28×10^{9}	1.00×10^{11}	7.04×10^{12}	4.71×10^{14}	3.07×10^{16}
$\Vert z_h\Vert_{L^2(Q_T)}$	1.804	2.083	2.309	2.462	2.559
$\Vert z_h(\cdot\,,T)\Vert_{L^2(0,1)}$	1.18×10^{-1}	4.08×10^{-2}	6.46×10^{-3}	4.98×10^{-3}	1.41×10^{-5}
$\Vert v_h\Vert_{L^2(q_T)}$	0.97	1.002	1.023	1.035	1.041
$\Vert y_h\Vert_{L^2(Q_T)}$	2.01×10^{-1}	1.998×10^{-1}	1.990×10^{-1}	1.986×10^{-1}	1.984×10^{-1}
$\Vert y_h(\cdot\,,T)\Vert_{L^2(0,1)}$	1.13×10^{-3}	3.00×10^{-4}	7.59×10^{-5}	1.89×10^{-5}	4.74×10^{-6}
$\Vert y - y_h\Vert_{L^2(Q_T)}$	6.47×10^{-3}	3.52×10^{-3}	1.59×10^{-3}	5.35×10^{-4}	-
$\Vert v - v_h\Vert_{L^2(q_T)}$	1.39×10^{-1}	7.42×10^{-2}	3.31×10^{-2}	1.11×10^{-2}	-

Table 1: $\omega = (0.2, 0.8)$, $y_0(x) \equiv \sin(\pi x)$, $a(x) = a_0 \equiv 10^{-1}$ - $\alpha = 1/2$.

$\Delta x, \Delta t$	1/20	1/40	1/80	1/160	1/320
$\kappa(\mathcal{M}_h)$	1.36×10^{9}	1.05×10^{11}	6.87×10^{12}	4.67×10^{14}	3.07×10^{16}
$\Vert z_h\Vert_{L^2(Q_T)}$	9.58	16.18	24.22	33.46	44.11
$\Vert z_h(\cdot\,,T)\Vert_{L^2(0,1)}$	6.84×10^{-1}	1.90×10^{-1}	1.92×10^{-2}	8.05×10^{-3}	1.63×10^{-5}
$\Vert v_h\Vert_{L^2(q_T)}$	1.596	2.005	2.334	2.571	2.729
$\Vert y_h\Vert_{L^2(Q_T)}$	1.881×10^{-1}	1.837×10^{-1}	1.827×10^{-1}	1.827×10^{-1}	1.829×10^{-1}
$\Vert y_h(\cdot\,,T)\Vert_{L^2(0,1)}$	4.09×10^{-3}	1.65×10^{-3}	5.65×10^{-4}	1.68×10^{-4}	4.62×10^{-5}
$\Vert y - y_h\Vert_{L^2(Q_T)}$	7.92×10^{-2}	5.01×10^{-2}	2.70×10^{-2}	1.07×10^{-2}	-
$\Vert v - v_h\Vert_{L^2(q_T)}$	1.580	1.064	0.613	0.258	-

Table 2: $\omega = (0.3, 0.6)$, $y_0(x) \equiv \sin(\pi x)$, $a(x) = a_0 \equiv 10^{-1}$ - $\alpha = 1/2$.

we check that the control function v_h and the corresponding solution y_h are independent of α, so that the rates of convergence are very similar.

A relevant feature of these Tables is that they show that the condition number $\kappa(\mathcal{M}_h)$ of the matrix $\mathcal{M}_h$ depends polynomially on $h = (\Delta x, \Delta t)$. The condition number is defined here as follows:

$$\kappa(\mathcal{M}_h) = |||\mathcal{M}_h|||_2 |||\mathcal{M}_h^{-1}|||_2,$$

where the norm $|||\mathcal{M}_h|||_2$ stands for the largest singular value of $\mathcal{M}_h$. Thus, for $\alpha = 1/2$, we get $\kappa(\mathcal{M}_h) = \mathcal{O}(h^{-6.12})$ and $\kappa(\mathcal{M}_h) = \mathcal{O}(h^{-6.09})$ respectively for $\omega = (0.2, 0.8)$ and $\omega = (0.3, 0.6)$, while for $\alpha = 0$ we get $\kappa(\mathcal{M}_h) = \mathcal{O}(h^{-1.44})$ and $\kappa(\mathcal{M}_h) = \mathcal{O}(h^{-1.58})$ (notice that $\kappa(\mathcal{M}_h)$ is not too sensitive to ω).

This polynomial evolution contrasts with the exponential growth observed when we do not introduce the change (18) and we consider directly the formulation (15). Table 5 provides some numerical values in both situations and definitively highlights (even if the computation of κ may be unaccurate) the influence of the change of variable on the condition number.

If we use an iterative method to solve the linear system, we see that the influence in terms of iterations is still significative, although less important. For the same data considered in Table 5, Table 6 gives the number of iterates needed by GMRES (without restart and without preconditioner) obtained with change of variable (see #GMRES($\mathcal{M}_h$)) and without it (#GMRES($\mathcal{M}_{1,h}$)). The tolerance is taken equal to $\sigma = 10^{-6}$.

The computed state and control y_h and v_h for $h = (1/80, 1/80)$ and $\omega = (0.3, 0.6)$ are displayed in Figures 3 and 4.

From these results, we see that the finite dimensional formulation (52) provides an efficient and robust method to approximate null controls for the heat equation (1). Let us mention two

$\Delta x, \Delta t$	$1/20$	$1/40$	$1/80$	$1/160$	$1/320$
$\kappa(\mathcal{M}_h)$	1.69×10^{12}	5.05×10^{12}	1.22×10^{13}	2.88×10^{13}	1.06×10^{14}
$\Vert z_h\Vert_{L^2(Q_T)}$	1.005	1.113	1.191	1.239	1.266
$\Vert z_h(\cdot\,,T)\Vert_{L^2(0,1)}$	5.37×10^{-10}	2.61×10^{-10}	6.27×10^{-11}	7.07×10^{-12}	2.87×10^{-13}
$\Vert v_h\Vert_{L^2(q_T)}$	0.971	1.003	1.023	1.035	1.041
$\Vert y_h\Vert_{L^2(Q_T)}$	2.011×10^{-1}	1.998×10^{-1}	1.990×10^{-1}	1.986×10^{-1}	1.984×10^{-1}
$\Vert y_h(\cdot\,,T)\Vert_{L^2(0,1)}$	6.17×10^{-4}	1.56×10^{-4}	3.83×10^{-5}	9.44×10^{-6}	2.35×10^{-6}
$\Vert y - y_h\Vert_{L^2(Q_T)}$	6.27×10^{-3}	3.43×10^{-3}	1.56×10^{-3}	5.28×10^{-4}	-
$\Vert v - v_h\Vert_{L^2(q_T)}$	1.36×10^{-1}	7.26×10^{-2}	3.25×10^{-2}	1.09×10^{-2}	-

Table 3: $\omega = (0.2, 0.8)$, $y_0(x) \equiv \sin(\pi x)$, $a(x) = a_0 \equiv 10^{-1}$ - $\alpha = 0$.

$\Delta x, \Delta t$	$1/20$	$1/40$	$1/80$	$1/160$	$1/320$
$\kappa(\mathcal{M}_h)$	3.70×10^{12}	1.12×10^{13}	3.33×10^{13}	1.01×10^{14}	3.03×10^{14}
$\Vert z_h\Vert_{L^2(Q_T)}$	4.664	7.725	10.98	14.30	17.63
$\Vert z_h(\cdot\,,T)\Vert_{L^2(0,1)}$	3.98×10^{-9}	1.36×10^{-9}	3.05×10^{-10}	1.19×10^{-11}	3.40×10^{-13}
$\Vert v_h\Vert_{L^2(q_T)}$	1.597	2.023	2.348	2.58	2.733
$\Vert y_h\Vert_{L^2(Q_T)}$	1.879×10^{-1}	1.834×10^{-1}	1.826×10^{-1}	1.827×10^{-1}	1.829×10^{-1}
$\Vert y_h(\cdot\,,T)\Vert_{L^2(0,1)}$	4.96×10^{-3}	1.82×10^{-3}	5.91×10^{-4}	1.71×10^{-4}	4.65×10^{-5}
$\Vert y - y_h\Vert_{L^2(Q_T)}$	7.52×10^{-2}	4.82×10^{-2}	2.62×10^{-2}	1.04×10^{-2}	-
$\Vert v - v_h\Vert_{L^2(q_T)}$	1.57	1.04	0.59	0.25	-

Table 4: $\omega = (0.3, 0.6)$, $y_0(x) \equiv \sin(\pi x)$, $a(x) = a_0 \equiv 10^{-1}$ - $\alpha = 0$.

drawbacks:

- First, for any fixed h, the controlled state computed by solving (1) numerically does not satisfy exactly the null controllability condition at time $t = T$; this is mainly explained by the fact that, in (17), this requirement appears as a Neumann condition.
- Secondly (and above all), the method requires a finite element approximation that must be C^1 in space (in higher dimension, this involves the use of specific and complex finite elements; see [9]).

We will try to circumvent these two points in the next Section.

4 A second method: solving a mixed variational formulation

Let us introduce the new variables $m = \rho^{-1}L^*p$ and $r = \rho_0^{-1}p$ and let us rewrite (15) in the form

$$\left\{\begin{array}{l} \displaystyle\iint_{Q_T} m\,\overline{m}\,dx\,dt + \iint_{q_T} r\,\overline{r}\,dx\,dt = \int_0^1 \rho_0(x,0)y_0(x)\,\overline{r}(x,0)\,dx \\ \forall(\overline{m},\overline{r}) \text{ with } \rho^{-1}L^*(\rho_0\overline{r}) - \overline{m} = 0 \text{ and } \overline{r} \in \rho_0^{-1}P;\ \ \rho^{-1}L^*(\rho_0 r) - m = 0 \text{ and } r \in \rho_0^{-1}P. \end{array}\right. \tag{56}$$

Let us introduce the spaces $M = L^2(Q_T)$, $R := \rho_0^{-1}P$ and $\tilde{M} := (T-t)^{1/2}M$, the bilinear forms

$$a(\,(m,r),(\overline{m},\overline{r})\,) = \iint_{Q_T} m\,\overline{m}\,dx\,dt + \iint_{q_T} r\,\overline{r}\,dx\,dt \quad \forall(m,r),(\overline{m},\overline{r}) \in M \times R$$

$\Delta x, \Delta t$	$1/20$	$1/40$	$1/80$	$1/160$
$\kappa(\mathcal{M}_h)$	3.70×10^{12}	1.12×10^{13}	3.33×10^{13}	1.01×10^{14}
$\kappa(\mathcal{M}_{1,h})$	3.52×10^{15}	2.56×10^{27}	2.13×10^{50}	2.48×10^{95}

Table 5: $\omega = (0.3, 0.6)$ - **First line** : $\kappa(\mathcal{M}_h)$ with $\alpha = 0$ - $(z_h, \overline{z}_h)_{Z_h} = (\mathcal{M}_h\{z_h\}, \{z_h\})$ - $\kappa(\mathcal{M}_h) \approx O(h^{-1.58})$ - **Second line**: $\kappa(\mathcal{M}_{1,h})$ - $(p_h, \overline{p}_h)_{P_h} = (\mathcal{M}_{1,h}\{p_h\}, \{p_h\})$ - $\kappa(\mathcal{M}_{1,h}) \approx O(e^{h^{-0.87}})$

$\Delta x, \Delta t$	$1/20$	$1/40$	$1/80$	$1/160$
$2N_xN_t$	462	1722	6642	26082
#GMRES($\mathcal{M}_h$)	385	1297	4141	7201
#GMRES($\mathcal{M}_{1,h}$)	411	1501	5890	11567

Table 6: $\omega = (0.3, 0.6)$ and $\alpha = 0$ - Iterations numbers of the GMRES algorithm to solve the variational formulation (22) (**Second line**) and the variational formulation (15) (**Third line**).

and

$$b(\,(\overline{m},\overline{r}),\mu\,) = \iint_{Q_T} (\rho^{-1}L^*(\rho_0\overline{r}) - \overline{m})\,\mu\,dx\,dt \quad \forall(\overline{m},\overline{r}) \in M \times R, \quad \forall \mu \in \tilde{M}$$

and the linear form

$$\langle \ell, (\overline{m},\overline{r})\rangle = \int_0^1 \rho_0(x,0) y_0(x)\,\overline{r}(x,0)\,dx \quad \forall(\overline{m},\overline{r}) \in M \times R.$$

Then, it is not difficult to check that $a(\cdot\,,\cdot)$, $b(\cdot\,,\cdot)$ and ℓ are well-defined and continuous and the announced mixed formulation is the following :

$$\left\{ \begin{array}{rcll} a(\,(m,r),(\overline{m},\overline{r})\,) + b(\,(\overline{m},\overline{r}),\lambda\,) & = & \langle \ell,(\overline{m},\overline{r})\rangle & \forall(\overline{m},\overline{r}) \in M \times R \\ b(\,(m,r),\mu\,) & = & 0 & \forall \mu \in \tilde{M} \\ (m,r) \in M \times R, \quad \lambda \in \tilde{M}. & & & \end{array} \right. \tag{57}$$

We can now state and prove an existence and uniqueness result:

THEOREM 4.1 *There exists a unique solution (m, r, λ) to (57). Moreover, $y := \rho^{-1}m$ is, together with $v := -\rho_0^{-1} r|_{q_T}$, the unique solution to (6).*

PROOF: It is clear that, if (m, r, λ) solves (57), then $m = \rho^{-1}L^*(\rho_0 r)$, $p = \rho_0 r$ is the unique solution to (15) and, consequently, the unique solution to (6) is given by $y = \rho^{-1}m$ and $v = -\rho_0^{-1} r|_{q_T}$.

Let us introduce the space

$$\begin{aligned} V &= \{\,(m,r) \in M \times R : b(\,(m,r),\mu\,) = 0 \;\; \forall \mu \in \tilde{M}\,\} \\ &= \{\,(m,r) \in M \times R : m = \rho^{-1}L^*(\rho_0 r)\,\}. \end{aligned}$$

In order to prove that (57) possesses exactly one solution, we will apply a general result concerning mixed variational problems. More precisely, we will check that

- $a(\cdot\,,\cdot)$ is coercive on V, that is:

$$a(\,(\overline{m},\overline{r}),(\overline{m},\overline{r})\,) \geq \kappa_1 \|(\overline{m},\overline{r})\|^2_{M\times R} \quad \forall(\overline{m},\overline{r}) \in V, \quad \kappa_1 > 0. \tag{58}$$

- $b(\cdot\,,\cdot)$ satisfies the usual "inf-sup" condition with respect to $M \times R$ and $\tilde{M}$, i.e.

$$\kappa_2 := \inf_{\mu\in\tilde{M}} \sup_{(\overline{m},\overline{r})\in M\times R} \frac{b(\,(\overline{m},\overline{r}),\mu\,)}{\|(\overline{m},\overline{r})\|_{M\times R}\|\mu\|_{\tilde{M}}} > 0. \tag{59}$$

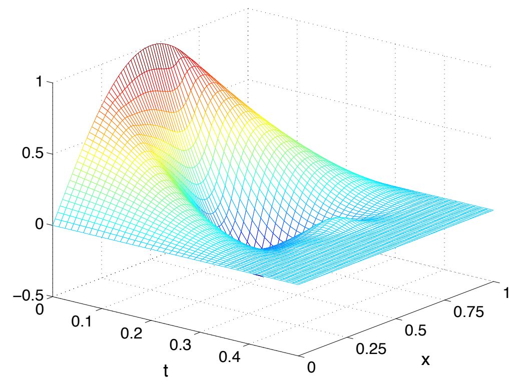

Figure 3: $\omega = (0.3, 0.6)$. The state y_h.

This will suffice to ensure existence and uniqueness; see for instance [7].

The proofs of (58) and (59) are straightforward. Indeed, we first notice that, for any $(\overline{m}, \overline{r}) \in V$, $\overline{m} = \rho^{-1} L^*(\rho_0 \overline{r})$ and thus

$$\begin{aligned} a(\,(\overline{m}, \overline{r}), (\overline{m}, \overline{r})\,) &= \frac{1}{2} \iint_{Q_T} |\overline{m}|^2 \, dx \, dt \\ &+ \frac{1}{2} \iint_{Q_T} \rho^{-2} |L^*(\rho_0 \overline{r})|^2 \, dx \, dt + \iint_{q_T} |\overline{r}|^2 \, dx \, dt \\ &\geq \frac{1}{2} \, \|(\overline{m}, \overline{r})\|^2_{M \times R} \end{aligned}$$

This proves (58).

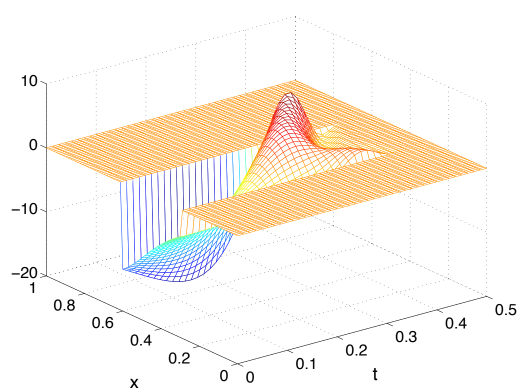

Figure 4: $\omega = (0.3, 0.6)$. The control v_h.

On the other hand, for any $\mu \in \tilde{M}$ there exists $(\tilde{m}, \tilde{r}) \in M \times R$ such that

$$b(\,(\tilde{m}, \tilde{r}), \mu\,) = \iint_{Q_T} (T-t)^{-1} |\mu|^2 \, dx \, dt \quad \text{and} \quad \|(\tilde{m}, \tilde{r})\|_{M \times R} \geq C \|\mu\|_{\tilde{M}}.$$

For instance, it suffices to take $(\tilde{m}, \tilde{r}) = ((T-t)^{-1/2}\mu, 0)$. Consequently,

$$\sup_{(\overline{m},\overline{r})\in M\times R} \frac{b(\,(\overline{m},\overline{r}),\mu\,)}{\|(\overline{m},\overline{r})\|_{M\times R}\|\mu\|_{\tilde{M}}} \geq \frac{b(\,(\tilde{m},\tilde{r}),\mu\,)}{\|(\tilde{m},\tilde{r})\|_{M\times R}\|\mu\|_{\tilde{M}}} \geq \frac{1}{C}$$

and we also have (59). □

As usual, solving (57) is equivalent to finding the *saddle-points* of a *Lagrangian.* In this case, the Lagrangian is given by

$$\begin{aligned} \mathcal{L}(\overline{m},\overline{r};\mu) &= \frac{1}{2}\,a(\,(\overline{m},\overline{r}),(\overline{m},\overline{r})\,) + b(\,(\overline{m},\overline{r}),\mu\,) - \langle \ell, (\overline{m},\overline{r})\rangle \\ &= \frac{1}{2}\left(\iint_{Q_T} |\overline{m}|^2\,dx\,dt + \iint_{q_T} |\overline{r}|^2\,dx\,dt\right) + \iint_{Q_T} (\rho^{-1}L^*(\rho_0\overline{r}) - \overline{m})\,\mu\,dx\,dt \\ &- \int_0^1 \rho_0(x,0)y_0(x)\,\overline{r}(x,0)\,dx \end{aligned} \tag{60}$$

for all $(\overline{m},\overline{r},\mu)\in M\times R\times \tilde{M}$. This can be viewed as the starting point of a large family of iterative methods for the solution of (57).[1]

Remark 3 The variable r coincides with the variable z (given by (18)) for $\alpha = 0$. Therefore, the term $\rho L^*(\rho^{-1}\overline{r})$ in the bilinear form $b(\cdot\,,\cdot)$ possesses a singularity of the kind $(T-t)^{-1/2}$, see (20)–(21), that we may easily cancel by replacing the multiplier μ by $(T-t)^{\gamma}\mu$, for any $\gamma \geq 1/2$. □

4.1 A non-conformal mixed finite element approximation

For any $h = (\Delta x, \Delta t)$ as before, let us consider again the associated uniform quadrangulation $\mathcal{Q}_h$. We now introduce the following finite dimensional spaces:

$$M_h = \{\, z_h \in C^0(\overline{Q}_T) : z_h|_K \in (\mathbb{P}_{1,x}\otimes\mathbb{P}_{1,t})(K) \;\; \forall K\in\mathcal{Q}_h \,\}, \quad Q_h = M_h \;\text{ and} \tag{61}$$

$$\tilde{M}_h = \{\, \mu_h \in Q_h : \mu_h|_{t=T} \equiv 0 \,\}. \tag{62}$$

We are now dealing with usual C^0 finite element spaces. We have $M_h \subset M$ but, contrarily, $Q_h \not\subset R = \rho_0^{-1}P$ (of course, this is the price we have to pay in order to use C^0 finite elements).

Let us introduce the bilinear form $b_h(\cdot,\cdot)$, with

$$\left\{ \begin{array}{l} b_h(\,(\overline{m}_h,\overline{r}_h),\mu_h\,) = \displaystyle\iint_{Q_T} \left(\rho^{-1}(-(\rho_0\overline{r}_h)_t + A\rho_0\overline{r}_h)\mu_h + a(\rho_0\overline{r}_h)_x(\rho^{-1}\mu_h)_x - \overline{m}_h\mu_h\right)\,dx\,dt \\ \forall (\overline{m}_h,\overline{r}_h)\in M_h\times Q_h,\;\forall \mu_h\in\tilde{M}_h. \end{array} \right.$$

Let us also set

$$\langle \ell_h, (\overline{m}_h,\overline{r}_h)\rangle = \int_0^1 \rho_0(x,0)y_0(x)\,\overline{r}_h(x,0)\,dx \quad \forall (\overline{m}_h,\overline{r}_h)\in M_h\times Q_h.$$

Then the mixed finite element approximation of (57) is the following:

$$\left\{ \begin{array}{rcll} a(\,(m_h,r_h),(\overline{m}_h,\overline{r}_h)\,) + b_h(\,(\overline{m}_h,\overline{r}_h),\lambda_h\,) &=& \langle \ell,(\overline{m}_h,\overline{r}_h)\rangle & \forall(\overline{m}_h,\overline{r}_h)\in M_h\times Q_h \\ b_h(\,(m_h,r_h),\mu_h\,) &=& 0 & \forall \mu_h\in\tilde{M}_h \\ (m_h,r_h)\in M_h\times Q_h, \quad \lambda_h\in\tilde{M}_h. && & \end{array} \right. \tag{63}$$

[1] In fact, one of these methods will be considered below, in Section 5 in the context of a problem similar to (57) in higher spatial dimension.

The good property of this approach is that there is no weight growing exponentially as $t \to T^-$. The worst behavior is found in the computation of $\rho^{-1}(\rho_0 \overline{r}_h)_t\, \mu_h$, which behaves at most like $(T-t)^{-1/2}$, but this singularity is weak, numerically acceptable and removable (see Remark 3).

In practice, the numerical experiments we have performed show that (63) possesses exactly one solution $(m_h, r_h, \lambda_h) \in M_h \times Q_h \times \tilde{M}_h$ for each h that is stable and converges as $h \to 0^+$.

4.2 Numerical experiments (II)

We present in this Section some numerical experiments obtained by solving (63).

Three unknown functions, m_h, r_h and μ_h, are involved in the formulation (to be compared with (52), where only the variable z_h appears). The order of the corresponding matrix, which is sparse and symmetric, is of order $3N_xN_t$. Once again, its components, as well as those in the right hand side, are computed with exact integration formulae. Moreover, as before, the linear system is solved using the Gauss method. Once the triplet (m_h, r_h, μ_h) is computed, the numerical solution (y_h, v_h) is given directly by

$$y_h = \pi_h(\rho^{-1})m_h, \quad v_h = -\pi_h(\rho_0^{-1})r_h, \quad (x,t) \in Q_T. \tag{64}$$

First, we consider again the data of Section 3.3, that is, $y_0(x) \equiv \sin(\pi x)$, $a \equiv a_0 = 1/10$ and $T = 1/2$. We also take $\gamma = 1/2$ (see Remark 3). Tables 7 and 8 give the norms of v_h and y_h for various h and $\omega = (0.2, 0.8)$ and $\omega = (0.3, 0.6)$, respectively. The numerical values agree with those obtained with the previous method, see Tables 1 and 2. If we compare closer, the case $\omega = (0.3, 0.6)$ suggests a faster convergence of the norms $\|v_h\|_{L^2(q_T)}$ and $\|y_h\|_{L^2(Q_T)}$. Assuming again that $h = (1/320, 1/320)$ provides a reference solution, we see that $\|v - v_h\|_{L^2(q_T)} = \mathcal{O}(h^{1.23})$ and $\|y - y_h\|_{L^2(Q_T)} = \mathcal{O}(h^{1.23})$ for $\omega = (0.2, 0.8)$ and $\|v - v_h\|_{L^2(q_T)} = \mathcal{O}(h^{1.15})$ and $\|y - y_h\|_{L^2(Q_T)} = \mathcal{O}(h^{1.02})$ for $\omega = (0.3, 0.6)$. Very similar values are observed for $\gamma = 0$, for which the coefficient B_1 is weakly singular at $t = T$.

The main difference observed with respect to the method described in Section 3 is the size of the condition numbers $\kappa(\mathcal{M}_h)$, which is significantly reduced. Once again, the $\kappa(\mathcal{M}_h)$ behave polynomially with respect to h.

We also report in Table 10 the number of iterates leading to the convergence of GMRES versus h. These values can be compared to those in Table 6. Let us emphasize that, here, as a consequence of (64), the null controllability condition is exactly satisfied, that is, $y_h(\cdot\,, T) = 0$ on $(0, 1)$.

As we have seen, the measure of the support $|\omega|$ may affect the convergence of the approximation. Contrarily, due to the regularizing effect of the heat operator, the regularity of the initial condition has no impact in practice. More determinant are the norm (and the sign) of the potential A, the size of the controllability time T and, of course, the size of the diffusion coefficient.

Let us consider a much more stiff situation. The diffusion coefficient will be now a non-constant C^1 function: we take $D_1 = (0, 0.45)$, $D_2 = (0.55, 1)$, $a_1 = 1$ and $a_2 = 1/15$ and we assume that a is the C^1 function that coincides with a polynomial of the third order in $(0,1) \setminus (D_1 \cup D_2)$ and satisfies

$$a(x) \equiv a_i \ \text{ in } \ D_i.$$

In particular, $\min(a_1, a_2) \le a(x) \le \max(a_1, a_2)$ in $(0, 1)$; a is displayed in Figure 1-Right.

We take $\omega = (0.2, 0.4)$ (where the diffusion is higher), $\beta_0 = \beta_{0,0.3}$, $T = 1/2$ and we localize y_0 in D_2, where the diffusion is low: $y_0(x) \equiv e^{-100(x-3/4)^2}\, 1_{(0,1)}$. Finally, we take $A \equiv -1$ (of course, the effect of A is opposite to diffusion, which enhances the action of the control).

$\Delta x, \Delta t$	$1/20$	$1/40$	$1/80$	$1/160$	$1/320$
$\kappa(M_h)$	1.47×10^5	8.30×10^5	6.48×10^6	5.02×10^7	-
$\Vert v_h\Vert_{L^2(q_T)}$	0.974	1.006	1.025	1.036	1.041
$\Vert y_h\Vert_{L^2(Q_T)}$	2.001×10^{-1}	1.996×10^{-1}	1.989×10^{-1}	1.986×10^{-1}	1.984×10^{-1}
$\Vert y - y_h\Vert_{L^2(Q_T)}$	6.32×10^{-3}	3.21×10^{-3}	1.41×10^{-3}	4.75×10^{-4}	-
$\Vert v - v_h\Vert_{L^2(q_T)}$	1.27×10^{-1}	6.56×10^{-2}	2.90×10^{-2}	9.72×10^{-3}	-

Table 7: $\omega = (0.2, 0.8)$, $y_0(x) \equiv \sin(\pi x)$, $a(x) = a_0 \equiv 10^{-1}$ - $\gamma = 1/2$.

$\Delta x, \Delta t$	$1/20$	$1/40$	$1/80$	$1/160$	$1/320$
$\kappa(M_h)$	1.39×10^5	8.78×10^5	6.62×10^6	4.76×10^7	-
$\Vert v_h\Vert_{L^2(q_T)}$	1.865	2.339	2.651	2.830	2.936
$\Vert y_h\Vert_{L^2(Q_T)}$	1.836×10^{-1}	1.814×10^{-1}	1.817×10^{-1}	1.822×10^{-1}	1.826×10^{-1}
$\Vert y - y_h\Vert_{L^2(Q_T)}$	7.13×10^{-2}	3.82×10^{-2}	1.78×10^{-2}	6.33×10^{-3}	-
$\Vert v - v_h\Vert_{L^2(q_T)}$	1.56	0.957	0.489	0.182	-

Table 8: $\omega = (0.3, 0.6)$, $y_0(x) \equiv \sin(\pi x)$, $a(x) = a_0 \equiv 10^{-1}$ - $\gamma = 1/2$.

$\Delta x, \Delta t$	$1/20$	$1/40$	$1/80$	$1/160$	$1/320$
$\kappa(M_h)$	4.18×10^5	5.02×10^5	6.04×10^5	1.17×10^6	-
$\Vert v_h\Vert_{L^2(q_T)}$	0.976	1.007	1.026	1.036	1.041
$\Vert y_h\Vert_{L^2(Q_T)}$	2.008×10^{-1}	1.996×10^{-1}	1.989×10^{-1}	1.986×10^{-1}	1.984×10^{-1}
$\Vert y - y_h\Vert_{L^2(Q_T)}$	6.24×10^{-3}	3.17×10^{-3}	1.41×10^{-3}	4.73×10^{-4}	-
$\Vert v - v_h\Vert_{L^2(q_T)}$	1.25×10^{-1}	6.48×10^{-2}	2.87×10^{-2}	9.67×10^{-3}	-

Table 9: $\omega = (0.2, 0.8)$, $y_0(x) \equiv \sin(\pi x)$, $a(x) = a_0 \equiv 10^{-1}$ - $\gamma = 0$.

$\Delta x, \Delta t$	$1/20$	$1/40$	$1/80$	$1/160$	$1/320$
$\kappa(M_h)$	1.43×10^6	3.29×10^6	8.75×10^6	4.67×10^7	-
$3N_xN_t$	693	2593	9963	39123	-
#GMRES(M_h)	457	1535	5678	10350	-
$\Vert v_h\Vert_{L^2(q_T)}$	1.849	2.335	2.650	2.833	2.936
$\Vert y_h\Vert_{L^2(Q_T)}$	1.84×10^{-1}	1.814×10^{-1}	1.817×10^{-1}	1.822×10^{-1}	1.826×10^{-1}
$\Vert y - y_h\Vert_{L^2(Q_T)}$	7.21×10^{-2}	3.83×10^{-2}	1.79×10^{-2}	6.35×10^{-3}	-
$\Vert v - v_h\Vert_{L^2(q_T)}$	1.587	0.962	0.491	0.183	-

Table 10: $\omega = (0.3, 0.6)$, $y_0(x) \equiv \sin(\pi x)$, $a(x) = a_0 \equiv 10^{-1}$ - $\gamma = 0$.

$\Delta x, \Delta t$	$1/40$	$1/80$	$1/160$	$1/320$	$1/640$
$\kappa(M_h)$	1.05×10^8	8.28×10^9	2.32×10^{11}	7.06×10^{13}	-
$3N_xN_t$	861	3321	13041	51681	205761
#GMRES(M_h)	631	2912	10211	33091	-
$\Vert v_h\Vert_{L^2(q_T)}$	14.44	19.70	23.48	25.41	26.01
$\Vert y_h\Vert_{L^2(Q_T)}$	3.58×10^{-1}	4.67×10^{-1}	5.59×10^{-1}	6.18×10^{-1}	6.43×10^{-1}
$\Vert y - y_h\Vert_{L^2(Q_T)}$	6.30×10^{-1}	3.21×10^{-1}	9.15×10^{-2}	4.76×10^{-2}	-
$\Vert v - v_h\Vert_{L^2(q_T)}/\Vert v\Vert_{L^2(q_T)}$	1.21	0.45	0.23	0.09	-

Table 11: $\omega = (0.2, 0.4)$ - Numerical norms for a stiff case. $\gamma = 0$.

Table 11 collects some numerical values. The numerical convergence as $h \to 0$ is observed. Using now the solutions associated to $h = (1/640, 1/640)$ as a reference solution, we see that $\|v - v_h\| = \mathcal{O}(h^{0.94})$ and $\|y - y_h\| = \mathcal{O}(h^{1.1})$.

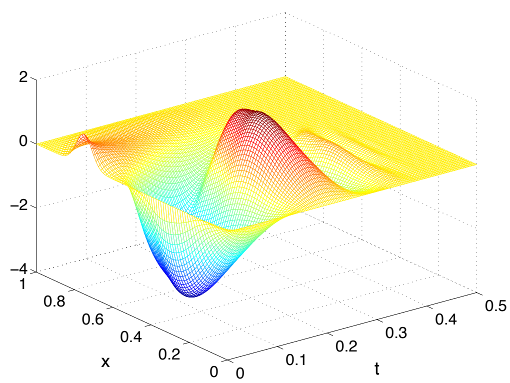

Figure 5: $\omega = (0.2, 0.4)$ - $y_0(x) = e^{-300(x-3/4)^2}$. The state y_h on Q_T

In Figures 5 and 6, the computed controlled state y_h and null control v_h are displayed for $h = (1/80, 1/80)$. The action of the control is here much stronger than in previous examples; in particular, $\|v_h\|_{L^\infty(Q_T)} \approx 2.58 \times 10^2$. As a consequence, y_h takes relatively large values for $t < T$: $\|y_h\|_{L^\infty(Q_T)} \approx 3.24$, while $\|y_0\|_{L^\infty(0,1)}$ is only equal to one.

This situation can be amplified for (weakly explosive) semilinear heat equations; see [15] for more details.

Finally, let us mention the work [34], that also rely on a variational reformulation of the controllability and allow to obtain both boundary and inner controls.

5 Further comments and concluding remarks

5.1 Other related papers and results

First, let us mention [32], where the null controllability for the heat equation with constant diffusion is proved for finite difference schemes in one spatial dimension on uniform meshes.

In higher dimensions, discrete eigenfunctions may be an obstruction to the null controllability; see [42], where a counter-example for finite differences due to O.Kavian is described.

A result of null controllability for a constant portion of the lower part of the discrete spectrum is given in [5]. In [25], in the context of approximate controllability, a relaxed observability inequality is given for general semi-discrete (in space) schemes, with the parameter ε of the order of Δx. The work [6] extends the results in [25] to the full discrete situation and proves the convergence of full discrete (approximated) controls toward a semi discrete one, as the time step Δ_t tends to zero. Let us also mention [12], where the authors prove that any controllable parabolic equation, be it discrete or continuous in space, is null controllable after time discretization upon the application of an appropriate filtering of the high frequencies.

Notice that, in order to find a solution to (6), we can apply methods of two kinds :

- The primal methods considered in this paper, that provide an optimal couple (y, v) satisfying the constraint $(y, v) \in \mathcal{C}(y_0, T)$ and usually rely on the characterization of optimality.

- Dual methods, in the spirit of the pioneering contribution of Carthel, Glowinski and Lions in [8] (see also [22]), relying on appropriate reformulations of (6) as unconstrained problems, that use new (dual) variables. They will be the subject of a forthcoming paper.

5.2 The lack of regularity of $p(\cdot\,,T)$

An important feature of the problem satisfied by p is that it gives no information on the regularity of $p(\cdot\,,T)$. There is an argument that justifies this lack of information even when a is constant and ρ and ρ_0 are regular and bounded in $\overline{Q}_T$. It is the following.

Assume that $\omega \neq (0,1)$ and set $j(w) = J(y_w, w)$, where y_w is the (unique) solution to (1) with v replaced by w and let us set $Kw = z_w(\cdot\,,T)$, where z_w is the solution to (9). Then K can be viewed as a linear continuous and surjective mapping on $L^2(q_T)$ with values in a "very small" Hilbert space $R(K)$, a dense subspace of $L^2(0,1)$. From the *Lagrange multipliers theorem,* we know that (y,v) is solves (6) if and only if

- There exists $\lambda \in R(K)'$ such that
$$(j'(v), w)_{L^2(q_T)} + \langle \lambda, z_w(\cdot\,,T)\rangle = 0 \quad \forall w \in L^2(q_T), \tag{65}$$
where $\langle \cdot\,,\cdot\rangle$ stands for the duality pairing for $R(K)'$ and $R(K)$ and
- y solves (1).

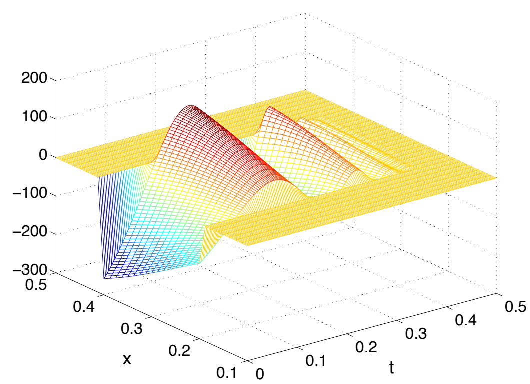

Figure 6: $\omega = (0.2, 0.4)$ - $y_0(x) = e^{-300(x-3/4)^2}$. The control v_h on $(0.1, 0.5) \times (0,T)$

Let (y,v) be an optimal pair and let p be the solution to (17); we know that (y,v) and p satisfy (14). Let us assume that $p(\cdot\,,T) \in L^2(0,1)$ and let $\lambda \in R(K)'$ satisfy (65). In principle, there is no reason to have $\lambda \in L^2(0,1)$ However, it is clear from (14) and (17) that

$$\begin{aligned}(j'(v), w)_{L^2(q_T)} + \langle \lambda, z_w(\cdot\,,T)\rangle &= \iint_{q_T} \left(p + \rho_0^2 v\right) w\,dx\,dt + \langle \lambda - p(\cdot\,,T), z_w(\cdot\,,T)\rangle \\ &= \langle \lambda - p(\cdot\,,T), z_w(\cdot\,,T)\rangle\end{aligned}$$

for all $w \in L^2(q_T)$. Consequently, we should have $\lambda = p(\cdot\,,T)$, which is in contradiction with the fact that λ does not necessarily belong to $L^2(0,1)$. Thus, except in the particular case where the control acts on the whole space domain, the function $p(\cdot\,,T)$ (that can be viewed as a multiplier associated to the constraint $y(\cdot\,,T) = 0$) does not belong to $L^2(0,1)$.

5.3 The role of the weights

The explicit introduction of y in the functional J in 6 allows to give expressions of the optimal control and state in terms of the solution p to (15). With $\rho = 0$, this would have not been possible. The exponential behavior of these weights gives a meaning to the variational formulation (15), reinforces the controllability requirement (through the weight ρ) and regularizes the behavior of the control near $t = T$ (through ρ_0), in contrast with the evolution of the control of minimal L^2-norm, that is highly oscillatory near T.

Carleman estimates ensure the well-posedness of the variational formulation for these specific weights, that blow up exponentially as $t \to T$. Numerically, using other weights in (15) leads to non-convergent sequences.

5.4 Numerical analysis and error estimates

The variational formulation, within the framework of finite elements, leads to strong convergence results for the approximate sequence $\{v_h\}$ as h goes to zero. To our knowledge, this is the first convergence result for the numerical approximation to the null controllability problem for the heat equation. After an apprropiate regularity analysis of the solution to 15, one may further obtain estimates of $\|v - v_h\|_{L^2(q_T)}$ in terms of $|h|$ and a suitable norm of p; we refer to [16].

The mixed approach provides numerical results for which the null controllability requirement is very well satisfied, even when the diffusion function is only piecewise constant. This is in contrast with the existing literature, mainly devoted to the approximate controllability issue. It will be interesting to analyze rigorously (63) from the viewpoints of stability and convergence. In particular, a relevant question is whether inequalities like (58) and (59) hold at the finite dimensional level, with constants κ_1 and κ_2 independent of h.

5.5 Some extensions

The methods used in this paper can be extended to cover null controllability problems for linear heat equations in higher spatial dimensions. More precisely, let $\Omega \subset \mathbf{R}^N$ be a regular, bounded, connected open set and let us consider the linear system

$$\begin{cases} y_t - \nabla \cdot (a(x)\nabla y) + A(x,t)\, y = v 1_{\mathcal{O}}, & (x,t) \in \Omega \times (0,T) \\ y(x,t) = 0, & (x,t) \in \partial\Omega \times (0,T) \\ y(x,0) = y_0(x), & x \in \Omega \end{cases} \tag{66}$$

where $a \in C^1(\overline{\Omega})$ with $a(x) \geq a_0 > 0$, $A \in L^\infty(\Omega \times (0,T))$, $\mathcal{O} \subset \Omega$ is a (small) non-empty open set, $v \in L^2(\mathcal{O} \times (0,T))$ is the control and $y_0 \in L^2(\Omega)$ is the initial state. The null controllability problem for (66) is to find, for each $y_0 \in L^2(\Omega)$, a control v such that the associated solution satisfies

$$y(x,T) = 0, \quad x \in \Omega.$$

The situation is more involved: first, notice that a result similar to Theorem 2.1 holds, but stronger regularity is needed in order to get Carleman estimates; secondly, observe that the analog of the space P_h in (32) is considerably more complex in this general case. Fortunately, this can be avoided using mixed formulations, as in Section 4.

In order to illustrate the situation, let us present the results of an experiment. We solve numerically the null controllability problem for (66) with $N = 2$, $\Omega = (0,1)\times(0,1)$, $\mathcal{O} = (0.2,0.6)\times(0.2,0.6)$, $T = 1$, $a(x) \equiv 1$, $A(x,t) \equiv 1$ and $y_0(x) \equiv 1000$. The space-time domain and the mesh

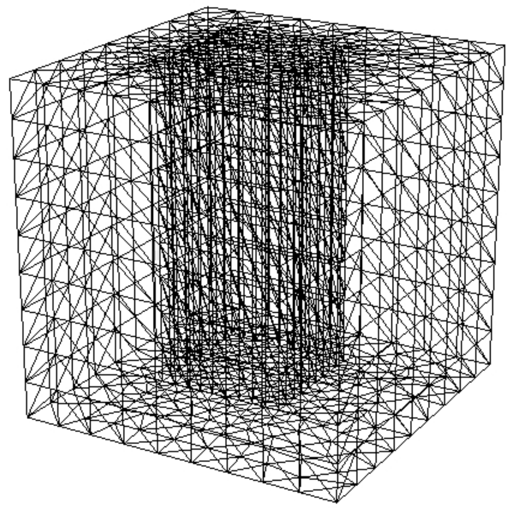

Figure 7: The space-time domain and the mesh: 2 800 vertices. Total number of unknwons (the values of m_h, r_h and λ_h at the nodal points, see (63)): $6\,846 \times 3 = 20\,538$.

are displayed in Figure 7. We have used a mixed formulation similar to (63), where M_h, Q_h and $\tilde{M}_h$ are standard finite element P_2-Lagrange spaces. The resulting system, in view of its size and structure, has been solved with the Arrow-Hurwicz method, that provides good results, better than a direct solver; see for instance [19, 20] (recall that solving (63) is equivalent to the computation of the saddle-points of a Lagrangian, see the related argument in Section 4). The iterates have been stopped for a relative error of two consecutive iterates less than 10^{-5}. The computed control and state are shown in Figures 8–10. The computations have been performed with the FreFem++ package, see `http://www.freefem.org/ff++`. For more information, a detailed analysis and other similar numerical experiments, see [17].

The previous methods can also be extended to cover many other controllable systems for which appropriate Carleman estimates are available: non-scalar parabolic systems, Stokes and Stokes-like systems, etc.; we refer again to [17]. It is also possible to extend the previous arguments and methods to the boundary null controllability case and to the exact controllability to trajectories (with distributed or boundary controls). Actually, as first noticed in [18] and using in part the results by [37] and [40], the approach may also work for linear equations of the hyperbolic kind, where the practical computation of exact controls remains a challenge, see [10]. This work also opens the possibility to address the numerical solution of nonlinear control problems, the optimization of the control support ω, etc.

References

[1] G. Alessandrini and L. Escauriaza, *Null-controllability of one-dimensional parabolic equations,* ESAIM Control Optim. Calc. Var. 14 (2008), no. 2, 284–293.

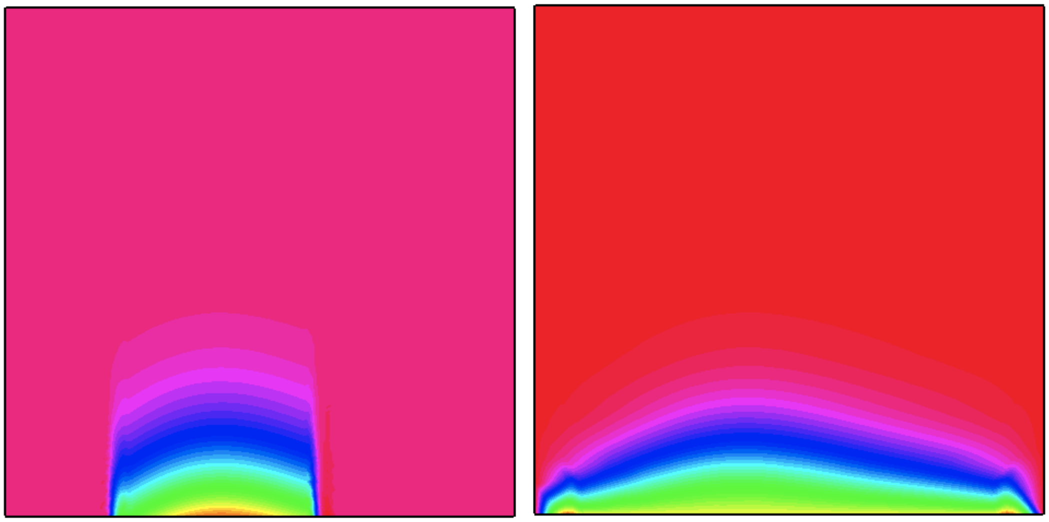

Figure 8: Control and state iso-lines at $x_1 = 0.28$. Minimal and maximal values for u_h: -9.89×10^2, 9.45×10^1. Minimal and maximal values for y_h: -2.84×10^3, 3.70×10^1.

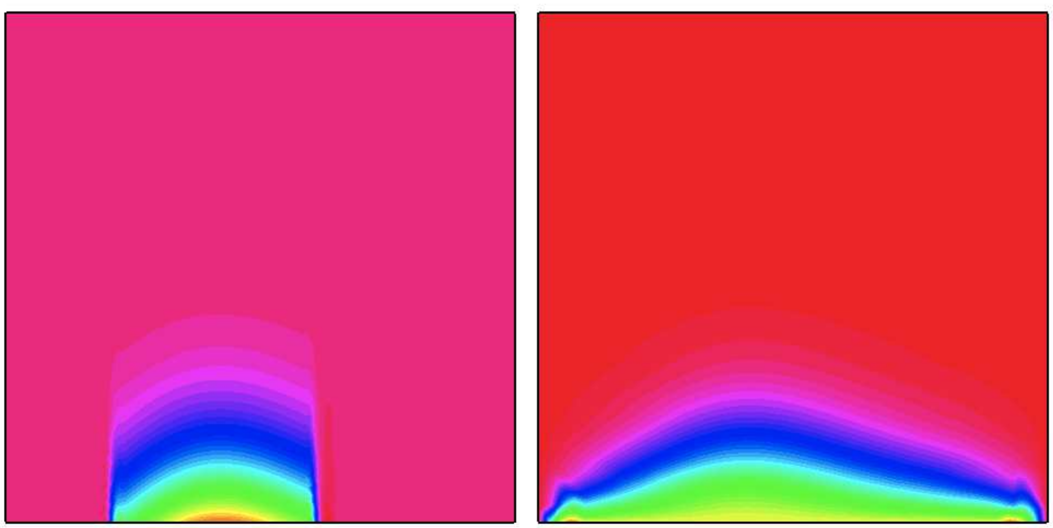

Figure 9: Control and state iso-lines at $x_1 = 0.44$. Minimal and maximal values for u_h: -1.26×10^3, 1.17×10^2. Minimal and maximal values for y_h: -2.84×10^3, 3.70×10^1.

[2] F. Ben Belgacem and S.M. Kaber, *On the Dirichlet boundary controllability of the 1-D heat equation: semi-analytical calculations and ill-posedness degre*, Inverse Problems 27 (2011), no. 5.

[3] A. Benabdallah, Y. Dermenjian and J. Le Rousseau, *Carleman estimates for the one-dimensional heat equation with a discontinuous coefficient and applications to controllability and an inverse problem,* J. Math. Anal. Appl. 336 (2007), no. 2, 865–887.

[4] A. Benabdallah, Y. Dermenjian and J. Le Rousseau, *Carleman estimates for stratified media,* J. Funct. Anal. 260 (2011), no. 12, 3645–3677.

[5] F. Boyer, F. Hubert and J. Le Rousseau, *Discrete Carleman estimates for elliptic operators in arbitrary dimension and applications*, SIAM J. Control Optim. 48 (2010), no. 8, 5357–5397.

[6] F. Boyer, F. Hubert and J. Le Rousseau, *Discrete Carleman estimates for elliptic operators and uniform controllability of semi-discretized parabolic equations,* J. Math. Pures Appl. (9) 93 (2010), no. 3, 240–276.

[7] F. Brezzi, M. Fortin, *Mixed and hybrid finite element methods,* Springer Series in Computational Mathematics, 15, Springer-Verlag, New York, 1991.

[8] C. Carthel, R. Glowinski and J.-L. Lions, *On exact and approximate boundary controllability for the heat equation: a numerical approach*, J. Optimization, Theory and Applications 82(3), (1994) 429–484.

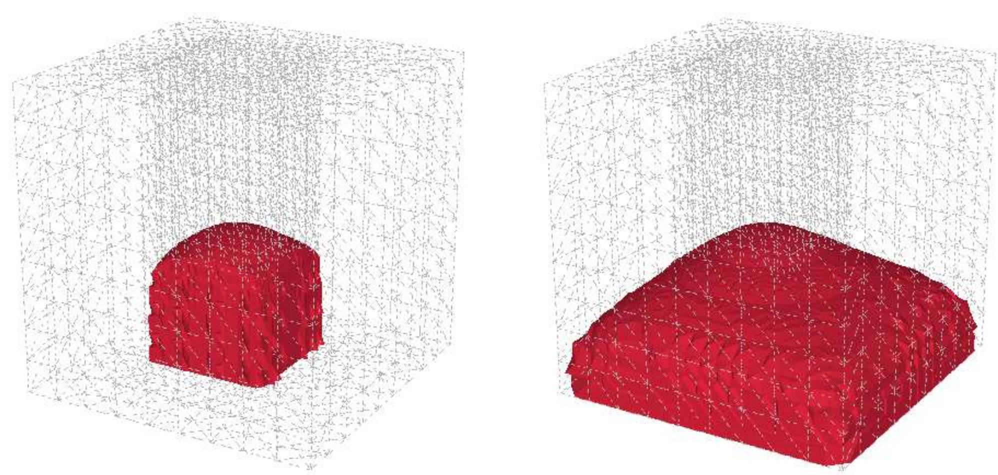

Figure 10: The surfaces $u_h(x,t) = 0$ and $y_h(x,t) = 0$ in the (x,t) space. The region below $u_h = 0$ is the real support of the computed control.

[9] P.G. Ciarlet, *The finite element method for elliptic problems*, Studies in Mathematics and its Applications, Vol. 4, North-Holland Publishing Co., Amsterdam, 1978.

[10] N. Cîndea, E. Fernández-Cara, A. Münch, *Numerical null controllability of the wave equation with a primal approach: convergence results,* in preparation.

[11] J-M. Coron, *Control and Nonlinearity*, Mathematical Surveys and Monographs, Volume 136, AMS, Providence, Rhode Island, 2007.

[12] S. Ervedoza, J. Valein, *On the observability of abstract time-discrete linear parabolic equations*, Rev. Mat. Complut., 23 (2010), no. 1, 163–190.

[13] C. Fabre, J.-P. Puel and E. Zuazua, *Approximate controllability of the semilinear heat equation,* Proc. Roy. Soc. Edinburgh Sect. A 125 (1995), no. 1, 31–61.

[14] E. Fernández-Cara and S. Guerrero, *Global Carleman inequalities for parabolic systems and applications to controllability,* SIAM J. Control Optim. 45 (2006), no. 4, 1399–1446.

[15] E. Fernández-Cara and A. Münch, *Numerical null controllability of a semi-linear 1D heat via a least squares reformulation,* C.R. Acad. Sci. Serie I, . 349 (2011), 867–871.

[16] E. Fernández-Cara and A. Münch, *Analysis and error estimates of the numerical solution to the null controllability problem for the 1D heat equation,* in preparation.

[17] E. Fernández-Cara, A. Münch and D.A. Souza, *On the numerical null controllability of the heat equation and the Stokes system,* in preparation.

[18] A.V. Fursikov and O. Yu. Imanuvilov, *Controllability of Evolution Equations,* Lecture Notes Series, No. 34. Seoul National University, Korea, (1996) 1–163.

[19] R. Glowinski, *Finite element methods for incompressible viscous flow,* Handbook of numerical analysis, Vol. IX, 3-1176, North-Holland, Amsterdam, 2003.

[20] R. Glowinski and P. Le Tallec, *Augmented lagrangian and operator-splitting methods in non-linear mechanics,* SIAM Studies in Applied Mathematics 9, (1989).

[21] R. Glowinski and J.-L. Lions, *Exact and approximate controllability for distributed parameter systems,* Acta Numerica (1996), 159–333.

[22] R. Glowinski, J.-L. Lions and J. He, *Exact and approximate controllability for distributed parameter systems: a numerical approach* Encyclopedia of Mathematics and its Applications, 117. Cambridge University Press, Cambridge, 2008.

[23] O. Yu. Imanuvilov, *Controllability of parabolic equations (Russian),* Mat. Sb. 186 (1995), no. 6, 109–132; translation in Sb. Math. 186 (1995), no. 6, 879–900.

[24] S. Kindermann, *Convergence Rates of the Hilbert Uniqueness Method via Tikhonov regularization*, J. of Optimization Theory and Applications 103(3), (1999) 657-673.

[25] S. Labbé and E. Trélat, *Uniform controllability of semi-discrete approximations of parabolic control systems*, Systems and Control Letters 55 (2006) 597-609.

[26] I. Lasiecka and R. Triggiani, *Exact controllability of semilinear abstract systems with applications to waves and plates boundary control,* Appl. Math. & Optim., 23 (1991), 109–154.

[27] J. Le Rousseau, *Carleman estimates and controllability results for the one-dimensional heat equation with BV coefficients,* J. Differential Equations 233 (2007), no. 2, 417–447.

[28] J. Le Rousseau and L. Robbiano, *Carleman estimate for elliptic operators with coefficients with jumps at an interface in arbitrary dimension and application to the null controllability of linear parabolic equations,* Arch. Ration. Mech. Anal. 195 (2010), no. 3, 953–990.

[29] J. Le Rousseau and L. Robbiano, *Local and global Carleman estimates for parabolic operators with coefficients with jumps at interfaces,* Invent. Math. 183 (2011), no. 2, 245–336.

[30] G. Lebeau and L. Robbiano, *Contrôle exact de l'équation de la chaleur,* Comm. Partial Differential Equations 20 (1995), no. 1–2, 335–356.

[31] J-L. Lions, *Contrôlabilité exacte, perturbations et stabilisation de systèmes distribués,* Recherches en Mathématiques Appliquées, Tomes 1 et 2, Masson, Paris 1988.

[32] A. López and E. Zuazua, *Some new results to the null controllability of the 1-d heat equation*, Séminaire sur les Equations aux dérivees partielles, 1997-1998, Exp. No. VIII, 22p., Ecole Polytech. Palaiseau (1998).

[33] S. Micu and E. Zuazua, *On the regularity of null-controls of the linear 1-d heat equation,* C.R.Acad. Sci. Paris, Ser. I (2011).

[34] A. Münch and P. Pedregal, *Numerical null controllability of the heat equation through a variational approach,* to appear in Inverse Problems.

[35] A. Münch and F. Periago, *Optimal distribution of the internal null control for the 1D heat equation,* J. Differential Equations, 250 95-111 (2011).

[36] A. Münch and E. Zuazua, *Numerical approximation of null controls for the heat equation: ill-posedness and remedies,* Inverse Problems, 26(8) 085018 (2010).

[37] J.-P. Puel, *Global Carleman inequalities for the wave equations and applications to controllability and inverse problems,* Cours Udine Mod1-06-04-542968, Udine, 2011.

[38] D.L. Russell, *A unified boundary controllability theory for hyperbolic and parabolic partial differential equations,* Studies in Appl. Math. 52, 1973, 189–221.

[39] D.L. Russell, *Controllability and stabilizability theory for linear partial differential equations. Recent progress and open questions,* SIAM Rev. 20, 1978, 639–739.

[40] X. Zhang, *Explicit observability inequalities for the wave equation with lower order terms by means of Carleman inequalities,* SIAM J. Control. Optim., 39 (2000) 812-834.

[41] E. Zuazua, *Exact boundary controllability for the semilinear wave equation,* In "Nonlinear Partial Differential Equations and their Applications", Vol. X (H. Brezis & J.-L. Lions eds.), Pitman, New York 1991.

[42] E. Zuazua, *Control and numerical approximation of the wave and heat equations*, International Congress of Mathematicians, Madrid, Spain, Vol. III (2006) 1389–1417.